Combined Newton-Kurchatov method for solving nonlinear operator equations

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Roman Iakymchuk [1,*], **Stepan Shakhno** [2,**], and **Halyna Yarmola** [2,***]

[1] KTH Royal Institute of Technology, Teknikringen Str. 70, 114 28, Stockholm, Sweden
[2] Ivan Franko National University of Lviv, Universitetska Str. 1, 79000, Lviv, Ukraine

We investigate local and semi-local convergence of the combined Newton-Kurchatov method under the classical and generalized Lipschitz conditions for solving nonlinear equations. The convergence order of the method is examined and the uniqueness ball for the solution of the nonlinear equation is proved. Numerical experiments are conducted on test problems.

1 Introduction

Let us consider the nonlinear equation

$$H(x) \equiv F(x) + G(x) = 0, \tag{1}$$

where F and G are nonlinear operators defined on a convex subset D of a Banach space X with their values in a Banach space Y. F is a Fréchet-differentiable operator and G is a continuous operator; the differentiability of G is not required.

For solving the problem (1), we proposed in [2] the two-point iterative process

$$x_{n+1} = x_n - [F'(x_n) + G(2x_n - x_{n-1}; x_{n-1})]^{-1} H(x_n),\ n = 0, 1, \ldots, \tag{2}$$

which is built on the Newton method and the Kurchatov method of linear interpolation [1]. We studied the semi-local convergence of the method (2) and showed that its convergence order is $(1 + \sqrt{5})/2 \approx 1.618\ldots$, which is lower than the convergence order of the underlying methods. In this work, we extend our studies to the local convergence of the method (2) under the classical and generalized Lipschitz conditions for the first-order derivatives of the operator F and for the first- and second-order divided differences of the operator G. Moreover, we investigate the semi-local convergence of the method (2) under the classical Lipschitz conditions and determine its radius of the convergence ball and the quadratic convergence order.

2 Local convergence analysis under the generalized Lipschitz conditions

Let us denote $B(\tilde{x}, R) = \{x : \|x - \tilde{x}\| < R\}$ an open ball of radius R with a center at the point $\tilde{x} \in D$.

Theorem 2.1 *Let $F,\ G : D \subseteq X \to Y$ be continuous nonlinear operators, where X and Y are Banach spaces. Suppose, that: 1) $H(x) = 0$ has a solution $x^* \in D$ and there exists a Fréchet derivative $H'(x^*)$ that is invertible; 2) F has the derivative of the first-order and G has divided differences of the first- and second-order on $B(x^*, 3r) \subset D$, satisfying the generalized Lipschitz conditions on $B(x^*, 3r)$*

$$\|H'(x^*)^{-1}(F'(x) - F'(x^\tau))\| \le \int_{\tau\rho(x)}^{\rho(x)} L_1(u)du,$$

$$\|H'(x^*)^{-1}(G(x;y) - G(u;v))\| \le \int_0^{\|x-u\|+\|y-v\|} L_2(z)dz, \quad \|H'(x^*)^{-1}(G(u;x;y) - G(v;x;y))\| \le \int_0^{\|u-v\|} N(z)dz,$$

where $0 \le \tau \le 1$; $x, y, u, v \in B(x^*, 3r)$; $x^\tau = x^* + \tau(x - x^*)$; $\varrho(x) = \|x - x^*\|$; L_1, L_2, N are positive nondecreasing functions; 4) $r > 0$ satisfies the equation

$$\Big(\frac{1}{r}\int_0^r L_1(u)udu + \int_0^r L_2(u)du + 2r\int_0^{2r} N(u)du\Big) / \Big(1 - \Big(\int_0^r L_1(u)du + \int_0^{2r} L(u)du + 2r\int_0^{2r} N(u)du\Big)\Big) = 1.$$

Then for all $x_{-1},\ x_0 \in B(x^*, r)$ the iterative process (2) is correctly defined and the generated sequence $\{x_n\}_{n\ge0}$, which belongs to $B(x^*, r)$, converges to x^* and satisfies the inequality

$$\|x_{n+1} - x^*\| \le \frac{\frac{1}{\rho(x_n)}\int_0^{\rho(x_n)} L_1(u)udu + \int_0^{\rho(x_n)} L_2(u)du + \int_0^{\|x_n - x_{n-1}\|} N(u)du\|x_n - x_{n-1}\|}{1 - \Big(\int_0^{\rho(x_n)} L_1(u)du + \int_0^{2\rho(x_n)} L_2(u)du + \int_0^{\|x_n - x_{n-1}\|} N(u)du\|x_n - x_{n-1}\|\Big)}\|x_n - x^*\|.$$

* Corresponding author:e-mail: riakymch@pdc.kth.se, phone: +46 (0) 70 166 74 09
** E-mail: s_shakhno@lnu.edu.ua, phone: +38 032 239 43 91
*** E-mail: h_yarmola@lnu.edu.ua, phone: +38 032 239 43 91

Corollary 2.2 *The Newton-Kurchatov method (2) has the quadratic convergence order.*

3 Semi-local convergence analysis under the classical Lipschitz conditions

Theorem 3.1 *Let $F,\ G : D \subseteq X \to Y$ be continuous nonlinear operators, where X and Y are Banach spaces. Suppose that: 1) $A_0 = F'(x_0) + G(2x_0 - x_{-1}; x_{-1})$, where $x_{-1}, x_0 \in U_0 = \{x : \|x - x_0\| \le r_0\} \subset D$, is invertible operator; 2) F has the derivative of the first-order and G has divided differences of the first- and second-order on $V_0 = \{x : \|x - x_0\| \le 3r_0\} \subset D$, satisfying the Lipschitz conditions on V_0*

$$\|A_0^{-1}(F'(x) - F'(y))\| \le 2l_0\|x - y\|,$$
$$\|A_0^{-1}(G(x; y) - G(u; v))\| \le p_0(\|x - u\| + \|y - v\|), \quad \|A_0^{-1}(G(x; y; z) - G(u; y; z))\| \le q_0\|x - u\|;$$

3) $a,\ c,\ r_0$ are non-negative numbers such that $\|x_0 - x_{-1}\| \le a$, $\|A_0^{-1}H(x_0)\| \le c$, $c < a$, $r_0 < \dfrac{1 - 2q_0a^2 - (l_0 + p_0)c}{2(l_0 + p_0)}$, $r_0 \ge \dfrac{c}{1-\gamma}$, $\gamma = \dfrac{(l_0 + p_0)c + q_0a^2}{1 - q_0a^2 - 2(l_0 + p_0)r_0}$, $0 \le \gamma < 1$.

Then the following inequalities hold for all $n \ge 0$

$$\|x_n - x_{n+1}\| \le t_n - t_{n+1}, \quad \|x_n - x^*\| \le t_n - t^*,$$

where $t_{-1} = r_0 + a,\ t_0 = r_0,\ t_1 = r_0 - c,\ t_{n+1} - t_{n+2} = \dfrac{(l_0 + p_0)(t_n - t_{n+1}) + q_0(t_{n-1} - t_n)^2}{1 - q_0a^2 - 2(l_0 + p_0)(t_0 - t_{n+1})}(t_n - t_{n+1})$, $n \ge 0$, $\{t_n\}_{n\ge0}$ is a non-negative, nonincreasing sequence that converges to a certain t^* such that $r_0 - c/(1 - \gamma) \le t^* < t_0$; the sequence $\{x_n\}_{n\ge0}$, generated by the iterative process (2), is well defined, remains in U_0 and converges to the solution x^*.

Theorem 3.2 *Let $F(x^*) + G(x^*) = 0$. Suppose that: 1) $A_0 = F'(x_0) + G(2x_0 - x_{-1}; x_{-1})$, where $x_{-1}, x_0 \in U_0$, is invertible operator; 2) F has the derivative of the first-order and G has divided differences of the first-order in D, satisfying the Lipschitz conditions*

$$\|A_0^{-1}(F'(x) - F'(y))\| \le 2l_0\|x - y\|,\ \|A_0^{-1}(G(x; y) - G(u; v))\| \le p_0(\|x - u\| + \|y - v\|),$$

where $x, y, u, v \in D$; 3) r_0 satisfies the inequality $r_0 < (1 - 2p_0a)/(2(l_0 + p_0))$.

Then the problem (1) has the unique solution $x^* \in U_0$. If there exists $r_1 > r_0$, that $U_1 = \{x : \|x - x_0\| \le r_1\} \subset D$ and $2p_0a + (l_0 + p_0)(r_0 + r_1) < 1$, then the problem (1) has the unique solution $x^* \in U_1$.

4 Numerical experiments

We carried out a set of experiments and present in Tab. 1 the results for the following system of nonlinear equations:

$$\begin{cases} 3x^2y - y^2 - 1 + |x - 1| = 0, \\ x^4 + xy^3 - 1 + |y| = 0. \end{cases}$$

We used initial approximations $x_0 = (d, 0)$, $x_{-1} = (5d, 5d)$ and stopping conditions $\|x_{n+1} - x_n\|_\infty \le \epsilon$, $\|H(x_{n+1})\|_\infty \le \epsilon$.

Table 1: *The amount of iterations required to find the approximation to the solution of the problem (1) with the accuracy $\epsilon = 10^{-15}$.*

d	Newton-Kurchatov method (2)	Kurchatov method [1]	Newton-type method [3]
1	7	8	33
10	14	16	41
100	22	25	49

The obtained results confirm that the Newton-Kurchatov method (2) has the highest convergence speed.

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