From thermonuclear fusion to Hamiltonian chaos

D. F. Escande

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D F Escande[1,a]

Aix-Marseille Université, CNRS, PIIM, UMR 7345, case 321, campus Saint-Jérôme, FR-13013 Marseille, France

Abstract. This paper aims at a historical and pedagogical presentation of some important contributions of the research on thermonuclear fusion by magnetic confinement to the study of Hamiltonian chaos. This chaos is defined with the help of Poincaré maps on a simple two-wave Hamiltonian system. A simple criterion for computing the transition to large scale chaos is introduced. A renormalization group approach for barriers in phase space is described pictorially. The geometrical structure underlying chaos is introduced, and then described in the adiabatic limit of Hamiltonian chaos. The issue of chaotic transport is discussed in simple limit cases.

1 Introduction

The fusion of some isotopes of hydrogen releases energy, which offers the prospect of a new energy source. The energetically easiest reaction uses deuterium and tritium, and each fusion reaction yields a neutron and an alpha particle with a release of 17.6 MeV. After the second world war, this motivated several several countries to start research projects aiming at the use of *controlled thermonuclear fusion* as an energy source. A broad international cooperation started on a large part of these projects after their declassification in 1958, those using the confinement of charged particles in magnetic bottles to heat them to the high temperatures required for fusion. At such temperatures, matter is in the so-called plasma state where it is fully ionized. The most successful magnetic bottles are toroidal ones, especially if their magnetic field lines wind upon nested toroidal magnetic surfaces[1]. The shape of these lines turned out to be ruled by low dimensional classical mechanics, forcing plasma physicists to face an issue mathematicians had been studying for half a century: Hamiltonian chaos.

Hamiltonians are ubiquitous in physics, since they are useful both to quantize physical systems and to describe their classical mechanics. Classical mechanics textbooks start with regular dynamics, but many of those published since the eighties provide an introduction to chaotic dynamics too. Some aspects of this dynamics are now popular like the "butterfly effect", i.e. the strong dependence of chaotic dynamics on initial conditions due to the exponential separation in time of initially nearby

[a] e-mail: `dominique.escande@univ-amu.fr`

[1] See http://fusionwiki.ciemat.es/wiki/Flux_surface.

orbits. "*Hamiltonian chaos*" is the chaos of non dissipative systems described by a Hamiltonian. Systems without chaos are an exception... except in textbooks!

Hamiltonian chaos is relevant in various topics of magnetic fusion: magnetic field line topology, dynamics of particles in magnetic fields, turbulent transport, radiofrequency heating, ray dynamics, etc... Therefore, the study of Hamiltonian chaos was naturally stimulated by magnetic fusion[2]. Subsequently, this elicited interest among all plasma physicists, who brought important contributions to its understanding and description, especially for low dimensional dynamics. They benefited from numerical calculations made possible by the development of computers, which enabled them to visualize dynamics, to compute their features, and to back up the development of non rigorous approaches. Their main contributions are the topic of this paper.

It is apparently paradoxical that a community devoted to N-body physics devoted so much efforts to deal with low dimensional classical mechanics. Per se, this topic was not fashionable among most physicists who were attracted toward quantum mechanics, or toward statistical mechanics and kinetic theory, as to classical physics. However, in plasma physics low dimensional dynamics comes naturally because there is often a separation of time scales in the considered issues. Also because it is simpler to start with simplified models where the field-particle self-consistency of plasmas is absent. Sometimes the problem is intrinsically low-dimensional, as stated above for the "dynamics of magnetic field lines".

This paper aims at a historical and pedagogical presentation of some important contributions of the research on thermonuclear fusion by magnetic confinement to the study of Hamiltonian chaos. In order to stay within a reasonable length, this paper has necessarily a restricted and subjective scope. The interested reader can find a hopefully exhaustive account of the contributions of plasma physics to nonlinear dynamics and chaos in the Topical Review [Escande 2016]. This reference is repeatedly quoted in the present review with the short nickname "REV" with the idea that the reader can find in a corresponding section a complete set of references for many topics where only the most recent relevant one is provided here.

Section 2 tells the beginning of the story, section 3 defines Hamiltonian chaos with the help of Poincaré maps on a simple two-wave Hamiltonian system. Section 4 introduces a simple criterion for computing the transition to large scale chaos. Section 5 describes pictorially a renormalization group approach for barriers in phase space. Section 6 is devoted to the description of chaos with maps. Section 7 discusses the issue of chaotic transport in simple limit cases. It also introduces the geometrical structure underlying chaos whose section 8 provides the description in the adiabatic limit of Hamiltonian chaos.

2 How did the story start?

The theory of one of the toroidal magnetic bottles, the stellarator invented in 1950[3], immediately ran into a difficulty because of its magnetic field: it could not be given for granted that its field lines were regular. Therefore magnetic field lines might wander from the center of the plasma to its edge, and hot particles might be rapidly lost to the wall! This forced theoreticians to investigate the nature of magnetic field lines of stellarators. In a torus, this nature can be checked by looking at their successive intersections with the plane of section corresponding to a given toroidal angle: either

[2] [Morrison 2000] makes at length the point that plasma physics re-ignited research in classical dynamics.

[3] In such a bottle, the whole magnetic field is produced by external current-carrying windings. See https://en.wikipedia.org/wiki/Stellarator .

these intersections lie within the trace of a toroidal magnetic surface, a closed line, which means regularity, or they fill a two-dimensional domain, which means chaos.

A massless charged particle streaming freely along a given magnetic field line crosses such a surface almost periodically in time. By analogy, it is natural to consider this line as the orbit of a Hamiltonian system parameterized by a time which is the toroidal angle. Then, studying the nature of magnetic field lines boils down to the study of the nature of orbits in a torus, and their successive intersections build the so-called Poincaré map of their dynamics. Magnetic flux conservation makes such a map area-preserving, implying another relationship between magnetic field lines and Hamiltonian systems. The dynamics ot these lines was fully recognized as Hamiltonian after a long process only (see section II of [Morrison 2000] and section 2.1 of REV). Therefore, investigating the nature of magnetic field lines of stellarators meant investigating typical low dimensional Hamiltonian dynamics. This started the contribution of thermonuclear fusion and plasma physics to the theory of chaos. In order to tell the story, it is necessary to introduce some concepts of Hamiltonian dynamics: nonlinear resonance, trapping island, conflicting resonances, barriers in phase space. Fortunately, this can be done in an intuitive way by appealing to the nonlinear pendulum.

3 Simple model Hamiltonian

At the end of the seventies, Doveil was working on ion acoustic waves in a multipole plasma device where these waves are dispersive. This led him to study numerically the (chaotic) dynamics of ions when several waves are present. For simplicity he focused on the two-wave case. The corresponding Hamiltonian describes the motion of one particle in the presence of two longitudinal waves[4]

$$H(p,q,t) = \frac{p^2}{2} - A\cos q - B\cos k(q-t), \tag{1}$$

where, by an appropriate choice of the units, the mass of the particle is 1, the phase velocity of the second wave is 1, and the wave-number of the first wave is 1; the reference frame is that of the first wave. H is the corresponding particle energy[5]. We take both A and B positive.

3.1 Wave-particle resonance

We first consider the case of a single wave $B = 0$. Then, the equations of motion may be compacted into $\ddot{q} = -A\sin q$, which is the equation of motion of a nonlinear pendulum with length l in a gravity field with acceleration g, when $A = g/l$ (the moment of inertia of the pendulum ml^2 being set equal to 1), q is the angle from the vertical, and $q = 0$ is the position of the stable equilibrium. The orbit may be computed by quadrature from the equations of motion by expressing p as a function $P(q)$ for a given initial condition (q_0, p_0). The dynamics is said to be integrable.

[4] Such waves occur naturally in plasma physics also as Langmuir waves, i. e; vibrations of the electrons with respect to the ions. In solid state physics these waves are quantized as plasmons.

[5] The equations of motion are generated by

$$\dot{q} = \frac{\partial H}{\partial p} = p,\ \dot{p} = -\frac{\partial H}{\partial q} = -A\sin q - kB\sin k(q-t). \tag{2}$$

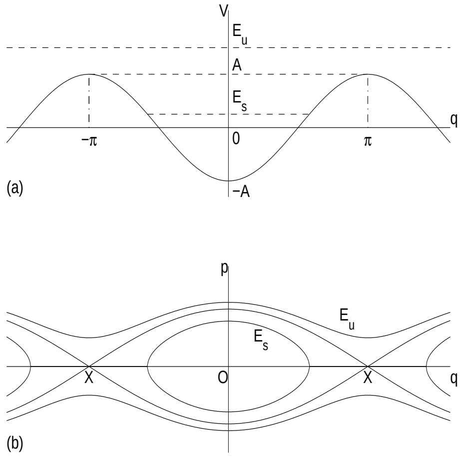

Fig. 1. (a) Potential and (b) phase portrait of the pendulum hamiltonian 1 ($B = 0$). (after Elskens and Escande 2003)

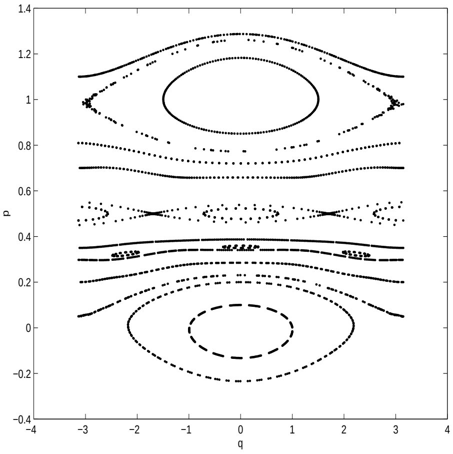

Fig. 2. Poincaré map of the dynamics of hamiltonian 1 for $s = 0.5$, $A = B$, $k = v = 1$. (after Elskens and Escande 2003)

The mechanical potential $-A \cos q$ of equation (1) imposes two kinds of motions to the particle. Either it is passing (energy E_u in figure 1a), and its velocity is only modulated by the potential troughs and hills, or it is trapped inside the potential troughs (energy E_s in figure 1a). Figure 1b displays the phase portrait of this dynamics, which is exactly that of a nonlinear pendulum when two successive maxima of the potential are identified. A separatrix ($H = A$) with the shape of an eye of cat separates trapped orbits, and passing ones with positive and negative velocities. It goes through the X-point corresponding to the unstable equilibrium of the pendulum. The center of the eye-of-cat is an O-point related to the stable equilibrium.

Close to the origin in Fig. 1b, the dynamics is that of a harmonic oscillator whose pulsation, called the bounce frequency, is $\omega_B = A^{1/2}$. At $q = 0$ the separatrix has a half width in velocity $\Delta p = 2A^{1/2}$. Particles inside the resonant domain, trapped particles, are said to be resonant with the wave; indeed, they have a time-averaged velocity which is equal to the wave velocity (0 here)[6]. Δp is the typical width of this resonance process. If H is large, p has only small modulations about $2H^{1/2}$: the particle is almost free. Therefore the separatrix turns out to be the boundary between distorted free particle orbits and distorted harmonic oscillator orbits. The above phase portrait shows that a longitudinal wave has a strong influence only on particles with a velocity close to its phase velocity: *wave-particle interaction is local in velocity.* The wave-particle (or pendulum) resonance[7] turns out to be the paradigm of nonlinear resonances in classical mechanics (Chirikov, 1979; Escande, 1985).

[6] For the nonlinear pendulum, the separatrix separates libration and rotation. In a more general frame of reference, the wave-particle resonance condition for a particle with time-averaged velocity v, and a longitudinal wave with pulsation ω and wave number k is $\omega = kv$. For the second wave of Hamiltonian (1), $v = 1$.

[7] In Hamiltonian mechanics, the word "resonance" is used with a geometrical meaning. The above eye-of-cat is called "island", "resonance island", or "resonance". When two successive maxima of the mechanical potential are not identified, the eye of cat structure repeats periodically in space, and creates what is called an island chain. Then, the word "resonance" is often used to qualify the whole chain. It is also used for brevity to qualify the resonant term of interest in the Hamiltonian, or the set of trapped orbits. The precise meaning in each case is clarified by the context.

The role of the two waves may be exchanged by defining a new position $q' = k(q-t)$, a new time $t' = -kt$, and a new momentum $p' = 1-p$. This yields a Hamiltonian $H'(p', q', t')$, named *equivalent hamiltonian*, which is H with (k, A, B) substituted with $(1/k, B, A)$. Therefore, the case $A = 0$ and $B \neq 0$ is trivially deduced from the previous one.

Figure 1b displays the continuous curves generated by orbits of particles. If one looks at a given orbit with a stroboscope with period $T = 2\pi/k$, the orbit appears as a series of successive dots. Unless the atypical case where the period of the orbit is commensurable with T, these dots progressively fill up the continuous curves of figure 1b in both single-wave cases $A = 0$ and $B = 0$. If the $A = 0$ and $B \neq 0$ case is displayed in the (p, q) coordinates, the corresponding eye-of-cat is centered at $p = 1$, the phase velocity of the second wave. These stroboscopic plots are another instance of the Poincaré map introduced above for magnetic field lines. They become very useful when both waves are simultaneously present.

3.2 Transition to chaos

We now consider the case where both waves have a finite amplitude in Hamiltonian (1). Define the stochasticity parameter

$$s = 2\sqrt{A} + 2\sqrt{B} \tag{3}$$

as the sum of the individual wave trapping widths.

Barriers in phase space If the wave amplitudes are small, the analysis of 3.1 is still expected to be useful, due to the property of locality in p for the influence of one resonance in phase space. Indeed if $s \ll 1$, we may expect the Poincaré map of the system to display separated trapping domains with velocities 0 and 1, with the passing orbits in between being slightly squeezed due to the presence of the eyes-of-cat. For k rational, this naive picture is supported by the Kolmogorov-Arnold-Moser (KAM) theorem [Kolmogorov 1954,Moser 1962,Arnold 1963a] which states that, if the velocity u of a torus[8] is a typical irrational for $s = 0$, then for s small enough this torus persists, and a positive measure of such tori are preserved[9] .

Numerical calculations of orbits enable visualizing the trace of these KAM tori (indeed, since u is irrational, the periods of their orbits are not commensurable with T). We now consider the case where $k = A/B = 1$. Then H is its own equivalent Hamiltonian, and the Poincaré map is symmetrical with respect to $p = 1/2$. In the following we avoid plotting all symmetrical orbits in order to avoid too crowded figures. Figure 2 displays the Poincaré map of the orbits for $s = 0.5$. We recognize KAM tori which are slightly distorted with respect to those corresponding to $s = 0$.

Stochastic layers and conflicting resonances By increasing s to $s = 0.68$, an element of discontinuity shows up at the edge of the trapping regions where a separatrix would be expected to exist. The corresponding numerical Poincaré map (figure 3) displays

[8] "Torus" means "set of orbits with the same time averaged velocity u". This set may be considered as a torus, since such an orbit is characterized by initial position and time, which are equivalent to angles, since the Hamiltonian is periodic in q and t.

[9] In a naive way this theorem states "As you may expect, most integrable orbits are slightly perturbed by a small perturbation of the dynamics". This sounds trivial, but the proof was a breakthrough in mathematics!

orbits whose points do not look like belonging to a curve, but rather seem to fill a layer. This is especially visible close to the X-point. Furthermore, when using a display following the successive points of the orbit, one sees an intriguing feature : sometimes the orbit behaves like a passing one with $|u|$ small, which keeps a definite sign to p and crosses the abscissa of the X-point, and sometimes it behaves like a trapped orbit, which rotates about the O-point and crosses the ordinate of the X-point ; it switches apparently unpredictably from one behaviour to the other : the orbit is chaotic. Another orbit started close enough to the first one displays the same behaviour. It first diverges exponentially from the first orbit, when two regular orbits would diverge linearly from each other ; at some close encounter with the X-point the two orbits separate in a more radical way : one adopts the passing behaviour and the other one the trapped behaviour. Together they provide a more precise definition of a layer where orbits are typically chaotic (or stochastic). For this reason this layer is called a stochastic layer (for some time "stochasticity" was a challenger to "chaos"). A close inspection of other chains of islands shows that their apparent separatrix is in fact also a thin stochastic layer.

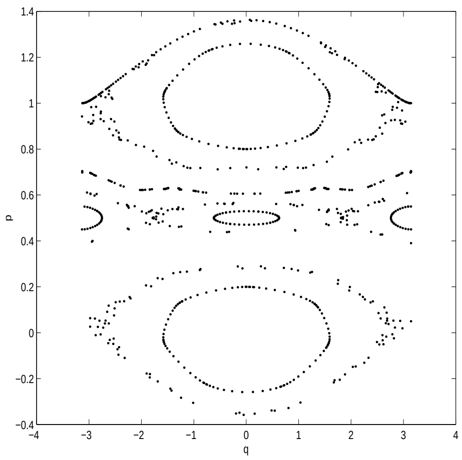

Fig. 3. Poincaré map of the dynamics of hamiltonian 1 for $s = 0.68$, $A = B$, $k = v = 1$. (after Elskens and Escande 2003)

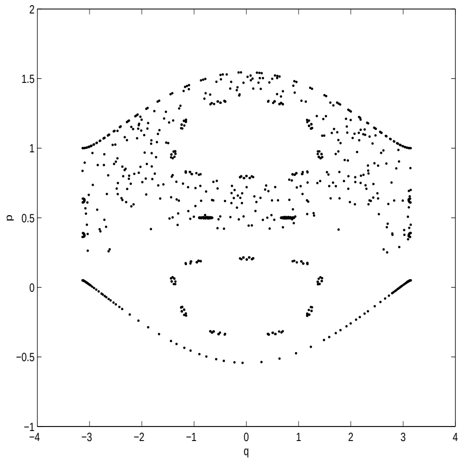

Fig. 4. Poincaré map of the dynamics of hamiltonian 1 for $s = 1$, $A = B$, $k = v = 1$. (after Elskens and Escande 2003)

A stochastic layer is bounded by KAM tori : one inside the trapping domain, and one in each of the two passing domains.

Let us define $H_2(p, w; q, \tau) = w + H(p, q, \tau)$, a two-degree of freedom time-independent Hamiltonian. Its equations of motion show that τ is nothing but t, and that p and q obey the dynamics of H. If the latter is integrable, so is the dynamics of H_2. Since H_2 has two degrees of freedom, its integrability means the existence of a second constant of motion on top of the energy H_2. Each constant of the motion defines a 3-dimensional manifold in the 4-dimensional phase space. This constrains any orbit to lie on a two-dimensional manifold, the intersection of two 3-dimensional manifolds. Consequently, their trace in the Poincaré map is one-dimensional. Therefore, the existence of two-dimensional stochastic layers is really incompatible with integrability.

When $s = 1$ (figure 4), a chaotic orbit connects the neighbourhoods of $p = 0$ and of $p = 1$. Therefore, the passing KAM tori between the two resonances are no longer present. This is a dramatic change with respect to the previous cases, and one says that large scale chaos is present. When following the successive points of the orbit, it looks like being trapped sometimes in wave 1 and sometimes in wave 2, switching from one behaviour to the other unpredictably. These intermittent trappings sound for the particle like being in resonance with the two waves, and for two waves (resonances) like being in conflict about the particle control. It turns out that passing KAM tori between the two resonances figures 2 and 3 were *barriers in phase space* opposing to transport. The chain of trapped islands in the lower eye-of-cat of figure 4 is the analog for the trapped pendulum orbits of passing chains of islands.

4 Resonance-overlap criterion

This structural change in the dynamics when going from $s = 0.68$ to $s = 1$, substantiates the basic intuition underlying Chirikov's resonance overlap criterion [Chirikov 1959] which rests on the simple idea that large scale chaos should occur when an orbit may be trapped in both waves, if considered separately. This corresponds to the overlap of the two unperturbed separatrices, i.e. to $s = 1$. In 1959, the criterion was derived to predict when particles escape from a magnetic mirror trap due to the interaction of resonances between the Larmor rotation of charged particles and their slow oscillations along the lines of force. For a general Hamiltonian system, Chirikov defines the resonance overlap parameter s as the sum of the half-widths of the two resonances divided by the difference in their phase velocities, the exact generalization of definition (3). The resonance overlap criterion is intuitive and easy to implement. However it must be used with caution : it yields a correct order of magnitude estimate for both k and A/B of order 1, but it becomes wrong when either k or A/B go to zero or infinity (see figure 10 of [Escande 1981b] or figure 2.19 of [Escande 1985]). In particular, it is obvious that no chaos occurs when $AB = 0$, since the dynamics is integrable.

This simple to implement criterion became rapidly famous among physicists[10], and especially after Chirikov's review paper[11] [Chirikov 1979]. Rechester and Stix when dealing with magnetic chaos due to weak asymmetry in a tokamak, used this criterion to compute the width of narrow chaotic ("stochastic") layers next to the separatrix of an integrable system when it is perturbed[12] [Rechester 1976].

In reality, this criterion is useful for systems with many degrees of freedom too. Indeed, it can be directly applied to determine the energy border for strong chaos in the Fermi-Pasta-Ulam system when only a few long wave modes are initially excited [Chirikov 1966,Chirikov 1973]. In [Escande 1994] one computes the Gibbsian probability distribution of the overlap parameter s corresponding to two nearby resonances of the Hamiltonian of a chain of rotators. Requiring the support of this distribution to be above the threshold of large scale chaos, gives the right threshold in energy above which the Gibbsian estimate of the specific heat at constant volume agrees with the time average of the estimate given by the fluctuations of the kinetic energy: one has a self-consistent check of the validity of Gibbs calculus using the observable s!

[10] An indication of the importance of this criterion is obtained when typing "resonance overlap criterion" in Google Scholar: 209,000 references are obtained; [Chirikov 1979] is quoted 4,200 times according to the Web of Science. Though not quoting Chirikov's seminal work, reference [Rosenbluth 1966] contributed to publicize the concept of resonance overlap too.

[11] This paper brings also a wealth of information about Hamiltonian chaos which is very useful for physicists of the eighties to get acquainted with chaos theory ("stochasticity theory" at that time). This was the case for the author of the present paper.

[12] Here again, Chrikov's authorship of the resonance overlap criterion is overlooked. Rechester and Stix' estimates are improved in [Escande 1982a].

5 Renormalization approach

As we saw in figures 2 and 3, the Poincaré map corresponding to the motion of one particle in two longitudinal waves displays resonance islands. They are related to higher order nonlinear resonances which become explicit by canonical transformations. Following the philosophy of Chirikov's review paper [Chirikov 1979], it is then tempting to apply the resonance overlap criterion to two neighboring such resonances. The criterion is easier to apply if the corresponding Hamiltonian is approximated by a simpler one. *Surprisingly*, the simplification leads to a Hamiltonian describing the motion of one particle in two longitudinal waves! The passage from the initial two-wave Hamiltonian to the transformed one corresponds to the transform of Kadanoff's block-spin *renormalization group* (see figure 5)... where Chirikov's criterion is absent [Escande 1981a,Escande 1981b]! When the renormalization mapping is iterated, either the amplitudes of the two waves go to infinity or to zero (figure 6). This discrete dynamics is ruled by a hyperbolic fixed point whose stable manifold[13] separates the two asymptotic behaviours. For (A, B) below the stable manifold of the X-point, the system eventually lands in the domain where KAM theorem applies. Then the KAM torus of interest is preserved at small scale, and consequently at all scales.

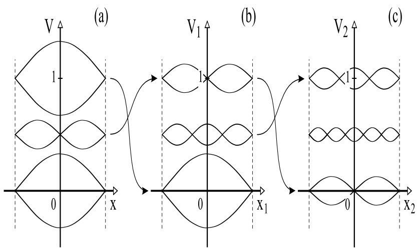

Fig. 5. Renormalization scheme. The finite resolution of the numerical microscope exhibits only three chains of islands at each successive step of the renormalization process. (v, x) corresponds to (p, q) for this process. (after Escande 2010)

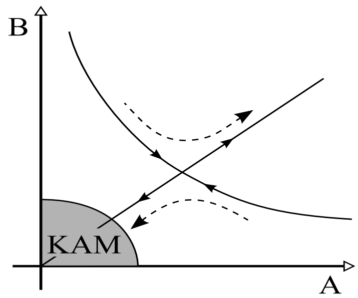

Fig. 6. Mapping of the two resonance amplitudes by the renormalization transform. (after Escande 2010)

The practical implementation of the renormalization technique for more general Hamiltonians is then described in [Escande 1984] and in sections 3.1 and 4.1 of the review paper [Escande 1985]. Appendix B of the latter reference shows how to derive a renormalization for any KAM torus trapped into a resonance island. A one parameter renormalization scheme is derived for "stochastic layers" in [Escande 1982a]. All these schemes are approximate ones in a physicist sense: the approximations are not mathematically controlled.

Later on, several mathematical works tried and coped with this shortcoming. A way to make the 1981 renormalization scheme rigorous is indicated in [MacKay 1995]. The ideas proposed originally in [Escande 1981a,Escande 1981b] lead to approximate renormalization transformations showing the universality of the mechanism of break-up of invariant tori, and enabling a very precise determination of the corresponding threshold for Hamiltonian systems with two degrees of freedom [Chandre 2002]. In 2004, Koch brought this type of approach to a complete rigorous proof [Koch 2004]:

[13] The set of initial conditions attracted toward the X-point by the renormalization map.

it is a computer assisted proof of the existence of a fixed point with non-trivial scaling for the break-up of golden mean KAM tori: this fixed point is the rigorous version of the hyperbolic point in figure 6[14].

For Hamiltonians with zero or one primary resonance, one cannot apply the resonance overlap criterion or the above renormalization approach. Reference [Codaccioni 1982] shows how to compute the threshold of large scale chaos by using the blow-up of the width of chaotic layers[15] as computed in [Rechester 1976].

6 Working with maps

6.1 Standard map

Having a Hamiltonian description of magnetic field lines is nice, but when coming to the numerical calculation of Poincaré maps, the integration of orbits from differential equations is a formidable task for the computers of the sixties! This motivated physicists to derive explicit area preserving maps corresponding to a full step of the Poincaré map. This started in 1952 with Kruskal[16] who introduced and iterated area preserving maps for stellarator magnetic field lines [Kruskal 1952]. The paradigm of area preserving maps is the standard map[17] acting in the (J, q) plane

$$q_{n+1} - q_n = J_n, \;\; J_{n+1} - J_n = K_{\rm std} \sin q_{n+1}, \tag{4}$$

where n is an index numbering the steps in the iteration of the map, and $K_{\rm std}$ is a parameter. This map appeared first in 1960 in the context of electron dynamics in the microtron[18] [Kolomenskii 1960], a type of particle accelerator concept originating from the cyclotron in which the accelerating field is not applied through large D-shaped electrodes, but through a linear accelerator structure. This map was independently proposed and numerically studied in a magnetic fusion context by Taylor in 1968[19], and by Chirikov[20] in 1969 (also for particle accelerators) [Chirikov 1969].

[14] While KAM theorem sounded as a trivial statement, renormalization brings two important practical, but non trivial, results: the threshold of breakup of a given KAM torus and critical exponents related to this breakup [MacKay 1984a,Escande 1985]

[15] One of the considered cases is the polynomial Hénon-Heiles Hamiltonian [Hénon 1964]. However, the technique of [Codaccioni 1982] is unable to detect integrability. Indeed, it predicts a blow-up of the width of a chaotic layer also for the integrable Hamiltonian obtained from Hénon-Heiles' one by changing a sign in its formula!

[16] Kruskal is quoted several time in this paper. Indeed, he made essential contributions to nonlinear dynamics and chaos. He was also quite influential. In particular in astrophysics, as can be seen in the acknowledgements of the Hénon-Heiles paper where he is thanked [Hénon 1964]. Indeed, this famous work was performed while Hénon was in Princeton. It describes the non-linear motion of a star around a galactic center where the motion is restricted to a plane, and uncovers this motion can be chaotic. Arnold was acutely aware of the power of the techniques used by Hénon and actually asked him to exhibit numerically the chaos of force-free magnetic field lines (see [Hénon 1966]).

[17] Also called Chirikov-Taylor map.

[18] According to reference 2 of [Melekhin 1975], it appeared ten years earlier in Kolomenskii's PhD thesis at the Lebedev Intritute.

[19] The map was not published, but is in the 1968-9 Culham Progress report, and is quoted in [Froeschlé 1970]. See Taylor's account of the story of his discovery in section 2.1 of REV.

[20] Chirikov is quoted repeatedly in this paper for contributions in many different problems of nonlinear dynamics and chaos. A summary of his main contributions can be found in [Bellissard 1999], published in a special issue of Physica D in his honor.

We now introduce a Hamiltonian germane to Hamiltonian (1)

$$H(p,q,t)=\frac{p^2}{2}+A\sum_{m=-M}^{M}\cos(q-mt+\varphi_m) \tag{5}$$

where M is a positive integer and the φ_m's are fixed phases. It describes the one-dimensional motion of a particle in a set of longitudinal waves having equally spaced phase velocities, and the same wave-number and amplitude. The standard map may be viewed as the Poincaré map of Hamiltonian (5) when all φ_m's vanish and M is infinite. Indeed, the infinite sum in the corresponding equations of motion define the periodic Dirac function and yield a periodically impulsive force on the particle. Integrating them yields the standard map with $J = 2\pi p$ and $K_{\mathrm{std}} = 4\pi^2 A$. Therefore, the corresponding Poincaré map looks very similar to a periodization in p of the maps obtained for Hamiltonian (1) and $k = A/B = 1$. However the calculation is much faster, since the stroboscopic/Poincaré map is explicit.

6.2 Greene residue criterion

In the seventies, Greene was interested in the nature of magnetic field lines in stellarators, and in their corresponding return area-preserving map. He naturally focussed on the simplest example of such maps, the standard map, which was easy enough to iterate on computers of that time. In the same way as it is natural to focus on higher order nonlinear resonances in a Hamiltonian description, it is natural to focus on periodic orbits with a long period in area-preserving maps. These periodic orbits correspond to O-points and X-points of resonance islands in figures 2 and 3. At the end of the 70's, while studying the stable periodic orbits approximating a given KAM torus when troncating the continuous fraction expansion of its winding number at high order, Greene noted they become unstable when the KAM torus breaks up; this instability corresponds to the change of value of a quantity characterizing the orbit, called "residue". This led him to his famous "residue criterion" which provides a method for calculating, to very high accuracy, the parameter value for the destruction of the last torus [Greene 1979] (see more information in section 2.2.2 of REV). Defining the threshold of large scale chaos in a given domain of phase space means finding the threshold of break-up of the most robust KAM torus. The continued fraction expansion of their winding number was found numerically to have a special form exhibited in [Greene 1986].

Greene's work was placed in a renormalization group setting by MacKay, then his student [MacKay 1983]. This work is closely related to the approximate renormalization described above; in particular, the renormalization transform exhibits a hyperbolic structure like in figure 6, with a one-dimensional unstable manifold, but an infinite-dimensional stable manifold. This triggered a dialog between the latter renormalization and the rigorous one under the auspices of Greene's criterion during almost a decade (see section 2.2.2 of REV). The existence of a fixed point with a non-trivial scaling for MacKay's renormalization was finally rigorously proved in 2010 [Arioli 2010].

6.3 Further results

Reference [MacKay 1989] derives a simple criterion for non-existence of invariant tori. When applied to Hamiltonian (1), it gives results in close agreement with those of Greene's residue method.

Till now we considered maps such that the period of the motion on a torus is a monotonic function of the action of the torus: they are twist maps. We now deal with *non-twist systems* where the period goes through an extremum on a given torus; in such systems, the overlap criterion fails and KAM theorem cannot be applied. They were introduced in 1984 by Howard, motivated by multifrequency electron-cyclotron-resonance heating in plasmas [Howard 1984]. He derived an accurate analytic reconnection threshold of the approximate separatrices of the pairs of islands corresponding to actions symmetrical with respect to that of the extremum period. Motivated by the study of magnetic chaos in systems with reversed shear configurations, del-Castillo-Negrete and Morrison proposed a prototype map called the standard non-twist map, and published a detailed renormalization group study of the non-twist transition to chaos (see [del-Castillo-Negrete 1997] and references therein) : there is a new universality class in this transition. This stimulated a series of rigorous mathematical results: in [Delshams 2000], the proof of persistence of critical circles and a partial justification of Greene's criterion as generalized in del-Castillo-Negrete and Morrison work. In [Gonzalez-Enriquez 2014], the bifurcations of KAM tori are studied by using the classification of critical points of a potential as provided by Singularity Theory. This approach is applicable to both the close-to-integrable case and the far-from-integrable case where a bifurcation of invariant tori has been detected numerically.

In view of the many techniques, which can be used for area preserving maps, it is interesting to construct a *finite time mapping* corresponding to a given Hamiltonian flow. In the nineties, motivated by various plasma physics issues, Abdullaev developed to this end his mathematically rigorous "mapping method" based on Hamilton-Jacobi theory and classical perturbation theory, which works for Hamiltonians that are the sum of an integrable part and of a small perturbation (see [Abdullaev 2002] and references therein, and section 2.2.3 of REV). This method can be used to perform symplectic integration, with an accuracy controlled by the product of the perturbation parameter and of the mapping time step [Abdullaev 2002]. Also to derive the canonical separatrix mapping describing the dynamics near a separatrix [Abdullaev 2004b], while keeping the canonical variables of the corresponding Hamiltonian, an improvement with respect to Chirikov's separatrix mapping [Chirikov 1979]. The new mapping is consistent with the rescaling invariance described in section 7.3.

7 Chaotic transport

7.1 Quasilinear diffusion

We now consider the many-wave ($M \gg 1$)dynamics defined by Hamiltonian (5). For an orbit being at (p_0, q_0) at $t = 0$ with $-1 < p_0 < 1$, first order perturbation expansion in the wave amplitudes of the equation of motion yields $\Delta p(t) \equiv p(t) - p_0 = \sum_{m=-M}^{M} \frac{A}{\Omega_m}[\cos(q_0 + \varphi_m) - \cos(\Omega_m t + q_0 + \varphi_m)]$, with $\Omega_m = p_0 - m$; if $\Omega_0 = 0$, one replaces the corresponding term by its limit for $\Omega_0 \to 0$. At this order, averaging over the wave phases yields $\langle \Delta p(t)^2 \rangle = \sum_{m=-M}^{M} \left(\frac{A}{\Omega_m}\right)^2 [1 - \cos(\Omega_m t)]$. For $t \ll M^{-1}$, Δp grows linearly with time, and $\langle \Delta p^2 \rangle$ grows quadratically, as all waves act with a constant force on the orbit. We now assume $M^{-1} \ll t \ll 1$. Because of the first inequality, the sum in $\langle \Delta p(t)^2 \rangle$ may be turned into an integral, and because of the second one the bounds for integration may be considered as infinite, which yields $\langle \Delta p(t)^2 \rangle \simeq \frac{2D_{\mathrm{QL}}}{\pi} \int_{-\infty}^{\infty} \frac{1-\cos(\Omega t)}{\Omega^2} \mathrm{d}\Omega \simeq 2\ D_{\mathrm{QL}}\ t$, where $D_{\mathrm{QL}} = \frac{\pi A^2}{2}$, is the *quasilinear diffusion coefficient*. As a result $\langle \Delta p(t)^2 \rangle$ has a diffusion-like behaviour, and the diffusion coefficient takes on the quasilinear value. The adjective *quasilinear* to characterize this diffusion coefficient is related to the fact that $\Delta p(t)$ is computed by a linear approximation in the calculation. A similar calculation can be made for $\Delta q(t) = q(t) - p_0$, and shows that $\langle \Delta q(t)^2 \rangle \simeq \frac{2}{3} D_{\rm QL} t^3$. For the perturbative calculation to make sense, the range of t is restricted by the condition for the orbit to remain close to the unperturbed one. This translates into condition $\langle \Delta q^2(t) \rangle \ll 4\pi^2$, namely $t \ll 4\tau_{\rm spread}$ where $\tau_{\rm spread} = (D_{\rm QL})^{-1/3}$. So, for M large enough, over a time smaller than both 1 and $\tau_{\rm spread}$, the quasilinear estimate is correct, even for orbits which turn out to be eventually chaotic.

As an Ansatz If the correlation time of the force acting on the particle is small, it is natural to make a quasilinear estimate of transport with the idea that over each short correlation time, a perturbative calculation of the orbit is correct. Such an estimate was made popular in 1962 by two papers on the bump-on-tail instability[21] published in two successive issues of Nuclear Fusion [Vedenov 1962,Drummond 1962] (see also section 2.3.1 of REV).

In 1966, Rosenbluth, Sagdeev, and Taylor were interested in transport when magnetic field surfaces are destroyed and magnetic field lines are chaotic in a magnetic bottle. They stated that if there is resonance overlap, "then a Brownian motion of flux lines and rapid destruction of surfaces results" and made a quasilinear estimate of transport [Rosenbluth 1966]. Then, quasilinear estimates were made systematically for chaotic transport for almost three decades without questioning its validity, except for the standard map, which exhibits a diffusive behavior with a diffusion constant oscillating as a function of the control parameter of the map about the quasilinear value (see [Meiss 1983] and references therein, and section 2.3.1 of REV). However, in 1998 the diffusion properties of the standard map were shown to be nonuniversal in the framework of the wave-particle interaction, because this map corresponds to a spectrum of waves whose initial phases are all correlated [Bénisti 1998b].

Quasilinear or not? For a chaotic motion, the perturbative approach used in the original derivation of the quasilinear equations cannot be justified. Therefore, the quasilinear description might not be correct to describe the saturation of the bump-on-tail instability. After showing its inconsistency, in 1984 Laval and Pesme proposed a new Ansatz to substitute the quasilinear one, and predicted that the velocity diffusion coefficient should be renormalized by a factor 2.2 during the saturation of the instability (see [Laval 1984] and references therein, and section 2.3.1 of REV). This motivated Tsunoda, Doveil, and Malmberg (TDM) to perform an experiment with an electron beam in a traveling wave tube[22], in order to avoid the noise present in

[21] This instability is the one-dimensional beam-plasma instability with a beam density much lower than the plasma one. This system is a paradigm for wave-particle interactions in plasma turbulence. The beam free energy is converted into plasma waves triggered by linear instabilities. These unstable waves have phase velocities close to the beam particle velocities and are thus resonant with the beam. The waves experience a strong nonlinear behaviour and beam particles are submitted to chaotic motion.

[22] In the bump-on-tail instability, while waves experience a strong nonlinear behaviour and beam particles are submitted to chaotic motion, the plasma bulk can be treated linearly as a dielectric that mainly supports the unstable waves. This situation is the same as in a Traveling Wave Tube (TWT), where an electron beam interacts with waves propagating along a so-called slow wave structure (most often a helix with a pitch much smaller than its circumference) with phase velocities close to the beam electrons velocities. Along the axis of the TWT, where the beam is launched, the waves are essentially electrostatic with an electric field parallel to their wave-vector as in the 1D beam-plasma system. Therefore, the physics in the TWT is the same as in a beam-plasma system, but with the considerable advantage that

beam-plasma systems (see [Tsunoda 1991] and references therein, and section 2.3.1 of REV). This experiment came with a surprising result: quasilinear predictions looked right, while quasilinear assumptions were proved to be completely wrong. Indeed no renormalization was measured, though mode-mode coupling was not negligible at all[23]. This set the issue: is there a rigorous way to justify quasilinear estimates for chaotic dynamics?

This issue was first tackled by the author of this paper by considering the self-consistent motion of a finite number of waves and particles corresponding to the beam-plasma problem (see section 2.3.1 and 4.1.1 of REV). However, the motion of a particle in a prescribed spectrum of waves was mysterious too and deserved a thorough study. For uncorrelated phases, it was natural to expect the diffusion coefficient to converge to its quasilinear estimate from below when the resonance overlap of the waves increases. In 1990, numerical simulations of the motion of one particle in a spectrum of waves, in particular the case of Hamiltonian (5), performed by Verga came with a *surprising result*: for intermediate overlaps, the diffusion coefficient exceeds its quasilinear value by a factor about 2.5 [Cary 1990].

Figure 8(a) displays a cartoon of the variance $\langle \Delta v^2 \rangle$ of the velocities of particles all released with the same initial velocity in a prescribed spectrum of Langmuir waves (here $v = p$). The initial quasilinear diffusion corresponds to the green segment on the left, and the red segment on the right to the saturation occurring for a small value of the overlap parameter s of nearby resonances. When s is large enough for chaos to become dominant, numerical calculations revealed [Cary 1990] that after a time about $\tau_{\rm spread}$, $\langle \Delta v^2(t) \rangle$ grows with a slope in between the quasilinear one and 2.3 times this value[24] (in the range bounded by the brown and blue curves in figure 8(a)).

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noise is greatly reduced, and a much better control of waves is possible. Contrary to industrial TWTs, commonly used in telecommunication to amplify radio-frequency waves since Second World War, in the TDM experiment the beam is warm (with a large velocity dispersion controlled by passing the beam through a system of three grids with appropriate biasing). Then a wide spectrum of waves are excited: all the waves with a phase velocity falling in the velocity range associated with a positive slope in the electron beam velocity distribution are unstable. Moreover an arbitrary waveform generator (AWG) allows externally controlling the frequency, the amplitude, and the phase of each launched wave. Common AWGs work at frequencies around 50 MHz, which sets the somewhat awkward dimensions of the tube: 2.7 m long helix with 2.1 cm diameter, and 1 mm pitch.

[23] For averaged quantities, the experiment validated the predictions of quasilinear theory, which neglects any mode-coupling effect. Each unstable wave grew with an averaged growth-rate given by linear theory. Saturation occured through chaotic diffusion of the beam electrons inside the overlapping resonant trapping domains of all unstable waves, leading to the observed plateau formation for the averaged beam distribution function at the output of the TWT. However, when no averaging was made, the experiment displayed strong mode coupling effects: two modes, with neighbouring frequencies (and therefore almost same phase velocity) had completely different axial behaviours that resulted from interference (constructive or destructive) between the linearly unstable mode and the contribution induced at the same frequency by the coupling of all other modes. Therefore, mode coupling could not be neglected, in contrast with quasilinear assumptions, but quasilinear predictions were right.

[24] The necessity to go beyond $\tau_{\rm spread}$ to see the chaotic diffusion is a caveat for the numerical measurement of a chaotic diffusion coefficient. This minimum time comes from the locality in velocity of wave-particle interaction [Bénisti 1997,Elskens 2003]. Indeed it can be shown that at a given moment the waves making particle dynamics chaotic have a phase velocity within $\Delta v \sim 1/\tau_{\rm spread} = (D_{\rm QL})^{1/3}$ from the particle velocity [Bénisti 1998a]. Those out of this range act perturbatively. If waves have random phases, after visiting several "resonance boxes" of width Δv, a particle feels as having been acted upon by a series of independent chaotic dynamics, which triggers a diffusive behavior. This decorrelation makes it possible

Figure 8(b) summarizes in a sketchy way the various regimes as to the value of the diffusion coefficient D measured over many τ_{spread}'s. For small values of s, this "time asymptotic" value vanishes because of the saturation of $\langle \Delta v^2 \rangle$ after time 1, related to the discreteness of the wave spectrum. This corresponds to the horizontal red segment. When s grows above the chaotic threshold, D takes on positive values, but first below the quasilinear one (red growing curve in figure 8(b)). For intermediate values of s, D takes on superquasilinear values (blue curve in figure 8(b)). For large values of s, D takes on the quasilinear value (brown curve in figure 8(b)).

Diffusion or not? This result triggered a series of works aiming at understanding whether the diffusion picture makes sense and when the quasilinear estimate is correct. In 1997, Bénisti showed numerically that the diffusion picture is right, provided adequate averages are performed on the dynamics [Bénisti 1997][25]; however, this picture is wrong if one averages only over the initial positions of particles with the same initial velocity: chaotic does not mean stochastic! This motivated Elskens to look for mathematical proofs using probabilistic techniques, and led him to two results: individual diffusion and particle decorrelation are proved for the dynamics of a particle in a set of waves with the same wavenumber and integer frequencies if their electric field is gaussian [Elskens 2010], and if their phases have enough randomness [Elskens 2012]. The intuitive reason for the validity of the diffusive picture is given in [Bénisti 1997]: it is due to the locality in velocity of the wave-particle interaction[26], which makes the particle to be acted upon by a series of uncorrelated dynamics when experiencing large scale chaos. This locality of the wave-particle interaction was rigorously proved in [Bénisti 1998a]. On taking into account that the effect of two phases on the dynamics is felt only after a long time when there is strong resonance overlap, it can be approximately proved that the diffusion coefficient is larger than quasilinear, but converges to this value when the resonance overlap goes to infinity [Bénisti 1997,Escande 2002b].

For the advection of particles in drift waves or in 2-dimensional fluid turbulence, care must be exerted when trying to define a corresponding diffusive transport. Then one must define the Kubo number K which is the ratio of the correlation time of the stochastic potential as seen by the moving object to the (nonlinear) time where the dynamics is strongly perturbed by this potential (trapping time, chaos separation time, Lyapunov time, ...) [Ottaviani 1992,Vlad 2004]. At a given time, the potential has troughs and peaks. If the potential is frozen, particles bounce in these troughs and peaks. When $K \ll 1$, the particles typically run only along a small arc of the trapped orbits of the instantaneous potential during a correlation time (see figure 7b). During the next correlation time they perform a similar motion in a potential completely uncorrelated with the previous one. These uncorrelated random steps yield

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to numerically measure the diffusion coefficient by following the dynamics either of a single particle for a series of random outcomes of the wave phases, or of many particles for a single typical outcome of the phases.

[25] See also section 6.2 of [Elskens 2003], and [Escande 2007,Escande 2008].

[26] The idea of locality was already present in the resonance broadening concept introduced in 1966 by Dupree [Dupree 1966]. He improved upon the too simple idea of the original quasilinear theory that the particles are subjected only to the field of the wave with which they are resonant. Dupree assumed that instead of the perturbed ballistic motion of quasilinear theory, the particle motion is diffusive since $t = 0$. He found that the waves to be included in order to compute the diffusion coefficient are in a range $\Delta v \sim 1/\tau_{\text{spread}} = (D_{\text{QL}})^{1/3}$ about the particle velocity. In [Bénisti 1998a], locality is found without assuming diffusion and exists even before the asymptotic diffusion regime sets in for intermediate values of A, but the scaling of D may differ from A^2 for some wave spectra.

a 2 dimensional Brownian motion with a diffusion coefficient which can be computed with a quasilinear estimate.

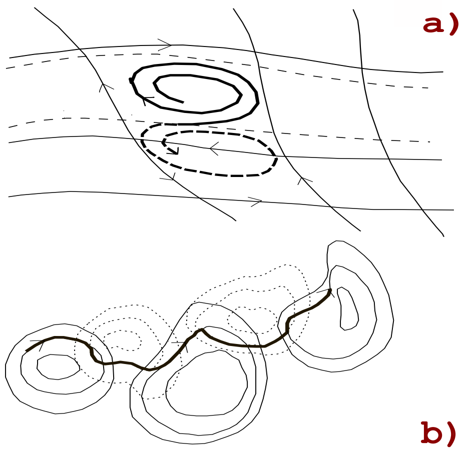

Fig. 7. (a) $K \gg 1$: as the net of intersecting separatrices evolves in time (from solid to dashed curves), the lower domain enlarges at the expense of the upper one. Hence, a trajectory may go from the upper to the lower one (thick to dashed curve). (b) $K \ll 1$: jumps among small trapping arcs in a quickly evolving potential topography. Solid and dashed contours stand for potential hills and wells. (after Escande and Sattin 2007)

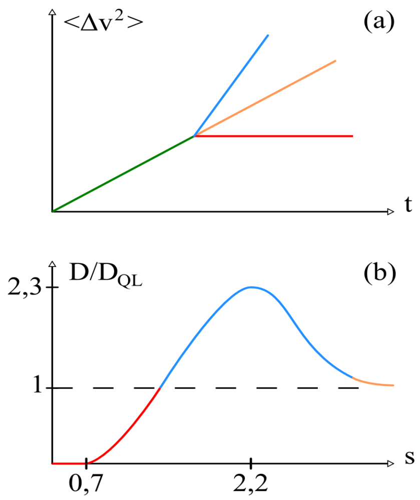

Fig. 8. Regimes of diffusion. (a) $\langle \Delta v^2 \rangle$ vs. time; initial quasilinear regime: green line; asymptotic saturation: red line; superquasilinear regime: blue line; time-asymptotic quasilinear regime: brown line. (b) D/D_{QL} vs. s; same color code as in (a), except for the red growing segment that corresponds to the weakly chaotic regime. (after Escande 2013)

7.2 Diffusion with trajectory trapping

If $K \gg 1$, a quasi-adiabatic picture works: the particles make a lot of bounces before the potential changes its topography (see figure 7a). The change of topography forces particles to jump to a nearby trough or peak. The successive jumps produce a random walk whose order of magnitude of the corresponding diffusion coefficient can be easily computed [Ottaviani 1992,Vlad 2004,Escande 2007,Escande 2008]. In a series of works, Vlad and coworkers clarified the issue of diffusion with trajectory trapping. The just described simple picture is almost correct for a Gaussian spatial correlation function of the potential [Vlad 2004]. More generally, the frozen potential displays as well "roads" crossing the whole chaotic domain. This enables long flights in the dynamics that bring some dependence upon K in the estimate for the diffusion coefficient. The correct calculation of this coefficient is a much harder task. To this end, one may group together the trajectories with a high degree of similarity, and one starts the averaging procedure over these groups. This yields the decorrelation trajectory method [Vlad 1998] and the nested subensemble approach [Vlad 2004]. These techniques are extensively used for the study of the transport in magnetically confined plasmas, for the study of astrophysical plasmas, and of fluids (see [Vlad 2015] and references therein, and section 2.3.2 of REV). Reference [Vlad 1998] computes numerically the diffusion coefficient of particles in a spectum of waves scaling like k^{-3}, and for Kubo numbers up to 210^5. For $1 < K < 10^4$, the data fit very well the scaling $K^{0.64}$ provided by the decorrelation trajectory method (see section 2.3.2 of REV for more details).

7.3 Further results

In reality, when the diffusive picture is correct, there is a pinch or dynamic friction part on top of the diffusive part, and the correct model is the Fokker-Planck equation [Escande 2007,Escande 2008]. For the advection of particles in drift waves or in 2-dimensional turbulence, the sign of this pinch part depends on K [Vlad 2006].

Consider a one–dimensional Hamiltonian which is the sum of an integrable part displaying a hyperbolic fixed point X and of a time–periodic perturbation with amplitude ϵ. Its phase–space near X turns out to be invariant with respect to a rescaling of the conjugate coordinates along the eigenvectors of X, of ϵ, and of the phase of the perturbation. In the middle of the nineties, Abdullaev and Zaslavsky showed it numerically [Zaslavsky 1995], and proved it rigorously [Abdullaev 1995], which implies a periodicity of the statistical properties of chaotic transport in narrow stochastic layers ([Abdullaev 2006] and references therein).

7.4 Transport through cantori

In the eighties it became clear among plasma physicists that chaotic transport is intrinsically more intricate than a diffusion, especially if one considers a single realization of the physical system of interest. In particular, it may be strongly inhomogeneous in phase space due to localized objects restricting it: the cantori described now.

A prerequisite for this description is the definition of *homoclinic intersections*. When $B \neq 0$ in equation (1), the pendulum separatrices no longer exist. As sketched in figure 9, the stable and unstable manifolds of the X-points do not coincide, but intersect an infinite number of times, making what are called *homoclinic intersections*. The closed domain defined by the arcs of the two manifolds joining nearby intersections is called a *homoclinic lobe*. Figure 9 displays the homoclinic intersections for a slightly perturbed nonlinear pendulum. The arcs of the stable and unstable manifolds up to H_0 are close to the unperturbed separatrix. So, it is natural to define the boundary of the trapping domain by these two arcs. We now consider the lobe L corresponding to the nearby homoclinic points H_0 and H_1. Due to conservation of orientation in Hamiltoinan mechanics, the (pre)images of this lobe correspond to homoclinic points H_{2n} and H_{2n+1}. Therefore the close enough antecedents of L are certainly in the trapped domain, but its successor and its close followers are not. Therefore, the lobe area tells the amount of orbits which transit from trapped to untrapped during a Poincaré time, and vice-versa for the family of lobes related to points H_{2n+1} and H_{2n+2}.

When a KAM torus breaks up, it becomes a Cantor set called a cantorus [Aubry 1978,Percival 1980]. In 1984, MacKay, Meiss, and Percival showed that a cantorus is a leaky barrier for chaotic orbits, and that the flux through the cantorus between two successive iterates of the Poincaré map can be computed as the area of a turnstile built in a way similar to homoclinic lobes for X-points. [MacKay 1984a]. The above-described renormalization theories for KAM tori provide a critical exponent for this area, as a consequence of renormalization dynamics close to its X-point [MacKay 1984a]. The area of a turnstile can be obtained from the actions of homoclinic orbits [MacKay 1987]. A new description of transport in a chaotic domain can be obtained through Markov models combining the fluxes through the discrete set of the most important noble cantori (see [Meiss 2015] for an exhaustive set of references, and section 2.3.5 of REV).

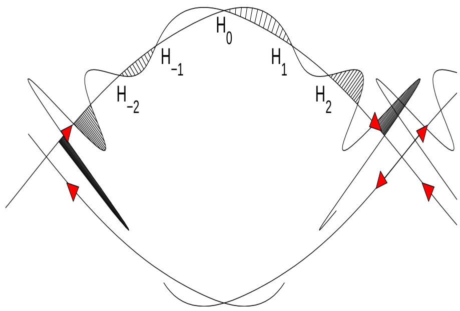

Fig. 9. Homoclinic intersections for a perturbed nonlinear pendulum. A homoclinic point H_n is mapped into H_{n+2} by the Poincaré map. The dashed lobes are mapped into each other. The two lower arcs of the stable and unstable manifolds are close to the unperturbed separatrix. (after Elskens and Escande 2003)

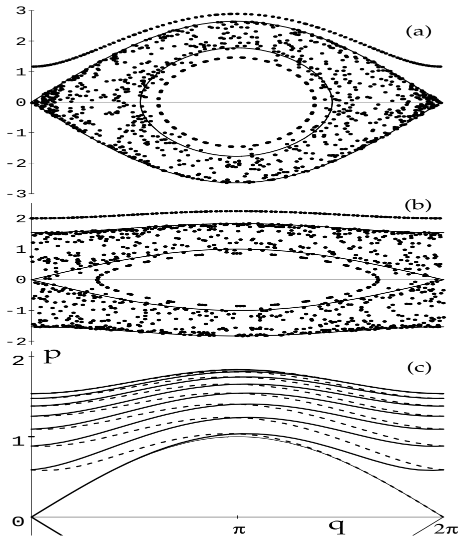

Fig. 10. Poincaré sections of the modulated pendulum for $\epsilon = 0.1$, $\mu = 0.75$, $\mu' = 1$ at (a) $\lambda = \pi$ and (b) $\lambda = 0$. (c) Beginning of the stable (dotted line) and unstable (solid line) manifolds of the X-point at $\lambda = 0$. (after Elskens and Escande, 1993)

As shown in the right of figure 9, homoclinic lobes of branches of a stable and of an unstable manifolds intersect. Therefore, the infinitely many lobes resulting from their iteration create a homoclinic tangle (or trellis). Similar tangles may be defined for the X-points of the other wave, or of the higher order resonances evidenced in figures 2 and 3. Therefore, we must envision the coexistence of (un)stable manifolds related to different X-points. Because of the finite area of the chaotic domains, homoclinic lobes must interpenetrate and the flux through cantori can be eventually traced back to the area of homoclinic lobes of nearby resonances. Finally, there is another kind of intersections, called heteroclinic intersections, between the stable manifold of one resonance X-point and the unstable manifold of the X-point of another resonance. The existence of sizable chaotic domains was related in section 3.2 to the break-up of KAM tori. Now we may relate it to the heteroclinic intersection of manifolds coming from the X-points of the two waves of Hamiltonian (1). After the break-up of a KAM torus, *these manifolds go through the small holes of the corresponding cantorus*. In this respect, the Chirikov resonance overlap criterion may be viewed as an approximate heteroclinic intersection criterion, computed by approximating the manifolds of the X-points of the two waves by branches of the corresponding separatrices. An orbit belonging to one of the manifolds must follow all its folds, and looks rather erratic though deterministic. The stochastic layers found numerically show that this behaviour is quite general, though the mathematical knowledge about the orbits in chaotic layers is still limited. We can now understand why in figure 3 the apparent separatrices are in fact thin stochastic layers.

8 Adiabatic description

When dealing with configurations for the magnetic confinement of charged particles, one often finds that the motion of a particle in such a configuration has multiple scales. For instance, section 4 considered the case of magnetic mirror traps where a particle has a fast Larmor rotation and slow oscillations along the lines of force. If the dynamics is not in a regime of large scale chaos, it is natural to take advantage of the time scale separation to describe the motion. This leads to tractable analytical calculations if the fast degree of freedom is nearly periodic compared to the slow one: one makes a (classical) adiabatic theory of the motion. In reality, the adiabatic ideas carry over to some non strictly adiabatic cases: this is neo-adiabatic theory. The applications of these ideas are now described.

Classical adiabatic theory was formalized in 1936 [Krylov 1936]. In the 1950s and early 1960s, Kruskal was working on asymptotics and on the preservation or destruction of magnetic flux surfaces. His unpublished work motivated Lenard and Gardner to develop a theory of adiabatic invariance to all orders, and he then developed a Hamiltonian version of adiabatic theory where adiabatic invariants are related to proper action variables (see [Kruskal 1962] and references therein, and section 3.1.1 of REV). Adiabatic motion in plasma physics was also a source of inspiration for pure mathematicians, as can be seen in [Arnold 1963b] which deals, in particular, with magnetic traps, and quotes Kruskal's work in his section devoted to adiabatic invariants.

Neo-adiabatic theory Several problems in plasma physics where there is a slow variation of the system of interest cannot be addressed by classical adiabatic theory. This is in particular the case when this slow variation induces a transition from trapped to passing orbits in magnetic configurations of magnetic fusion or of the magnetosphere. Then orbits cross a separatrix. Since the period of a motion diverges on a separatrix, whatever slow be the evolution of the mechanical system, classical adiabatic theory breaks down to describe this crossing. However, it turns out that one can still take advantage of a separation of time scales for most crossing orbits: those which do not stick too long to the X-point. In 1986, four (groups) of authors came up with the calculation of the change of adiabatic invariant due to separatrix crossing among which a group of plasma physicists (see section 3.1.2 of REV and [Bazzani 2014]). The approaches are very similar and constitute what is now called neo-adiabatic theory ([Bazzani 2014] provides an extensive list of papers on this theory). The theory provides also explicit formulas for the trapping probabilities in a resonance region.

Adiabatic description of Hamiltonian chaos Section 7.1 considered the case of diffusive transport of a particle in strongly overlapping longitudinal waves. The diffusive picture was justified by the locality in velocity of the wave-particle interaction. This locality is quantified by a width in velocity which grows with the overlap parameter. Then, the diffusive picture is justified if this width is much smaller than the range of the phase velocities of the waves with strong resonance overlap. In the opposite case, the locality in velocity of the wave-particle interaction corresponds to a motion where the trapping time in the frozen potential of all waves is much smaller than the time scale of variation of this potential (this corresponds to the case of a large Kubo number introduced in section 7.1). At a given time, the frozen potential displays one or more separatrices which are pulsating with time. This issue is of interest to plasma physicists[27].

The simplest case corresponds to a single pulsating separatrix, as occurs for the dynamics of a nonlinear pendulum in a slowly oscillating gravity field whose Hamiltonian is

$$H_{\mathrm{ps}}(p, q, \lambda) = \frac{p^2}{2} + g(\lambda)(\cos q - 1), \tag{6}$$

where $\lambda = \epsilon t$ and $g(\lambda) = A - \mu \cos \lambda$. When λ is frozen at a given value, the corresponding nonlinear pendulum has a separatrix. When $\lambda = \epsilon t$, this separatrix pulsates. So does the corresponding cat's eye whose area is minimal ($\mathcal{A}_{\mathrm{m}}$) for $\lambda = 0$ and maximal ($\mathcal{A}_{\mathrm{M}}$) for $\lambda = \pi$. Numerical simulations reveal that the domain $\mathcal{S}$ swept by the separatrix in the Poincaré map looks like a chaotic sea where no island is visible [Menyuk 1985,Elskens 1993][28]. This is shown in figures 10a and b, which display two Poincaré surfaces of section for the dynamics of Hamiltonian (7) at $\epsilon t = 0 \bmod 2\pi$ and $\epsilon t = \pi \bmod 2\pi$, for $A = 1$, $\mu = 0.75$ and $\epsilon = 0.1$. Three typical orbits are displayed, as well as the light solid lines indicating the boundaries of $\mathcal{S}$. The two orbits outside $\mathcal{S}$ look regular, but that inside $\mathcal{S}$ looks chaotic, and seems to fill in this domain in a fairly uniform way.

As a result one might think the limit of infinite overlap[29] to correspond to some "pure" chaos. A fact pushing in this direction is a theorem telling that in the domain swept by the separatrix, the homoclinic tangle is tight when ϵ goes to 0[30] [Elskens 1991]. This is illustrated by figure 10c, which displays the beginning of the upper branches the stable (unstable) manifold of the X-point at $\epsilon t = 0$. These branches look like touching the upper boundary of $\mathcal{S}$. They intersect transversally, which shows the absence of a separatrix and the presence of a homoclinic tangle. The meshes of the trellis are kind of parallelograms whose sides are pieces of stable and unstable manifolds. When ϵ goes to 0, the homoclinic trellis becomes tighter and tighter with a number of branches per manifold scaling like $1/\epsilon$. In figure 10, ϵ is not very small but the chaotic sea related to the trellis fits well within $\mathcal{S}$. We notice that no large size island is visible in the chaotic sea.

However, another theorem tells the total area covered by islands in the same domain remains finite for symmetric frozen potentials when the slowness of the system increases [Neishtadt 1997] (this area decreases for asymmetric system, though). This shows the dynamics is not hyperbolic at all in the chaotic sea related to the motion of a nonlinear pendulum in a slowly modulated gravity field: chaotic does not

[27] In 1997, the understanding of adiabatic chaos led to finding a way of mitigating its effects, such as in the work on omnigenous stellarators, stellarators where all orbits are confined [Cary 1997]

[28] See also section 5.5.2 of [Elskens 2003].

[29] Expanding the potential of Hamiltonian (6) as a sum of cosines yields

$$H_3(p, q, t) = \frac{p^2}{2} + A \cos q - \frac{\mu}{2}[\cos(q + \epsilon t) + \cos(q - \epsilon t)] - (A - \mu \cos \epsilon t), \tag{7}$$

where the last term plays no role in the equations of motion. This Hamiltonian can be interpreted as describing the motion of a particle in three longitudinal waves: a static one, and two propagating waves with phase velocities $\pm\epsilon$. The resonance overlap parameter of the two waves upper or lower waves is $s = \frac{2\sqrt{A}+\sqrt{2\mu}}{\epsilon}$. The small ϵ limit of interest is thus the limit of strong overlap of these three waves. Figures 10a and b correspond to $s \simeq 32$.

[30] When resonance overlap diminishes, at some moment the heteroclinic intersection between manifolds of the two resonances vanishes. This occurs at a threshold approximately given by the resonance overlap criterion if the two resonances are not too different in size and wavelength [Escande 1981b,Escande 1985].

mean stochastic! This also shows that chaos is not pure at all, and that the numerical simulation of orbits may provide a misleading information[31]. In the adiabatic limit, successive separatrix crossings are not independent, which significantly affects transport [Bruhwiler 1989,Cary 1989]. However the separation of nearby orbits is intuitive, since two such orbits may be separated when coming close to the X-point, one staying untrapped and the other one becoming trapped.

9 Conclusion

The research on magnetic fusion triggered studies of Hamiltonian chaos, which rapidly propagated to the whole of plasma physics. This paper has reviewed some important and graphical aspects of this chaos resulting from these studies. The transition to chaos may be understood as the collision of eyes-of-cat in phase space, especially at small scale, as revealed by the renormalization picture. However, this transition corresponds in reality to heteroclinic intersections. To homoclinic ones too, as clearly exhibited in the adiabatic limit of Hamiltonian chaos. In the small Kubo number limit, chaotic transport can be understood by using the locality of the action of nonlinear resonances symbolized by the finite with of their eye-of-cat. In the large Kubo limit, this transport can be interpreted with an adiabatic view of the dynamics.

It is not by chance that plasma physics contributed so much to chaos and nonlinear dynamics: it deals with complex system where statistical effects are obvious, but out of reach of thermodynamics and standard statistical mechanics. Therefore, it was very important to try and understand chaotic mechanics. The irruption of numerical calculation was precious for this task. It helped the development of concepts, which are important to progress in the understanding of the complexity of plasmas [Escande 2013].

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[31] If the mathematical model is thought as the approximation of a true physical system, the dynamics of the latter undergoes actually perturbations like noise. These perturbations are likely to smear out the many minuscule islands of [Neishtadt 1997]. Then, the above numerical simulation gives the right physical picture. This sets the important issue of the structural stability of mathematical models when embedded into more realistic ones: numerical simulations might be more realistic than the mathematical model they approximate!

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