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Differential Flatness and Control of Protocentric Aerial Manipulators with Any Number of Arms and Mixed Rigid-/Elastic-Joints

Burak Yüksel¹, Gabriele Buondonno² and Antonio Franchi³

Abstract—In this paper we introduce a particularly relevant class of aerial manipulators that we name protocentric. These robots are formed by an underactuated aerial vehicle, a planar-Vertical Take-Off and Landing (PVTOL), equipped with any number of different parallel manipulator arms with the only property that all the first joints are attached at the Center of Mass (CoM) of the PVTOL, while the center of actuation of the PVTOL can be anywhere. We prove that protocentric aerial manipulators (PAMs) are differentially flat systems regardless the number of joints of each arm and their kinematic and dynamic parameters. The set of flat outputs is constituted by the CoM of the PVTOL and the absolute orientation angles of all the links. The relative degree of each output is equal to four. More amazingly, we prove that PAMs are differentially flat even in the case that any number of the joints are elastic, no matter the internal distribution between elastic and rigid joints. The set of flat outputs is the same but in this case the total relative degree grows quadratically with the number of elastic joints. We validate the theory by simulating object grasping and transportation tasks with unknown mass and parameters and using a controller based on dynamic feedback linearization.

I. INTRODUCTION

Systems consisting of aerial vehicles and manipulator arms have been increasingly studied within the last years, as they enjoy the great workspace of the flying robot base, and the dexterity of the manipulator arm attached to it. The major implementation of such systems are aerial physical systems, e.g., control frameworks for aerial vehicles together with a light-scale, the authors of [7] showed the experimental results in advance (algebraically) the nominal state and the input system characteristics of the fixed-base manipulators, e.g., differential flatness properties. This property allows to know in advance (algebraically) the nominal state and the input trajectories along which the system will evolve while tracking a desired output trajectory [13], which is very useful especially in the planning phase. Moreover it is well known that differential flatness implies input-to-state linearizability via dynamic feedback in an open and dense set of the state space and that a flat output is exactly linearizing [14]. Exact linearizability of grounded manipulator arms have been studied, when the joint connection is rigid [15], elastic [16] or both of them [17]. More interesting studies on controlling arms with compliance can be found in [18].

The differential flatness and control of quadrotor UAVs have been studied previously by different groups, e.g. [19]. In [20], the authors studied the case of a planar-Vertical Take-Off and Landing (PVTOL) vehicle equipped with a one DoF rigid arm. This result has been recently extended generalized in a few directions by [11].

As for the flying vehicle we consider the case of a (vertical) planar-VTOL (PVTOL) aerial platform, similar to previous studies (see, e.g., [21], [22]) in the aerial robotics

Fig. 1: Sketch of a protocentric aerial manipulator (PAM) for \( m = 3 \). On the left up relative and absolute angles of the motor and the link of the first joint of the first manipulator are depicted, where the length of the \( z \) axes are made different just for illustration purposes.

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field. This reduced system does not only capture the nonlinear features and the underactuation of a 3D system, but also allows to generalize the obtained results in a later stage. Furthermore, many practical aerial problems are, fundamentally, 2D problems immersed in a 3D world.

In this paper, we present a generic dynamic model of a PVTOL equipped with multiple arms, each having a possibly different number of links with any distribution of rigid or elastic joints. We consider the case of protocentric aerial manipulators (PAM), where all the arms are attached to the CoM of the PVTOL. We show that PAMs are differentially flat systems both when all the joints of the arms are rigid and also when any number of them are elastic. A surprising fact is that, contrarily to the case of a robotic arm attached to a fixed base [16], the total relative degree grows quadratically with the number of elastic joints, due to the underactuation of the flying base. This fact makes the control of aerial manipulators with elastic joints a very challenging task. Finally, we present an exact linearizing controller for tracking in the case that all the joints are rigid.

The detailed derivations of the model and proofs are presented as a technical report in [23] due to the page limit.

II. DEFINITIONS AND ASSUMPTIONS

In order to model a generic Protocentric Aerial Manipulator (PAM), we start with the following definitions:

- A PAM is constituted by a PVTOL with attached motors m ≥ 1 manipulating arms (see Fig. 1 where m = 3); an arbitrary arm is called the μ-th arm, where μ ∈ \{1, 2, 3, ..., m\}.
- The μ-th arm is constituted by \( n^\mu \) joint/motor/link elements; an arbitrary joint, motor, or link is called the \( v^\mu \)-th joint/motor/link, motor, or link, where \( v^\mu \in \{1, 2, 3, ..., n\} \).
- The μ-th arm has \( k^\mu \) elastic joints; an arbitrary elastic joint is called the \( k^\mu \)-th elastic joint, where \( k^\mu \in \{1, 2, 3, ..., k\} \). Similarly, we define \( k = \sum_{\mu=1}^{m} k^\mu \).
- It is always \( 0 \leq k^\mu \leq n^\mu \) and \( 0 \leq k \leq n \).

With this convention, we call, e.g., the mass of the \( v^\mu \)-th link of the \( \mu \)-th arm as \( m_{v^\mu} \), or the motor rotational inertia of the \( v^\mu \)-th link of the \( \mu \)-th arm is called as \( J_{m, v^\mu} \) (i.e., the subscript corresponds to the joint, and the superscript to the arm).

The following assumptions are then made:

A1. Only the 2D dynamics of a PVTOL aerial vehicle with \( m \) different fully actuated robotic arms is considered.

A2. All the joints are actuated via a motor, and the rotational center of this motor is the same with the center of the revolute joint that is attached to it.

A3. [Protocentricity] The first joint of each robotic arm is placed at the Center of Mass (CoM) of the PVTOL, i.e., \( P_{C_0} = P_{M_1} = P_{M_2} = \cdots = P_{M_m} \) (see also Fig. 1).

A4. Each motor is attached to the next link in the chain either rigidly or via some elastic joint.

We denote with \( R_W : \{\hat{p}_W, \hat{x}_W, \hat{z}_W \} \) and \( R_\nu : \{\hat{p}_{C_\nu}, \hat{x}_{C_\nu}, \hat{z}_{C_\nu} \} \) the world (inertial) frame and the frame attached to the PVTOL, respectively, where \( P_{C_\nu} \) is the Center of Mass (CoM) of the PVTOL. Define \( P_{M, v^\mu} \) as the center of the \( v^\mu \)-th motor. The \( v^\mu \)-th joint and motor rotate about an axis parallel to \( \hat{z}_W \times \hat{x}_W \) and passing through \( P_{M, v^\mu} \). The \( v^\mu \)-th motor frame is denoted with \( R_{M, v^\mu} : \{\hat{p}_{M, v^\mu}, \hat{x}_{M, v^\mu}, \hat{z}_{M, v^\mu} \} \) and it is rigidly attached to the output shaft of the \( v^\mu \)-th motor. We consider also the \( v^\mu \)-th link frame \( R_{v^\mu} : \{\hat{p}_{C_{v^\mu}}, \hat{x}_{v^\mu}, \hat{z}_{v^\mu} \} \), where \( P_{C_{v^\mu}} \) is the CoM of the \( v^\mu \)-th link. Finally we denote with \( P_{E_{v^\mu}} \) the terminal point of the end-effector of the \( \mu \)-th arm, and with \( P_C \) the CoM of the whole robotic system (i.e., the PVTOL plus all the arms).

Given an angle \( \theta_{v^\mu - \mu} = \theta - \theta_{m,v^\mu} = \theta_{v^\mu} - \theta_{m,v^\mu} \) is constantly zero if the \( v^\mu \) joint is rigid and can be any if it is elastic.

The constant position of \( P_{M,v^\mu} \) and of \( P_{M,v^\mu+1} \) in \( R_{\nu} \) are denoted with \( -\hat{d}_{\nu,v^\mu} = [-d_{\nu,v^\mu}]T \in \mathbb{R}^2 \) and with \( \hat{d}_{\nu,v^\mu} = [d_{\nu,v^\mu,1}, d_{\nu,v^\mu,2}]T \in \mathbb{R}^2 \), respectively. The (time-varying) positions of \( P_{C_\nu}, P_{C_{v^\mu}}, P_{M,v^\mu}, \) and \( P_{E_{v^\mu}} \) in \( R_W \) are denoted with \( \hat{p}_{C_\nu} = [x_{C_\nu}, z_{C_\nu}]T \in \mathbb{R}^2, \hat{p}_{C_{v^\mu}} = [x_{C_{v^\mu}}, z_{C_{v^\mu}}]T \in \mathbb{R}^2, \) \( \hat{p}_{M,v^\mu} = [x_{M,v^\mu}, z_{M,v^\mu}]T \in \mathbb{R}^2, \) and \( \hat{p}_{E_{v^\mu}} = [x_{E_{v^\mu}}, z_{E_{v^\mu}}]T \in \mathbb{R}^2 \), respectively. The mass and moment of the inertia of the PVTOL and the \( v^\mu \)-th motor and link are denoted with \( m_0 \in \mathbb{R}_{>0}, J_0 \in \mathbb{R}_{>0}; m_{m,v^\mu} \in \mathbb{R}_{>0}, J_{m,v^\mu} \in \mathbb{R}_{>0}; m_{v^\mu} \in \mathbb{R}_{>0}, J_{v^\mu} \in \mathbb{R}_{>0} \), respectively. The gravitational constant is \( g \in \mathbb{R}^+ \). Also \( m_0 = m_0 + \sum_{\nu=1}^{n} (m_{m,v^\nu} + m_{v^\nu}) \) is the total mass of the overall system.

The point \( P_{G} \) is the center of actuation of the PVTOL (green point in Fig 1). The constant position of \( P_G \) is denoted with \( \hat{d}_{G} = [d_{G}, d_{G}]T \in \mathbb{R}^2 \). The PVTOL is actuated by means of: i) a total thrust force \( -u_{\nu} z_0 \in \mathbb{R}^2 \) applied at \( P_G \), where \( u_{\nu} \in \mathbb{R} \) is its magnitude, and ii) a total torque (moment) \( u_{\nu} (z_0 \times x_0) \in \mathbb{R}^2 \) applied also at \( P_G \), where \( u_{\nu} \in \mathbb{R} \) is the torque intensity. Furthermore, an individual motor for each joint applies a torque \( \tau_{v^\mu} (z_{v^\mu} \times x_{v^\mu}) \) at \( P_{M,v^\mu} \) to the joint, where \( \tau_{v^\mu} \in \mathbb{R} \) is its intensity.

III. CASE R: DYNAMICS WITH RIGID JOINTS ONLY

Let us first consider the case in which all the joints are rigid, i.e., \( k = 0 \). The aerial manipulator has therefore \( 3 + n \) degrees of Freedom (DoFs) corresponding to the generalized coordinates \( \{q_1^T, q_{m}^T\} \in \mathbb{R}^{3+n} \) where \( q_1 \) are the PVTOL coordinates, and \( q_{m} \) are the arm-side coordinates:

\[
q_1 = [p_0^T, \theta_0]T \in \mathbb{R}^3
\]

\[
q_{m} = [q_1^T, \cdots, q_m^T]T \in \mathbb{R}^n, \quad q_{m}^T = [\theta_{1,1} \cdots \theta_{m,n}]T \in \mathbb{R}^{n^m}.
\]

Then using the Lagrange equation and after some straightforward algebra (see Sec. II-A of [23] for details), we can find the generalized inertia matrix as

\[
M = \begin{bmatrix}
M_p & M_{pr} \\
M_{pr} & M_{prh}
\end{bmatrix} \in \mathbb{R}^{(3+n) \times (3+n)}
\]

\[
M_p = \text{diag}(m_0, m_1, J_0), \quad M_{pr} = \begin{bmatrix} T_{pr1} & \cdots & T_{prm}\end{bmatrix}^T
\]

\[
M_{prh} = \begin{bmatrix}
\sum_{\nu=1}^{n} (\theta_{\nu,1})^2 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & \sum_{\nu=1}^{n} (\theta_{\nu,m})^2
\end{bmatrix} \in \mathbb{R}^{m \times 3}.
\]
where $m_i$ is the total mass, $M_p \in \mathbb{R}^{3 \times 3}$ is the PVTOL side inertia matrix, $M_i(q_i) \in \mathbb{R}^{n \times n}$ is the manipulator side inertia matrix, and $M_{pi}(q_i) \in \mathbb{R}^{n \times 3}$ represents the inertial couplings between the PVTOL and the manipulator arms. More details on the computation are given in Sec.II-B of [23].

The gravitational forces are the following

$$
g = \left[ g_p^T \ g_i^T \right]^T \in \mathbb{R}^{(3+n)}, \quad g_p = \begin{bmatrix} 0 & -m_s g & 0 \end{bmatrix}^T \in \mathbb{R}^3 \tag{2}
$$

where $g_i = \begin{bmatrix} g_{i1}^T \cdots \ g_{in}^T \end{bmatrix}^T \in \mathbb{R}^n$ and for the $\nu$-th manipulator;

$$
g_{\mu} = \begin{bmatrix} -g_m(\theta_{0,\mu}) \ e_2 \\
0_l \end{bmatrix} \in \mathbb{R}^{n_{\mu}},
$$

with $e_2 = [0 \ 1]^T$. The Coriolis/centrifugal forces are found as

$$
c = \begin{bmatrix} \Sigma_{i=1}^{n} \Sigma_{j=1}^{n}  \dot{\theta}_{\nu,ij} \dot{\theta}_{\nu,ij} \ 0 \\
0_l \ c_i(q_i, \dot{q}_i) \end{bmatrix} \in \mathbb{R}^{(3+n) \times 1}, \tag{3}
$$

where $m_{0,\mu} = \frac{\partial m_i}{\partial \theta_{\mu}} \in \mathbb{R}^{2 \times 1}$ and $c_i(q_i, \dot{q}_i) \in \mathbb{R}^n$ are the arm side Coriolis forces. All the explicit steps for computing $g$ and $c$ can be found in Sec. II-C of [23].

Finally, the generalized forces are

$$
f = \begin{bmatrix} -u_t \sin(\theta_0) \\
-u_t \cos(\theta_0) \\
d_{G_i} u_t + u_t - \Sigma_{i=1}^{n} \tau_{ij} \end{bmatrix} = g \in \mathbb{R}^{n(3+n)}, \tag{4}
$$

$$
T = \begin{bmatrix} \dot{\tau}_{1}^T \\
\vdots \\
\dot{\tau}_{n}^T \end{bmatrix} \in \mathbb{R}^n,
$$

$$
\dot{\tau}_\mu = \begin{bmatrix} \tau_{1,\mu} - \tau_{2,\mu} \\
\vdots \\
\tau_{n-1,\mu} - \tau_{n,\mu} \\
\tau_{n,\mu} \end{bmatrix} \in \mathbb{R}^{n_{\mu}}, \tag{5}
$$

which leads to a control input matrix of the following form

$$
G = \begin{bmatrix} v(\theta_0) & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
G_p & 0 & G_r \end{bmatrix} \in \mathbb{R}^{(n+3) \times (n+2)}, \tag{6}
$$

where $v = -\dot{\theta}_0 \in \mathbb{R}^3$, and all the other parts are explicitly given in Sec. II-D of [23]. The control input vector is

$$
u = \begin{bmatrix} u_t \ u_t \ \tau_1^T \ \tau_2^T \ \cdots \ \tau_{n}^T \end{bmatrix} \in \mathbb{R}^{(n+2)}, \tag{7}
$$

where $\tau_\mu = \begin{bmatrix} \tau_{1,\mu} \ \tau_{2,\mu} \ \cdots \ \tau_{n,\mu} \end{bmatrix} \in \mathbb{R}^{n_{\mu}}$. Then finally the system dynamics can be written in the following form

$$
\ddot{\mathbf{q}} = G \nu + C + g = G \nu. \tag{8}
$$

The following result holds:

**Proposition 1.** $y = [p_0 \ q_i^T]^T \in \mathbb{R}^{(n+2)}$ is a flat output for the protocentric aerial manipulator with all rigid joints ($k = 0$). The relative degree of each entry of $y$ is 4, and the total relative degree is 4n+8.

**Proof.** (Sketch) See Sec. I-A of [23] for the full proof. From the dynamics of the CoM of overall system one can show that $u_t$ and $\dot{\theta}_0$ can be computed as functions of $y$, $\dot{y}$ and $\ddot{y}$. (See (1)-(2) of [23]). The torque of the $v^\mu$-th motor is

$$
\tau_{v\mu} = \dot{\tau}_{v\mu} + m_{i,\mu}^T (\dot{\theta}_{0,\mu}) p_0 + c_{r,\mu} (q_\mu, \dot{q}_\mu) + g_{r,\mu} (\dot{q}_0, \dot{q}_\mu) + \sum_{i=1, j \neq v}^{n} m_{ij,\mu} (\dot{\theta}_{0,\mu}) (\dot{\theta}_{0,j}) \dot{\theta}_{0,\mu}, \tag{9}
$$

where $c_{r,\mu}$ and $g_{r,\mu}$ are the $v^\mu$-th elements of vectors $c_i$ and $g_i$, corresponding to the Coriolis and gravitational forces acting on the center of the $v^\mu$-th link, respectively (for $v^\mu = n^\mu$ it is $\tau_{v\mu+1} = 0$). This means we can write the control torque of the $v^\mu$-th joint in the form of $\tau_{v\mu} = \tau_{v\mu} (y, \dot{y}, \ddot{y})$. Finally the PVTOL torque is computed as

$$
u_t = \sum_{j=0}^{n} \tau_{j} - d_{G_i} u_t, \tag{10}
$$

which means one can compute all the inputs and the states of the system as the functions of flat outputs and a finite number of its derivatives.

Also, since $\theta_0$ is a function of $\dot{y}$, then $\dot{\theta}_0$ is a function of $\ddot{y}$, and so is $\nu_t$, implying the relative degree of the system is four times the dimension of $\dot{y}$, i.e. $r = 4(2+n) = 8+4n$.

**IV. CASE E: DYNAMICS WITH RIGID/ELASTIC JOINTS**

We consider now the case $k \geq 1$, i.e., when at least one joint is elastic (see Fig. 2). The coordinates for the $\nu$-th manipulator are $q_\mu = [q_{i,\mu} \ q_\nu^T]^T \in \mathbb{R}^{n^\mu+k}$. Then $q_\mu$ contains the orientations of the links and $q_\nu \in \mathbb{R}^{n^\mu}$ contains the orientations of the motors that are connected to their links via an elastic joint. The full generalized coordinates are $q = [q_{i,\mu} q_\nu^T]^T \in \mathbb{R}^{3 \times n^\mu+k}$ where $q_{i,\mu} = [q_{i,1}^T \cdots q_{i,n}^T]^T \in \mathbb{R}^{n^\mu}$ and $q_\nu = [q_{\nu}^T q_{\nu}^T q_{\nu}^T]^T \in \mathbb{R}^{k}$. Define the set of sets $N = \{N_1, N_2, \ldots, N^{n^\mu}\}$, where $N^\mu = \{1, 2, \ldots, n^\mu\}$, and the set of sets $K = \{K_1, K_2, \ldots, K^{k}\}$, where $K^\mu \subset N$ is the sorted set of indexes of the elastic joints of the $\nu$-th arm. We denote with $x^\mu$ the $x^\mu$-th element of $K^\mu$, where $k^\mu = 1, \ldots, |K^\mu| = k^\mu$. Therefore, it is $q_{\nu} = \{\theta_{0,\mu} \theta_{m_1(\nu,\nu-1)} \theta_{m_2(\nu,\nu-2)} \cdots \theta_{m_{n^\mu\nu}(\nu,\nu)}\}^T \in \mathbb{R}^{k^\mu}$.

For each arm $\mu$ we define i) the diagonal matrix $S_{NN} \in \mathbb{R}^{n^\mu \times n^\mu}$ whose $v^\mu$-th diagonal element is equal to 1 if $v^\mu \in K^\mu$ and zero otherwise, and ii) the selection matrix $S_{NK} \in \mathbb{R}^{k^\mu \times n^\mu}$ obtained from $S_{NN}$ by removing all the zero row vectors. Then we define the block diagonal matrices

$$
S_N = \text{diag}(S_{N1}, S_{N2}, \ldots, S_{N^{n^\mu}}) \in \mathbb{R}^{n^\mu \times n^\mu},
$$

$$
S_K = \text{diag}(S_{K1}, S_{K2}, \ldots, S_{K^{k^\mu}}) \in \mathbb{R}^{k^\mu \times n^\mu}.
$$

See Sec. II-E of [23] for examples of this notation.

Let us then first rewrite the generalized inertia matrix as
where \( M_p \) and \( M_{pr} \) are defined in (1), the inertial terms of the elastically connected motors are summarized in

\[
D_k = \text{diag}\{D_{k1}, D_{k2}, \ldots, D_{kn}\} \in \mathbb{R}^{k \times k},
\]

and \( M_{E} \) is the link side inertia matrix, which is expressed as

\[
M_{E} = M_l - S_0 D_k S_k \in \mathbb{R}^{n \times n},
\]

where \( D_{k\mu} = \text{diag}\{J_{\mu}, J_{\mu}, \ldots, J_{\mu}\} \in \mathbb{R}^{n \times n} \). Those forces are identically identical to (8). If it is elastic, then we first compute,

\[
\nu \theta = \text{terms necessary}. \amalg \nu \theta \text{ as a function of } \nu \theta \text{ in Sec. I-B of [23] for the full proof and}
\]

\[
\begin{align*}
\tau_{\nu \theta} & = \nu \theta \bar{\theta} \nu \theta + \nu \theta \nu \theta \nu \theta \nu \theta + \\
& \quad + \nu \theta \nu \theta \nu \theta \nu \theta + \\
& \quad + \nu \theta \nu \theta \nu \theta \nu \theta
\end{align*}
\]

Notice the similarity with (8). We observe that (13) can also be employed for \( \nu \theta = \mu \theta \), simply setting \( \tau_{\nu \theta} = 0 \) equal to zero. Then, \( \tau_{\nu \theta} \) can be easily computed from

\[
\tau_{\nu \theta} = \nu \theta \bar{\theta} \nu \theta + \nu \theta \nu \theta \nu \theta \nu \theta
\]

From the third equation of the system dynamics we have

\[
u \theta \tau_{\nu \theta} = \nu \theta \bar{\theta} \nu \theta + \nu \theta \nu \theta \nu \theta \nu \theta
\]

in which \( \nu \theta \) is taken from either (8) or (14), depending on the type of the actuation.

In order to briefly explain the relative degree formula, notice the different relative degree of the dependencies of \( \nu \theta \) given in (14) on the flat outputs for different values of \( \nu \theta \). Assume for instance that both the \( (\nu \theta - 1) \)-th and the \( n \mu \)-th link are elastic. Then first, from (14) for \( \nu \theta = \mu \theta \), we see that \( \nu \theta \) is a function of \( \nu \theta \nu \theta \nu \theta \nu \theta \), while \( \nu \theta \nu \theta \nu \theta \nu \theta \) is a function of \( \nu \theta \nu \theta \nu \theta \nu \theta \), making \( \nu \theta \) itself a function of \( \nu \theta \nu \theta \nu \theta \nu \theta \). Second, from (14), \( \nu \theta \nu \theta \nu \theta \nu \theta \) is a function of \( \nu \theta \nu \theta \nu \theta \nu \theta \), making \( \nu \theta \nu \theta \nu \theta \nu \theta \) a function of \( \nu \theta \nu \theta \nu \theta \nu \theta \). Knowing from the first step above that \( \nu \theta \nu \theta \nu \theta \nu \theta \) is a function of \( \nu \theta \nu \theta \nu \theta \nu \theta \) and \( \nu \theta \nu \theta \nu \theta \nu \theta \), we find \( \nu \theta \nu \theta \nu \theta \nu \theta \) as a function of \( \nu \theta \nu \theta \nu \theta \nu \theta \) and \( \nu \theta \nu \theta \nu \theta \nu \theta \), which are the sixth time derivatives.

This can be generalized by recalling that \( \kappa \theta \) is the number of elastic joints in link \( \mu \), and defining \( \kappa \theta = \max(1, \kappa \theta) \), then \( r = 4 + 4 \mu \max(1, \kappa \theta) + \sum_{\mu=1}^{\mu} (2 + 2k) \mu^2 \), where it can be seen a quadratic dependence on the number of elastic joints. The term \( \max(1, \kappa \theta) \) returns the value \( \kappa \theta \) for the manipulator arm with the highest number of elastic joint. To fix the ideas, see also the examples in Sec. I-B of [23].

V. CONTROL FOR CASE R

In this section we present the exact linearizing controller for the system given in (7). We purposely limit our computation to the Case R, since the high relative degree involved in Case E may cause the controller to be unpractical.

Now, based on the findings of Proposition 1, we take \( y = [p_{\mu}^T \ q_{\mu}^T]^T \in \mathbb{R}^{n + n} \) as control variables, leaving out the PVTOL orientation \( \theta_0 \). We approach the control problem by studying the system with \( \theta_0 \) removed. We can decompose the inertia matrix \( M \) by defining the following quantities:

\[
M = \begin{bmatrix}
M_p & 0 & M_r^T & 0 \\
M_r & 0 & 0 & 0 \\
0 & J_0 & 0 & 0 \\
0 & 0 & 0 & M_r
\end{bmatrix}
\]

where \( 0 \) is a zero vector or matrix of proper dimensions, \( M_p = \text{diag}(m, m) \), and \( M_r \) is simply constituted by the first
two columns of $M_{pr}$. Similarly, for $G$:

$$
G = \begin{pmatrix}
\begin{bmatrix}
\nu & 0 \\
0 & G_{rr}
\end{bmatrix}
\end{pmatrix},
\tilde{G} = \begin{pmatrix}
\begin{bmatrix}
\nu & 0 \\
0 & G_{rr}
\end{bmatrix}
\end{pmatrix},
$$

where $\nu = [-\sin(\theta_0) \ -\cos(\theta_0)]^T$. This allows us to write:

$$
\tilde{M}(y)\ddot{y} + \tilde{n}(y, \dot{y}) = \tilde{G}(\theta_0)\tilde{u},
$$

where $\tilde{n}$ is $n = c + g$ with the 3rd element removed, and $\tilde{u} = [\tau_1 \ \tau_2]^T$, where $\tau = [\tau_1 \ \tau_2]^T$. Now we can differentiate (16), yielding (dependencies omitted):

$$
\tilde{M}\ddot{y} + \tilde{n}(\theta) + \tilde{n} = \tilde{G}\tilde{u} + \tilde{v}u_t,
$$

where we have evidenced $\tilde{v} = [v^T \ 0]^T$. Differentiating further:

$$
\tilde{M}\ddot{y} + 2\tilde{M}\ddot{y} + \tilde{n} = \tilde{G}\tilde{u} + 2\dot{\tilde{u}}u_t + \tilde{v}u_t,
$$

but:

$$
\ddot{v} = \begin{bmatrix}
-\cos(\theta_0) \ \dot{\theta}_0 \\
\sin(\theta_0) \ \dot{\theta}_0
\end{bmatrix} = \begin{bmatrix}
h^T \\
0
\end{bmatrix},
$$

where $h = [-\cos(\theta_0) \ \sin(\theta_0)]^T$. From the 3rd row of (7)

$$
\dot{\theta}_0 = \frac{1}{J_0} (d_G u_t + u_r + G_{rp}\tau),
$$

We substitute (20) in (19), and (19) in the last term of (18):

$$
\tilde{v}u_t = \frac{u_t}{J_0} h \cdot (d_G u_t + u_r + G_{rp}\tau) - \nu u_t \dot{\theta}_0 = \gamma + \frac{u_t}{J_0} h \cdot u_r,
$$

where we have introduced the new symbol $\gamma$ for compactness. This finally allows us to write:

$$
\tilde{M}\ddot{y} + 2\tilde{M}\ddot{y} + \tilde{n} = \tilde{G}\tilde{u} + 2\dot{\tilde{u}}u_t - \gamma = \tilde{G}\tilde{u},
$$

where:

$$
\tilde{G} = \begin{pmatrix}
\begin{bmatrix}
\nu & 0 \\
0 & G_{rr}
\end{bmatrix}
\end{pmatrix},
\tilde{G} = \begin{pmatrix}
\begin{bmatrix}
\nu & 0 \\
0 & G_{rr}
\end{bmatrix}
\end{pmatrix},
\ddot{\tilde{u}} = \begin{pmatrix}
\begin{bmatrix}
\nu & 0 \\
0 & G_{rr}
\end{bmatrix}
\end{pmatrix},
\dot{\tilde{u}} = \begin{pmatrix}
\begin{bmatrix}
\nu & 0 \\
0 & G_{rr}
\end{bmatrix}
\end{pmatrix}.
$$

Matrix $\tilde{G} \in \mathbb{R}^{(2+n) \times (2+n)}$ is the decoupling matrix and it is clearly invertible, as long as $u_t \neq 0$, since $|\tilde{G}| = -\frac{\nu^2}{G_{rr}}$.

The relative degree of the extended system is clearly $r = 4(2+n) = 8 + 4n$. Notice the overall new states of the system can be described with $x = [q]^T \in \mathbb{R}^{3(2+n)}$, $\dot{q} \in \mathbb{R}^{3(2+n)}$, $\ddot{q} \in \mathbb{R}^{(1+2n)}$, where $\ddot{q} \in \mathbb{R}^{(1+2n)}$, $\ddot{q} \in \mathbb{R}^{(1+2n)}$, meaning that the total number of states is $\tilde{n} = 2(3+n) + 2(n+1) = 8 + 4n = r$. Thus, no internal dynamics is left, consistently with the notion that the system is flat. The virtual control input can be computed as:

$$
\ddot{u} = \tilde{G}^{-1} (\tilde{M}\ddot{y} + 2\tilde{M}\ddot{y} + \tilde{M}\ddot{n} - 2\dot{\tilde{u}}u_t - \gamma),
$$

where, for a desired trajectory denoted as $y_d$

$$
\dddot{y} = \dddot{y}_d + \mathbf{K}_2(\dot{y}_d - \dddot{y}) + \mathbf{K}_1(\dddot{y}_d - \dddot{y}) + \mathbf{K}_0(y_d - \dddot{y}) + \mathbf{K}_1 \int_0^t (y_d - \dddot{y})\,dr.
$$

The $\mathbf{K}_i$’s are diagonal positive definite matrices, assigned according to the usual linear pole-placement strategies. Specifically, if $\mathbf{K}_{ij}$ is the $i$-th diagonal element of $\mathbf{K}_i$, then each polynomial

$$
p_j(x) = x^5 + K_{3,j}x^4 + K_{2,j}x^3 + K_{1,j}x^2 + K_{0,j}x + K_{-1,j},
$$

must be Hurwitz, i.e. all its roots must have negative real parts; the introduction of an integral error term provides some ability to reject disturbances, such as carried loads and parameter uncertainty (see Sec. VI for its implementation). The inverse of $G$ is easily obtained:

$$
\mathbf{G}^{-1} = \begin{pmatrix}
\begin{bmatrix}
-\sin(\theta_0) & -\cos(\theta_0) \\
\frac{1}{J_0} \cos(\theta_0) & \frac{1}{J_0} \sin(\theta_0)
\end{bmatrix}
\end{pmatrix},
$$

$$
\mathbf{G}_{rr}^{-1} = \begin{pmatrix}
\begin{bmatrix}
G_{rr}^{-1}
\end{bmatrix}
\end{pmatrix},
\mathbf{G}_{rr}^{-1} = \begin{pmatrix}
1 & \cdots & 1 \\
\ddots & \ddots & \ddots \\
0 & \cdots & 1
\end{pmatrix} \in \mathbb{R}^{n \times n}.
$$

It should be noticed, the algorithm makes apparent use of higher-order derivatives of the flat outputs, $\ddot{y}$ and $\dddot{y}$, which are difficult or impossible to estimate directly. However, these can be computed from $\ddot{u}$ and $\dddot{u}$, obtained from integration of appropriate components of the virtual input $\ddot{u}$:

$$
\dddot{y} = \mathbf{M}^{-1} (\mathbf{G}\ddot{u} - \dddot{n}),
$$

$$
\dddot{y} = \mathbf{M}^{-1} (\mathbf{G}\ddot{u} + \dot{\dddot{u}}u_t - \dddot{M}\dddot{y} - \dddot{n}).
$$
VI. NUMERICAL VALIDATION

In this section we present simulations results for testing the proposed controller in a realistic situation in which measurement noises and sampling errors are taken into account. We consider a PAM with $m = 2$, $n = 4$ and $n^2 = 3$. Hence the flat output is $y = [\theta_0 \theta_{01} \cdots \theta_{04} \theta_{02}]^T \in \mathbb{R}^4$. System and simulation parameters are given in Table I of [23]. A pick and place task is chosen for the robot. This is divided in 5 phases: (i) the robot follows a desired trajectory, (ii) the two arms grasp two individual point mass objects with unknown mass for the controller (each mass is 0.25kg), (iii) the objects are carried to another location, where they are unloaded, (iv) phase (i) and (ii) are repeated while following a different trajectory, (v) phase (iii) is repeated while following a different trajectory, and then arms return to the initial configuration. We encourage the reader to watch the attached video for a better understanding.

Notice that the tracking performance is almost perfect, despite the uncertainties. At the time of grasping there are small errors on the tracking of the flat outputs, which are due to the unknown masses. However these errors goes to zero again thanks to the integral terms defined in the controller (see Sec. V), and their effects on both PVTOL CoM and end-effector positions are negligible.

VII. CONCLUSION

In this paper we have introduced a particularly relevant class of aerial manipulators that we named protocentric. We have shown that protocentric aerial manipulators are differentially flat systems regardless of the number of arms and the presence of rigid- or elastic-joints in the arms. We have then proposed a controller for the case of rigid joints only and we have validated the controller with simulations.

In this study we observed that if the aerial vehicle is under-actuated, the number of the compliant actuators increases the relative degree quadratically. This requires much smoother trajectories to be tracked, which would eventually lead to very slow robot motions. Hence choosing a fully actuated aerial vehicle might be beneficial if it is equipped with an arm that has multiple compliant joints. We will investigate this further in our future studies.

In the future we also plan also to extend our theory to the 3D case, to use of sensor-based calibration methods as, e.g., in [24] to retrieve the system parameters on the fly, and to investigate the use of decentralized multi-robot schemes [25]–[28] for the control of multiple aerial manipulators.

REFERENCES