Integrability and non-integrability in Hamiltonian mechanics

V. Kozlov

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Wagner: Allein die Welt! des Menschen Herz und Geist! Möcht' jeglicher doch was davon erkennen.
Faust: Ja, was man so erkennen heisst!
Goethe "Faust"

Contents

1. In 1834 Hamilton expressed the differential equations of classical mechanics, the Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q}, \qquad L\colon \mathbf{R}^n\{\dot{q}\} \times \mathbf{R}^n\{q\} \to \mathbf{R}$$

in the "canonical form":

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}. \tag{1}$$

Here $p = \partial L/\partial \dot{q} \in \mathbf{R}^n$ is the generalized momentum and the Hamiltonian function $H = p\dot{q} - L|_{p,q}$ is the "total energy" of the mechanical system. "In part he had been anticipated by the great French mathematicians: for Poisson, in 1809, had taken the step introducing a function[(1)]

$$\sum_{r=1}^{n} p_r \dot{q}_r - T^{1)},$$

and expressing it in terms of $q_1, q_2, \ldots, q_n$, and had actually derived half of Hamilton's equations: Lagrange in 1810 had obtained a particular set of equations (for the variation of elements) in the Hamiltonian form the disturbing function taking the place of H. Moreover, the theory of non-linear partial differential equations of the first-order had led to systems of ordinary differential equations possessing this form: as was shown by Pfaff in 1814–15 and Cauchy in 1819 (completing the earlier work of Lagrange and Monge), the equations of the characteristics of a partial differential equation

$$f(x_1, x_2, \ldots, x_n, p_1, p_2, \ldots, p_n) = 0, \tag{2}$$

where

$$p_s = \frac{\partial z}{\partial x_s},$$

are

$$\frac{dx_1}{\partial f/\partial p_1} = \frac{dx_2}{\partial f/\partial p_2} = \ldots = \frac{dx_n}{\partial f/\partial p_n} = \frac{dp_1}{-\partial f/\partial x_1} = \frac{dp_2}{-\partial f/\partial x_2} = \ldots = \frac{dp_n}{-\partial f/\partial x_n}.$$

Hamilton's investigation was extended to the cases when the kinetic potential contains the time, etc. by Ostragradskii in 1845–50 and by Donkin in 1854" (Whittaker [55]).[(2)]

2. The problem of integration of Hamiltonian systems (not then written in canonical form) had already been discussed in works of the brothers Bernoulli, Clairaut, D'Alembert, Euler and, of course, Lagrange, in connection with the application of the ideas and principles of Newton to various problems of mechanics. Only those problems that could be solved by means

[(1)] T is the kinetic energy of the system.

[(2)] "It would be rather desirable to make a detailed critical study of the historical development. In fact, the traditional references to the origin of the fundamental mathematical notions in analytical dynamics are almost always incorrect" (Wintner [54]).

of finitely many algebraic operations and "quadratures", the computation of integrals of known functions, were regarded as "soluble" (integrable). However, most of the actual problems of dynamics (say, the n-body problem) turned out to be "non-integrable" (more precisely, not integrated). Only in the simplest cases when the system had just one degree of freedom ($n = 1$) or, decomposed into several independent one-dimensional systems, did the integration turn out to be possible, due to the presence of integrals of the type of conservation of the total energy (H = const).

3. Hamilton (in 1834) and Jacobi (in 1837) developed a general method of integrating the equations of dynamics, based on the introduction of special canonical coordinates.

The idea of the Hamilton-Jacobi method appears in the work of Pfaff and Cauchy (and, even earlier, in the investigations of Lagrange and Monge) on the theory of characteristics. The essence of this is the following: a transformation of independent variables $p, q \to P, Q$ of the form

$$p=\frac{\partial S}{\partial q}, \quad Q=\frac{\partial S}{\partial P}; \quad S(P, q): R^{2n} \to R$$

takes the canonical equations (1) to the canonical equations

$$\dot{P}=-\frac{\partial K}{\partial Q}, \quad \dot{Q}=\frac{\partial K}{\partial P} \tag{3}$$

with the Hamiltonian function

$$K(P, Q)=H(p, q)|_{P, Q}.$$

If K does not depend on Q, then (3) can be integrated immediately: $P = P_0$, $Q = Q_0 + t\frac{\partial K}{\partial P}\Big|_{P_0}$. Thus, the problem of integrating the canonical equations (1) reduces to a search for a "generating" function $S(P, q)$, satisfying the non-linear Hamilton-Jacobi equation

$$H\left(\frac{\partial S}{\partial q}, q\right)=K(P),$$

which is a particular case of (2).

If a problem is solved by the Hamilton-Jacobi method, then the functions $P_1(p, q), \ldots, P_n(p, q)$ are first integrals, which, as is easy to verify, are in involution, that is, their Poisson brackets

$$\{P_i, P_j\}=\sum_s\left(\frac{\partial P_i}{\partial q_s}\frac{\partial P_j}{\partial p_s}-\frac{\partial P_i}{\partial p_s}\frac{\partial P_j}{\partial q_s}\right)$$

are identically zero. This idea was developed by Bour [63] and Liouville [71] in 1855. By means of the Hamilton-Jacobi method they proved that a Hamiltonian equation with n degrees of freedom can be integrated if n independent integrals in involution are known. This is essentially an invariant statement of the Hamilton-Jacobi method. Within the framework of this circle of ideas are works of Jacobi, Liouville, Kovalevskaya, Clebsch, and other authors in which a number of new problems in dynamics, some of which are very non-trivial, were solved. In later works the attention was concentrated on the qualitative investigation of the motion of Hamiltonian systems that can be solved by the Hamilton-Jacobi method, first of all by the method of separation of the variables. In scientific usage the "action-angle" variables, specifically for integrable systems, made their appearance. These "were introduced by Delauney (see [66]) for the discussion of astronomical perturbations. Later, they were found to be admirably suited to the older form of quantum mechanics, for the Bohr-Sommerfeld quantization consisted in making each action-variable an integral multiple of Planck's constant" (Synge [52]). Initially conditions for quantization were stated for systems with separated variables [11], but it gradually became clear that in the most general case the compatible levels of a complete set of integrals in involution, in the compact case, are homeomorphic to many-dimensional tori, that the motion in them in the corresponding "angle" variables is conditionally periodic, and that the "action" variables are the integrals $\frac{1}{2\pi}\oint p\,dq$ over independent cycles, covering the tori in various ways (see, for example, [57], [54]; there are modern accounts in the books [7], [16]). Systems with a complete set of integrals in involution are now called completely integrable.

4. On the other hand, the efforts of Clairaut, Lagrange, Poisson, Laplace, and Gauss, directed towards an approximate solution of applied problems of celestial mechanics, lead ultimately to the creation of perturbation theory. It was proposed to search for solutions of the equations of motion in the form of series in powers of a small parameter (for example, in the solar system such a parameter is the ratio of the mass of Jupiter to the mass of the Sun). Afterwards Delauney, Hilden and Lindstedt modified perturbation theory by using the Hamilton-Jacobi method. Let $H = H_0 + \epsilon H_1 + \epsilon^2 H_2 + \dots$ ($\epsilon \ll 1$) and suppose that the "unperturbed" problem with the Hamiltonian H_0 is integrable. One then looks for a generating function S in the form of a series $S_0 + \epsilon S_1 + \dots$ satisfying the equation

$$(4)\qquad H_0\left(\frac{\partial S}{\partial q}, q\right) + \varepsilon H_1\left(\frac{\partial S}{\partial q}, q\right) + \dots = K_0(P) + \varepsilon K_1(P) + \dots,$$

where the functions K_i are for the present unknown. The functions S_0 and K_0, by the assumptions, can be found from (4) with $\epsilon = 0$. The K_i and S_i, $i \geqslant 1$, are found consecutively: the resulting arbitrariness in their definition can be removed by a condition on the absence of so-called "secular" terms.

Thus, the perturbed problem can be regarded as "solved" if the series of perturbation theory are well-defined and convergent. Their convergence would lead to a number of important consequences (in particular, the eternal stability of the solar system). To anticipate, we mention a disappointing result due to Poincaré: in general, because of the presence of the so-called small divisors, the series of perturbation theory diverge. Moreover, the series of an improved perturbation theory proposed by Poincaré and Bolinom, in which solutions are expanded in power series in

$\sqrt{\epsilon}$ not ϵ, also diverge. We note that if the series of perturbation theory do converge then the equations of motion have a complete set of integrals in involution, which can be expressed as convergent power series in ϵ (or $\sqrt{\epsilon}$).

Subsequently Whittaker, Cherry, and Birkhoff later (in 1916–1927) obtained similar results for Hamiltonian systems in neighbourhoods of equilibrium positions and periodic trajectories. They showed that, in general, there is a canonical transformation specified by a formal power series, after which the Hamiltonian equations integrate simply. Hamiltonian systems with convergent Birkhoff transformations are sometimes called "integrable in the sense of Birkhoff". In this case also there is a complete set of independent commuting integrals of special form.

5. As we see, each new generation interprets in its own way the essence of the problem of integration of Hamiltonian systems. However, a common feature of the diverse approaches to this problem is the presence in Hamiltonian systems of independent integrals—"conservation laws". Unfortunately, in a typical situation, integrals not only cannot be found, but do not exist at all, since the trajectories of Hamiltonian systems, generally speaking, do not lie on integral manifolds of a small number of dimensions.

The first rigorous results on non-integrability of Hamiltonian systems are due to Poincaré. In [47] (1890) he proved the non-existence of analytic integrals that can be represented in the form of convergent power series in a small parameter. Hence, in particular, there follows the divergence of the series of the various versions of perturbation theory. Poincaré also mentioned qualitative phenomena in the behaviour of phase trajectories that prevent the appearance of new integrals. Among them are the creation of isolated periodic solutions and the bifurcation of asymptotic surfaces. Poincaré applied his general method to varous versions of the n-body problem. It turned out that, apart from the known classical conservation laws, the equations of motion do not have new analytic integrals relative to the masses of the planets. The non-integrability of the n-body problem for fixed values of their masses has not yet been proved.(1)

Even earlier, in 1887, Bruns proved the absence of new algebraic integrals in the three-body problem (for all values of the point masses). Afterwards similar results were obtained by Husson (1906) and other authors in the dynamics of a rigid body with a fixed point. We can, however, agree with

(1)Here we must make two reservations. Firstly, the investigations of Alekseev on final motions in the three-body problem imply the non-integrability of the restricted three-body problem when two of the masses are equal [1]. Secondly, the question is of integrals on the whole phase space of the problem. A complete set of integrals always exists locally and, consequently, may exist in larger domains, where the motion is not recurrent. Apparently, an example is the domain of positive energy in the many-body problem (a conjecture of Alekseev).

Wintner ([54], §129), that these "elegant negative results do not have any dynamical significance" in view of their non-invariance under changes of variables.

The truth is that in practically all integrated problems the first integrals turn out to be either rational functions or simply polynomials. Also, solutions, as functions of complex time, often turn out to be meromorphic. As examples we can quote Jacobi's problem on the motion of a point on a triaxial ellipsoid, Kovalevskaya's spinning top and Clebsch's case of the motion of a rigid body in an ideal fluid. In addition, the investigations of Kovalevskaya and Lyapunov on the classical problem of the rotation of a heavy top showed that the general solution of the equations of motion are single-valued functions of time only when there is an additional polynomial integral. In this connection there arose the interesting problem of the relation between the existence of single-valued holomorphic integrals and branching of solutions in the complex time plane. Its formulation dates back to Painlevé.

In 1941–1954 Siegel investigated the question of integrability of Hamiltonian systems close to stable positions of equilibrium. He proved that in a typical situation the Hamiltonian equations do not have a complete set of analytic integrals and the Birkhoff transformation diverges. Siegel's proof of the divergence of the Birkhoff transformation dates back in principle to the investigations of Poincaré: it is based on a careful analysis of the families of non-degenerate long-periodic solutions.

After the work of Poincaré it became clear in the 20-th century that the impossibility of extending local integrals to integrals "in the large" is connected with the complex behaviour of phase trajectories on the level sets of those integrals (not unlike the energy integral), which are known but are not present in sufficient numbers. To put it simply, on an integral level there must exist trajectories that are everywhere dense in some domain in it (see the discussion of these problems, for example, in [52] and [54]). Levi-Civita had proposed to call m-imprimitive systems having m but not $m+1$ integrals "in the large". A direct application of the idea of complex behaviour of phase trajectories to the problem of integrability can be found in the above papers of Alekseev.

6. Recently some of the possibilities of the Poincaré method have been realized, which make it possible to prove non-integrability of a number of important problems of Hamiltonian mechanics, and also to find new phenomena of a qualitative nature that obstruct integrability. As a result an independent part of the theory of Hamiltonian systems has taken shape. In this paper the author wishes to continue the tradition of a "fairly popular account of the proofs of its basic result", of which Alekseev wrote in the preface to (the Russian translation of) Moser's book [41].

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CHAPTER I

HAMILTONIAN SYSTEMS

There are various approaches to an exposition of Hamiltonian mechanics. They can be found in the books [3], [7], [55], and [61]. In this chapter we recall the definitions of the fundamental objects of Hamiltonian mechanics, and also we consider several concrete Hamiltonian systems, which in what follows we shall use repeatedly as examples.

§1. Hamilton's equations

1. Let M be an even-dimensional manifold. The set of all infinitely differentiable functions $f: M \to \mathbf{R}$ is denoted by $C^\infty(M)$. A symplectic (canonical) structure Σ on M is a bilinear map
with the following properties:

1) $\{f, g\} = -\{g, f\}$ (skew-symmetry),

2) $\{fg, h\} = f\{g, h\} + g\{f, h\}$ (Leibniz' rule),

3) $\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0$ (the Jacobi identity),

4) if $m \in M$ is not a critical point for a function f, then there is a smooth function g such that $\{f, g\}(m) \neq 0$ (non-degeneracy).(1)

The pair (M, Σ) is called a symplectic (canonical) manifold. The function $\{f, g\}$ is called the Poisson bracket of f and g. It makes the linear space $C^\infty(M)$ into an infinite-dimensional Lie algebra over $\mathbf{R}$. Its centre consists solely of the constant functions.

Theorem (Darboux). *In a small neighbourhood of any point of M there are local coordinates $x_1, \ldots, x_n;\ y_1, \ldots, y_n$ ($2n = \dim M$) such that*

$$\{f, g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i} \frac{\partial g}{\partial y_i} - \frac{\partial f}{\partial y_i} \frac{\partial g}{\partial x_i} \right).$$

The coordinates x and y are called symplectic (canonical). A proof of Darboux's theorem can be found in [7] or [51].

2. Let $H: M \to \mathbf{R}$ be a smooth function. A Hamiltonian system on (M, Σ) with Hamiltonian H is the name for the differential equation

$$\dot{F} = \{F, H\} \quad \forall F \in C^\infty(M). \tag{1.1}$$

(1)The idea of an axiomatic definition of the bracket goes back apparently to Dirac [15].

A solution of it is a smooth map $m:\Delta\to M$ (Δ is an interval on $\mathbf{R}$) such that

$$\frac{dF(m(t))}{dt}=\{F,\ H\}(m(t))\quad \forall t\in\Delta.$$

In the symplectic coordinates x, y (1.1) is equivalent to the $2n$ canonical Hamiltonian equations:

$$\dot{x}_i=\{x_i,\ H\}=\frac{\partial H}{\partial y_i},\quad \dot{y}_i=\{y_i,\ H\}=-\frac{\partial H}{\partial x_i}\qquad (1\leqslant i\leqslant n).$$

These equations can be written in more compact form if we introduce the skew-symmetric matrix

$$\mathfrak{J}=\begin{Vmatrix} 0 & E\\ -E & 0\end{Vmatrix},$$

where E is the $n\times n$ unit matrix. If $(x,\ y)=z$, then

(1.2) $$\dot{z}=\mathfrak{J}\frac{\partial H}{\partial z}.$$

M is called the state space, or phase space, of (1.1), and (dim M)/2 is the number of its degrees of freedom.

3. A diffeomorphism $\varphi:M\to M$ is called canonical if it preserves the Poisson bracket: $\{f,\ g\}(m)=\{f,\ g\}(\varphi m)$. Of course, the canonical diffeomorphisms of a symplectic manifold $(M,\ \Sigma)$ form a group.(1) The phase flow g^t_H of any Hamiltonian system on M is a one-parameter subgroup of canonical diffeomorphisms of M.

In local symplectic coordinates the canonical condition for $\varphi:x,\ y\to X,\ Y$ may be expressed by either of the two following equivalent conditions:

1) for each closed contour γ

$$\oint_\gamma y\,dx=\oint_\Gamma Y\,dX\ \left(=\oint_\gamma Y(x,\ y)\,dX(x,\ y)\right),$$

where Γ is the image of γ under φ.

2) $J^*\mathfrak{J}J=\mathfrak{J}$, where J is the Jacobian matrix of φ.

In the new coordinates $(X,\ Y)=Z$, (1.2) again has Hamiltonian form

$$\dot{Z}=\mathfrak{J}\frac{\partial K(Z)}{\partial Z},$$

where $K(Z)=H(z)$.

A symplectic structure on M can be specified by a symplectic atlas: a set of mutually compatible charts, where the transition from chart to chart is a smooth canonical map. For example, let $M=T^*N$ be the cotangent bundle of a smooth manifold N. A symplectic structure on T^*N is specified by a collection of local coordinates x, y, where x are local coordinates on N and y are the components of linear differential forms from T^*_xN in the basis dx.

(1) "...whenever you have to do with a structure endowed entity Σ, try to determine its group of automorphisms... You can expect to gain deep insight into the constitution of Σ in this way." (Weyl "Symmetry".)

It is helpful to study canonical diffeomorphisms by the apparatus of generating functions. For example, let $\det \| \partial X / \partial x \| \neq 0$. In this case we can solve (at least locally) the equation $X = X(x, y)$ for x and regard X and y as "independent" coordinates. Then $x = x(X, y)$, $Y = Y(X, y)$. If we put

$$S = \int_{X_0, y_0}^{X, y} x\,dy + Y\,dX$$

(the value of the integral is independent of the path of integration), then

$$x = \frac{\partial S}{\partial y}, \quad Y = \frac{\partial S}{\partial X}.$$

Then $S(X, y)$ is called a generating function of the canonical map φ. If, for example, φ is the identity map, then $S = Xy$.

4. Suppose that the smooth functions H and F commute (are "in involution"): $\{H, F\} = 0$. Then F is a first integral of the canonical system with Hamiltonian H and vice versa. The phase flows g_H^t and g_F^s of these systems also commute on M.

Since

$$\{\{F, G\}, H\} = \{\{F, H\}, G\} - \{\{G, H\}, F\},$$

the integrals of any Hamiltonian system form a subalgebra of the Lie algebra of all smooth functions on M (Poisson's theorem).

5. A natural mechanical system is a triple (N, T, V), where N is a smooth manifold (the state space), T is a Riemannian metric on N (the kinetic energy), and V is a smooth function on N (the potential of a force field). The motions of this system are smooth maps $q(t): \mathbf{R} \to N$ that are extremals of the action functional:

$$\int_{t_1}^{t_2} L(\dot{q}(t), q(t))\,dt,$$

where $\dot{q}(t)$ is the tangent vector to N at $q(t)$, $L = T + V$ is the Lagrangian. A time change of the local coordinates q on N is described by the Euler-Lagrange equation:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q}.$$

We consider the natural map $TN \to T^*N$ generated by the Riemannian metric: $(q, \dot{q}) \to (q, p)$, where

$$p = \frac{\partial T}{\partial \dot{q}}.$$

Obviously, p is a linear form on T_qN. Since the quadratic form T is positive definite, the linear map $\dot{q} \to p$ is an isomorphism of the linear spaces T_qN and T_q^*N.

We consider the total energy of the system, $H: T^*N \to \mathbf{R}$, which is defined by the formula

$$H(p, q) = p\dot{q} - L|_{\dot{q} \to p} = \frac{\partial T}{\partial \dot{q}} \dot{q} - T - V = T - V|_{p, q}.$$

Theorem (Poisson-Hamilton). *The functions $p(t)$ and $q(t)$ satisfy the canonical equations*

$$\dot{p}=-\frac{\partial H}{\partial q}, \quad \dot{q}=\frac{\partial H}{\partial p}.$$

A similar construction is valid for the more general "seminatural" systems, when the Lagrangian function contains additional terms that are linear in the velocities.

It is often necessary to consider non-autonomous Hamiltonian systems when the Hamiltonian explicitly depends on time.

§2. The motion of a rigid body

1. In many problems of mechanics the rotation of a rigid body in three-dimensional Euclidean space can be described by equations of the following form:

$$\dot{M}=M\times\omega+e\times u, \quad \dot{e}=e\times\omega, \tag{2.1}$$

where $\omega = \partial H/\partial M$, $u = \partial H/\partial e$, and $H(M, e)$ is a known function on $\mathbf{R}^6 = \mathbf{R}^3\{M\}\times \mathbf{R}^3\{e\}$. The vectors ω and M are called the angular velocity and the kinetic momentum of the body. The physical meaning of e and u depend on the concrete statement of the problem.

For example, let us consider the rotation of a heavy rigid body with a fixed point. In this case e is a vertical unit vector and $u = \epsilon r$ is the product of the weight of the body by the radius vector of the centre of mass. The function H, the total energy, has the following form:

$$\frac{1}{2}\langle M, J^{-1}M\rangle+\varepsilon\langle r, e\rangle,$$

where J^{-1} is a positive definite self-adjoint operator. The equations (2.1) are usually written on the following form:

$$J\dot{\omega}=J\omega\times\omega+\varepsilon e\times r, \quad \dot{e}=e\times\omega.$$

These are called the Euler-Poisson equations ([3], [14]). Since J is self-adjoint, in some orthogonal frame ξ_1, ξ_2, ξ_3 connected with the body its matrix (also denoted by J) can be brought to diagonal form: $J = \operatorname{diag}(J_1, J_2, J_3)$. The eigendirections of J are called the axes of inertia and the eigenvalues, the numbers J_1, J_2, J_3, the principal moments of inertia of the body. This problem contains six parameters J_1, J_2, J_3, and ϵr_1, ϵr_2, ϵr_3 (r_s are the coordinates of the centre of mass relative to the axes of inertia).

In the problem of the motion of a rigid body in an infinite ideal liquid, H is a positive definite quadratic form

$$\langle AM, M\rangle/2 + \langle BM, e\rangle + \langle Ce, e\rangle/2.$$

The vectors e and u are usually called the impulsive force and the impulsive momentum and the equations (2.1) are named after Kirchhoff. The matrices A, B, and C are symmetric: without loss of generality we may assume that $A = \mathrm{diag}(a_1, a_2, a_3)$. Thus, in the general case the quadratic form H contains 15 parameters. If the rigid body has three mutually perpendicular planes of symmetry (say, a triaxial ellipsoid), then $B = 0$ and $C = \mathrm{diag}(c_1, c_2, c_3)$.

2. The equations (2.1) have three integrals: $F_1 = H$, $F_2 = \langle M, e \rangle$, and $F_3 = \langle e, e \rangle$. In the problem of the rotation of a rigid body around a fixed point $F_3 = 1$, obviously. The integral levels $I_{23} = \{F_2 = f_2, F_3 = f_3 > 0\} \subset \mathbf{R}^6$ are diffeomorphic to the (co)tangent bundle of the two-dimensional sphere.

We define in $\mathbf{R}^6\{M, e\}$ the bracket $\{,\}$ by putting

$$(2.2) \quad \{M_1, M_2\} = -M_3, \ldots, \{M_1, e_1\} = 0, \{M_1, e_2\} = -e_3,$$
$$\{M_1, e_3\} = e_2, \ldots, \{e_i, e_j\} = 0.$$

Taking the operation $\{,\}$ to be bilinear, skew-symmetric and satisfying Leibniz' rule we can compute the "Poisson bracket" of any two smooth functions on $\mathbf{R}^6$ by using (2.2). The bracket (2.2) satisfies the Jacobi identity. The equations (2.1) can be expressed in the following Hamiltonian form:

$$\dot{M}_s = \{M_s, H\}, \quad \dot{e}_s = \{e_s, H\} \qquad (1 \leqslant s \leqslant 3).$$

However, the bracket $\{,\}$ thus defined is degenerate: any smooth function commutes with the integrals F_2 and F_3. This circumstance permits us to restrict the bracket $\{,\}$ to the integral levels I_{23}. Let $x \in I_{23}$ and let f and g be smooth functions on I_{23}. We extend them to smooth functions F and G on the whole of $\mathbf{R}^6\{M, e\}$ and put

$$\{f, g\}_*(x) = \{F, G\}(x).$$

This is well-defined (independent of the method of extension) and the bracket $\{,\}_*$ is non-degenerate and gives a symplectic structure on I_{23}.

***Theorem* 1.** *The equations* (2.1) *on I_{23} can be expressed in the form of a Hamiltonian equation $\dot{f} = \{f, h\}_*$, where h is the restriction of H to I_{23}* [45].

This construction looks particularly simple when $f_2 = 0$. We put $M = p \times e$. If $f_3 > 0$ and $f_2 = \langle M, e \rangle = 0$, then the vector p exists and is unique up to a shift along e. Let $K(p, e) = H(p \times e, e)$.

***Theorem* 2** [33]. *The functions $p(t)$ and $e(t)$ satisfy the canonical equations*

$$\dot{p} = -\frac{\partial K}{\partial e}, \quad \dot{e} = \frac{\partial K}{\partial p}.$$

In $\mathbf{R}^6\{p, e\}$ there is a "standard" symplectic structure, generated by the Poisson bracket: $\{p_i, p_j\} = 0, \{e_i, e_j\} = 0$, $\{p_i, e_j\} = \delta_{ij}$ $(1 \leqslant i, j \leqslant 3)$. In this structure (2.2) holds for the Poisson brackets of M_i, e_j. The vectors e, p, and M have a simple interpretation: e is the radius vector of a point in three-dimensional space, p is its momentum and M is its kinetic momentum (taken with the opposite sign). We emphasize that the coordinates $(e_1, e_2, e_3) = e$ are "surplus". When $f_2 \neq 0$, the change of variables $M = p \times e$ must be somewhat "rectified".

Proof of Theorem 2. We first calculate

$$\dot{e}=\frac{\partial K}{\partial p}=\frac{\partial H}{\partial M}\frac{\partial M}{\partial p}=e\times\omega.$$

Since $M = p \times e$,

$$\dot{M}=\dot{p}\times e+p\times\dot{e}=-\frac{\partial K}{\partial e}\times e+p\times(e\times\omega),$$

$$\frac{\partial K}{\partial e}=\frac{\partial H}{\partial e}+\frac{\partial H}{\partial M}\frac{\partial M}{\partial e}=u+\omega\times p.$$

Hence,

$$\dot{M}=-u\times e+e\times(\omega\times p)+p\times(e\times\omega)=M\times\omega+e\times u.$$

As required.

3. On I_{23} we introduce special canonical coordinates $L, G;\ l, g \bmod 2\pi$ (Fig.1), which are convenient in what follows. For simplicity we restrict ourselves to the case when $f_2 = 0$. In $\mathbf{R}^3\{e\}$ we consider the sphere $\langle e, e\rangle = f_3 > 0$. We introduce the node line, the intersection of the planes passing through $e = 0$ and perpendicular to the vectors M and ξ_3. Let l and g be the angles between ξ_1 and ξ_y and between ξ_y and e (ξ_y is the "direction" vector of the node line).

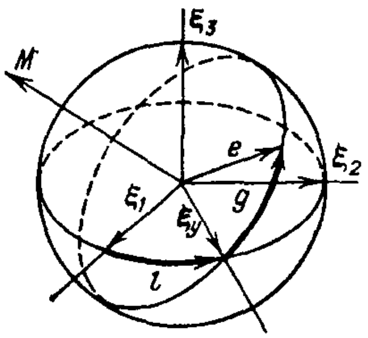

Fig. 1

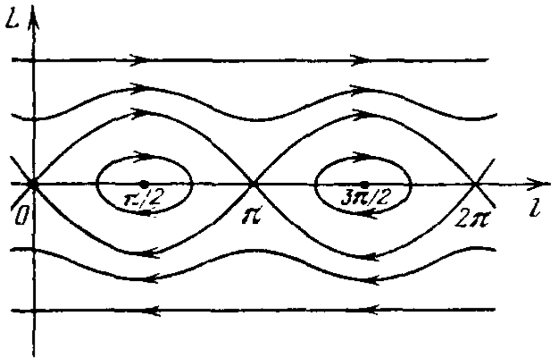

Fig. 2

We put, finally, $L = \langle M, \xi_3\rangle$ and $G = |M|$. The Hamiltonian $K: I_{23} \to \mathbf{R}$ can be expressed as a function of L, G, l, and g that is 2π-periodic in l and g.

Theorem 3 [62]. *The functions L, G, l, and $g|_t$ satisfy the canonical equations*

$$\dot{L}=-\frac{\partial K}{\partial l},\quad \dot{l}=\frac{\partial K}{\partial L},\quad \dot{G}=-\frac{\partial K}{\partial g},\quad \dot{g}=\frac{\partial K}{\partial G}.$$

We omit the proof of this theorem, which is based on simple formulae of vector analysis.

Let $e=\sum e_i\xi_i$. Then

$$e_1/\sqrt{f_3}=\cos l\cos g-\frac{L}{G}\sin l\sin g,\quad e_2/\sqrt{f_3}=\sin l\cos g+\frac{L}{G}\cos l\sin g,$$

$$e_3/\sqrt{f_3}=\sqrt{1-\left(\frac{L}{G}\right)^2}\sin g.$$

When $f_2 \neq 0$, this formula becomes somewhat complicated (details can be found in [32]).

4. The case when the total energy reduces to a quadratic form $\langle M, J^{-1}M\rangle/2$ is called the Euler problem. It is realized, for example, in the rotation of a heavy rigid body around a fixed point, when the centre of mass coincides with the point of suspension. Let ω_1, ω_2, ω_3 be the projections of the angular velocity ω onto the eigendirections of J. Then

$$J_1\omega_1=\sqrt{G^2-L^2}\sin l, \quad J_2\omega_2=\sqrt{G^2-L^2}\cos l, \quad J_3\omega_3=L.$$

Consequently,

$$H=\frac{1}{2}\langle J\omega, \omega\rangle=\frac{G^2-L^2}{2}\left(\frac{\sin^2 l}{J_1}+\frac{\cos^2 l}{J_2}\right)+\frac{L^2}{2J_3}. \tag{2.3}$$

The Hamiltonian of the Euler problem has the same form even for non-zero values of f_2. Since G is a first integral, integration of the equations of motion reduces to the solution of the one-dimensional Hamiltonian system with the Hamiltonian function (2.3), in which the variable $G = G_0$ is a parameter. The phase portrait of this system is illustrated in Fig. 2 (under the assumption that $J_1 < J_2 < J_3$). The phase trajectories are contained in the ring $C = \{L, l: |L| \leqslant G_0, l \bmod 2\pi\}$. This ring can be regarded as a cross-section of the three-dimensional level sets of the integral of the modulus of the angular momentum, $\{G = G_0\} \subset I_{23}$, by the plane $g = 0$. Since $\dot{g} \neq 0$ for $G \neq 0$, any trajectory intersects C. Thus, there arises a natural map of C onto itself. It preserves the area element $dLdl$ and rotates the boundaries of the ring in opposite directions. To the fixed points of this map here correspond the periodic solutions, the constant rotations of the rigid body around the axes of interia. The rotations around the middle axis (with moment of inertia J_2) are unstable.

§3. The oscillations of a pendulum

1. Suppose that the point of suspension of a mathematical pendulum of length l performs an oscillation with the periodic law $\epsilon\xi(t)$, ϵ = const. If x is the angle of deviation of the pendulum from the vertical, then the kinetic energy is

$$T=\frac{v^2}{2}=\frac{1}{2}(l^2\dot{x}^2+\varepsilon^2\dot{\xi}^2+2\varepsilon l\dot{x}\dot{\xi}\sin x).$$

Let g be the acceleration of free fall. Then the potential energy of the pendulum is

$$U=-g(l\cos x+\varepsilon\xi(t)).$$

The Lagrange equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}}=\frac{\partial L}{\partial x}, \qquad L=T-U,$$

has the following form:

$$\ddot{x}+\omega^2(1+\varepsilon f(t))\sin x=0, \tag{3.1}$$

where $\omega^2 = g/l$ and $f = \ddot{\xi}/g$ is a periodic function of time.

This equation, of course, is Hamiltonian: the canonical coordinates are $x \bmod 2\pi$, $p = \dot{x}$, and the Hamiltonian function is

$$H = \frac{p^2}{2} - \omega^2 (1 + \varepsilon f) \cos x. \tag{3.2}$$

The state space is the circle $S^1\{x \bmod 2\pi\}$, and the phase space is the cylinder $S^1 \times \mathbf{R}\{p\}$.

For $\epsilon = 0$ we have an integrable problem with one degree of freedom (a mathematical pendulum of constant length).

2. In many problems of mechanics there occur equations resembling (3.1). Let us consider, for example, the planar oscillations of a satellite in an elliptical orbit. The equation of oscillations can be expressed in the following form:

$$(1 + e \cos \nu) \frac{d^2\delta}{d\nu^2} - 2e \sin \nu \frac{d\delta}{d\nu} + \mu \sin \delta = 4e \sin \nu. \tag{3.3}$$

Here e is the eccentricity of the orbit and μ is a parameter characterizing the mass distribution of the satellite. The meaning of the variables δ and ν is clear from Fig. 3.

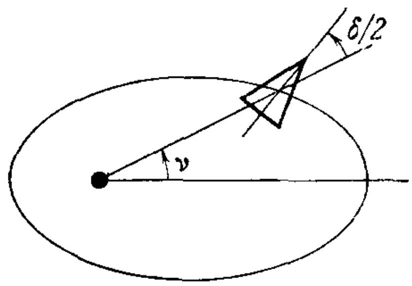

Fig. 3

This equation can be expressed in Hamiltonian form (Burov):

$$\frac{dp}{d\nu} = -\frac{\partial H}{\partial \delta}, \quad \frac{d\delta}{d\nu} = \frac{\partial H}{\partial p},$$

$$H = \frac{1}{2}\left[\frac{p}{1 + e\cos\nu} - 2(1 + e\cos\nu)\right]^2 - (1 + e\cos\nu)\,\mu\cos\delta.$$

For satellite motion in almost circular orbits ($e \ll 1$) the equation (3.3) is close to the equation of oscillations of an ordinary pendulum.

§4. The restricted three-body problem

Suppose that the Sun $\mathcal{S}$ and Jupiter $\mathcal{Y}$ rotate around a common centre of mass with circular orbits. The units of length, time, and mass are taken so that the angular velocity of rotation, the sum of the masses of $\mathcal{S}$ and $\mathcal{Y}$, and also the gravitational constant are 1. It is easy to see that then the distance $\mathcal{S}\mathcal{Y}$ is also 1.

The equations of motion of an asteroid $\mathcal{A}$ in a moving system of coordinates can be described in the form of two equations

$$\ddot{x} - 2\dot{y} = \frac{\partial V}{\partial x}, \quad \ddot{y} + 2x = \frac{\partial V}{\partial y}, \tag{4.1}$$

where $V = (x^2+y^2)/2+(1-\mu)/\rho_1+\mu/\rho_2$, μ is the mass of Jupiter, and ρ_1 and ρ_2 are the distances from $\mathcal{A}$ to $\mathcal{S}$ and $\mathcal{J}$. The equations (4.1) have the integral

$$H = \frac{\dot{x}^2+\dot{y}^2}{2} - V(x, y),$$

the so-called Jacobi integral. These equations can be expressed in canonical form: the Hamiltonian function H is the total energy of the asteroid.

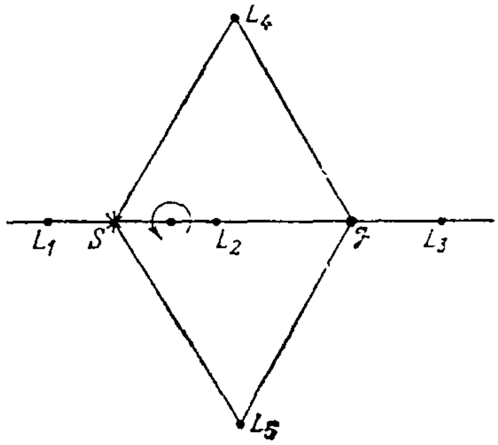

Fig. 4

It is well known that (4.1) has five positions of equilibrium $L_1 - L_5$, the so-called libration points. The equilibrium positions L_1–L_3 on the line from the Sun to Jupiter were discovered by Euler. They are always unstable. The remaining two positions of equilibrium L_4 and L_5 (which were discovered by Lagrange) complement the points $\mathcal{S}$ and $\mathcal{J}$ to the vertices of equilateral triangles. The equilibrium positions L_4 and L_5 are stable in the linear approximation if $\mu(1-\mu) < 1/27$. The problem of their Lyapunov stability turned out to be considerably more complicated. By means of a theorem of Kolmogorov on the preservation of conditionally periodic motions, various authors have shown that the triangles of libration points are stable for all μ (satisfying the stability condition in linear approximation), except for two values $\mu_1 = 0.0242938...$ and $\mu_2 = 0.013560...$ If $\mu = \mu_1$ or μ_2, then the frequences of linear oscillations are in resonance $1:2$ or $1:3$. Markeev has proved the Lyapunov instability of the triangles of libration points for these exceptional values of the parameter [37].

§5. Some problems of mathematical physics

1. From hydromechanics it is known [36] that the motion of n point (cylindrical) vortices in the plane (in space) can be described by the following system of $2n$ differential equations:

$$(5.1) \qquad \begin{cases} \Gamma_s \dot{x}_s = -\dfrac{\partial H}{\partial y_s}, \quad \Gamma_s \dot{y}_s = \dfrac{\partial H}{\partial x_s} \quad (1 \leqslant s \leqslant n), \\ H = \dfrac{1}{\pi} \sum\limits_{s \neq k} \Gamma_s \Gamma_k \log\left((x_s - x_k)^2 + (y_s - y_k)^2\right). \end{cases}$$

Here (x_s, y_s) are the Cartesian coordinates of the s-th vortex with intensity Γ_s. It is assumed that all the Γ_s are non-zero. The equations (5.1) are canonical: a symplectic structure in $\mathbf{R}^{2n}\{x, y\}$ is given by the Poisson bracket

$$\{f, g\} = \sum_s \frac{1}{\Gamma_s}\left(\frac{\partial f}{\partial y_s}\frac{\partial g}{\partial x_s} - \frac{\partial f}{\partial x_s}\frac{\partial g}{\partial y_s}\right).$$

In addition to the Hamiltonian H they have another three independent integrals:

$$P_x = \sum \Gamma_s x_s,\quad P_y = \sum \Gamma_s y_s,\quad M = \frac{1}{2}\sum \Gamma_s (x_s^2 + y_s^2).$$

It is easy to verify that

$$\{P_x, P_y\} = -\sum \Gamma_k = \text{const},\quad \{P_x, M\} = -P_y,\quad \{P_y, M\} = P_x.$$

If the sum of the intensities of the system of vortices is zero, then P_x and P_y commute.

2. Kontopoulos in his paper [64] on galactic models considered some Hamiltonian systems in neighbourhoods of positions of equilibrium that admit resonance relations between frequences. The simplest such system with the Hamiltonian

$$H = \frac{1}{2}\left(y_1^2 + y_2^2 + x_1^2 + x_2^2 + 2x_1^2 x_2 - \frac{2}{3}x_2^3\right)$$

was investigated in detail by Hénon and Heiles by means of numerical calculations [69]. In this problem the frequences of small oscillations are equal to each other. In Gustavson's paper [68] there is an interesting discussion of the numerical results of Hénon-Heiles in connection with the construction of formal integrals of Hamiltonian systems.

3. The study of homogeneous two-component models of the Yang-Mills equations is connected with the investigation of the Hamiltonian system with the Hamiltonian

$$H = \frac{1}{2}(p_1^2 + p_2^2) + q_1^2 q_2^2$$

(see [16], [17]).

CHAPTER II

INTEGRATION OF HAMILTONIAN SYSTEMS

Differential equations, including Hamiltonian equations, are usually divided into the integrable and the non-integrable. "When, however, one attempts to formulate a precise definition of integrability, many possibilities appear, each with a certain intrinsic theoretic interest."(1) In this chapter we give a brief list of the various approaches to integrability of Hamiltonian systems, "not forgetting the dictum of Poincaré, that a system of differential equations is only more or less integrable".(1)

(1)D. Birkhoff "Dynamical systems".

§1. Quadratures

1. Integration by quadratures is the search for solutions by "algebraic" operations (including the inverting of functions) and "quadratures", the calculation of the integrals of known functions. This definition of integrability formally has a local character. The solution by quadratures of a differential equation on a manifold means its integration in any local coordinates. We assume that the transition from one system of local coordinates to another is an "algebraic" operation. The following result connects the integration by quadratures of Hamiltonian systems with the existence of a sufficiently large set of first integrals.

***Theorem* 1.** *Let M be a symplectic manifold. Suppose that the system with the Hamiltonian $H: M \times \mathbf{R} \to \mathbf{R}$ has $n = \dim M/2$ first integrals $F_1, \ldots, F_n$: $M \times \mathbf{R} \to \mathbf{R}$ $(F_t' + \{F, H\} = 0)$ such that $\{F_i, F_j\} = \sum c_{ij}^k F_k$, $c_{ij}^k = \text{const}$. If*

1) on the set $M_f = \{(x, t) \in M \times \mathbf{R} : F_i(x, t) = f_i,\ 1 \leqslant i \leqslant n\}$ the functions $F_1, \ldots, F_n$ are independent,

2) $\sum c_{ij}^k f_k = 0$ for all $i, j = 1, \ldots, n$,

3) the Lie algebra $\mathfrak{A}$ of linear combinations $\sum \lambda_s F_s$, $\lambda_s \in \mathbf{R}$, is soluble, then the solutions of the Hamiltonian system that lie on M_f can be found by quadratures [30].

Corollary. *If a Hamiltonian system with n degrees of freedom has n independent integrals in involution (the algebra $\mathfrak{A}$ is commutative), then it can be integrated by quadratures.*

This result was first proved by Bour for automonous canonical equations [63] and later was generalised by Liouville to the non-autonomous case [71]. Suppose that H and $F_1, \ldots, F_n$ do not depend on time. Then H is also a first integral, for example, $H = F_1$. The theorem on integrability by quadratures still holds, of course, in that case (the condition $\{H, F_i\} = 0$ can be replaced by the weaker condition $\{H, F_i\} = \lambda_i H$, $\lambda_i = \text{const}$; $1 \leqslant i \leqslant n$).

The proof of Theorem 1 is based on a lemma due to Lie.

Lemma. *Suppose that n vector fields $X_1, \ldots, X_n$ are linearly independent in a small domain $U \subset \mathbf{R}^n\{x\}$ and generate a soluble Lie algebra under the commutation operation, and that $[X_1, X_i] = \lambda_i X_1$. Then the differential equation $\dot{x} = X_1(x)$ is integrable by quadratures in U (see* [58], [60]).

We prove this result in the very simple case $n = 2$. In the general case the proof is similar.

The equation $\dot{x} = X_1(x), x \in U$, can be integrated if we can find a first integral $F(x)$ such that $F'(x) \neq 0$ in U. We remark that by the straightening-out theorem such a function obviously exists (at least locally). If $X_1 F = 0$, then $X_2 F$ is again an integral, since $X_1(X_2 F) = X_2(X_1 F) + \lambda_2 X_1 F = 0$.

Obviously, $X_2F = f(F)$, where $f(y)$ is a smooth function, $f \neq 0$. We put

$$G(F) = \int_0^F \frac{dF}{f(F)}.$$

Since $X_1G = 0$ and $X_2G = G'X_2F = X_2F/f(F) = 1$, a solution of the system of equations

$$X_1F = a_{11}\frac{\partial F}{\partial x_1} + a_{12}\frac{\partial F}{\partial x_2} = 0,$$

$$X_2F = a_{21}\frac{\partial F}{\partial x_1} + a_{22}\frac{\partial F}{\partial x_2} = 1$$

exists. Calculating F'_{x_1} and F'_{x_2} we find F by an additional integration. Since $X_2F = 1$, we see that $F' \neq 0$, as required.

To prove Theorem 1 (in the autonomous case) we consider the n Hamiltonian vector fields $\mathfrak{J}F'_i$. By the conditions 1 and 2, they are tangent to $M_f = \{x : F_i(x) = f_i,\ 1 \leqslant i \leqslant n\}$ and are independent everywhere on M_f. Since $\{F_i,\ F_j\} = \sum c^k_{ij}F_k$, obviously, $[\mathfrak{J}F'_i,\ \mathfrak{J}F'_j] = \mathfrak{J}\{F_i,\ F_j\}' = \sum c^k_{ij}\mathfrak{J}F'_k$. Consequently, the tangent vector fields $\mathfrak{J}F'_i$ form a soluble algebra, and $[\mathfrak{J}H', \mathfrak{J}F'_i] = \lambda_i\mathfrak{J}H'$. Theorem 1 now follows from Lie's lemma.

The non-autonomous case can be reduced to the autonomous one by the following general construction. The Hamiltonian equations

$$\dot{x} = \frac{\partial H(x,\ y,\ t)}{\partial y}, \quad \dot{y} = -\frac{\partial H(x,\ y,\ t)}{\partial x}$$

can be expressed in the form of a canonical system in an extended space of variables $x,\ y,\ h,\ t$ with the Hamiltonian $K(x,\ y,\ h,\ t) = H(x,\ y,\ t) - h$:

$$\dot{x} = \frac{\partial K}{\partial y}, \quad \dot{y} = -\frac{\partial K}{\partial x}, \quad \dot{h} = \frac{\partial K}{\partial t}, \quad \dot{t} = -\frac{\partial K}{\partial h}.$$

If we denote by $\{,\}_*$ the Poisson bracket in the extended symplectic space $\mathbf{R}^{2n}\{x,\ y\} \times \mathbf{R}^2\{t,\ h\}$ then

$$\{F_i(x,\ y,\ t),\ F_j(x,\ y,\ t)\}_* = \{F_i,\ F_j\} = \sum c^k_{ij}F_k,$$

$$\{F_i(x,\ y,\ t),\ K(x,\ y,\ h,\ t)\}_* = \{F_i,\ H-h\}_* = \frac{\partial F_i}{\partial t} + \{F_i,\ H\} = 0.$$

It remains to observe that the functions $F_1, \ldots, F_n$ and K are independent.

2. As a simple example we consider the problem on the motion on a line of three points with an attracting force inversely proportional to the cube of the distances between them. Let m_i be the masses, x_i the coordinates, and $p_i = m_i\dot{x}_i$ the moment of the points. The potential energy of interactions is

$$U = \sum_{i<j} \frac{a_{ij}}{(x_i - x_j)^2}, \qquad a_{ij} = \text{const.}$$

The functions $F_1 = \sum p_i^2/2m_i + U$, $F_2 = \sum p_i x_i$ and $F_3 = \sum p_i$ are independent and $\{F_1,\ F_3\} = 0, \{F_2,\ F_3\} = -F_3, \{F_1,\ F_2\} = 2F_1$. Since the corresponding Lie algebra $\mathfrak{A}$ is soluble, the motions on the zero levels of the total energy and the momentum can be found by quadratures. This possibility is not hard to realize directly. We note that in the case of equal masses m_i and coefficients a_{ij} $(i < j)$ we can find a complete set of integrals in involution.

3. Let M be a symplectic manifold and $F_1, \dots, F_n$ independent functions on M generating a finite-dimensional subalgebra of the Lie algebra $C^\infty(M)$ (that is, $\{F_i, F_j\} = \sum c_{ij}^k F_k$, $c_{ij}^k = \text{const}$). At each point $x \in M$ the vectors $\sum \lambda_i \Im F_i'$, $\lambda_i \in \mathbf{R}$, form an n-dimensional linear subspace $\Pi(X)$ of $T_x M$. The distribution of the planes $\Pi(X)$ is "involutive" (if $X, Y \in \Pi$, then $[X, Y] \in \Pi$). Consequently, by Frobenius' theorem, through each point $x \in M$ there passes a maximal integral manifold N_x of Π. The manifolds N_x can be embedded in M in a very complicated way; in particular, they need not be closed. If $n = \dim M/2$, then among the integral manifolds of Π there are closed surfaces $M_f = \{x \in M : F_i(x) = f_i,\ \Sigma c_{ij}^k f_k = 0\}$. If $x \in M_f$, then N_x is a connected component of M_f. In the special case when $F_1, \dots, F_n$ commute pairwise M is foliated into the closed manifolds M_f.

§2. Complete integrability

***Theorem* 1.** *Let $F_1, \dots, F_n : M \to \mathbf{R}$ be smooth functions in involution: $\{F_i, F_j\} = 0$ $(1 \leqslant i, j \leqslant n)$ and* $\dim M = 2n$. *If*

1) *they are independent on M_f,*

2) *the fields $\Im F_i'$ $(1 \leqslant i \leqslant n)$ are unconstrained on M_f,*

then

1) *each connected component of M_f is diffeomorphic to $\mathbf{R}^k \times T^{n-k}$ (T^1 is a circle),*

2) *on $\mathbf{R}^k \times T^{n-k}$ there are coordinates $y_1, \dots, y_k, \varphi_1, \dots, \varphi_{n-k}$* mod 2π *such that in these coordinates the Hamiltonian equation $\dot{x} = \Im F_i'$ takes the following form:*

$$\dot{y}_m = c_{mi}, \quad \dot{\varphi}_s = \omega_{si} \qquad (c,\ \omega = \text{const}).$$

The proof of this theorem is by now too well known for us to repeat it here (see [7], [16]). Hamiltonian systems with each of the Hamiltonian functions $F_1, \dots, F_n$ are called completely integrable.

The most interesting case is when M_f is compact. Then $k = 0$, consequently, $M_f \cong T^n$. The uniform motion on $T^n\{\varphi \bmod 2\pi\}$ according to the rule $\varphi_i = \varphi_i^0 + \omega_i t$ $(1 \leqslant i \leqslant n)$ is called conditionally-periodic. The numbers $\omega_1, \dots, \omega_n$ are its frequencies. The torus with the set of frequencies $\omega_1, \dots, \omega_n$ is called non-resonant if from $\sum k_i \omega_i = 0$, with integers $k_1, \dots, k_n$, it follows that all $k_i = 0$. On non-resonant tori the phase trajectories are everywhere dense. In the resonant case they fill out tori of lower dimension.

Small neighbourhoods of invariant tori $M_f \cong T^n$ in M are diffeomorphic to the direct product $D \times T^n$, where D is a small domain in $\mathbf{R}^n$. It turns out that in $D \times T^n$ one can always introduce symplectic coordinates I, φ $(I \in D,\ \varphi \in T^n)$ such that in these variables the Hamiltonian function of a completely integrable system depends only on I (see [7]). Here

$$\dot{I} = -\frac{\partial H}{\partial \varphi} = 0, \quad \dot{\varphi} = \frac{\partial H}{\partial I} = \omega(I).$$

Consequently, $I = I_0$, $\omega(I) = \omega(I_0) = \text{const}$. The variables I, which "enumerate" the invariant tori in $D \times T^n$, are called "action" variables, and the uniformly changing coordinates φ "angle" variables. The Hamiltonian system is called non-degenerate (in $D \times T^n$) if

$$\left|\frac{\partial \omega}{\partial I}\right| = \left|\frac{\partial^2 H}{\partial I^2}\right| \neq 0$$

in D. In this case almost all invariant tori (in the sense of Lebesgue measure) are non-resonant, while the resonant tori are everywhere dense in $D \times T^n$.

The system is called properly degenerate if

$$\left|\frac{\partial \omega}{\partial I}\right| \equiv 0.$$

The reason for degeneracy may be that the number of first integrals on the whole phase space is greater than n (but, of course, not all of them in involution). Such is the case, for example, in Kepler's and in Euler's problem. This situation is described by generalizations of Liouville's theorem. We denote by $F_1, \ldots, F_{n+k}$ the independent first integrals of a system with Hamiltonian H and, as before, let $M_f = \{F_i = f_i\}$. We assume M_f to be connected and compact.

Theorem on generalized action-angle variables (Nekhoroshev [43]). *Suppose that the first $n-k$ functions F_i are in involution. Then in a neighbourhood of M_f there are canonical coordinates I, p, φ* mod 2π, *and q such that*

$$I_s = I_s(F_1, \ldots, F_{n-k}),$$

and p and q depend on all the F_i.

Theorem on the finite-dimensional algebra of integrals. *Suppose that the F_i generate a finite-dimensional algebra of integrals, that is, $\{F_i, F_j\} = \sum c_{ij}^k F_k$ and the rank of the matrix of Poisson brackets*

$$\| \{F_i, F_j\} \|$$

is $2k$. Then the manifolds M_f in general position are $(n-k)$-dimensional tori.

In the paper [39] by Mishchenko and Fomenko, where this theorem is proved and applied, there is also the conjecture that the assumption on the algebra of integrals being finite-dimensional can be removed. In fact, shortly afterwards, Strel'tsov generalized the preceeding two results and showed that if $\{F_i, F_j\} = f_{ij}(F_1, \ldots, F_{n+k})$ and the rank of $\| \{F_i, F_j\} \|$ is $2k$, then in a neighbourhood of M_f there are first integrals G_i satisfying Nekhoroshev's generalization. This result was announced in [40]. As noted by Tatarinov (unpublished), all of these generalizations of Liouville's theorem fall under the following observation: part of the integrals ($2k$ of them) cut out canonical submanifolds in M of dimension $2(n-k)$; in each of these a proper Poisson bracket can be specified, for example, by Dirac's formula [15]; then the restrictions of the remaining $(n-k)$ integrals on these submanifolds satisfy the usual Liouville theorem.

§3. Examples of completely integrable systems

1. The equations of rotation of a heavy rigid body around a fixed point are Hamiltonian in the integral manifolds $I_{23}=\{F_2=f_2, F_3=1\}$. One integral always exists: the energy integral. Thus, for the complete integrability of the equations on I_{23} it is sufficient to know one other independent integral. We list the known cases of integrability. As we have already noted the problem of a heavy top contains 6 parameters: the three eigenvalues of the inertia operator, J_1, J_2, J_3, and the three coordinates of the centre of mass relative to its eigenaxes r_1, r_2, r_3.

1) Euler's case (1750): $r_1=r_2=r_3=0$. The new integral is $M^2=\langle J\omega, J\omega\rangle$.

2) Lagrange's case (1788): $J_1=J_2$, $r_1=r_2=0$. The new integral is $M_3=J_3\omega$.

3) Kovalevskaya's case (1889): $J_1=J_2=2J_3$, $r_3=0$. The integral, which she found, is

$$(\omega_1^2-\omega_2^2-\nu e_1)^2+(2\omega_1\omega_2-\nu e_2)^2,$$

where $\nu=\varepsilon r/J_3$, $r^2=r_1^2+r_2^2$.

4) Goryachev-Chaplygin's case (1900): $J_1=J_2=4J_3$, $r_3=0$ and $f_2=\langle M, e\rangle=0$. In contrast to 1)–3) here we have an integrable case on a single integral level I_{23}.

We note that all these integrable cases form manifolds in the six-dimensional parameter space J_i, r_i of one and the same dimension 3.

2. The equations of motion in the first two cases have been studied in detail from various points of view in the classical works of Euler, Poinsot, Lagrange, Poisson, and Jacobi. The Kovalevskaya case is non-trivial in many ways. She found it from the condition for meromorphicity of the solutions of the Euler-Lagrange equations in the complex time plane. Recently, Perelomov obtained the Kovalevskaya integral by means of a representation of Lax [73]. The Goryachev-Chaplygin case is somewhat simpler: it can be integrated by separation of the variables. Let us show this.

In the special canonical coordinates L, G, l, and g the Hamiltonian function has the following form:

$$H=\frac{G^2+3L^2}{8J_3}+\mu\left(\frac{L}{G}\cos l\sin g+\sin l\cos g\right),\quad \mu=\varepsilon r.$$

We consider the canonical transformation

$$L=-p_1-p_2,\ G=p_2-p_1,\ q_1=-l-g,\ q_2=g-l.$$

In the new symplectic coordinates p and q

$$H=\frac{p_1^3-p_2^3}{2J_3(p_1-p_2)}-\mu\left(\frac{p_1\sin q_1}{p_1-p_2}+\frac{p_2\sin q_2}{p_1-p_2}\right).$$

Putting this expression equal to h and multiplying by p_1-p_2 we see that it separates:

$$hp_1-p_1^3/2J_3+\mu p_1\sin q_1=hp_2-p_2^3/2J_3-\mu p_2\sin q_2.$$

We put

$$(3.1)\qquad p_1^3/2J_3-\mu p_1\sin q_1-Hp_1=\Gamma,\quad p_2^3/2J_3+\mu p_2\sin q_2-Hp_2=\Gamma.$$

Here Γ is a first integral of the equations of motion. In the special canonical variables it has the following form:

$$\Gamma=\frac{L(L^2-G^2)}{8J_3}+\frac{L^2-G^2}{2G}\mu\sin l\cos g,$$

and in the traditional Euler-Poisson variables ω, e

$$\Gamma=-2J_3^2\gamma,\ \gamma=\omega_3(\omega_1^2+\omega_2^2)+\nu\omega_1e_3\ \ (\nu=\mu/J_3).$$

We write down a closed system of equations for the change of variables p_1, p_2:

$$\dot p_1=-\frac{\partial H}{\partial q_1}=-\frac{\mu p_1}{p_1-p_2}\cos q_1,\ \dot p_2=-\frac{\partial H}{\partial q_2}=-\frac{\mu p_2}{p_1-p_2}\cos q_2,$$

or, taking account of (3.1),

$$(3.2)\qquad \dot p_1=\frac{\pm\sqrt{\Phi(p_1)}}{p_1-p_2},\quad \dot p_2=\frac{\pm\sqrt{\Phi(p_2)}}{p_1-p_2},$$

where $\Phi(z)=\mu^2z^2-(\Gamma+Hz-z^3/2J_3)^2$ is a polynomial of degree 6. The solutions of these equations can be expressed in terms of hyperelliptic functions of time. The variables p_1 and p_2 are changed in disjoint intervals $[a_1, b_1]$ and $[a_2, b_2]$, where a_i and b_i are adjacent roots of the polynomial $\varphi(z)$ between which it takes positive values.

We introduce angle variables φ_1, φ_2 mod 2π by the formulae

$$(3.3)\qquad \varphi_i=\frac{\pi}{\tau_i}\int_{a_i}^{p_i}\frac{dz}{\pm\sqrt{\Phi(z)}},\quad \tau_i=\int_{a_i}^{b_i}\frac{dz}{\sqrt{\Phi(z)}}.$$

In the new variables (3.2) takes the following form:

$$(3.4)\qquad \dot\varphi_i=\frac{\pi}{2\tau_i(p_1(\varphi_1)-p_2(\varphi_2))}\quad (i=1,2),$$

where $p_i(z)$ are the real hyperelliptic functions of period 2π, defined by (3.3).

Since the trajectories of (3.4) on $T^2\{\varphi \bmod 2\pi\}$ are straight lines, the ratio of the frequencies of the corresponding conditionally-periodic motions is τ_1/τ_2, the ratio of the periods of the hyperelliptic integral

$$\int_{z_0}^{z}\frac{dz}{\sqrt{\Phi(z)}}.$$

This remarkable fact holds even for the equations of Kovalevskaya's problem. Details can be found in [32].

3. The problem of the motion of a rigid body in an ideal fluid is much richer in integrable cases (see [53]). We mention two of them: they were discovered by Clebsch (1871) and Steklov (1893). In Clebsch's case it is assumed that $B = 0$, $C = \operatorname{diag}(c_1, c_2, c_3)$ and

$$a_1^{-1}(c_2-c_3)+a_2^{-1}(c_3-c_1)+a_3^{-1}(c_1-c_2)=0.$$

An additional integral of the Kirchhoff equations has the form

$$M_1^2+M_2^2+M_3^2-a_1e_1^2-a_2e_2^2-a_3e_3^2.$$

In Steklov's case $B = \operatorname{diag}(b_1, b_2, b_3)$, $C = \operatorname{diag}(c_1, c_2, c_3)$, where

$$b_j=\mu\,(a_1a_2a_3)\,a_j^{-1}+\nu,\quad c_1=\mu^2a_1\,(a_2-a_3)^2+\nu',\ \ldots,\ (\mu,\ \nu,\ \nu'=\text{const}).$$

An additional integral is

$$\sum_j (M_j^2-2\mu\,(a_j+\nu)\,M_je_j)+\mu^2\,((a_2-a_3)^2+\nu'')\,e_1^2+\ldots$$

The parameters ν, ν', ν'' are not essential: their appearance is connected with the presence of the classical Kirchhoff integrals F_2 and F_3.

4. The problem of the motion of n point vortices in a plane is completely integrable for $n \leqslant 3$. The case $n = 1$ is trivial, for $n = 2$ independent commuting integrals are, for example, the functions H and M, for $n = 3$, the functions H, M, and $P_x^2+P_y^2$. In the problem of four vortices there are as many independent integrals as there are degrees of freedom. However, they do not all commute.

We consider in detail the special case when the sum of the intensities Γ_s is zero. Then the integrals P_x and P_y are in involution. If their constants are zero, then the equations of motion of four vortices turn out to be Liouville integrable. The idea of the solution is based on the application of a suitable canonical transformation, which is standard in celestial mechanics in connection with the "exceptional" motions of the centre of mass in the n-body problem. To be definite let, $\Gamma_1 = \Gamma_2 = -\Gamma_3 = -\Gamma_4 = -1$. We consider the linear canonical transformation $x, y \to \alpha, \beta$ given by

$$\begin{aligned}
x_1 &= -\beta_4, & y_1 &= -\alpha_3-\alpha_4+\beta_2,\\
x_2 &= \beta_3-\beta_4, & y_2 &= \alpha_3-\beta_1+\beta_2,\\
x_3 &= \alpha_1+\alpha\ \ -\beta_4, & y_3 &= \beta_2,\\
x_4 &= -\alpha_1+\beta_3-\beta_4, & y_4 &= -\beta_1+\beta_2.
\end{aligned}$$

In the new coordinates $P_x = \alpha_2$, $P_y = \alpha_4$. Consequently, the Hamiltonian function H does not depend on the conjugate variables β_2 and β_4. Thus, the number of degrees of freedom is reduced by 2: we have obtained a family of Hamiltonian systems with two degrees of freedom depending on the two parameters α_2 and α_4. The variables $\alpha_1, \alpha_3, \beta_1, \beta_3$ are symplectic coordinates. When $\alpha_2 = \alpha_4 = 0$, M is an integral of the "reduced" system. Consequently, this Hamiltonian system with two degrees of freedom is completely integrable. In particular, the functions $\alpha_1, \alpha_3, \beta_1, \beta_3\,|_t$ can be found by quadratures.

The remaining "cyclic" coordinates, β_2 and β_4, in view of the formulae

$$\dot{\beta}_2=\frac{\partial K}{\partial\alpha_2},\quad \dot{\beta}_4=\frac{\partial K}{\partial\alpha_4};\quad K(\alpha,\beta)=H(x,y)|_{\alpha,\beta}$$

can be found by a simple integration. As far as the author knows, this possibility has not been realized.

5. Other interesting examples of completely integrable systems can be found, for example, in Moser's paper [42]. In the same place some modern methods of integration of Hamilton's equations are discussed.

§4. Perturbation theory

1. Suppose that the direct product $M = D \times T^n\{\varphi \bmod 2\pi\}$, D a domain in $\mathbf{R}^n\{I\}$, is equipped with the standard symplectic structure and that $H(I, \varphi, \epsilon): M \times (-\epsilon_0, \epsilon_0) \to \mathbf{R}$ is an analytic function such that $H(I, \varphi, 0) = H_0(I)$. The canonical equations with Hamiltonian H_0 can be integrated directly:

$$\dot{I}=-\frac{\partial H_0}{\partial\varphi}=0,\quad \dot{\varphi}=\frac{\partial H_0}{\partial I}=\omega(I)\Rightarrow I=I^0,\quad \varphi=\varphi^0+\omega(I^0)\,t.$$

According to Poincaré, the investigation of the complete system

$$\dot{I}=-\frac{\partial H}{\partial\varphi},\quad \dot{\varphi}=\frac{\partial H}{\partial I};\quad H=H_0(I)+\varepsilon H_1(I,\varphi)+\ldots \tag{4.1}$$

for small values of ϵ is a basic problem of dynamics [48].

The idea of classical perturbation theory consists in the following: to find a canonical transformation $I, \varphi \to J, \psi$, depending analytically on ϵ,

$$I=\frac{\partial S}{\partial\varphi},\quad \psi=\frac{\partial S}{\partial J},\quad S(J,\varphi,\varepsilon)=S_0+\varepsilon S_1+\ldots,$$

such that

1) $S_0 = J\varphi$ (it is close to the identity),
2) the functions $S_k(J, \varphi)$ are periodic in φ with period 2π for all $k \geqslant 1$,
3) in the new variables $H = K(J, \epsilon)$.

Consequently, any function $f(I, \varphi, \epsilon)$ that is 2π-periodic in φ is also 2π-periodic in ψ in the new variables J, ψ.

If such a transformation can be found, the Hamiltonian equations (4.1) are completely integrable. Here the n functions $J_s = J_s(I, \varphi, \epsilon)$, $J_s(I, \varphi, 0) = I_s$ $(1 \leqslant s \leqslant n)$ form a complete set of independent integrals in involution.

2. The function $S_1(J, \varphi)$ satisfies the equality

$$\left\langle\frac{\partial H_0}{\partial J},\ \frac{\partial S_1}{\partial\varphi}\right\rangle+H_1(J,\varphi)=K_1(J), \tag{4.2}$$

where $K_1(J)$ is, for the present, unknown. We expand the "perturbing" function H_1 in a multiple Fourier series:

$$H_1=\sum_{m\in Z^n} H_m(J)\exp i\langle m,\varphi\rangle.$$

If (4.2) has a solution that is periodic in φ, then

$$K_1(J) = \frac{1}{(2\pi)^n} \int_{T^n} H_1(J, \varphi)\, d\varphi.$$

Let

$$S_1 = \sum_{m \neq 0} S_m(J) \exp i \langle m, \varphi \rangle.$$

Then

$$S_m(J) = \frac{H_m(J)}{i \langle m, \omega(J) \rangle}.$$

In the subsequent analysis a major role is played by the secular set $\mathfrak{B} \subset D$, the set of $J \in D$ for which

$$\sum_{m \neq 0} \left| \frac{H_m(J)}{\langle m, \omega(J) \rangle} \right|^2 = \infty.$$

In particular, those $J \in D$ for which $\langle m, \omega(J) \rangle = 0$, $m \neq 0$ and $H_m(J) \neq 0$ belong to $\mathfrak{B}$. By Bessel's inequality,

$$\sum_{m \in Z^n} S_m^2 < \infty$$

and the generating function S_1 is not defined on the set $\mathfrak{B} \times T^n \subset D \times T^n$.

In essence the secular set consists of those tori of the unperturbed integrable problem that split under a perturbation of order ϵ. In a typical situation $\mathfrak{B}$ is everywhere dense in D and this is connected with a well-known difficulty, the phenomena of "small divisors", which obstruct not only convergence, but even the formal construction of a number of the classical schemes in perturbation theory.

3. ***Theorem* 1.** *Suppose that* (4.1) *has n first analytic integrals*

$$F_i : D \times T^n \times (-\varkappa, \varkappa) \to R$$

such that

1) *for all values of ϵ the functions $F_1, \ldots, F_n$ are in involution,*
2) $F_i(I, \varphi, 0) = f_i(I)$, $1 \leqslant i \leqslant n$,
3) *the Jacobian*

$$\frac{\partial(f_1, \ldots, f_n)}{\partial(I_1, \ldots, I_n)} \neq 0$$

in D. Then on $G \times T^n \times (-\alpha, \alpha)$, where G is a compact subdomain of D and α is small, there is an analytic generating function $S(J, \varphi, \epsilon)$ satisfying §4.1, 1)–3).

If the equations (4.1) have integrals that are formally analytic in ϵ (power series in ϵ with analytic coefficients in $D \times T^n$) and satisfy the conditions of the theorem, then we can construct (at least formally) the series of perturbation theory defined for $(J, \varphi) \in D \times T^n$. Let us prove this.

Let $F_s(I, \varphi, \varepsilon) = f_s(I) + \sum \varepsilon^k F_{sk}(I, \varphi)$. We consider the system of equations

$$(4.3) \qquad F_s\left(\frac{\partial S}{\partial \varphi}, \varphi, \varepsilon\right) = f_s(J) + \sum_{k \geqslant 1} \varepsilon^k f_{sk}(J) \quad (1 \leqslant s \leqslant n)$$

with at present unknown analytic functions $f_{sk}: D \to \mathbf{R}$. For $\epsilon = 0$ (4.3) is satisfied if we put $S_0 = J\varphi$. Since $F_s(I, \varphi, 0) = f_s(I)$ and

$$\frac{\partial(f_1, \ldots, f_n)}{\partial(I_1, \ldots, I_n)} \neq 0,$$

the f_{sk} given there define a formal series

$$(4.4) \qquad I(\varphi, \varepsilon) = \frac{\partial S}{\partial \varphi} = J + \varepsilon \frac{\partial S_1}{\partial \varphi} + \ldots,$$

satisfying (4.3). We claim that the differential form $I(\varphi, \varepsilon) d\varphi = \frac{\partial S}{\partial \varphi} d\varphi$ is exact. To prove this we need a simple lemma.

Lemma. *Suppose that*

$$F_s(p, q) = c_s, \ 1 \leqslant s \leqslant n,$$

is a given system of equations in $\mathbf{R}^{2n}\{p, q\}$ *and that* $p_s = f_s(q, c_1, \ldots, c_n)$ *is a solution of it. If the functions* $F_1, \ldots, F_n$ *commute (in the standard symplectic structure on* $\mathbf{R}^{2n}$*), then for fixed values of c the form* $\sum f_s(q, c) dq_s$ *is a complete differential.*

Proof. The functions $G_s(p, q) = p_s - f_s(q, F_1(p, q), \ldots, F_n(p, q))$ are obviously constant. Since $F_1, \ldots, F_n$ commute,

$$\{G_s, G_m\} = \frac{\partial f_m}{\partial q_s} - \frac{\partial f_s}{\partial q_m} = 0,$$

as required.

For an arbitrary choice of $f_{sk}(J)$ the functions $S_k(J, \varphi)$ are multivalued on T^n. This can be removed by choosing the f_{sk} in a suitable way. First let $k = 1$. From (4.3) we obtain

$$(4.5) \qquad \left\langle \frac{\partial f_s}{\partial J}, \frac{\partial S_1}{\partial \varphi} \right\rangle = f_{s1}(J) - F_{s1}(J, \varphi).$$

If we put

$$f_{s1} = \frac{1}{(2\pi)^n} \int_{T^n} F_{s1}(J, \varphi) \, d\varphi,$$

then from (4.5) we obtain a periodic solution S_1. When $k \geqslant 1$ we have for the definition of S_k and f_{sk} an equation of the form (4.5) whose right-hand side contains the known functions S_m and f_{sm} $(m < k)$.

In the new canonical coordinates J, ψ the functions $F_1, \ldots, F_n$ depend only on J and ϵ. Since these functions are first integrals of the Hamiltonian system (4.1) and are independent, the same is true for $J_1, \ldots, J_n$. Consequently, the Hamiltonian function H does not depend on the angles ψ:

$$\frac{\partial H}{\partial \psi} = -\dot{J} = 0.$$

This proves the theorem.

§5. Normal forms

1. We consider the Hamiltonian system

$$\dot{z} = \Im H', \quad z = (p, q) \in \mathbf{R}^{2n}$$

in a neighbourhood of $z = 0$. Suppose that the real-analytic function H can be represented by a convergent power series in z, beginning with terms of the second degree: $H = \sum_{k\geqslant 2} H_k$. Then $z = 0$ is, obviously, a position of equilibrium.

Of special interest is the case when the eigenvalues of the linearized system $\dot{z} = \Im H_2'$ are purely imaginary and distinct. It is well-known [18], that then there is a linear canonical transformation of coordinates $p, q \to x, y$ that takes the quadratic form H_2 to

$$\frac{1}{2} \sum_s \alpha_s (x_s^2 + y_s^2). \tag{5.1}$$

The eigenvalues are precisely $\pm i\alpha_1, \ldots, \pm i\alpha_n$.

***Theorem* 1** (Birkhoff). *If $\alpha_1, \ldots, \alpha_n$ are independent over the rationals, then there is a formal canonical transformation $x, y \to \xi, \eta$ (given by a formal power series $S(x, \eta) = x\eta + \sum_{m\geqslant 3} S_m(x, \eta) : \xi = S_\eta', \ y = S_x'$) that transforms $H(x, y)$ into a Hamiltonian $K(\rho)$, a formal power series in $\rho_s = \xi_s^2 + \eta_s^2$* [9].

If the series $\sum S_m$ converges, then the equations with Hamiltonian H are completely integrated: $\rho_1, \ldots, \rho_n$ are power series in x and y that form a complete set of independent integrals in involution. The converse is also true.

***Theorem* 2** (Rüssman). *If a system with Hamiltonian $H = \sum_{k\geqslant 2} H_k$ has n analytic integrals in involution*

$$G_m = \frac{1}{2} \sum \varkappa_{ms} (x_s^2 + y_s^2) + \sum_{k\geqslant 3} G_{mk}$$

and $\det \| \varkappa_{ms} \| \neq 0$, *then the Birkhoff transformation converges* [74].

Normalization of a Hamiltonian system in a neighbourhood of a stable position of equilibrium is closely connected with the classical scheme of perturbation theory. For by introducing a small parameter ϵ by $x \to \epsilon x$, $y \to \epsilon y$ and passing to polar coordinates I, φ by the formulae

$$x_s = \sqrt{2I_s} \sin \varphi_s, \quad y_s = \sqrt{2I_s} \cos \varphi_s,$$

we obtain a Hamiltonian system

$$\dot{I}_s = -\frac{\partial H}{\partial \varphi_s}, \quad \dot{\varphi}_s = \frac{\partial H}{\partial I_s}$$

with Hamiltonian $H = \sum_{m\geqslant 0} \varepsilon^m H_m^* (I, \varphi)$, $H_0^* = \sum \alpha_s I_s$,

$$H_m^* = H_{m+2}(x, y) \,|_{I, \varphi},$$

that is 2π-periodic in φ. If the frequencies $\alpha_s = \dfrac{\partial H_0}{\partial I_s}$ are rationally independent, then there are formal series of classical perturbation theory corresponding to the Birkhoff transformation. Rüssman's theorem can be derived by the same device from Theorem 1 of §4.

2. In applications H usually depends on certain parameters $\epsilon \in D$ (where D is a domain in $\mathbf{R}^m$). We take $H(z, \epsilon)$ to be analytic in z and ϵ and $H'(0, \epsilon) = 0$ for all ϵ. If for all ϵ the eigenvalues of the linearized system are purely imaginary and distinct, then by a suitable linear symplectic transformation that is analytic in ϵ, the form H_2 can be reduced to the "normal" form (5.1). The coefficients α_s, of course, are analytic in ϵ. The following theorem is an insignificant improvement of Rüssman's result.

***Theorem* 3.** *Suppose that there exist n integrals in involution*

$$G_m(x, y, \varepsilon) = \frac{1}{2} \sum \varkappa_{ms}(\varepsilon)(x_s^2 + y_s^2) + \sum_{k \geqslant 3} G_{mk}(x, y, \varepsilon),$$

that are analytic in ϵ and such that $\det \| \varkappa_{ms}(\varepsilon) \| \neq 0$ *for all $\epsilon \in D$. Then there is an analytic canonical transformation $x, y \to \xi, \eta$ that is analytic in ϵ and takes $H(x, y, \epsilon)$ to the Hamiltonian $K(\rho_1, \ldots, \rho_n, \epsilon)$, $\rho_s^2 = \xi_s^2 + \eta_s^2$.*

If the series $\sum G_{mk}$ are formal (not necessarily convergent), then we can find a formal canonical transformation "normalizing" the Hamiltonian H. In particular, under the conditions of the theorem, the Birkhoff transformation exists also for rationally dependent sets of frequencies $\alpha_1, \ldots, \alpha_n$.

The transformation to normal form can be carried out not only in neighbourhoods of positions of equilibrium, but also, for example, in neighbourhoods of periodic trajectories. All that has been said above remains valid with necessary changes in that case.

CHAPTER III

TOPOLOGICAL OBSTRUCTIONS TO COMPLETE INTEGRABILITY OF NATURAL SYSTEMS

§1. The topology of the state space of an integrable system

1. We consider a mechanical system with two degrees of freedom (see Ch.I, §1). We assume that its state space M is a compact orientable analytic surface. The topological structure of such surfaces is well known: they are spheres with a certain number $\varkappa$ of handles attached. The number $\varkappa$ is a topological invariant of the surface, it is called its genus.

The motions of a natural system are described by the Hamiltonian equations in the cotangent bundle T^*M, which is its phase space. The bundle T^*M has a natural structure as a four-dimensional analytic manifold. We assume that the Hamiltonian function $H: T^*M \to \mathbf{R}$ is everywhere analytic. Since $H = T(p, q) + U(q)$ and $T(p, q)$ is a quadratic form in

$p \in T_q^*M$ for all $q \in M$, the functions $T(p, q)$ (kinetic energy) and $U(q)$ (potential energy) are analytic on T^*M and M, respectively. The solutions of the canonical system

$$(1.1) \qquad \dot{p} = -\frac{\partial H}{\partial q}, \quad \dot{q} = \frac{\partial H}{\partial p}$$

are analytic maps from $\mathbf{R}\{t\}$ to T^*M. On their trajectories the total energy $H = T + U$, of course, is constant.

***Theorem* 1.** *If the genus of M is not equal to* 0 *or* 1 *(that is, if M is not diffeomorphic to the sphere S^2 or the torus T^2), then the equation* (1.1) *does not have a first integral that is analytic on T^*M and independent*[(1)] *of the energy integral* [31].

Numerous examples are know of integrable systems whose configuration spaces are homeomorphic to S^2 or T^2 (say, the motion of an inertial material particle on a "standard" sphere or torus).

In the infinitely differentiable case Theorem 1, generally speaking, is not valid: for any smooth surface M one can give a "natural" Hamiltonian $H = T + U$ such that Hamilton's equations (1.1) on T^*M have an additional infinitely differentiable integral independent of (more precisely, not everywhere dependent on) H. For let us consider the standard sphere S^2 in $\mathbf{R}^3$ and suppose that M is obtained from S^2 by attaching any number of handles to some small domain N on S^2. Let H be the Hamiltonian function for the problem of the motion of an inertial particle ($U \equiv 0$) on M, embedded in $\mathbf{R}^3$. Outside N the particle obviously moves along great circles of S^2. Consequently, in the phase space T^*M there is an invariant domain that is diffeomorphic to the direct product $D \times T^2$ foliated into two-dimensional invariant tori. The points of D "enumerate" these tori. Let $f: D \to \mathbf{R}$ be a smooth function that vanishes outside some subdomain G lying wholly within D. Corresponding to f there is a smooth function F on $D \times T^2$ that is constant on the invariant tori of $D \times T^2$. It extends to a smooth function on the whole of T^*M if we put $F = 0$ outside $G \times T^2$. Obviously, F is a first integral of the canonical equations (1.1) and the functions H and F (for suitable f) are not everywhere dependent.

2. Theorem 1 is a consequence of a stronger result establishing the non-integrability of the equations of motion for fixed sufficiently large values of the total energy. The precise statement is as follows. For all values $h > \max_M U$ the level of total energy $I_h = \{x \in T^*M : T + U = h\}$ is a three-dimensional analytic manifold having the natural structure of a fibre space with base M and fibre S^1. Local coordinates on I_h are q, φ, where q are coordinates on M and φ is the angular variable on the "fibre"

[(1)]Analytic functions are called independent if they are independent at some point (they are then independent almost everywhere).

$S_q^1 = \{p \in T_q^*M : T(p, q) + U(q) = h\}$, which is a circle in the cotangent plane. Since the initial Hamiltonian vectorfield $\mathfrak{I}H'$ is tangent to I_h, on I_h there arises a certain analytic system of differential equations.

***Theorem* 2.** *If the genus of M is not equal to* 0 *or* 1, *then for all $h > \max_M U$ the flow on I_h does not have a non-constant analytic integral.*

3. In the infinitely differentiable case, under the assumptions of Theorems 1 and 2 we can assert that new smooth integrals satisfying certain supplementary conditions are absent.

***Theorem* 3.** *If the genus of a smooth surface M is not equal to* 0 *or* 1, *then for all $h > \max_M U$ the phase flow on I_h does not have an infinitely differentiable first integral $f(p, q): I_h \to \mathbf{R}$ such that*

α) *it has finitely many critical values, and*

β) *the points $q \in M$ for which the set $\{f(q, \varphi) = c\}$ is finite or is the whole fibre S_q^1 are everywhere dense in M.*

In the analytic case the conditions α) and β) are automatically satisfied. Here condition β), obviously, holds for all $q \in M$. But α) is non-trivial: a proof can be found in [75].

More generally, if a compact orientable smooth surface M is not homeomorphic to the sphere or the torus, then the equations of motion do not have a new integral $F(p, q)$ that is an infinitely differentiable function on T^*M, is analytic for fixed $q \in M$ on the cotangent plane T_q^*M, and has finitely many distinct critical values. Functions that are polynomial in the velocity are an extensive class of examples of integrals that are analytic in the momenta p. The collection of distinct critical values of a smooth function on a compact manifold is finite if, for example, all the critical points are isolated or if the critical points form a non-degenerate critical manifold.

The examples of §1.1 do not contradict Theorem 3: β) obviously does not hold for points $q \in M$ that are sufficiently remote from the "singular" domain N.

4. Theorems 1–3 also hold in the case of non-orientable compact surfaces if, in addition, the projective plane $\mathbf{RP}^2$ and the Klein bottle K are excluded. For the standard regular double covering $N \to M$, where N is an orientable surface, induces a certain mechanical system on N, which has an additional integral if the system on M has a new integral. It remains to remark that when M is not homeomorphic to $\mathbf{RP}^2$ or K, then the genus of N is greater than 1.

§2. Proof of the theorem on non-integrability

1. According to the Maupertuis principle of least action, the trajectories of the motions of a mechanical system that lie on integral level surfaces I_h with total energy $h > \max_M U$ are geodesic lines of the Riemannian space (M, ds), where the metric ds is defined by the form $(ds)^2 = 2(h - U)T(dt)^2$.

We fix a point $q \in M$ satisfying β). Since (M, ds) is a smooth two-dimensional compact orientable Riemannian manifold and not homeomorphic to the sphere, by a theorem of Gaidukov [12], for any non-trivial class of freely homotopic paths in M there are geodesic semitrajectories Γ emanating from and approaching asymptotically some closed geodesic from the given homotopy class. The geodesic Γ itself may be a closed curve. In what follows, the geodesic semitrajectory Γ is called a Γ_q-geodesic.

Suppose that the reduced system has on I_h an infinitely differentiable first integral $F(q, \varphi)$. Any of its non-critical levels is a union of a certain number of two-dimensional invariant tori. In the cotangent plane T_q^*M we consider the circle S_q^1 consisting of vectors p such that $T(p, q) + U(q) = h$. To each $p \in S_q^1$ there corresponds a unique motion $q(t)$, $p(t)$ with the initial conditions $q(0) = q$, $p(0) = p$. F is constant on this motion. The momentum p is called critical if the corresponding value of F is critical. We claim that there are infinitely many distinct critical momenta. If the number of critical momenta is finite, then S_q^1 splits into finitely many open sectors $\Delta_1, \ldots, \Delta_n$ such that any momentum $p \in \Delta_i$, $1 \leqslant i \leqslant n$, is non-critical.

With each $p \in \Delta_i$ we can associate a unique invariant torus T_p^2 on which the solution $q(t)$, $p(t)$ of (1.1) with the initial conditions $q(0) = q$, $p(0) = p$ lies. Since there are non-critical values of F for $p \in \Delta_i$, the natural map

$$f_i\colon\ \Delta_i \times T^2 \to D_i = \bigcup_{p \in \Delta_i} T_p^2$$

is continuous. Let $\pi : T^*M \to M$ be the projection of the cotangent bundle T^*M onto M. We put $X_i = \pi(D_i) \subset M$. The continuous map $\pi \circ f_i : \Delta_i \to X_i$ induces a homomorphism of the homology groups $g_i : H_1(\Delta_i \times T^2) \to H_1(X_i)$. Since $X_i \subset M$, there is a natural homomorphism $\varphi_i : H_1(X_i) \to H_1(M)$. We denote by G_i the subgroup of $H_1(M)$ that is the image of $H_1(\Delta_i \times T^2) \to H_1(M)$ under the homomorphism $\varphi_i \circ g_i : H_1(\Delta_i \times T^2) \to H_1(M)$. The elements of $H_1(M)$ are homology classes of cycles, and in each class there is a connected cycle. Freely homotopic cycles, obviously, are homologous. Γ_q-geodesics corresponding to non-critical initial momenta are, of course, closed. For certain critical initial momenta the Γ_q-geodesics may turn out not to be closed. These geodesics are "winding" on certain cycles γ generating one-dimensional subgroups $\{n\gamma,\ n \in Z\} \subset H_1(M)$. By hypothesis, the number of critical momenta is finite. Consequently, the number of such subgroups is also finite. We denote them by $N_1, \ldots, N_m$. If $\alpha \in H_1(M)$ does not lie in the union $N_1 \cup \ldots \cup N_m$, then in the class of homologous cycles α there is at least one closed Γ_q-geodesic. Since Γ_q-geodesics under the maps $\pi \circ f_i$ go over to certain closed curves in the domains $\Delta_1 \times T^2, \ldots, \Delta_n \times T^2$, the set $H_1(M) \setminus \cup N_i$ is wholly covered by the subgroups $G_1, \ldots, G_n$. Since $H_1(\Delta_i \times T^2) \approx H_1(T^2) \approx Z^2$ $(1 \leqslant i \leqslant n)$, the G_i are Abelian subgroups of rank not exceeding 2. It is well known that if the genus of M is $\varkappa$, then $H_1(M) \approx Z^{2\varkappa}$. Since M is not homeomorphic to a sphere or a torus, $2\varkappa \geqslant 4$, and from dimension arguments it follows that $H_1(M)$ cannot be covered by finitely many one-dimensional and two-dimensional subgroups. This contradiction proves that the collection of critical momenta is infinite.

According to α) the number of distinct critical values of the function $F: I_h \to \mathbf{R}$ is finite. Consequently, for the value of $q \in M$ fixed above the function $F(q, \varphi)$, $\varphi \in S^1_q$, takes the same value infinitely often. But then, by β), $F(q, \varphi)$ is constant on S^1_q (that is, does not depend on φ). The surface M is connected and compact, hence, any two of its points can be joined by a minimal geodesic [38]. Since F is constant along each motion, it takes the same value at all points $q \in M$ satisfying β). Since, by assumption, the set of such points is everywhere dense in M, F = const by continuity.

This proves the theorem.

2. Another proof of Theorem 1 based on the introduction of a complex-analytic structure on M, is in the paper [34] by Kolokol'tsov. There is also a description of the two-dimensional systems with first integrals that are quadratic in the velocity.

§3. Unsolved problems

1. Does the existence of new analytic integrals impose restrictions on the topology of the analytic manifold M when dim $M > 2$? In particular, can any many-dimensional analytic manifold be the state space of a completely integrable analytic natural mechanical system?

We remark that on the manifold T^*M with the natural canonical structure there are always completely integrable (not natural) Hamiltonian systems. For n independent analytic functions $f_s: M \to \mathbf{R}$ $(1 \leqslant s \leqslant n,\ n = \dim M)$ are independent as functions on T^*M and are in involution. It would seem that this property holds for arbitrary (or, at least, compact) analytic symplectic manifolds.

2. Let k be the Gaussian curvature of the Maupertuis Riemannian metric $(ds)^2 = 2(h - U)T(dt)^2$ on M. By the Gauss-Bonnet formula

$$\frac{1}{2\pi} \int_M k\, d\sigma = \chi(M),$$

where $\chi(M)$ is the Euler characteristic of the compact surface M. If the genus of M is greater than 1, then $\chi(M) < 0$, consequently, the mean curvature is negative. If the curvature is negative everywhere, then the dynamical system on I_h is a Y-system, consequently, is ergodic on I_h [2]. This result holds also in the many-dimensional case (we need only require that the curvature is negative in all two-dimensional directions). Here the differential equations of motion on I_h do not even have continuous integrals, since almost all trajectories are everywhere dense on I_h. Of course, a curvature that is negative in the mean is by no means always negative everywhere. It would be of interest to study the connection between complete integrability of a natural system and the geometric properties of the Riemannian space (M, ds) (not only with the coarser topologies).

"Mais ce n'est pas aux géodésiques des surface à courbure opposées que les trajectoires du problème des trois corps[1] sont comparable; c'est, au contraire, aux géodésiques des surfaces convexes ... malheureusement, le problème est beaucoup plus difficile ... J'ai done dû une boruer à quelques résultats partiels ..." (Poincaré [49]).

Chapter IV

NON-INTEGRABILITY OF NEARLY INTEGRABLE HAMILTONIAN SYSTEMS

In this chapter we investigate the integrability of the "fundamental problem" of dynamics:

$$(1) \qquad \dot{I} = -\frac{\partial H}{\partial \varphi}, \quad \dot{\varphi} = \frac{\partial H}{\partial I}; \quad H = H_0(I) + \varepsilon H_1(I, \varphi) + \dots$$

§1. Poincaré's method

1. We introduce into the discussion the Poincaré set $\mathfrak{P}$, which is related to the secular set $\mathfrak{B}$ (defined in Ch.II, §4). Let

$$H_1 = \sum_{m \in Z^n} H_m(I) e^{i(m, \varphi)}.$$

The Poincaré set $\mathfrak{P}$ is the set of values $I \in D$ for which there exist $n-1$ linearly independent vectors $k_1, \dots, k_{n-1} \in \mathbf{Z}^n$ such that

1) $\langle k_s, \omega(I) \rangle = 0,\ 1 \leqslant s \leqslant n-1$,

2) $H_{k_s}(I) \neq 0$.

In the case of two degrees of freedom, obviously, $\mathfrak{P} \subset \mathfrak{B}$.

We denote by $\mathfrak{A}(V)$ the class of functions that are analytic in a domain $V \subset \mathbf{R}^n$. A set $M \subset V$ is called a key set (or set of uniqueness) for $\mathfrak{A}(V)$ if any analytic function that vanishes on M vanishes identically on V. Thus, if two analytic functions coincide on M, then they coincide on the whole of V. For example, a set of points of an interval $\Delta \subset \mathbf{R}$ is a key set for $\mathfrak{A}(\Delta)$ if and only if it has a limit point in the interior of Δ. The sufficiency of this condition is obvious, the necessity follows from Weierstrass' theorem on infinite products. We note that if M is a set of uniqueness for the class $C^\infty(V)$, then M is dense in V.

***Theorem* 1.** *Suppose that the unperturbed system is non-degenerate:* $\det \| \partial^2 H_0 / \partial I^2 \| \neq 0$ *in* D; *that* $I^0 \in D$ *is a non-critical point of* H_0; *and that in any neighbourhood* U *of it the Poincaré set* $\mathfrak{P}$ *is a key set for* $\mathfrak{A}(U)$. *Then the Hamiltonian equations* (1) *do not have an integral* F *independent of* H *that can be expressed as a formal power series* $\sum_{s \geqslant 0} F_s(I, \varphi)\varepsilon^s$ *with coefficients analytic in* $D \times T^n$ *(see* [48], [25]*).*

[1] And of many other problems of classical mechanics.

A formal series $\sum f_s \varepsilon^s$ is regarded as being zero if all $f_s = 0$. The series $F = \sum_{k \geqslant 0} F_k \varepsilon^k$ is a formal integral of the canonical equations with the Hamiltonian $H = \sum_{m \geqslant 0} H_m \varepsilon^m$ if

$$\{H, F\} = \sum_{s \geqslant 0} \left(\sum_{k+m=s} \{H_m, F_k\} \right) \varepsilon^s = 0.$$

Two series $\sum f_s \varepsilon^s$ and $\sum g_s \varepsilon^s$ are regarded as dependent when all second-order minors of their Jacobian matrices are identically zero as formal series in powers of ϵ.

For the proof of Theorem 1 we need the following lemma.

***Lemma* 1.** *Suppose that the functions $F_s: D \times T^n \to \mathbf{R}$ are continuously differentiable and that the series $\sum_{s \geqslant 0} F_s(I, \varphi)\varepsilon^s$ is a formal integral of* (1) *with a non-degenerate function H_0. Then*

1) *$F_0(I, \varphi)$ does not depend on φ.*
2) *H_0 and F_0 are dependent on $\mathfrak{W}$.*

Proof. The condition $\{H, F\} = 0$ is equivalent to the sequence of equations

$$\{H_0, F_0\} = 0, \quad \{H_0, F_1\} + \{H_1, F_0\} = 0, \; \ldots \tag{1.1}$$

From the first equation it follows that F_0 is an integral of the unperturbed equation with Hamilton function H_0. Suppose that the torus $I = I^*$ is non-resonant. Then $F_0(I^*, \varphi)$ does not depend on φ, since any trajectory fills out a non-resonant torus densely. To complete the proof of 1) it remains to take into account that F_0 is continuous and the set of non-resonant tori of a non-degenerate integrable system is everywhere dense.

Let Φ_0, Φ_1 $D \times T^n \to \mathbf{R}$ be continuously differentiable functions, Φ_0 not depending on φ. Then

$$\frac{1}{(2\pi)^n} \int_{T^n} \{\Phi_0, \Phi_1\} e^{-i\langle m, \varphi\rangle} \, d\varphi = i \left\langle \frac{\partial \Phi_0}{\partial I}, m \right\rangle \Phi_m(I),$$

where

$$\Phi_m(I) = \frac{1}{(2\pi)^n} \int_{T^n} \Phi_1(I, \varphi) e^{i\langle m, \varphi\rangle} \, d\varphi.$$

Taking account of this remark, from the second equation of (1.1) we obtain a sequence of equalities:

$$\left\langle m, \frac{\partial H_0}{\partial I} \right\rangle F_m(I) = \left\langle m, \frac{\partial F_0}{\partial I} \right\rangle H_m(I), \qquad m \in \mathbf{Z}^n.$$

Let $I \in \mathfrak{W}$. Then at this point the vectors $\partial H_0/\partial I$ and $\partial F_0/\partial I$ are obviously dependent.

Proof of Theorem 1. Since at $I^0 \in D$ among the derivatives $\partial H_0/\partial I_1, \ldots, \ldots, \partial H_0/\partial I_n$ is at least one non-zero, in a small neighbourhood U of this point we can take $H_0, I_2, \ldots, I_n$ as local coordinates (if $\partial H_0/\partial I_1 \not\equiv 0$).

By Lemma 1, the functions H_0 and F_0 are dependent on the Poincaré set. Since the minors of the Jacobian matrix

$$\frac{\partial (H_0, F_0)}{\partial (I_1, \ldots, I_n)}$$

are analytic in U and $\mathfrak{B} \cap U$ is a key set, the functions H_0 and F_0 are dependent throughout U, consequently, in the new coordinates $F_0 = F_0(H_0)$.

Since $F - F_0(H) = \epsilon\Phi$, we see that Φ is a formal integral of (1). Let $\Phi = \sum_{s\geqslant 0} \Phi_s \varepsilon^s$. Then, by Lemma 1, Φ_0 does not depend on the angle variables φ, and Φ_0 is dependent on H_0 in U. Consequently, $\Phi_0 = \Phi_0(H_0)$ and again $\Phi - \Phi_0(H) = \epsilon\Psi$. But then $F = F_0(H) + \epsilon\Phi_0(H) + \epsilon^2\Psi$. Repeating the operation as often as necessary we find that the expansion of all second-order minors of the Jacobian matrix

$$\frac{\partial (H, F)}{\partial (I, \varphi)}$$

in series in powers of ϵ begins with terms of arbitrarily high order. Hence, H and F are dependent.

2. ***Theorem* 2.** *Suppose that H_0 is non-degenerate in D and that the Poincaré set $\mathfrak{B}$ is everywhere dense in D. Then the equations* (1) *have no formal integral $\sum F_s \varepsilon^s$, independent of H, with infinitely differentiable coefficients $F_s : D \times T^n \to \mathbf{R}$.*

The result is simple to prove by the method of §3.1.

3. We now consider the non-autonomous canonical system of equations

$$(1.2) \qquad \dot{I} = -\frac{\partial H}{\partial \varphi}, \quad \dot{\varphi} = \frac{\partial H}{\partial I}; \quad H = H_0(I) + \varepsilon H_1(I, \varphi, t) + \ldots$$

The Hamilton function H is assumed to be analytic and 2π-periodic in φ and t.

The equations (1.2) arise, for example, in the study of the autonomous system (1) when one of the angle coordinates φ is taken as the new time. For example, let $\partial H/\partial I_1 \neq 0$. Then (at least locally) we can solve the equation $H(I, \varphi, t, \epsilon) = h$ for I_1 and find that

$$I_1 = -K(I_2, \ldots, I_n, \varphi_2, \ldots, \varphi_n, \tau, \varepsilon, h), \quad \tau = \varphi_1.$$

Since $\dot{\varphi}_1 \neq 0$, the solutions $I_s(t)$ and $\varphi_s(t)$ ($s \geqslant 2$) of the original equations can be regarded as functions of τ. By Whittaker's theorem [7], [55], the functions $I_s(\tau)$ and $\varphi_s(\tau)$ ($2 \leqslant s \leqslant n$) satisfy the canonical equations

$$\frac{dI_s}{d\tau} = -\frac{\partial K}{\partial \varphi_s}, \qquad \frac{d\varphi_s}{d\tau} = \frac{\partial K}{\partial I_s}.$$

These have the form (1.2).

Again it is useful to introduce the Poincaré set $\mathfrak{B}_*$ as the set of points $I \in D$ satisfying the following conditions:

1) there exist n linearly independent vectors $k_s \in \mathbf{Z}^n$ and n integers m_s such that $\langle k_s, \omega(I)\rangle + m_s = 0$, $1 \leqslant s \leqslant n$,

2) the Fourier coefficients $H_{k_s m_s}(I)$ of the expansion of the perturbing function

$$H_1 = \sum_{(k,\ m)\in Z^{n+1}} H_{km}(I)\, e^{i((k,\ \varphi)+mt)}$$

are non-zero.

We note that if (1.2) are the Whittaker equations obtained from the autonomous equations (1) by a reduction of the order, then the Poincaré set $\mathfrak{W}_*$ of the reduced system is the projection onto the plane $\mathbf{R}^{n-1}\{I_2, \ldots, I_n\}$ of the intersection of the Poincaré set $\mathfrak{W}$ of the initial system with the level surface $H_0(I_1, \ldots, I_n) = h$.

***Theorem* 3.** *If H_0 is non-degenerate in D and $\mathfrak{W}_*$ is a key set for $\mathfrak{A}(D)$, then the equation* (1.2) *have no formal integral*

$$\sum_{s\geqslant 0} F_s(I,\ \varphi,\ t)\, \varepsilon^s$$

with analytic coefficients $F_s: D \times T^{n+1} \to \mathbf{R}$ [25].

The proof of Theorem 3 is based on a successive application of an auxiliary result similar to Lemma 1 of §3.1.

***Lemma* 2.** *Let $F_s: D \times T^{n+1} \to \mathbf{R}$ be continuously differentiable and let $\sum F_s\varepsilon^s$ be a formal integral of* (1.2) *with a non-degenerate function H_0. Then*

1) *$F_0(I, \varphi, t)$ does not depend on φ or t,*
2) *$dF_0 = 0$ on $\mathfrak{W}_*$.*

If the Poincaré set $\mathfrak{W}_*$ is everywhere dense in D, then the equations (1.2) obviously have no formal integral with continuously differentiable coefficients.

It is interesting to note that for $n = 1$ a theorem of Kolmogorov on the preservation of conditionally periodic motions [4] has the consequence that there exists a first integral, analytic in ϵ, with non-constant continuous coefficients. By way of contrast, in the many-dimensional case, for systems of general form, even a continuous integral seems impossible (see [6]).

§2. The creation of isolated periodic solutions—an obstruction to integrability

1. We recall some facts from the theory of periodic solutions of differential equations. We consider an autonomous system $\dot{x} = f(x)$; let $x(t, y)$ be the solution of it with the initial value $x(0, y) = y$. We assume that the system has an ω-periodic solution $x(t, x_0)$. Then

$$X(t) = \left\| \frac{\partial x}{\partial y}\bigg|_{x_0} \right\|$$

is the fundamental matrix of the linear system in variation

$$\dot{\xi} = \frac{\partial f}{\partial x}(x(t,\ x_0))\, \xi.$$

Obviously, $X(0) = E$. Here $X(\omega)$ is called the monodromy matrix for the ω-periodic solution $x(t, x_0)$. Its eigenvalues λ are called multipliers, and the numbers α defined by $\lambda = \exp(\alpha\omega)$ are called characteristic exponents. The multipliers λ may be complex, therefore, the characteristic numbers α are not uniquely determined. Since $(X(\omega) - E)f(x_0) = 0$ and $f(x_0) \neq 0$, in the autonomous case one of the multipliers λ is always equal to 1. By the theorem of Poincaré-Lyapunov [7], the characteristic exponents of an autonomous Hamiltonian system are pairwise equal in size and opposite in sign. Two of them are always zero. In the case of two degrees of freedom the remaining two characteristic exponents are either both real or both purely imaginary. If they are non-zero, then the periodic solution is called non-degenerate or isolated: on the corresponding three-dimensional energy level, in a small neighbourhood of the periodic trajectory, there are no other periodic solutions with period close to ω. A non-degenerate solution with real exponents is called hyperbolic, and with purely imaginary exponents, elliptic. A hyperbolic periodic solution is unstable, and an elliptic solution is stable in a first approximation.

We assume that the Hamiltonian system with two degrees of freedom $\dot{z} = \mathfrak{I}H'$ has, in addition to $H(z)$, an integral $F(z)$.

***Theorem* 1** (Poincaré). *If ζ is on the trajectory of a non-degenerate periodic solution, then the functions $H(z)$ and $F(z)$ are dependent at ζ* [48].

Proof. Since $F(z)$ is a first integral, $F(z(t, \zeta)) = F(\zeta)$ for all $t \in \mathbf{R}$. Differentiating this identity with respect to ζ we obtain

$$\frac{\partial F}{\partial z}\frac{\partial z}{\partial \zeta} = \frac{\partial F}{\partial \zeta}. \tag{2.1}$$

Since $z(t, \zeta)$ is a periodic solution with period ω, for $t = \omega$ we obtain from (2.1) the equality

$$(X(\omega) - E)\frac{\partial F}{\partial \zeta} = 0. \tag{2.2}$$

Similarly,

$$(X(\omega) - E)\frac{\partial H}{\partial \zeta} = 0. \tag{2.3}$$

Since the system is autonomous,

$$(X(\omega) - E)\,\mathfrak{I}\frac{\partial H}{\partial \zeta} = 0. \tag{2.4}$$

Since the periodic solution $z(t, \zeta)$ is non-degenerate, from (2.2)–(2.4) we conclude that the vectors $F'(\zeta)$, $H'(\zeta)$, and $\mathfrak{I}H'(\zeta)$ are linearly dependent:

$$\lambda_1 F' + \lambda_2 H' + \lambda_3 \mathfrak{I}H' = 0, \quad \sum |\lambda_i| \neq 0. \tag{2 5}$$

Obviously, $\langle F', \mathfrak{I}H' \rangle = 0$ and $\langle H', \mathfrak{I}H' \rangle = 0$. Taking the scalar product of (2.5) with $\mathfrak{I}H'$ we have $\lambda_3 \langle \mathfrak{I}H', \mathfrak{I}H' \rangle = 0$, hence, $\lambda_3 = 0$. But then it follows from (2.5) that H and F are dependent at ζ, as required.

Poincaré's theorem gives us a method of proving non-integrability: if the trajectories of non-degenerate periodic solutions densely fill out the phase space, or at least form a key set, then the Hamiltonian system has no additional analytic integral. Apparently, in Hamiltonian systems in general position the periodic trajectories are, in fact, everywhere dense (Poincaré [48]). This is still unproved. In the context of Poincaré's conjecture we mention the following result on geodesic flows on Riemannian manifolds of negative curvature: all periodic solutions are of hyperbolic type and the set of their trajectories densely fills out the phase space [2].

For Hamiltonian systems close to integrable ones, one can prove the existence of a large number of non-degenerate periodic solutions and from this derive the results of § 1.

2. Suppose that for $I = I^0$ the frequencies ω_1 and ω_2 of the unperturbed integrable problem are commensurable and that $\omega_1 \neq 0$. Then the perturbing function $H_1(I^0, \omega_1 t, \omega_2 t + \lambda)$ is periodic in t with some period T. We consider its mean value

$$\overline{H}_1(I^0, \lambda) = \lim_{s \to 0} \frac{1}{s} \int_0^s H_1(I^0, \omega_1 t, \omega_2 t + \lambda)\, dt = \frac{1}{T} \int_0^T H_1\, dt.$$

***Theorem* 2** (Poincaré). *Suppose that the following conditions are satisfied:*

1) $\det \| \partial^2 H_0 / \partial I^2 \| \neq 0$ *at the point* $I = I^0$,

2) *for some* $\lambda = \lambda^*$ *the derivative* $\partial \overline{H}_1 / \partial \lambda = 0$ *but* $\partial^2 \overline{H}_1 / \partial \lambda^2 \neq 0$.

Then for small $\epsilon \neq 0$ *there is a periodic solution of the perturbed Hamiltonian system* (1) *of period* T; *it depends analytically on* ϵ, *and for* $\epsilon = 0$ *it coincides with the periodic solution of the unperturbed system*

$$I = I^0, \quad \varphi_1 = \omega_1 t, \quad \varphi_2 = \omega_2 t + \lambda^*.$$

The two characteristic exponents $\pm\alpha$ *of this solution can be expanded in series of powers of* $\sqrt{\varepsilon}$:

$$\alpha = \alpha_1 \sqrt{\varepsilon} + \alpha_2 \varepsilon + \alpha_3 \varepsilon \sqrt{\varepsilon} + \dots,$$

where

$$\omega_1^2 \alpha_1^2 = \frac{\partial^2 \overline{H}_1}{\partial \lambda^2} (\lambda^*) \left(\omega_1^2 \frac{\partial^2 H_0}{\partial I_2^2} - 2\omega_1 \omega_2 \frac{\partial^2 H_0}{\partial I_1 \partial I_2} + \omega_2^2 \frac{\partial^2 H_0}{\partial I_1^2} \right).$$

A proof can be found in [48], [32].

The function $\overline{H}_1(I^0, \lambda)$ is periodic in λ with period 2π. Hence, there exist at least two values of λ for which $d\overline{H}_1 = 0$. In general, these critical points are non-degenerate. There are as many local minima (where $\partial^2 \overline{H}_1 / \partial \lambda^2 > 0$) as local maxima (where $\partial^2 \overline{H}_1 / \partial \lambda^2 < 0$). In a typical situation for $I = I^0$

$$\omega_1^2 \frac{\partial^2 H_0}{\partial I_2^2} - 2\omega_1 \omega_2 \frac{\partial^2 H_0}{\partial I_1 \partial I_2} + \omega_2^2 \frac{\partial^2 H_0}{\partial I_1^2} \neq 0. \tag{2.6}$$

Incidentally, the geometric condition indicates the absence of inflexions on the curve $H_0(I) = h$ at $I = I^0$. Thus, the equation $d\overline{H}_1 = 0$ has as many roots for which $\alpha_1^2 < 0$ as roots for which $\alpha_1^2 > 0$. Equivalently, for small values of $\epsilon \neq 0$ the perturbed system has as many periodic solutions of elliptic type as of hyperbolic type. In this situation it is usual to say that the disintegration of the unperturbed invariant torus $I = I^0$ creates pairs of isolated periodic solutions. By results of KAM-theory, the trajectories of typical elliptic periodic solutions "surround" invariant tori. Hyperbolic periodic solutions have two invariant surfaces (separatrices) filled out by solutions that approximate asymptotically to periodic trajectories as $t \to \pm\infty$. Various asymptotic surfaces may intersect, forming a rather tangled network in the intersection (see Fig. 5). The behaviour of the asymptotic surfaces will be discussed in detail in the next chapter.

Fig. 5

3. The main basis of the proof of non-integrability of the perturbed equations is Lemma 1 of § 1: if $F = F_0(I, \varphi) + \epsilon F_1(I, \varphi) + \dots$ is a first integral of the canonical equations (1), then F_0 does not depend on φ, and the functions H_0 and F_0 are dependent on the Poincaré set $\mathfrak{B}$. The first part of the lemma follows from the non-degeneracy of the unperturbed problem. Using Poincaré's theorem from § 1.1 we can prove the dependence of H_0 and F_0 on the set $\mathfrak{P}$ of unperturbed tori $I = I^0$, which satisfy the conditions of Theorem 2 and the inequality (2.6).

For the periodic solutions $\Gamma(\epsilon)$, arising from the family of periodic solutions situated in the resonant tori $I^0 \in \mathfrak{P}$ are non-degenerate. Therefore, as was proved in § 1.1, H and F are dependent at all points of $\Gamma(\epsilon)$. Let ϵ tend to zero. The periodic solution $\Gamma(\epsilon)$ goes over into one of the periodic solutions $\Gamma(0)$ of the unperturbed problem lying on the torus $I = I^0$, and the functions H and F become H_0 and F_0. By continuity they are dependent at all points of $\Gamma(0)$. Consequently, the rank of the Jacobian matrix

$$\frac{\partial (H_0, F_0)}{\partial (I, \varphi)}$$

is equal to 1 at points $(I, \varphi) \in \Gamma(0)$. In particular, at these points

$$\frac{\partial (H_0, F_0)}{\partial (I_1, I_2)} = 0.$$

To complete the proof it remains to note that the functions H_0 and F_0 do not depend on φ.

We mention that always $\mathfrak{P} \subset \mathfrak{W}$, however, in typical cases the sets $\mathfrak{P}$ and $\mathfrak{W}$ coincide. In addition, in the above arguments the integral F was assumed to be analytic in ϵ, but in § 1 it was proved that there are no integrals formally-analytic in ϵ. However, our aim was to clarify the geometry of the analytic computations of § 1.

For small values of the parameter $\epsilon \neq 0$ Theorem 2 guarantees the existence of a large but finite number of distinct isolated periodic solutions. Therefore, from this theorem one cannot deduce the non-integrability of perturbed systems for fixed values of $\epsilon \neq 0$. True, in the case of two degrees of freedom, which is what we are considering, the following result holds: if the unperturbed system is non-degenerate, then for small fixed values of $\epsilon \neq 0$ the perturbed Hamiltonian system has infinitely many distinct periodic trajectories. Unfortunately, nothing can be said about their isolation. This result can be deduced from Kolmogorov's theorem on the preservation of conditionally periodic motions and Poincaré's last geometric theorem [50].

§ 3. Applications of Poincaré's method

1. We return to the restricted three-body problem considered in Ch. I, § 4. We assume to begin with that the mass of Jupiter μ is zero. Then in the "fixed" space an asteroid rotates around a sun of unit mass in Keplerian orbits, say ellipses. Then it is convenient to go over from the rectilinear coordinates to the Delone canonical elements L, G, l, g; if a and e are the major semi-axis and the eccentricity of the orbit, then $L = \sqrt{a}$, $G = \sqrt{(a(1-e^2))}$, g is the length of the perihelion and l is the angle defined by the position of the asteroid in its orbit, the eccentric anomaly [48], [59]. It turns out that in the new coordinates the equations of motion of an asteroid are canonical with the Hamiltonian function $F_0 = -1/2L^2$. If $\mu \neq 0$, then the complete Hamiltonian F can be expanded in a series of increasing powers of μ: $F = F_0 + \mu F_1 + \ldots$ Since in a moving coordinate system connected with the Sun and Jupiter Keplerian orbits rotate with unit angular velocity, the Hamiltonian function depends on L, G, l, and $g-t$. We put $x_1 = L$, $x_2 = G$, $y_1 = l$, $y_2 = g-t$, and $H = F - G$. Here H depends on x_i and y_i only, and is 2π-periodic in the angular variables y_1 and y_2. As a result we have expressed the equations of motion of an asteroid in the form of the following Hamiltonian system:

$$(3.1) \quad \dot{x}_i = \frac{\partial H}{\partial y_i}, \quad \dot{y}_i = -\frac{\partial H}{\partial x_i}; \quad H = H_0 + \mu H_1 + \ldots, \quad H_0 = -\frac{1}{2x_1^2} - x_2.$$

The expansion of the perturbing function in a multiple trigonometric series in y_1 and y_2 was already studied by Lever'e (see, for example, [59]).

It takes the following form:

$$H_1 = \sum_{u=-\infty}^{\infty} \sum_{v=-\infty}^{\infty} H_{uv} \cos [u y_1 - v (y_1 + y_2)].$$

The coefficients H_{uv}, which depend on x_1 and x_2, are in general non-zero.

The Poincaré set $\mathfrak{P}$ for this problem consists of lines parallel to the x_2-axis: $u/x_1^3 - v = 0$, $H_{uv} \neq 0$. It is dense in the half-plane $x_1 > 0$. However, Poincaré's theorem on the absence of new analytic integrals cannot be directly applied because of the degeneracy of the unperturbed problem: $\det \| \partial^2 H_0 / \partial x^2 \| = 0$. This difficulty can be overcome by using the fact that the canonical equations with Hamiltonians H and $\exp H$ have the same trajectories (but not the same solutions). Consequently, these equations are simultaneously integrable or non-integrable. It remains to note that $\exp H = \exp H_0 + \mu (\exp H_0) H_1 + \dots$ and $\det \| \partial^2 \exp H_0 / \partial x^2 \| \neq 0$. Thus, we have found that the equations of the restricted three-body problem in the form (3.1) do not have an integral $\Phi = \sum \Phi_s \mu^s$, independent of H and formally-analytic in μ, whose coefficients are smooth functions on $D \times T^2\{y \bmod 2\pi\}$, where D is an arbitrary domain in the half-plane $x_1 > 0$.

Whittaker's procedure of reduction of the order is applicable to the autonomous system (3.1). We fix a constant energy $h < 0$ and solve the equation $H(x, y, \mu) = h$ for x_2. We find that

$$-x_2 = K(x_1, y_1, y_2, h, \mu) = K_0 + \mu K_1 + \dots, \qquad K_0 = \frac{1}{2x_1^2}.$$

If we take $y_2 = \tau$ as a new time variable, then the functions $x_1 = x(\tau)$ and $y_1 = y(\tau)$ satisfy Whittaker's equations

$$\frac{dx}{d\tau} = -\frac{\partial K}{\partial y}, \quad \frac{dy}{d\tau} = \frac{\partial K}{\partial x}. \tag{3.2}$$

For these equations the Poincaré set $\mathfrak{P}_*$ is also dense in the half-space $x > 0$. Since the unperturbed system is non-degenerate ($d^2 K_0 / dx^2 \neq 0$), all the conditions of Theorem 3 in §1 are satisfied. Thus, we may conclude that the equations (3.2) for all values of $h < 0$ do not have a first integral $\sum \Phi_s \mu^s$ with continuously differentiable coefficients in $\Delta \times T^2\{y, \tau \bmod 2\pi\}$, where Δ is any interval on the half-line $x > 0$.

We note that (3.1) and (3.2) have additional integrals in the form of convergent power series in μ with continuous (but not differentiable) coefficients.

2. "Let us proceed to another problem: that of the motion of a rigid body around a fixed point ... We can, therefore, ask whether in this problem the presented in this chapter oppose the existence of a single-valued integral other than those of the vis viva and of area" (Poincaré [48]).

To the group of symmetries, which consists of rotations of the body around a vertical line, there corresponds a linear integral $F_2 = \langle M, e \rangle$: the vertical projection of the kinetic momentum is constant. Fixing this constant, we reduce the number of degrees of freedom to two: on the four-dimensional integral levels $I_{23}=\{\langle M, e\rangle = f_2, \langle e, e\rangle = 1\}$ there arises a Hamiltonian system with two degrees of freedom. Its Hamiltonian function, the total energy of the body for a fixed value of the projection $\langle M, e\rangle$, is $H_0+\epsilon H_1$, where H_0 is the kinetic energy (the Hamiltonian function of the integrable Euler problem on the motion of an inertial body), and ϵH_1 is the potential energy of the body in a homogeneous gravitational force field (ϵ is the product of the weight of the body by the distance from the centre of mass to the point of suspension). We assume that ϵ is small. This is equivalent to the study of the rapid rotation of a body in a moderate force field. In the unperturbed integrable Euler problem we can introduce action-angle variables I and φ. The formulae for the transition from the special canonical variables L, G, l, g to the action-angle variables I and φ can be found, for example, in [32]. In the new variables $H = H_0(I)+\epsilon H_1(I, \varphi)$. The action variables I_1 and I_2 vary in the domain $\Delta = \{|I_1| \leqslant I_2,\ I_2 \geqslant 0\}$. The Hamiltonian $H_0(I_1, I_2)$ is a homogeneous function of degree 2 and is analytic in each of the four connected subdomains of Δ into which the domain is divided by the three lines π_1, π_2, and $I_1 = 0$. The equation of the lines π_1 and π_2 is $2H_0/I_2^2 = J_2^{-1}$. They are symmetric relative to the vertical axis and tend to the line $I_1 = 0$ as $J_2 \to J_1$ and to the pair of lines $|I_1| = I_2$ as $J_2 \to J_3$ (we recall that J_1, J_2, and J_3 are the principal moments of inertia of the body and $J_1 \geqslant J_2 \geqslant J_3$). The level lines of H_0 are illustrated in Fig. 6.

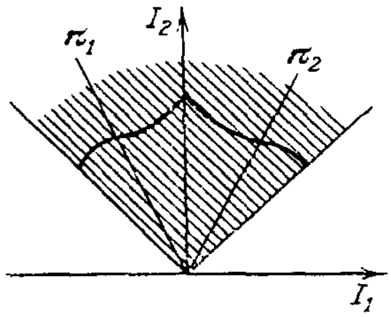

Fig. 6

The expansion of the perturbing function H_1 in a multiple Fourier series in the angle variables φ_1 and φ_2 is, in fact, contained in Jacobi's paper [70]:

$$\sum_{m\in Z} H_{m,1}e^{i(m\varphi_1+\varphi_2)} + \sum_{m\in Z} H_{m,-1}e^{i(m\varphi_1-\varphi_2)} + \sum_{m\in Z} H_{m,0}e^{im\varphi_1}.$$

It follows, in particular, that in this problem the sets $\mathfrak{B}$, $\mathfrak{W}$, and $\mathfrak{P}$ coincide. When $J_1 > J_2 > J_3$, the secular set consists of infinitely many lines passing through $I = 0$ and accumulating at the pair of lines π_1 and π_2. It can be shown that H_0 is non-degenerate in Δ. If H were analytic in I throughout Δ, then the results of § 1 would be applicable: the points I^0 lying on the lines π_1 and π_2 would satisfy the conditions of Theorem 1. The difficulty associated with the analytic singularities of the Hamiltonian function in the action-angle variables can be overcome by considering the problem of an additional analytic integral on the whole integral level set I_{23}. Using Poincaré's method we can prove the following result.

***Theorem* 1.** *If a heavy rigid body is dynamically asymmetric, then the equations of rotation do not have a formal integral $\sum F_s \epsilon^s$, independent of $H_0 + \epsilon H_1$, with coefficients analytic on I_{23}* [26].

This result gives a negative answer to a question posed by Poincaré [48].

CHAPTER V

BIFURCATION OF ASYMPTOTIC SURFACES

§1. Conditions for bifurcation

1. Let V be the smooth n-dimensional state space of a Hamiltonian system, T^*V its phase space and $H: T^*V \times \mathbf{R}\{t\} \to \mathbf{R}$ the Hamiltonian function. In the extended phase space $M = T^*V \times \mathbf{R}^2\{E, t\}$ the equations of motion are again Hamiltonian:

$$(1.1) \qquad \dot{x} = \frac{\partial K}{\partial y}, \quad \dot{y} = -\frac{\partial K}{\partial x}, \quad \dot{E} = \frac{\partial K}{\partial t}, \quad \dot{t} = -\frac{\partial K}{\partial E},$$

where $K = H(y, x, t) - E$, $x \in V$, $y \in T_x^* V$.

A smooth surface $\Lambda^{n+1} \subset M$ is called Lagrangian if for any closed contractible contour γ

$$\oint_\gamma y\,dx - E\,dt$$

($E = H(y, x, t)$ on Λ^{n+1}) is zero. Lagrangian surfaces are invariant under the action of the phase flow of the system (1.1) [16]. In the autonomous case Lagrangian surfaces $\Lambda^n \subset T^*V$ are given by the condition

$$\oint_\gamma y\,dx = 0 \qquad (\gamma \subset \Lambda^n,\ \partial\gamma = 0).$$

If a Lagrangian surface Λ^{n+1} has a one-to-one projection onto $D \times \mathbf{R}\{t\}$, $D \subset V$, then it can be represented as a graph

$$y = \frac{\partial S(x, t)}{\partial x}, \qquad H(y, x, t) = -\frac{\partial S(x, t)}{\partial t},$$

where $S: D \times \mathbf{R} \to \mathbf{R}$ is a smooth function. In the autonomous case Λ^n is given by the graph

$$y = \frac{\partial S}{\partial x}, \qquad x \in D.$$

The function $S(x, t)$ satisfies the Hamilton-Jacobi equation:

$$\frac{\partial S}{\partial t} + H\left(\frac{\partial S}{\partial x}, x, t\right) = 0.$$

In this section we are concerned with Lagrangian surfaces consisting of asymptotic trajectories. Naturally, such surfaces are called asymptotic.

2. We assume that the Hamiltonian function is 2π-periodic in t and depends on a further parameter $\epsilon : H = H(y, x, t, \epsilon)$. Suppose that $H(y, x, t, 0) = H_0(y, x)$ for $\epsilon = 0$ does not contain the time and satisfies the following conditions:

1) there exist two critical points y_-, x_- and y_+, x_+ of $H_0(y, x)$ at which the eigenvalues of the linearized Hamiltonian system

$$\dot{y} = -\frac{\partial H_0}{\partial x}, \quad \dot{x} = \frac{\partial H_0}{\partial y}$$

are real and non-zero. In particular, the 2π-periodic solutions $x_\pm(t) = x_\pm$, $y_\pm(t) = y_\pm$ are of hyperbolic type.

2) if Λ^+ (or Λ^-) is a stable (unstable) asymptotic manifold in T^*V passing through x_+, y_+ (or x_-, y_-), then $\Lambda^+ = \Lambda^-$. Hence, in particular, $H_0(y_+, x_+) = H_0(y_-, x_-)$.

3) There is a domain $D \subset V$ containing $x_\pm$ and such that in $T^*D \subset T^*V$ the equation of the surface $\Lambda^+ = \Lambda^-$ can be expressed in the following form: $y = \partial S_0/\partial x$, where S_0 is some analytic function in D. It is useful to consider the differential equation

$$\dot{x} = \frac{\partial H_0}{\partial y}\bigg|_{y(x)}, \quad y = \frac{\partial S_0}{\partial x}. \tag{1.2}$$

In a small neighbourhood of $x_\pm$ its solution tends to $x_\pm$ as $t \to \pm\infty$.

4) In D (1.2) has a doubly-asymptotic solution: $x_0(t) \to x_\pm$ as $t \to \pm\infty$ (Fig. 7).

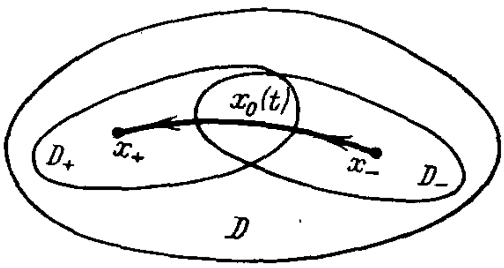

Fig. 7

The Hamiltonian system with the Hamiltonian function $H_0(y, x)$ must be regarded as the unperturbed system. In applications it is most frequently completely integrable. Let D_+ (or D_-) be a subdomain of D containing x_+ (or x_-) but not x_- (or x_+). For small ϵ the asymptotic surfaces Λ^+ and Λ^- do not vanish, but go over to the "perturbed" surfaces Λ^+_ϵ and Λ^-_ϵ. More precisely, in $D_\pm \times \mathbf{R}\{t\}$ the equation of the asymptotic surface $\Lambda^\pm_\epsilon$ can be written in the following form:

$$y = \frac{\partial S^\pm}{\partial x},$$

where $S^\pm(x, t, \epsilon)$ is 2π-periodic in t and is defined and analytic for $x \in D$ and small ϵ (Poincaré [48]). The functions $S^\pm$ must, of course, satisfy the Hamilton-Jacobi equation

$$\frac{\partial S^\pm}{\partial t} + H\left(\frac{\partial S^\pm}{\partial x}, x, t, \varepsilon\right) = 0. \tag{1.3}$$

By hypothesis, for $\epsilon = 0$ the surfaces Λ_0^+ and Λ_0^- coincide. However, as Poincaré [47] first noted, in general, for small $\epsilon \neq 0$, regarded as point sets in $T^*(D_+ \cap D_-) \times \mathbf{R}$ the surfaces no longer coincide. This phenomenon is called a bifurcation of the asymptotic surfaces. Obviously, Λ_ϵ^+ coincides with Λ_ϵ^- if and only if (1.3) has a solution $S(x, t, \epsilon)$ that is analytic in x throughout D.

3. ***Theorem*** 1 (Poincaré). *If $H_1(y_+, x_+, t) = H_1(y_-, x_-, t)$ and*

$$\int_{-\infty}^{\infty} \{H_0, H_1\}(y(x_0(t)), x_0(t), t)\, dt \neq 0, \tag{1.4}$$

then for small $\epsilon \neq 0$ the perturbed asymptotic surfaces Λ_ϵ^+ and Λ_ϵ^- do not coincide [47].

Proof. We assume that (1.3) has an analytic solution $S(s, t, \epsilon)$ that for small ϵ can be expressed as a convergent power series

$$S = S_0(x, t) + \varepsilon S_1(x, t) + \ldots$$

The function $S_0(x, t)$ must satisfy the equation

$$\frac{\partial S_0}{\partial t} + H_0\left(\frac{\partial S_0}{\partial x}, x\right) = 0.$$

Hence, $S_0 = -ht + W(x)$, where $h = H_0(y_\pm, x_\pm)$ and $W(x)$ is a solution of

$$H_0\left(\frac{\partial W}{\partial x}, x\right) = h.$$

Clearly, $W(x)$ coincides with the function $S_0(x)$ by § 1.2.

Let $H = H_0(y, x) + \epsilon H_1(y, x, t) + \ldots$ Then we obtain from (1.3) a quasilinear differential equation for S_1:

$$\frac{\partial S_1}{\partial t} + \left.\frac{\partial H_0}{\partial x}\right|_{y(x)} \frac{\partial S_1}{\partial x} + H_1(y(x), x, t) = 0. \tag{1.5}$$

Since (1.2) is autonomous, together with $x_0(t)$ it has the family of solutions $x_0(t + \alpha)$, $\alpha \in \mathbf{R}$. It follows from (1.5) that on these solutions

$$S_1(x_0(t+\alpha), t) = S_1(x_0(\alpha), 0) - \int_0^t H_1(y(x_0(t+\alpha)), x_0(t+\alpha), t)\, dt. \tag{1.6}$$

Without loss of generality we may assume that $H_1(y_\pm, x_\pm, t) = 0$ for all t. If this is not the case, then instead of H_1 we must take $H_1 - H_1(y_\pm, x_\pm, t)$. The Poisson bracket remains unchanged.

Since the Taylor expansion of H_1 in a neighbourhood of the points $x_\pm$, $y_\pm$ begins with linear terms in $x - x_\pm$, $y - y_\pm$, and since the functions $x_0(t) - x_\pm$, $y(x_0(t)) - y_\pm$ tend to zero exponentially as $t \to \pm\infty$, the integral

$$J(\alpha) = \int_{-\infty}^{\infty} H_1(y_0(t+\alpha), x_0(t+\alpha), t)\, dt \tag{1.7}$$

converges. From (1.5) it also follows that $S_1(x, t)$ at $x_\pm$ does not depend on t. By (1.6), the integral $J(\alpha)$ is equal to $S_1(x_+) - S_1(x_-)$, therefore, does not depend on α. To complete the proof it remains to calculate the derivative

$$\left.\frac{dJ}{d\alpha}\right|_{\alpha=0}=\int\limits_{-\infty}^{\infty}\sum\left(\frac{\partial H_1}{\partial x_s}\dot{x}_s+\frac{\partial H_1}{\partial y_s}\dot{y}_s\right)dt=\int\limits_{-\infty}^{\infty}\{H_0, H_1\}\,dt=0.$$

Remark. Another proof of Poincaré's theorem can be found in [6].

4. In the autonomous case the condition for bifurcation of asymptotic surfaces situated on a certain fixed energy level can be expressed as follows:

$$\int\limits_{-\infty}^{\infty}\{F_0, H_1\}\,dt\not\equiv 0, \tag{1.8}$$

where F_0 is the integral of the unperturbed system. If $dF_0 = 0$ at the points of unstable periodic trajectories, then the integral (1.8) necessarily converges.

§2. Bifurcation of asymptotic surfaces—an obstruction to integrability

1. We consider a Hamiltonian system with Hamiltonian $H(z, t, \epsilon) = H_0(z)+\epsilon H_1(z, t)+O(\epsilon^2)$ under the assumptions of §1. In particular, the unperturbed system has two hyperbolic equilibrium positions $z_\pm$, joined by a doubly-asymptotic solution $t \to z_0(t)$, $t \in \mathbf{R}$.

***Theorem* 1** (Bolotin). *Suppose that*

1) $\displaystyle\int\limits_{-\infty}^{\infty}\{H_0\{H_0, H_1\}\}(z_0(t), t)\,dt\neq 0,$

2) *for small ϵ the perturbed system has a doubly-asymptotic solution $t \to z_\epsilon(t)$, close to $t \to z_0(t)$.*

Then for small fixed values of $\epsilon \neq 0$, in any neighbourhood of a closed trajectory $z_\epsilon(t)$, the Hamiltonian equation $\dot{z} = \Im H'$ does not have a complete set of independent integrals in involution.

Remark. 1) can be replaced by the following condition: for some $m \geqslant 2$

$$\int\limits_{-\infty}^{\infty}\underbrace{\{H_0,\ldots\{H_0,}_{m} H_1\}\}(z_0(t), t)\,dt\neq 0.$$

If 1) holds, then the asymptotic surfaces necessarily do not coincide.
2) is, of course, not always satisfied. We give a sufficient condition for the existence of a family of doubly-asymptotic trajectories.

Let $H_0 = F_1, \ldots, F_n$ be commuting integrals of the unperturbed problem that are independent on $\Delta_0^+ = \Delta_0^-$. If

$$\int\limits_{-\infty}^{\infty}\{F_i, H_1\}(z_0(t), t)\,dt=0,$$

$$\det\left\|\int\limits_{-\infty}^{\infty}\{F_i\{F_j, H_1\}\}(z_0(t), t)\,dt\right\|\neq 0,$$

then there exists a family, analytic in ϵ, of asymptotic solutions $t \to z_\epsilon(t)$. This result is easy to derive from the implicit function theorem.

If we investigate the problem of the existence of independent involutive integrals $F_i(z, t, \epsilon)$, $1 \leqslant i \leqslant n$, that are analytic (or formally-analytic) in ϵ, then 2) can be omitted. In particular, if 1) is satisfied, then the series of perturbation theory diverge in a neighbourhood of bifurcated asymptotic surfaces (Poincaré [47]).

2. Using Birkhoff's method of normal forms we can find in a neighbourhood of unstable periodic solutions $z_\pm + O(\epsilon)$ a 2π-periodic in t formal canonical change of variables $z \to u$ that carries the Hamiltonian function $H(z, t, \epsilon)$ to a function $H^\pm(u, \epsilon)$ not depending on t. Because the characteristic exponents may be commensurable, the Birkhoff transformation may diverge. However, in the case of one degree of freedom ($n = 1$) the formal series of the change of variables $z \to u$ converge everywhere and depend analytically on ϵ.

Theorem 2. *We assume that the Birkhoff transformation converges and depends analytically on ϵ. If Theorem* 1, 1) *holds, then for small $\epsilon \neq 0$ the Hamiltonian equation does not have a complete set of independent analytic integrals in involution.*

In particular, 1) is for $n = 1$ a sufficient condition for integrability (Siegel [22]).

Proof of Theorem 2. We define a function $\mathbf{R}^\pm$ on $\Lambda_0^\pm$ by the formulae

$$\mathbf{R}^+(z) = -\int_0^\infty \{H_0\{H_0, H_1\}\}(z(t), t)\,dt, \quad \mathbf{R}^-(z) = \int_{-\infty}^0 \{H_0\{H_0, H_1\}\}(z(t), t)\,dt,$$

where $t \to z(t)$ is the asymptotic motion of the unperturbed system with the initial condition $z(0) = z$.

Lemma 1. *The functions $\mathbf{R}^\pm$ are defined by H_0, the family of surfaces $\Lambda_\epsilon^\pm$, and the canonical structure.*

For according to the results of the previous section, the functions

$$S^+(z) = -\varepsilon \int_0^{+\infty} (H_1(z(t), t) - H_1(z_+, t))\,dt,$$

$$S^-(z) = \varepsilon \int_{-\infty}^0 (H_1(z(t), t) - H_1(z_-, t))\,dt$$

are generating functions of the Lagrangian surfaces $\Lambda_\epsilon^\pm$ up to $O(\epsilon^2)$. But $\epsilon\mathbf{R}^\pm = \{H_0\{H_0, S^\pm\}\}$, as required.

The composition of the Birkhoff transformation with the powers of the map at a period allow us to extend $H^\pm$ from neighbourhoods of critical points $u_\pm(\epsilon)$ up to certain neighbourhoods $W_\pm$ of the asymptotic surfaces $\Lambda_\epsilon^\pm$. Since a possible bifurcation of the surfaces Λ_ϵ^+ and Λ_ϵ^- is of order ϵ, for small ϵ the neighbourhoods W_+ and W_- intersect.

Lemma 2. $\{H^+, H^-\} \not\equiv 0$ *for* $\epsilon \neq 0$.

Proof. We put $H^\pm(u, \epsilon) = H_0^\pm(u) + \epsilon H_1^\pm(u) + O(\epsilon^2)$. Since $H_0^\pm(u) = H_0(u)$,

$$\{H^+, H^-\} = \varepsilon \{H_0, H_1^- - H_1^+\} + O(\varepsilon^2).$$

Since Λ_0^- is an invariant asymptotic manifold of the Hamiltonian system $\dot{u} = \Im H_0'$, by Lemma 1,

$$\{H_0, H_1^-\}(u) = \int_{-\infty}^{0} \{H_0\{H_0, H_1^-\}\}(u_0(t))\, dt = R^-(u), \qquad u \in \Lambda_0^-.$$

Similarly

$$\{H_0, H_1^+\}(u) = \int_{0}^{+\infty} \{H_0\{H_0, H_1^+\}\}(u_0(t))\, dt = R^+(u), \qquad u \in \Lambda_0^*.$$

Consequently

$$\{H^+, H^-\} = \varepsilon \int_{-\infty}^{\infty} \{H_0\{H_0, H_1\}\}(z_0(t), t)\, dt + O(\varepsilon^2).$$

According to 1), for small $\epsilon \neq 0$ the Poisson bracket $\{H^+, H^-\} \not\equiv 0$.

In the new variables u the integrals $F_1, \ldots, F_n$ do not depend on t. For $\epsilon \neq 0$ let $F_1, \ldots, F_n$ be independent integrals at some point of $W_+ \cap W_-$. Since $\{H^\pm, F_i\} \equiv 0$, $\Im H^{\pm\prime}$ is a linear combination of the $\Im F_i'$. Since $\{F_i, F_j\} \equiv 0$, obviously, at this point $\{H^+, H^-\} = 0$. To complete the proof it remains to remark that the analytic function $\{H^+, H^-\}$ does not vanish on an everywhere dense set.

3. **Theorem 3.** *Let* $n = 1$. *If*

1) $\displaystyle\int_{-\infty}^{\infty} \{H_0, H_1\}(z_0(t), t)\, dt \neq 0,$

2) *for small* ϵ *the perturbed system has a doubly-asymptotic solution* $t \to z_\epsilon(t)$ *close to* $t \to z_0(t)$, *then for small* $\epsilon \neq 0$ *the Hamiltonian system* $\dot{z} = \Im H'$ *does not have an additional analytic integral* [65].

Proof. We consider the map at a period g of the section $t = t_0$ into itself. For small ϵ this map has two fixed hyperbolic points z_1 and z_2 with invariant separatrices $W_1^\pm$ and $W_2^\pm$ (see Fig. 8). By the conditions of the theorem, for $\epsilon \neq 0$ the separatrices W_1^+ and W_2^- intersect and do not coincide.

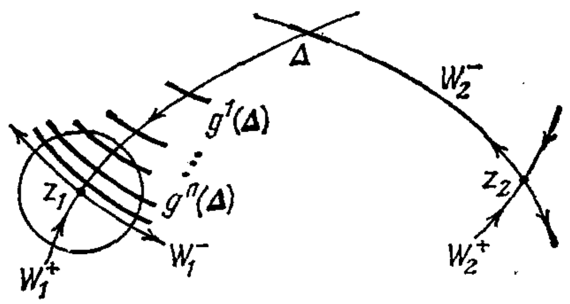

Fig. 8

Let V be a small neighbourhood of z_1 and Δ a small segment of W_2^- intersecting W_1^+. For sufficiently large n the segment $g^n(\Delta)$ lies wholly in V and again intersects W_1^+. By a theorem of Grobman-Hartman [44], in V the map g is topologically dual to a linear hyperbolic rotation. Consequently, as $n \to \infty$ the segment $g^n(\Delta)$ "stretches" along the separatrix W_1^- and approaches it unboundedly. Obviously, the union

$$\bigcup_{n=1}^{\infty} g^n(\Delta) \tag{2.1}$$

is a key set for the class of functions that are analytic in the section $t = t_0$.

Suppose now that the Hamiltonian equation has an analytic integral $f(z, t)$. The function $f(z, t_0)$ is invariant under g and constant on W_2^- (since the sequence $g^n(z)$, $z \in W_2^-$, converges to z_2 as $n \to \infty$). Consequently, the analytic function $f(z, t_0)$ is constant on the set (2.1) and is therefore constant for any t_0.

Remark. Poincaré divided the doubly-asymptotic solutions into two types: homoclinic (when $z_+ = z_-$) and heteroclinic (when $z_+ \neq z_-$). If $n = 1$, then for small ϵ the perturbed problem always has homoclinic solutions (if, of course, it has them for $\epsilon = 0$) [47].

§3. Some applications

1. We consider first the simplest problem of the oscillations of a pendulum with a vibrating point of suspension. The Hamiltonian function H is $H_0 + \epsilon H_1$, where

$$H_0 = p^2/2 - \omega^2 \cos x, \quad H_1 = -\omega^2 f(t) \cos x,$$

and $f(t)$ is a 2π-periodic function of time. When $\epsilon = 0$, then the upper position of the pendulum is an unstable equilibrium. The unperturbed problem has two families of homoclinic solutions:

$$\cos x_0 = \frac{2e^{\pm\omega(t-t_0)}}{e^{\pm 2\omega(t-t_0)} + 1}, \quad x_0 \to \pm\pi \quad \text{as} \quad t \to \pm\infty. \tag{3.1}$$

Since $\{H_0, H_1\} = -\omega^2 f(t)\dot{x} \sin x$, (1.8) to within a constant multiplier is equal to

$$\int_{-\infty}^{\infty} \dot{f}(t) \cos x_0 \, dt.$$

Let $f(t) = \sum f_n e^{int}$. Then (1.8) can be expressed as a series

$$\sum_{n \in Z} 2\, inf_n J_n e^{int_0}, \quad J_n = \int_{-\infty}^{\infty} \frac{e^{\pm\omega t} e^{int}}{e^{\pm 2\omega t} + 1} dt.$$

The integrals J_n are easily calculated by residues:

$$J_n = \frac{-ie^{-n\pi/2\omega}}{2\omega\,(1 + e^{\mp n\pi/\omega})} \neq 0.$$

Consequently, if $f(t) \neq$ const (that is, $f_n \neq 0$ for some $n \neq 0$), then (1.8) is non-zero on at least one doubly-asymptotic solution of the family (3.1). Thus, if $f(t) \neq$ const, then by the results of §2 this problem for sufficiently small (but fixed) $\epsilon \neq 0$ does not have an analytic first integral $F(p, x, t)$ that is 2π-periodic in x and t.

2. In the problem of rapid rotation of an asymmetric rigid body the Hamiltonian is $H = H_0 + \epsilon H_1$, where

$$H_0 = \frac{1}{2}\langle AM, M\rangle, \qquad H_1 = r_1 e_1 + r_2 e_2 + r_3 e_3; \qquad A = \operatorname{diag}(a_1, a_2, a_3).$$

The numbers a_1, a_2, a_3 are the inverses of the principal moments of inertia of the body. For $\epsilon = 0$ we have an integrable Euler system. In this "unperturbed" problem on all noncritical three-dimensional levels $I_{123} = \{F_1 = H_0 = f_1 > 0, F_2 = f_2, F_3 = 1\}$ there are two unstable periodic solutions: if $a_1 < a_2 < a_3$, then

$$(3.2) \qquad \begin{cases} M_1 = M_3 = 0, \quad M_2 = M_2^0 = \pm\sqrt{2f_1/a_2}, \quad e_2 = e_2^0 = \pm f_2/M_2^0, \\ e_1 = \alpha\cos(a_2 M_2^0)t, \quad e_3 = \alpha\sin(a_2 M_2^0)t; \quad \alpha^2 = 1 - (f_2/M_2^0)^2. \end{cases}$$

Since $\langle M, e\rangle^2 \leqslant \langle M, M\rangle\langle e, e\rangle$ and since the functions F_1, F_2, F_3 are independent on I_{123}, it follows that $\alpha^2 > 0$. The stable and unstable asymptotic surfaces of the periodic solutions (3.2) can be represented as the intersections of the manifold I_{123} with the hyperplanes $M_1\sqrt{(a_2 - a_1)} \pm M_3\sqrt{(a_3 - a_2)} = 0$. In the Euler problem the asymptotic surfaces are "doubled": they are completely filled out by doubly-asymptotic trajectories, which as $t \to \pm\infty$ approximate unboundedly to the periodic trajectories (3.2). The bifurcation of these surfaces was studied in [28], [22]. It turned out that on perturbation the asymptotic surfaces bifurcate always except in the "Hess-Appelrot case":

$$(3.3) \qquad r_2 = 0, \qquad r_1\sqrt{a_3 - a_2} \pm r_3\sqrt{a_2 - a_1} = 0.$$

In this case one pair of separatrices does not bifurcate, and the other does (Fig. 9). The reason for non-bifurcation is that under the condition (3.3) the perturbed problem, for all ϵ, has the "particular" integral $F = M_1\sqrt{(a_2 - a_1)} \pm M_3\sqrt{(a_3 - a_2)}$ ($\dot{F} = 0$ when $F = 0$). It can be shown that the closed invariant surfaces $H = f_1, F_2 = f_2, F_3 = 1, F = 0$, for small ϵ are just a pair of doubled separatrices of the perturbed problem (see [28]).

In the problem of rapid rotation of a heavy asymmetric top the bifurcated separatrices apparently do not always intersect. However, Theorem 2 of §2 is applicable, and with its help it can be established that there is no

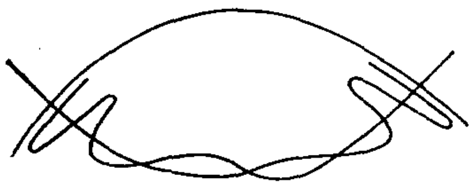

Fig. 9

additional analytic integral of the perturbed problem for small, but fixed, ϵ (Siegel [22]).

The behaviour of the solutions of the perturbed problem has been studied numerically in [67]. In Fig. 10 the results of the calculations for various values of ϵ are shown. It is fairly clear that the picture of the invariant curves of the unperturbed problem begins to be destroyed exactly in the neighbourhoods of the separatrices.

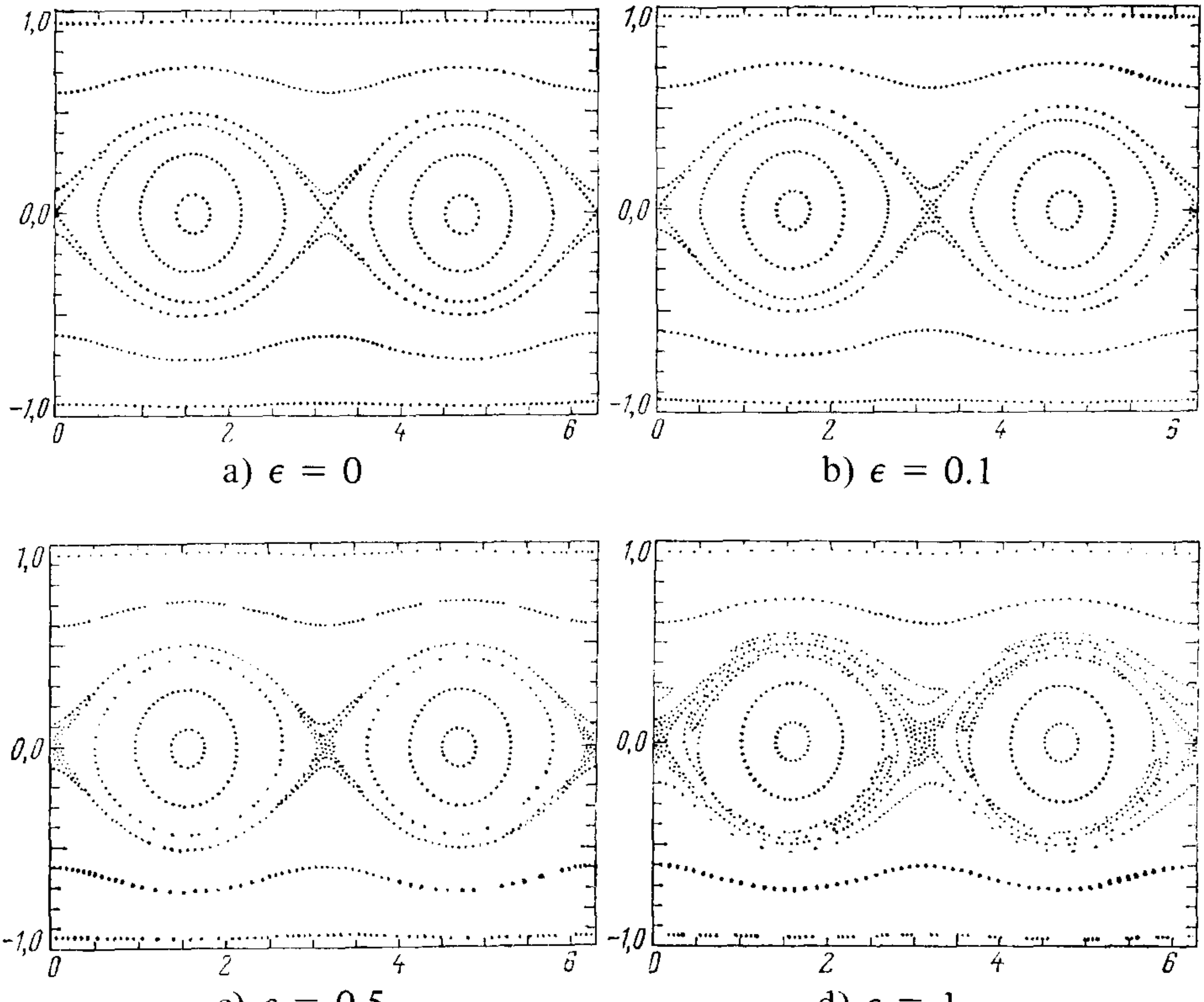

Fig. 10

3. We now consider the Kirchhoff equations

$$(3.4)\quad \begin{cases} \dot{M} = M \times \omega + e \times u, \quad \dot{e} = e \times \omega; \quad \omega = \dfrac{\partial H}{\partial M}, \quad u = \dfrac{\partial H}{\partial e} \\ H = \dfrac{1}{2}\langle AM, M\rangle + \langle BM, e\rangle + \dfrac{1}{2}\langle Ce, e\rangle, \end{cases}$$

which describe the rotation of a rigid body in an ideal fluid. The matrix $A = \operatorname{diag}(a_1, a_2, a_3)$ is diagonal and B and C are symmetric.

***Theorem* 1.** *Suppose that a_1, a_2, and a_3 are unequal. If the Kirchhoff equations have an additional integral independent of the functions $F_1 = H_1$, $F_2 = \langle M, e\rangle$, $F_3 = \langle e, e\rangle$ and analytic in $\mathbf{R}^6\{M, e\}$, then $B = \operatorname{diag}(b_1, b_2, b_3)$ and*

$$(3.5)\quad a_1^{-1}(b_2 - b_3) + a_2^{-1}(b_3 - b_1) + a_3^{-1}(b_1 - b_2) = 0.$$

If $B = 0$, *then the independent analytic integral exists only when* $C = \mathrm{diag}(c_1, c_2, c_3)$ *and*

$$a_1^{-1}(c_2 - c_3) + a_2^{-1}(c_3 - c_1) + a_3^{-1}(c_1 - c_2) = 0. \tag{3.6}$$

The matrix B in Steklov's integrable case is defined precisely by the condition (3.5), and (3.6) gives Clebsch's integrable case. It is interesting to note the coincidence that (3.5) and (3.6) are of the same form.

Corollary. *In general, the Kirchhoff equations are non-integrable.*

The proof of Theorem 1 is based on the phenomenon of bifurcation of the separatrices. We introduce a small parameter ϵ in (3.4), replacing e by ϵe. On the fixed integral level $I_{23} = \{F_2 = f_2,\ F_3 = f_3 > 0\}$ the equations (3.4) are Hamiltonian with $H_0 + \epsilon H_1 + \epsilon^2 H_2$, where H_0, H_1 and H_2 are the functions $\langle AM, M\rangle/2$, $\langle BM, e\rangle$, and $\langle Ce, e\rangle/2$ on I_{23}. This is equivalent to the case when the constant energy f_1 is much larger than f_2 and f_3. For $\epsilon = 0$ we have again the integrable Euler problem on the motion of a free rigid body under inertia.

Let F_0 be an analytic integral of the Euler problem. If the improper integral

$$J = \int_{-\infty}^{\infty} \{F_0, H_1\}\, dt, \tag{3.7}$$

calculated along solutions of the unperturbed problem that are asymptotic to periodic solutions (3.2), is not constant on the separatrices of the Euler problem, then by Theorem 2 of §2, for small $\epsilon \neq 0$ the Kirchhoff equations do not have on I_{123} a non-constant analytic integral.

The proof of Theorem 1 thus reduces to the verification that the integral (3.7) is not constant in which it is convenient to put $F_0 = \langle M, M\rangle/2$. When $B = 0$, then, of course, in (3.7) we must take H_2 instead of H_1. If $F_0 = \langle M, M\rangle/2$, then J exists only in the sense of the principal value. In this case we can put, for example, $F_0 = (\langle M, M\rangle - a_2^{-1}\langle AM, M\rangle)/2$.

As an example we obtain the Steklov condition (3.5) in the simplest case when $B = \mathrm{diag}(b_1, b_2, b_3)$. Since

$$\{F_0, H_1\} = (b_3 - b_2)(M_1M_2e_3 + M_1M_3e_2) + $$
$$+ (b_1 - b_3)(M_2M_3e_1 + M_1M_2e_3) + (b_2 - b_1)(M_1M_3e_2 + M_2M_3e_1),$$

we see that

$$J = (b_3 - b_2)(J_{123} + J_{132}) + (b_1 - b_3)(J_{231} + J_{123}) + (b_2 - b_1)(J_{132} + J_{231}),$$

where

$$J_{ijk} = \int_{-\infty}^{\infty} M_iM_je_k\, dt.$$

The integrals J_{ijk} satisfy the following linear equations:

$$\begin{cases} a_3J_{132} - a_2J_{123} + (a_3 - a_2)J_{231} = 0, \\ a_1J_{123} - a_3J_{231} + (a_1 - a_3)J_{132} = 0, \\ a_2J_{231} - a_1J_{132} + (a_2 - a_1)J_{123} = 0. \end{cases} \tag{3.8}$$

Let us derive, for example, the first relation. From Kirchhoff's equations for $\epsilon = 0$ it follows that

$$(M_1 e_1)^{\cdot} = a_3 M_1 M_3 e - a_2 M_1 M_2 e_3 + (a_3 - a_2) M_2 M_3 e_1.$$

Since $M_1 \to 0$ as $t \to \pm\infty$,

$$a_3 J_{132} - a_2 J_{123} + (a_3 - a_2) I_{231} = \int_{-\infty}^{\infty} (M_1 e_1)^{\cdot}\, dt = 0.$$

If $a_2 a_3 - a_1 a_2 - a_1 a_3 \neq 0$, then from (3.8) we obtain the two equalities

$$J_{132} = \frac{a_1 a_3 - a_1 a_2 - a_2 a_3}{a_2 a_3 - a_1 a_2 - a_1 a_3} J_{231}, \quad J_{123} = \frac{a_1 a_2 - a_1 a_3 - a_2 a_3}{a_2 a_3 - a_1 a_2 - a_1 a_3} J_{231}.$$

The integral J_{231} can be calculated by means of residues and it can be verified that it is non-zero (Onishchenko). If (3.5) is not satisfied, then $J \neq 0$ by the obvious equality

$$J\,(a_1 a_2 + a_1 a_3 - a_2 a_3)/2 a_1 a_2 a_3 J_{231} = a_1^{-1}(b_3 - b_2) + a_2^{-1}(b_1 - b_3) + a_3^{-1}(b_2 - b_1)$$

consequently, the perturbed separatrices are bifurcated. When $a_2 a_3 - a_1 a_2 - a_1 a_3 = 0$, then J is proportional to J_{123} or J_{132}. By arguments of symmetry and the preservation of the measure on I_{123} generated by the standard measure on $\mathbf{R}^6$, it follows that the perturbed separatrices intersect. Hence, the Kirchhoff equations are non-integrable on the invariant manifolds I_{123} and, in particular, on the whole phase space $\mathbf{R}^6$.

If (3.5) (or (3.6) for $B = 0$) does not hold, then one of the pair of separatrices of the Euler problem must bifurcate under perturbation. It is interesting to note that with a suitable choice of B and C one pair of separatrices remains doubled and the other is bifurcated. For example, suppose that $B = 0$ and that the elements of the symmetric matrix C satisfy the following conditions: $c_{12} = c_{23} = 0$,

$$\sqrt{a_2 - a_1}\, c_{13} \pm \sqrt{a_3 - a_2}\,(c_{22} - c_{11}) = 0,$$
$$\sqrt{a_2 - a_1}\,(c_{33} - c_{22}) \mp \sqrt{a_3 - a_2}\, c_{13} = 0.$$

Then for all ϵ the Kirchhoff equations have a "Hess-Appelrot particular integral" $F = M_1\sqrt{(a_2 - a_1)} \pm M_3\sqrt{(a_3 - a_2)}$. For small ϵ the separatrices of the Euler problem $I_{123} \cap \{F = 0\}$ remain separatrices of the perturbed periodic solutions (3.2).

4. By the method of bifurcation of asymptotic surfaces one can establish non-integrability of the problem of the motion of four-point vortices [21]. More precisely, we consider this problem in a restricted formulation: a vortex of zero intensity (that is, simply a particle in an ideal fluid) is moving in the "field" of three vortices of unit intensity. It turns out that the equation of motion of the zero vortex can be expressed in Hamiltonian form with a Hamiltonian that is periodic in time: these equations have hyperbolic periodic motions with intersecting separatrices. Therefore, the restricted problem of four vortices is not completely integrable, although (as in the unrestricted formulation) it has four independent integrals.

§4. Isolation of the integrable cases

1. When a Hamiltonian system depends on a parameter, then in a typical situation the integrable cases correspond to exceptional isolated values of the parameter. The proof of the isolation of the integrable cases in concrete problems may turn out to be a very difficult matter. We investigate this question for the Hamiltonian equation

$$\ddot{x} + \omega^2\{1 + \varepsilon f(t)\} \sin x = 0 \quad (\omega, \varepsilon = \text{const}), \tag{4.1}$$

which describes the oscillations of a mathematical pendulum. The analytic function $f(t)$ is taken to be non-constant and 2π-periodic in $t \in \mathbf{R}$. For $\epsilon = 0$ (4.1) is integrable and for small $\epsilon \neq 0$ it does not have an integral that is single-valued and analytic in the extended phase space $C\{\dot{x}, x \bmod 2\pi\} \times T^1\{t \bmod 2\pi\}$ (see §3). It will be shown below that this equation is integrable only for a finite set of values of ϵ in the interval $[-a, a]$, where $a = 1/\max_{\mathbf{R}} |f(t)|$.

For all $\epsilon \in [-a, a]$ the periodic solution $x(t) \equiv \pi$ (or, what is the same, $x(t) \equiv -\pi$), the vertical oscillations of the inverted pendulum, is hyperbolic. To prove this we put $x = \pi + y$. Then the equations in variations of the periodic solution $x(t) = \pi$ are

$$\ddot{y} - p(t)y = 0, \quad p(t) = \omega^2(1 + \varepsilon f(t)).$$

Since $p(t) \geqslant 0$ and $p(t) \not\equiv 0$, the multipliers of this solution are positive, one of them being larger than 1, the other smaller than 1 (Lyapunov). Thus, the solution $x(t) = \pi$ is, in fact, hyperbolic. It has two two-dimensional asymptotic surfaces Λ^+ and Λ^-, completely filled out by trajectories that approximate unboundedly to the points $x = \pm\pi$ as $t \to \pm\infty$. Since the Hamiltonian H is analytic, Λ_ϵ^+ and Λ_ϵ^- are regular analytic surfaces in $C \times T^1$, depending analytically on ϵ.

It turns out that the surfaces Λ_ϵ^+ and Λ_ϵ^- intersect for all $\epsilon \in (-a, a)$. This result, obviously, is equivalent to the existence of a homoclinic solution $x(t)$ ($x(t) \to \pm\pi$ as $t \to \pm\infty$). A proof can be derived, for example, from the following general result.

***Theorem* 1.** *Let (M, T, U) be a natural mechanical system where M is compact, the metric T does not depend on time, but the potential energy $U: M \times \mathbf{R}\{t\} \to \mathbf{R}$ is periodic in t. If $U(x, t) < U(x, t_0)$ for all $x \neq x_0$ and $t \in \mathbf{R}$, then there exists a doubly-asymptotic (homoclinic) solution $x(t)$ such that $x(t) \to x_0$ as $t \to \pm\infty$* [10].

In our case $M = S^1$, $T = \dot{x}^2/2$, and $U = -\omega^2(1 + \epsilon f)(1 + \cos x)$. If $-a < \epsilon < a$, then $U(x, t) < U(\pi, t)$ for all $0 \leqslant x < 2\pi$ and all t.

Since the surfaces Λ_ϵ^+ and Λ_ϵ^- do not coincide for small $\epsilon \neq 0$, the values of ϵ, $|\epsilon| \leqslant a + \delta (\delta > 0)$, for which $\Lambda_\epsilon^+ \equiv \Lambda_\epsilon^-$, are isolated. Since for $|\epsilon| \leqslant a$ the surfaces Λ_ϵ^+ and Λ_ϵ^- intersect, (4.1) is integrable only for isolated values of ϵ.

2. We now give an example of a Hamiltonian system that for everywhere dense sets of values of the parameter is both completely integrable and non-integrable. Thus, the integrable cases are not always isolated.

We consider in $\mathbf{R}\{y\}\times T^2\{x, t \bmod 2\pi\}$ the canonical equations

$$\dot{y} = -\frac{\partial H}{\partial x}, \quad \dot{x} = \frac{\partial H}{\partial y} \tag{4.2}$$

with the Hamiltonian function $H = \epsilon y - f(x, t)$, where $\epsilon \in \mathbf{R}$, $\epsilon > 0$, and $f(x, t)$ is a 2π-periodic analytic function of x and t.

We write (4.2) in the explicit form

$$\dot{x} = \varepsilon, \quad \dot{y} = \frac{\partial f}{\partial x} = F(x, t). \tag{4.3}$$

These equations obviously are integrable by quadratures:

$$x = \varepsilon t + x_0, \quad y = y_0 + \int_0^t F(\varepsilon s + x_0, s)\, ds.$$

We search for a first integral of (4.2) in the form $y + g(x, t)$, where $g: T^2 \to \mathbf{R}$ is an analytic function, which must satisfy the equation

$$\frac{\partial g}{\partial t} + \varepsilon \frac{\partial g}{\partial x} = -F(x, t). \tag{4.4}$$

Let

$$F = \sum F_{mn} e^{i(mx+nt)}, \quad g = \sum g_{mn} e^{i(mx+nt)}.$$

Then

$$g_{mn} = \frac{-F_{mn}}{i(m\varepsilon + n)}.$$

Since

1) $|F_{mn}| \leqslant c e^{-\rho(|m|+|n|)}$, $c, \rho > 0$ ($F: T^2 \to \mathbf{R}$ is analytic),

2) for almost all ϵ (in the sense of Lebesgue measure on $\mathbf{R}$)

$$|m\varepsilon + n| \geqslant \frac{k}{(|m| + |n|)^\gamma} \quad (k, \gamma > 0),$$

the series

$$\sum \frac{-F_{mn}}{i(m\varepsilon + n)} e^{i(mx+nt)}$$

represents an analytic solution of (4.4). Consequently, the canonical equations (4.2) are almost always integrable.

We claim that for a suitable choice of $f(x, t)$ the equations (4.2) are non-integrable for an everywhere dense set of ϵ: in these cases the equations (4.2) have solutions that are everywhere dense in the extended phase space $\mathbf{R} \times T^2$.

The proof is based on a certain ergodic property of "cylindrical cascades". Let $T: C \to C$ ($C = S^1 \times \mathbf{R}$) be the map given by the formula $T(x, y) = (x + \epsilon, y + h(x))$, where $\epsilon/2\pi$ is irrational and $h(x)$ is a 2π-periodic function with zero mean value:

$$\int_0^{2\pi} h(x)\, dx = 0.$$

The cascade $\{T^n\}$ is called ergodic if the sequence of points $T^n(a)$, $n \in \mathbf{N}$, is everywhere dense in C for some $a \in C$.

Theorem 2 (Krygin). *Let $\epsilon/2\pi$ be an irrational number such that the inequality*

$$\left|\frac{\varepsilon}{2\pi}-\frac{m}{n}\right|\leqslant\frac{1}{2^{|n|}n^2}$$

has infinitely many integral solutions. Then for some analytic function $h(x)$ the cylindrical cascade $T(x, y) = (x+\epsilon, y+h(x))$ is ergodic [35].

We note that the irrational numbers ϵ satisfying the conditions of Theorem 2 are everywhere dense in $\mathbf{R}$.

Naturally connected with (4.3) is the periodic map

$$T\colon\ x\to x+\varepsilon/2\pi,\quad y\to y+\int_0^{2\pi}F(\varepsilon s+x,\ s)\,ds. \tag{4.5}$$

Since the function

$$h(x)=\int_0^{2\pi}F(\varepsilon s+x,\ s)\,ds \tag{4.6}$$

is 2π-periodic in x and its mean

$$\int_0^{2\pi}h(x)\,dx=\int_0^{2\pi}\int_0^{2\pi}F(x,\ s)\,dx\,ds=0,$$

(4.5) generates a cylindrical cascade.

Choosing $f(x, t)$ suitably we can obtain an arbitrary analytic function $F(x, t)$ up to a 2π-periodic term $\varphi(t)$ with zero mean. However, the addition of $\varphi(t)$ has no influence on the map (4.5).

It seems that a stronger result might hold: for some $f\colon T^2\to\mathbf{R}$ there exist sets $M_\omega, M_\infty, \ldots, M_k, \ldots, M_0, M_{\not c}$, everywhere dense in $\mathbf{R}$, such that for $\epsilon\in M_\omega$ the equations (4.3) have an analytic integral, for $\epsilon\in M_\infty$ there is a smooth integral but not an analytic first integral, ..., for $\epsilon\in M_k$ there is an integral of class C^k, but no integrals of class C^{k+1}, ..., for $\epsilon\in M_{\not c}$ not even continuous integrals. Thus, we have proved that M_ω and $M_{\not c}$ are everywhere dense in $\mathbf{R}$.

We note in conclusion that the equations (4.3) were first studied by Poincaré in [46].

Chapter VI

NON-INTEGRABILITY IN THE NEIGHBOURHOOD OF A POSITION OF EQUILIBRIUM

§ 1. Siegel's method

We consider a canonical system of differential equations

$$\dot{x}_k=\frac{\partial H}{\partial y_k},\quad \dot{y}_k=-\frac{\partial H}{\partial x_k}\quad (1\leqslant k\leqslant n) \tag{1.1}$$

and assume that H is an analytic function in a neighbourhood of $x = y = 0$, where $H(0) = 0$ and $dH(0) = 0$. Let $H = \sum_{s\geqslant 2} H_s$, where H_s is a homogeneous polynomial in x and y of degree s.

Let $\lambda_1, \ldots, \lambda_{2n}$ be the eigenvalues of the linearized canonical system with the Hamiltonian H_2. We may assume that $\lambda_{n+k} = -\lambda_k$ $(1 \leqslant k \leqslant n)$. We consider the case when the numbers $\lambda_1, \ldots, \lambda_n$ are purely imaginary and independent over the field of rational numbers.

In this section we investigate the complete integrability of the equations (1.1) in the neighbourhood of the equilibrium position $x = y = 0$ and the convergence of the Birkhoff normalizing transformation.

1. We consider the set $\mathfrak{H}$ of all power series

$$H = \sum h_{ks} x^k y^s, \qquad k = (k_1, \ldots, k_n), \qquad s = (s_1, \ldots, s_n).$$

that converge in some neighbourhood of $x = y = 0$. We introduce the following topology $\mathcal{T}$ in $\mathfrak{H}$: a neighbourhood of a power series H^* with coefficients h^*_{ks} is the set of power series with coefficients h_{ks} for which $|h_{ks} - h^*_{ks}| < \epsilon_{ks}$, where ϵ_{ks} is an arbitrary sequence of positive numbers.

***Theorem* 1** (Siegel). *In any neighbourhood of any $H^* \in \mathfrak{H}$ there is a Hamiltonian H such that the corresponding canonical system* (1.1) *does not have an integral independent of H and analytic in a neighbourhood of* $x = y = 0$ [19].

Thus, integrable systems are everywhere dense in $\mathfrak{H}$. In particular, the Hamiltonian systems for which the Birkhoff transformation diverges are everywhere dense. Concerning the divergence of the Birkhoff transformation there is a stronger result.

***Theorem* 2** (Siegel). *The Hamiltonian functions H with convergent Birkhoff transformation form a subset of the first Baire category in the topology $\mathcal{T}$ on $\mathfrak{H}$* [20].

More precisely, Siegel proved the existence of a countably infinite set of analytic independent power series $\Phi_1, \Phi_2, \ldots$, in infinitely many variables h_{ks}, that are absolutely convergent for $|h_{ks}| < \epsilon$ (for all k, s) and such that if $H \in \mathfrak{H}$ is reduced to normal form by a convergent Birkhoff transformation, then at this point almost all Φ_s (except possibly finitely many) are zero. Since the functions Φ_s are analytic, their solutions are nowhere dense in $\mathfrak{H}$. Consequently, the set of points of $\mathfrak{H}$ satisfying at least one equation $\Phi_s = 0$ is of the first Baire category. If we attempt to investigate the convergence of the Birkhoff transformation in any concrete Hamiltonian system, then we must check infinitely many conditions. There is no known finite method for this, although all the coefficients of the Φ_s can be calculated explicitly.

2. Using Siegel's method we can prove the density of non-integrable systems in certain subspaces of $\mathfrak{H}$. As an example we consider the equation

$$\ddot{x} = -\frac{\partial U}{\partial x}, \qquad x \in \mathbf{R}^n, \tag{1.2}$$

which describes the motion of a material point in a force field with potential $U(x)$. This equation, of course, can be written in Hamiltonian form:

$$\dot{x}=\frac{\partial H}{\partial p}, \quad \dot{p}=-\frac{\partial H}{\partial x}; \quad H=p^2/2+U(x). \tag{1.3}$$

Let $U(0) = 0$ and $dU(0) = 0$. Then the point $x = 0$ is a position of equilibrium. We put $U = \sum_{s\geqslant 2} U_s$ and let $U_2 = \sum \omega_k^2 x_k^2/2$. We assume that the frequencies of small oscillations $\omega_1, \ldots, \omega_n$ are rationally independent.

We introduce the space $\mathfrak{U}$ of power series

$$\sum_{|k|\geqslant 2} u_k x^k,$$

that converge in some neighbourhood of $x = 0$. We equip $\mathfrak{U}$ with the topology $\mathcal{T}$ of §1.1. In §1.3 we shall prove the following theorem (modulo a certain lemma of Siegel).

***Theorem* 3.** *In $\mathfrak{U}$ with the topology $\mathcal{T}$ the points for which the equations* (1.2) *do not have an integral $F(\dot{x}, x)$ that is analytic in a neighbourhood of the point $\dot{x} = x = 0$ and independent of the energy integral $E = \dot{x}^2/2 + U(x)$ are everywhere dense.*

It seems that the points $U \in \mathfrak{U}$ for which the Birkhoff transformation to normal form converges, form a subset of the first category in $\mathfrak{U}$.

3. For simplicity we restrict ourselves to the case of two degrees of freedom ($n = 2$). Let $\omega_1 = 1$ and $\omega_2 = \omega$ be irrational.

We consider a canonical equation with Hamiltonian function of the following form:

$$H=i(x_1y_1+\omega x_2y_2)+\sum_{p+q\geqslant 3} h_{p_1p_2q_1q_2}x_1^{p_1}x_2^{p_2}y_1^{q_1}y_2^{q_2}. \tag{1.4}$$

The coefficients h_{pq} may be complex.

Let $\epsilon_{pq} < 1$ be an arbitrary sequence of positive numbers and ω an irrational number that can be approximated by rationals sufficiently well: the inequality

$$0<|r-\omega s|<\frac{\varepsilon_{pq}}{s^{s^4}}, \quad p=(r,\ 0) \quad q=(0,\ s) \tag{1.5}$$

must have infinitely many solutions in natural numbers r and s. The measure of the set of such numbers is zero, however, they are everywhere dense in $\mathbf{R}$.

Since ω is irrational, by Birkhoff's theorem (Ch. II) the canonical equations with the Hamiltonian function (1.4) have a formal integral

$$F(x,\ y)=x_1y_1+\sum_{p+q\geqslant 3} f_{p_1p_2q_1q_2}x_1^{p_1}x_2^{p_2}y_1^{q_1}y_2^{q_2}.$$

***Lemma* 1.** *In an ϵ_{pq}-neighbourhood of each function* (1.4) *there is a point H such that for the integers r, s in* (1.5) *the coefficients f_{r00s} admit the estimate*

$$|f_{r00s}|\geqslant s^{s^2}.$$

Corollary. The set of points H for which the Birkhoff transformation diverges is everywhere dense.

Proof of the lemma. Let $F = x_1y_1+\sum F_l$, where F_l is a homogeneous polynomial degree $l \geqslant 3$. The series F formally satisfies the equation

$$\sum_{k=1}^{2}\left(\frac{\partial F}{\partial x_k}\frac{\partial H}{\partial y_k}-\frac{\partial F}{\partial y_k}\frac{\partial H}{\partial x_k}\right)\equiv 0.$$

Equating the terms of the l-th order to zero we arrive at an equation for F_l:

$$x_1\frac{\partial F_l}{\partial x_1}-y_1\frac{\partial F_l}{\partial y_1}+\omega\left(x_2\frac{\partial F_l}{\partial x_2}-y_2\frac{\partial F_l}{\partial y_2}\right)+i\sum_{p+q=l}(p_1-q_2)\,h_{pq}x^py^q=\ \ldots,$$

where the right-hand side is some multinomial of degree l whose coefficients can be expressed in terms of the coefficients of the multinomials $F_3, \ldots, F_{l-1}$ and h_{pq} for $p+q<l$. For the terms $f_{r00s}x_1^ry_2^s$ of F_l we obtain the equation

$$f_{r00s}(r-\omega s)+irh_{r00s}=g_{r00s}. \tag{1.6}$$

Finally, g_{r00s} can be expressed in terms of the coefficients h_{pq} for $p+q<l$. Now let r and s be natural numbers satisfying (1.5). The coefficients h_{r00s} can be changed by not more than ε_{r00s}, so that $|\,irh_{r00s}-g_{r00s}\,|\geqslant\varepsilon_{r00s}$. Then by (1.5) and (1.6) we have the required estimate

$$|\,f_{r00s}\,|\geqslant s^{s^2}.$$

It is important to note that for the construction of the "perturbed" Hamiltonian function H we have "varied" only the coefficients of the form h_{r00s}.

We denote by $|F_l|_*$ the maximum of the absolute values of the coefficients of the form F_l.

Lemma 2 (Siegel). *Suppose that the canonical system*

$$\dot{x}_k=\frac{\partial H}{\partial y_k},\quad \dot{y}_k=-\frac{\partial H}{\partial x_k}\qquad (k=1,\ 2)$$

has a converging integral not depending on H. Then the sequence

$$\frac{\log|\,F_k\,|_*}{k\log k}\qquad (k=3,\ 4,\ \ldots)$$

is bounded [19].

In our case $\log|F_k|_*\geqslant s^2\log s$ if $k=r+s$. From (1.5) for $\epsilon<1$ we have an estimate for r: $r\leqslant\omega s+1$. Consequently, the sequence

$$\frac{\log|\,F_{r+s}\,|_*}{(r+s)\log(r+s)}\geqslant\frac{s^2\log s}{[(\omega+1)\,s+1]\log[(\omega+1)\,s+1]}$$

is unbounded as $s\to\infty$.

We return to the analysis of the canonical equations (1.3). In this case

$$H=H_2+H_3+\ \ldots,\quad H_2=\frac{1}{2}(p_1^2+x_1^2)+\frac{1}{2}(p_2^2+\omega^2x_2^2).$$

We make a linear canonical change of variables with complex coefficients:

$$p_1=\frac{\xi_1+i\eta_1}{\sqrt{2}},\quad x_1=\frac{i\xi_1+\eta_1}{\sqrt{2}}\quad p_2=\sqrt{\omega}\,\frac{\xi_2+i\eta_2}{\sqrt{2}},\quad x_2=\frac{1}{\sqrt{\omega}}\,\frac{i\xi_2+\eta_2}{\sqrt{2}}.$$

In the new variables $H = H_2+H_*$, where

$$H_2=i\,(\xi_1\eta_1+\omega\xi_2\eta_2),\quad H_*=\sum u_{k_1k_2}\frac{1}{(\sqrt{\omega})^{k_2}}\left(\frac{i\xi_1+\eta_1}{\sqrt{2}}\right)^{k_1}\left(\frac{i\xi_2+\eta_2}{\sqrt{2}}\right)^{k_2}.$$

The coefficients h_{pq} of the terms $\xi^p\eta^q$ are linear in the u_k, and

$$h_{r00s}=\frac{i^r u_{rs}}{(\sqrt{\omega})^s(\sqrt{2})^{r+s}}.$$

By varying the coefficients u_{rs} in the expansion of the potential energy $U(x)$, we vary, consequently, the required coefficients h_{r00s}.

§2. Non-integrability of systems depending on a parameter

1. Let $x = y = 0$ be a position of equilibrium of an analytic Hamiltonian system with the Hamiltonian function

$$H(x,\ y,\ \varepsilon)=H_2+H_3+\ \dots,\quad (x,\ y)\in R^{2n},\quad \varepsilon\in D\subset R^d.$$

We assume that for all $\epsilon \in D$ the frequencies of the linear oscillations $\omega(\varepsilon) = (\omega_1(\varepsilon), \dots, \omega_n(\varepsilon))$ do not satisfy any relation

$$\langle m,\ \omega\rangle = m_1\omega_1 + \dots + m_n\omega_n = 0$$

of order $|m_1| + \dots + |m_n| \leqslant m-1$. Then we can find a linear transformation $x,\ y \to p,\ q$ that is analytic in ϵ and such that in the new coordinates

$$H_2=\frac{1}{2}\sum_{i=1}^{n}\omega_i\rho_i,\quad H_k(\rho_1,\ \dots,\ \rho_n,\ \varepsilon),\quad k\leqslant m-1,$$

where $\rho_i=p_i^2+q_i^2$.

We now pass to canonical "action-angle" variables I, φ by the formulae

$$I_i = \rho_i/2,\quad \varphi_i = \arctan p_i/q_i\quad (1\leqslant i\leqslant n).$$

In the variables I, φ

$$H = H_2(I,\ \varepsilon) + \dots + H_{m-1}(I,\ \varepsilon) + H_m(I,\ \varphi,\ \varepsilon) + \dots$$

We express the trigonometric polynomial H_m as a finite Fourier series:

$$H_m=\sum h_k^{(m)}(I,\ \varepsilon)\, e^{i\langle k,\varphi\rangle}.$$

***Theorem* 1.** *We assume that* $\langle k,\ \omega(\varepsilon)\rangle \not\equiv 0$ *for all* $k \in \mathbf{Z}^n$, $k \neq 0$. *Suppose that for some* $\epsilon_0 \in D$ *the resonance relation* $\langle k_0,\ \omega(\epsilon_0)\rangle = 0$, $|k_0| = m$ *and* $h_{k_0}^{(m)}(I,\ \varepsilon) \not\equiv 0$, *is satisfied. Then the canonical equations with the Hamiltonian* $H = \sum H_s$ *do not have a complete set of (formal) integrals* $F_j = \sum_{s\geqslant 2} F_s^{(j)}$ *whose quadratic parts are independent for all* $\epsilon \in D$ [27].

We note that under the conditions of the theorem there may be independent integrals with dependent (for certain values of ϵ) quadratic parts in their MacLaurin expansions. Here is a simple example:

the canonical equations with the Hamiltonian

$$H=\frac{1}{2}(x_1^2+y_1^2)+\frac{\varepsilon}{2}(x_2^2+y_2^2)+2x_1y_1y_2+x_2y_1^2+x_1^2x_2$$

have the integral $F = x_1^2+y_1^2+2(x_2^2+y_2^2)$, which for $\epsilon = 2$ depends on the quadratic form H_2, however, all the conditions of the theorem are satisfied.

Theorem 1 is proved by Poincaré's method. First we prove a simple auxiliary result.

Lemma. *Let $\Phi(I, \varphi, \epsilon)$ be an analytic function in all the variables I, φ, ϵ and 2π-periodic in φ. If $\{H_2, \Phi\} \equiv 0$, then Φ does not depend on φ.*

For let

$$\Phi=\sum \Phi_k(I, \varepsilon)\, e^{i\langle k, \varphi\rangle}.$$

Since

$$\{H_2, \Phi\}=\sum i\langle k, \omega(\varepsilon)\rangle \Phi_k(I, \varepsilon)\, e^{i\langle k, \varphi\rangle}\equiv 0$$

and $\langle k, \omega(\epsilon)\rangle \not\equiv 0$ for $k \neq 0$, we see that $\Phi_k(I, \varepsilon) \neq 0$ only when $k = 0$.

Let $F(x, y, \epsilon) = \sum F_s(I, \varphi, \epsilon)$ be a formal analytic integral of the canonical equations with the Hamiltonian H. From the condition $\{H, F\} \equiv 0$ we obtain the series of equations

$$\{H_2, F_2\} = 0, \quad \{H_2, F_3\} + \{H_3, F_2\} = 0, \; \ldots$$
$$\ldots, \{H_2, F_m\} + \ldots + \{H_m, F_2\} = 0, \; \ldots$$

We claim that $F_2, \ldots, F_{n-1}$ do not depend on φ. For F_2 this has already been proved in the lemma. Since H_3 does not depend on φ, $\{H_3, F_2\} = 0$, therefore $\{H_2, F_2\} = 0$. According to the lemma F_3 also does not depend on φ, and so on. Taking account of this remark the equation for F_m can be written in the following form:

$$\{H_2, F_m\} + \{H_m, F_2\} = 0.$$

If

$$F_m=\sum f_k^{(m)}(I, \varepsilon)\, e^{i\langle k, \varphi\rangle},$$

then

$$\langle \omega(\varepsilon), k\rangle f_k^{(m)} = \left\langle \frac{\partial F_2}{\partial I}, k\right\rangle h_k^{(m)} \quad \forall k \in Z^n.$$

We put $k = k_0$, $\varepsilon = \varepsilon_0$. Then $\langle \omega, k\rangle = 0$ and $h_k^{(m)} \neq 0$. Consequently,

$$\left\langle \frac{\partial F_*}{\partial I}\bigg|_{\varepsilon_0}, k_0\right\rangle = 0.$$

If our equations have n integrals $F_1, \ldots, F_n$ then for $\varepsilon = \varepsilon_0$ we obtain the n linear equations

$$\left\langle \frac{\partial F_2^{(1)}}{\partial I}, k_0\right\rangle = \ldots = \left\langle \frac{\partial F_2^{(n)}}{\partial I}, k_0\right\rangle = 0.$$

Since $k_0 \neq 0$, the quadratic forms $F_2^{(1)}, \ldots, F_2^{(n)}$ are dependent for $\varepsilon = \varepsilon_0$ as required.

Although the proof of the theorem is simple, its use in concrete problems is beset by rather cumbersome calculations associated with the normalization of the Hamiltonians.

2. As a first example we consider the problem of the rotation around a fixed point of a dynamically symmetric rigid body ($J_1 = J_2$), whose centre of mass lies on the equatorial plane of the ellipsoid of inertia [27]. The majority of the integrable cases occur among this kind. The units of measurement of mass and length can be chosen so that $J_1 = J_2 = 1$ and the parameter ϵ, the product of the weight of the body by the distance from the centre of mass to the point of attachment, is also 1. The natural parameter in this problem is the moment of inertia J_3.

In all integral manifolds $I_{23} = \langle J\omega, e\rangle = f_2, \langle e, e\rangle = 1$ the reduced Hamiltonian system has two positions of equilibrium; they correspond to the uniform rotations of the body about the vertical axis in which the centre of gravity lies above (below) the point of suspension.

The angular velocity ω of such a rotation is connected with the area constant f_2 by the simple relation $f_2 = \pm J_3|\omega|$. Let us consider, to be definite, the case when the centre of mass is below the point of suspension.

In a neighbourhood of this equilibrium position the Hamiltonian function H of the reduced system with two degrees of freedom has the form $H_2 + H_4 + \ldots$ (terms of degree 3 are missing). The coefficients depend on two parameters $x = f_2^2$, $y = J_3^{-1}$. It can be shown that the characteristic roots of the secular equation are purely imaginary if $y > x/(x+1)$. We denote by Σ the subdomain of $\mathbf{R}^2\{x, y\}$, where this inequality is satisfied. The ratio of the frequencies is 3 when the parameters x and y are connected by the relation

$$\text{I:} \qquad 9x^2 - 82xy + 9y^2 + 118x - 82y + 9 = 0.$$

This is the equation of a hyperbola: its branches for $x > 0$, $y > 0$ lie wholly in Σ.

From the triangle inequality for moments of inertia ($J_1 + J_2 \geqslant J_3$) it follows that $y \geqslant \frac{1}{2}$. For any fixed $y_0 \geqslant \frac{1}{2}$ there is an x_0 such that (x_0, y_0) satisfies I. The condition of the vanishing of the coefficient $h_{1,-3}^{(4)}$ in the expansion of H_4 can be reduced to the following form:

$$\text{II:} \qquad 9x^4 - 10x^3y + x^2y^2 - 17x^3 + 58x^2y - 7xy^2 - \\ - 375x^2 - 86xy - 170y^2 + 541x + 1700y - 1530 = 0.$$

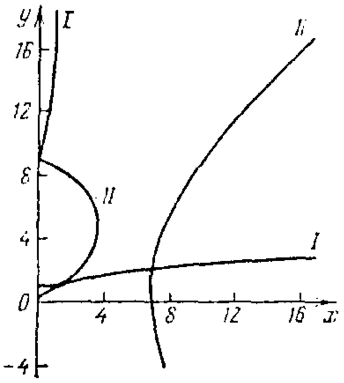

Fig. 11

The algebraic curves I and II intersect in two points (4/3, 1) and (7, 2), which correspond to the integrable systems of Lagrange ($J_1 = J_3$) and of Kovalevskaya ($J_1 = 2J_3$) (see Fig. 11).

3. Next we consider the planar circular restricted three-body problem. The equations of motion of an asteroid in a system of coordinates rotating with the Sun and Jupiter can be written in the Hamiltonian form:

$$\dot{x}_s = \frac{\partial H}{\partial y_s}, \quad \dot{y}_s = -\frac{\partial H}{\partial x_s} \qquad (s = 1,\ 2),$$

$$H = \frac{1}{2}(y_1^2 + y_2^2) + x_2 y_1 - x_1 y_2 - F(x_1,\ x_2,\ \mu),$$

$$F = \frac{1-\mu}{\sqrt{(x_1+\mu)^2 + x_2^2}} + \frac{\mu}{\sqrt{(x_1+\mu-1)^2 + x_2^2}}.$$

This Hamiltonian system has equilibrium positions at the points $x_1 = \frac{1}{2} - \mu$, $x_2 = \pm\sqrt{3}/2$, $y_1 = y_2 = 0$, which are called the Lagrange solutions or triangular libration points (see Ch. I). If $0 < 27\mu(1-\mu) < 1$, then the eigenvalues of the linearized system are purely imaginary and distinct; their ratio is a non-constant function of μ. In cases when commensurability of the third and fourth order holds, the coefficients $h_{1,2}^{(3)}$ and $h_{1,3}^{(4)}$ have been calculated by Markeev in an investigation of the stability of the triangular libration points [37]. These numbers are non-zero. It would seem that the same is true for all (or almost all) resonance ratios. From the theorem in §1.1 it follows, in particular, that in a neighbourhood of a libration point there is not even a formal Birkhoff normalizing transformation that is analytic in μ. "...it is so far unknown whether or not the differential equations of the restricted three body problem with fixed mass ratios can be reduced to normal form by a convergent transformation in a neighbourhood of the Lagrange solutions" (Siegel [20]).

Chapter VII

BRANCHING OF SOLUTIONS AND THE ABSENCE OF SINGLE-VALUED INTEGRALS

Let $M^{2n}_{\mathbf{C}}$ be a complex symplectic analytic manifold (the whole of M is covered by a set of complex charts from $\mathbf{C}^{2n}\{p, q\}$, where the transition maps from chart to chart are invertible holomorphic canonical transformations). Any complex analytic function $H(p, q, t): M^{2n} \times \mathbf{C} \to \mathbf{C}$ gives a certain complex Hamiltonian system

$$\frac{dp}{dt} = -\frac{\partial H}{\partial q}, \quad \frac{dq}{dt} = \frac{\partial H}{\partial p}.$$

It is natural to consider for this system the problem of the existence of additional holomorphic (or, more generally, meromorphic) first integrals. In the majority of integrated problems of Hamiltonian mechanics the known first integrals can be extended to the complex domain by a change of the canonical variables to certain holomorphic or meromorphic functions. In this chapter we show that branching of solutions of Hamiltonian systems in the complex time plane, in general, prevents the appearance of new single-valued integrals.

§1. Branching of solutions—an obstruction to integrability

Let $D_{\mathbf{C},\delta} = \{I \in \mathbf{C}^n : \operatorname{Re} I \in D \subset \mathbf{R}^n, |\operatorname{Im} I| < \delta\}$, where $T^n_{\mathbf{C}} = \mathbf{C}^n/2\pi\mathbf{Z}^n$ is the complex torus (over $\mathbf{R}$ this is $T^n \times \mathbf{R}^n$) with complex angle coordinates $\varphi_1, \ldots, \varphi_n$ mod 2π, and E is a neighbourhood of zero in $\mathbf{C}$. Let $H(I, \varphi, \epsilon)$: $D_{\mathbf{C},\delta} \times T^n_{\mathbf{C}} \times E \to \mathbf{C}$ be a holomorphic function taking real values for real values of I, φ, ϵ, and $H(I, \varphi, 0) = H_0(I)$.

The direct product $D_{\mathbf{C},\delta} \times T^n_{\mathbf{C}}$ is equipped with the simplest symplectic structure in which the Hamiltonian equations with the Hamiltonian H have the canonical form:

$$\frac{dI}{dt} = -\frac{\partial H}{\partial \varphi}, \quad \frac{d\varphi}{dt} = \frac{\partial H}{\partial I}; \quad H = H_0 + \varepsilon H_1 + \ldots \tag{1.1}$$

All solutions of this system with Hamiltonian H_0 are single-valued in the complex time plane $t \in \mathbf{C}$:

$$I = I^0, \quad \varphi = \varphi^0 + \omega(I^0)t.$$

For $\epsilon \neq 0$ the solutions of the "perturbed" equation (1.1), generally speaking, are no longer single-valued. Let γ be a closed contour in the complex time plane. According to a theorem of Poincaré, the solutions of (1.1) can be expanded in power series

$$\begin{cases} I = I^0 + \varepsilon I^1(t) + \ldots, \quad \varphi = \varphi^0 + \omega t + \varepsilon \varphi^1(t) + \ldots, \\ I^1(0) = \ldots = \varphi^1(0) = \ldots = 0, \end{cases} \tag{1.2}$$

that converge for sufficiently small values of ϵ if $t \in \gamma$ ([48], Ch. II; [13]).

We say that an analytic vector-valued function $f(t)$, $t \in \mathbf{C}$, is not single-valued along γ if it undergoes a jump $\Delta f = \xi \neq 0$ on a circuit of γ. If, for example, the function $I^1(t, I^0, \varphi^0)$ is unbounded along γ, then for small ϵ the perturbed solution (1.2) is also unbounded along γ. The jump ΔI^1, obviously, is equal to

$$\xi = \int_\gamma \Phi(t)\,dt, \quad \Phi(t) = -\left.\frac{\partial H_1}{\partial \varphi}\right|_{I^0,\ \varphi^0 + \omega(I^0)t}.$$

If for fixed I the function $H_1(I, \varphi)$ is holomorphic in $T^n_{\mathbf{C}}$, then, of course, $\xi = 0$. However, in important cases in practice this function has a singularity (say, a pole). Therefore, we regard $H(I, \varphi, \epsilon)$ as holomorphic only in a domain $D_{\mathbf{C},\delta} \times \Omega \times E$, where Ω is a connected domain in $T^n_{\mathbf{C}}$, containing the real torus $T^n_{\mathbf{R}}$ and the closed contour Γ, the image of γ under the map $\varphi = \varphi^0 + \omega(I^0)t$, $t \in \gamma$.

We fix the initial data I^0, φ^0 and deform γ continuously so that Γ does not intersect any singular point of H. Then, by Cauchy's theorem, the function $I^1(t)$ on going around the deformed contour changes again by the same quantity $\xi \neq 0$. On the other hand, since (1.2) is continuous in the initial data, $I^1(t, I^0, \varphi^0)$ is unbounded along γ holds for all values close to I^0, φ^0.

***Theorem* 1.** *Suppose that*

1) $\det \| \partial^2 H_0/\partial I^2 \| \not\equiv 0$ *on* $D_{\mathbf{C},\delta}$,

2) *for some initial data* I^0, φ^0 *the function* I^1 *is unbounded along the closed contour* $\gamma \subset \mathbf{C}$

Then the equations (1.1) *do not have a complete set of independent formal*[(1)] *integrals*

$$F_s = \sum_{i=0}^{\infty} F_i^s(I, \varphi)\varepsilon^i \quad (1 \leqslant s \leqslant n),$$

whose coefficients are single-valued holomorphic functions on the direct product $V \times \Omega \subset D_{\mathbf{C},\delta} \times T^n_{\mathbf{C}}$, *where* V *is a neighbourhood of* I^0 *in* $D_{\mathbf{C},\delta}$ ([29], [32]).

2. We point out the main features of the proof of the theorem. As always, we begin by showing that the $F^s(I, \varphi)$ do not depend on φ. Let $(I, \varphi) \in D \times T^n_{\mathbf{R}}$ and $F_0^s = \Phi_0^s + i\Psi_0^s$. Then Φ_0^s and Ψ_0^s are first integrals of the non-degenerate unperturbed system. According to Poincaré's lemma (Ch. IV, § 1), they do not depend on $\varphi \in T^n_{\mathbf{R}}$. When $\varphi \in \Omega$, the fact that F_0^s is constant follows from the connectedness of Ω and the uniqueness of the analytic continuation.

Next we prove that the functions $F_0^1(I), \ldots, F_0^n(I)$ are dependent in the domain $V \subset D_{\mathbf{C},\delta}$. For since $F_s(I, \varphi, \epsilon)$ is an integral of the canonical system (1.1), this function is constant on the solutions (1.2). Consequently, its values at the time $\tau \in \gamma$ and after a circuit of γ coincide:

$$\begin{aligned} F_0^s(I^0 + \varepsilon I^1(\tau) + \ldots) + \varepsilon F_1^s(I^0 + \varepsilon I^1(\tau) + \ldots, \varphi^0 + \omega\tau + \varepsilon\varphi^1(\tau) + \ldots) + \ldots \equiv \\ \equiv F_0^s(I^0 + \varepsilon(I^1(\tau) + \xi(I^0)) + \ldots) + \varepsilon F_1^s(I^0 + \ldots, \varphi^0 + \omega\tau + \ldots) + \ldots \end{aligned}$$

Expanding this identity in power series in ϵ and equating the coefficients of ϵ, we obtain

$$\left\langle \frac{\partial F_0^s}{\partial I}, \xi \right\rangle = 0, \quad 1 \leqslant s \leqslant n.$$

Since the jump ξ is non-zero in a neighbourhood of I^0, the Jacobian

$$\frac{\partial(F_0^1, \ldots, F_0^n)}{\partial(I_1, \ldots, I_n)} \equiv 0$$

on the whole domain V containing I^0.

On the other hand, applying Poincaré's method of Ch. IV we can prove the existence of independent integrals

$$\Phi_s(I, \varphi, \varepsilon) = \sum_{i \geqslant 0} \Phi_i^s(I, \varphi)\varepsilon^i$$

with coefficients holomorphic in $W \times \Omega$ (where W is a small subdomain of V) such that the functions Φ_0^s $(1 \leqslant s \leqslant n)$ are independent.

(1)We again suppose that the formal series $F = \sum F_i \varepsilon^i$ is an integral of the canonical equations (1.1) if formally $\{H, F\} \equiv 0$. It is easy to see that in this case the composition of the power series (1.2) and $\sum F_i \varepsilon^i$ is a power series with constant coefficients.

3. Again we consider the problem of a heavy asymmetric rigid body rotating rapidly around a fixed point. The Hamiltonian function H in this problem is $H_0(I)+\epsilon H_1(I, \varphi)$, $I \in \Delta \subset \mathbf{R}^2\{I\}$, $\varphi \in T^2$ (see Ch. IV, §3). The perturbing function H_1 can be expressed as a sum

$$h_1(I, \varphi_1)e^{i\varphi_2}+h_2(I, \varphi_1)e^{-i\varphi_2}+h_3(I, \varphi_1),$$

and for fixed $I \in \Delta$ the functions $h_s(I, z)$ $(1 \leqslant s \leqslant 3)$ are elliptic (doubly-periodic meromorphic functions of $z \in \mathbf{C}$). Consequently, the Hamiltonian H can be continued to a single-valued meromorphic function in $T^2_{\mathbf{C}}$.

Let $\varphi_0 = 0$ and $I_0 \in \mathfrak{B}$ (where $\mathfrak{B}$ is the secular set of the perturbed problem). We consider in the complex plane $t \in \mathbf{C}$ a closed contour γ, the boundary of a rectangle $ABCD$ (see Fig. 12).

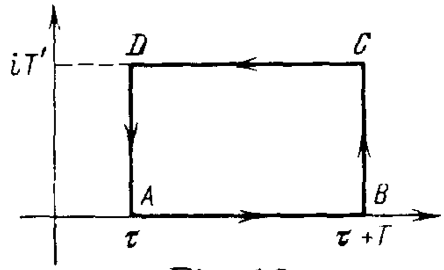

Fig. 12

Here T and iT' are, respectively, the real and purely imaginary periods of the elliptic functions $f_s(I^0, \omega, z)$, $\omega_1 = \partial H_0/\partial I_1$. The number τ is chosen so that these meromorphic functions do not have poles on γ. It can be shown that the function $I_2^1(t, I^0, o)$ is unbounded along γ [29]. Consequently, the solutions of the perturbed problem branch in the complex time plane and this situation prevents the appearance of a new single-valued integral.

4. Using the branching of solutions we can establish the absence of single-valued analytic integrals for small but fixed values of $\epsilon \neq 0$. We quote a result in this direction due to Ziglin [23].

Let $M^3 = \mathbf{C}^2 \times T^1_{\mathbf{C}}\{t \bmod 2\pi\}$ and let $H(z, t, \epsilon): M^3 \times E \to \mathbf{C}$ be a holomorphic function taking real values for real z, t and ϵ and such that $H(z, t, 0) = H_0(z)$. We consider the Hamiltonian system

$$\dot{z} = \mathfrak{J}H', \quad H = H_0(z)+\varepsilon H_1(z, t)+\dots \tag{1.3}$$

Let $z = z_0 \in \mathbf{C}^2$, $\operatorname{Im} z_0 = 0$, be a hyperbolic fixed point of the unperturbed system

$$\dot{z} = \mathfrak{J}H_0', \quad dH_0(z_0) = 0.$$

The eigenvalues $\pm\lambda$ of the linearized system have non-zero real parts ($\operatorname{Re}\lambda > 0$). The solution $z(t) = z_0$ can be regarded as periodic with period 2π. According to Poincaré, for sufficiently small $|\epsilon|$ the system (1.3) has a 2π-periodic solution $z = p(t, \epsilon)$, $p(t, 0) = z_0$. Continuing the solutions of (1.3) that are asymptotic to $p(t, \epsilon)$ as $t \to -\infty$ to functions maximally analytic in $t \in \mathbf{C}$ (possibly not single-valued), we obtain a two-dimensional complex surface Λ_ϵ^-, which we call the unstable complex asymptotic surface of the hyperbolic periodic solution $p(t, \epsilon)$.

We have seen in Ch. VI that the stable and unstable asymptotic surfaces Λ^+_ϵ and Λ^-_ϵ may intersect transversally in the real domain, and this leads to the absence of an analytic integral on $\mathbf{R}^2 \times T^1_{\mathbf{R}}$ (consequently, on the whole of $\mathbf{C}^2 \times T^1_{\mathbf{C}}$). In this case the complex asymptotic surface Λ^-_ϵ ($\Lambda^+{}_\epsilon$), in contrast to the real case, may have transversal self-intersections, which also prevent the existence of a holomorphic integral for (1.3).

We give a sufficient condition for self-intersection. Suppose that the asymptotic solution $z = z_a(t)$ of the unperturbed system ($\lim\limits_{t\to-\infty} z_a(t) = z_0$) has a single-valued analytic continuation along a closed continuous path $\gamma : [0, 1] \to \mathbf{C}$, $\gamma(0) = \gamma(1) \in \mathbf{R} \subset \mathbf{C}$. Then for sufficiently small $|\epsilon|$ the solution $z(t, t_0, \epsilon)$ of the perturbed system (1.3) with the initial condition $z(\gamma(0)+t_0, t_0, \epsilon) = z_a(\gamma(0))$ also has an analytic (but, in general, not single-valued) continuation along the "displaced" path $\gamma + t_0$. Let

$$h(t_0, \varepsilon) = H_0(z(\gamma(1) + t_0, t_0, \varepsilon)) - H_0(z_a(\gamma(0))) = \varepsilon h_1(t_0) + o(\varepsilon)$$

be the increment of $H_0(z(t, t_0, \epsilon))$ on a circuit of t along $\gamma + t_0$.

***Theorem* 2.** *If h_1 has a simple zero, then for sufficiently small $|\epsilon| \neq 0$ the complex surface Λ^-_ϵ has a transversal self-intersection, and the system* (1.3) *has no single-valued analytic first integral in M^3.*

We note that $h_1(t_0)$ can be calculated by the formula

$$\int_\gamma \frac{\partial H_1}{\partial t}(z_a(t), t + t_0)\, dt.$$

§2. The monodromy groups of Hamiltonian systems with single-valued integrals

1. In this section we are first concerned with the investigation of linear Hamiltonian equations with holomorphic coefficients.

Let $H = \langle z, A(t)z\rangle/2$ be a quadratic form in $z \in \mathbf{C}^{2n}$, and let $A(t)$ be a given $(2n \times 2n)$-matrix whose coefficients are holomorphic functions defined on some Riemann surface X. If, for example, the elements of $A(t)$ are functions meromorphic on $\mathbf{C}$, then X is the complex plane with some points (poles) removed. The linear Hamiltonian equations with the function H have the form

$$\dot z = \mathfrak{J}H' = \mathfrak{J}A(t)z. \tag{2.1}$$

Locally, for a given initial condition $z(t_0) = z_0$, there always exists a uniquely determined holomorphic solution. This can be continued along any curve in X, however, in general, the continuation is no longer a single-valued function on X. The branching of a solution of (2.1) is described by its monodromy group G: to each element σ of the fundamental group $\pi_1(X)$ there corresponds a $(2n \times 2n)$-matrix T_σ such that after a circuit round a closed path of homotopy class σ the value of $z(t)$ becomes $T_\sigma z(t)$.

If τ is another element of the group $\pi_1(X)$, then $T_{\tau\sigma} = T_\tau T_\sigma$. The correspondence $\sigma \to T_\sigma$ thus defines a group homomorphism $\pi_1(X) \to G$ (details can be found, for example, in [13], [56]).

A problem of interest to us is the presence of holomorphic integrals $F(z, t): \mathbf{C}^{2n} \times X \to \mathbf{C}$ for (2.1). Since any integral $F(z, t_0)$ is constant on the solutions of (2.1), for each $t_0 \in X$ the function $F(z, t_0)$ is invariant under the action of the monodromy group G. This property imposes severe restrictions on the form of first integrals: if G is sufficiently "rich", then the only invariant functions (integrals) are constants.

Since (2.1) is Hamiltonian, the monodromy transformation group is symplectic. The problem of integrals of groups of symplectic transformations has been studied by Ziglin in [24]. We briefly state his results.

2. According to the theorem of Poincaré-Lyapunov, the eigenvalues $\lambda_1, \dots, \lambda_{2n}$ of a symplectic transformation $g: \mathbf{C}^{2n} \to \mathbf{C}^{2n}$ split into pairs $\lambda_1 = \lambda_{n+1}^{-1}, \dots, \lambda_n = \lambda_{2n}^{-1}$. We call a transformation $g \in G$ non-resonant if from $\lambda_1^{m_1} \dots \lambda_n^{m_n} = 1$, with integers $m_1, \dots, m_n$, it follows that all $m_s = 0$. For $n = 1$ this condition means that λ is not a root of unity. Let T be the matrix of a non-resonant symplectic map g. Since no eigenvalue of T is 1, the equation $Tz = z$ has the trivial solution $z = 0$.

It is convenient to go over to a symplectic basis for the map g: if $z = (x, y)$, $x = (x_1, \dots, x_n)$, and $y = (y_1, \dots, y_n)$ are the coordinates in this basis, then $g: (x, y) \to (\lambda x, \lambda^{-1} y)$. A symplectic basis exists if all $\lambda_s \neq 1$ $(1 \leqslant s \leqslant n)$ (this result is proved, for example, in Siegel [18]).

Let $F(z) = \sum_{s \geqslant 1} F_s(z)$ be an integral of g. Then all the homogeneous forms F_s are also integrals. Let $F_s(x, y) = \sum_{k+l=s} f_{kl} x^k y^l$. Then, obviously,

$$\sum f_{kl} x^k y^l = \sum \lambda^{k-l} f_{kl} x^k y^l.$$

If g is non-resonant, then s is even and $f_{kl} = 0$ for $k \neq l$.

***Theorem* 1.** *Let $g \in G$ be non-resonant. If the Hamiltonian system has n independent holomorphic integrals $F(z, t): \mathbf{C}^{2n} \times X \to \mathbf{C}$, then any transformation $g' \in G$ has the same fixed points as g and takes the eigendirections of g into eigendirections. If no $k \geqslant 2$ eigenvalues of g' form a regular polygon in the complex plane with centre at zero, g' commutes with g* [24].

The latter condition is necessarily satisfied if g' is also non-resonant.

We now prove Theorem 1 for the simple case $n = 1$, which is important for applications. Suppose that the eigenvalues of g are not roots of unity and that $(x, y) = z$ is a symplectic basis for g. The eigendirections of g are the two lines $x = 0$ and $y = 0$. Above, it was shown that any homogeneous integral of g is of the form $c(xy)^s$, $s \in \mathbf{N}$. Let g' be another map of G. Since the function $(xy)^s$ is invariant under the action of g', the set $xy = 0$ is fixed by g'. Since g' is a non-degenerate linear map, the point $x = y = 0$ is fixed and g' either preserves the eigendirections of g or permutes them. In the first case g', obviously, commutes with g, and in the second case it has the form

$$x \to \alpha y, \quad y \to \beta x.$$

Since g' is symplectic, its matrix

$$S = \begin{Vmatrix} 0 & \alpha \\ \beta & 0 \end{Vmatrix}$$

satisfies the condition

$$S^* \mathfrak{I} S = \mathfrak{I},$$

hence $\alpha\beta = -1$. But in this case the eigenvalues of S are $\pm i$. The points $\pm i$ form precisely that exceptional regular polygon mentioned in the conclusion of the theorem, as required.

We consider the case when the elements of $A(t)$ are homogeneous doubly-periodic meromorphic functions of the time $t \in \mathbf{C}$, having only one pole inside the parallelogram of periods. We may take $A(t)$ to be a meromorphic function on the complex torus X obtained from the complex plane $\mathbf{C}$ by factoring out the lattice of periods. We consider two symplectic maps g and g' of a period of $A(t)$. We assume that their eigenvalues satisfy the conditions of Theorem 1. Then for (2.1) to have n independent analytic integrals it is necessary that g and g' commute. Consequently, to a circuit of a singular point (the element $gg'g^{-1}g'^{-1} \in G$) there corresponds the identity map of $\mathbf{C}^{2n}$.

3. We apply this argument to the linear differential equation

$$\ddot{z} + (\omega^2 + \varepsilon f(t))z = 0, \tag{2.2}$$

where ω and ϵ are real constants, $f(t)$ is an elliptic function with the periods 2π and $2\pi i$, having a unique pole of order 2 in the rectangle of periods. We may assume that f for real t takes real values. An example is the Weierstrass function $\wp$.

Now (2.2) can be interpreted as the linearized equation of the oscillations of a pendulum with an oscillating point of suspension in a neighbourhood of a position of equilibrium.

Let us find the eigenvalues of a map g in the monodromy group, generated by a circuit around the pole of f. For simplicity of writing, let the pole be at $t = 0$. The Laurent series of $f(t)$ in a neighbourhood of $t = 0$ has the form

$$\frac{\alpha}{t^2} + \sum_{n \geqslant 0} f_n t^n \qquad (\alpha \neq 0).$$

We look for linearly independent solutions of (2.2) in the form of a series

$$z(t) = t^\rho \sum_{n \geqslant 0} c_n t^n, \ \rho \in \mathbf{C}, \ c_0 \neq 0.$$

Since

$$\ddot{z}(t) = t^\rho \sum_{n\geqslant 0} (\rho+n)(\rho+n-1)\, c_n t^{n-2},$$

we have

$$\sum_{n\geqslant 0} (\rho+n)(\rho+n-1)\, c_n t^{n-2} + (\omega^2 + \varepsilon\alpha t^{-2} + \varepsilon \sum_{s\geqslant 0} f_s t^s) \sum_{n\geqslant 0} c_n t^n = 0.$$

Equating the coefficient of t^{-2} to zero we obtain the equation

$$(\rho(\rho - 1) + \varepsilon\alpha)c_0 = 0.$$

Since $c_0 \neq 0$,

$$\rho(\rho - 1) + \varepsilon\alpha = 0.$$

This equation gives us two values ρ_1 and ρ_2 to which there correspond two linearly independent solutions of (2.2). After a circuit of the pole these solutions are multiplied, respectively, by $e^{2\pi i\rho_1}$ and $e^{2\pi i\rho_2}$. The corresponding monodromy matrix is the identity if ρ_1 and ρ_2 are integers. In particular, $\epsilon\alpha$ must be an integer.

For $\epsilon = 0$ the eigenvalues of the monodromy matrix of (2.2) under the map with period 2π and $2\pi i$ are, respectively, $\lambda_{1,2} = e^{\pm 2\pi\omega i}$ and $\mu_{1,2} = e^{\pm 2\pi\omega}$. Obviously, $|\mu_{1,2}| \neq 1$ and $\lambda_{1,2} \neq \pm i$ if $\omega \neq 0$ and $\omega \neq \frac{1}{4} + k\pi$, $k \in \mathbf{Z}$. By continuity, if $\omega \neq \frac{1}{4} + k\pi$, then for small values $\epsilon \neq 0$ the eigenvalues $\mu_{1,2}$ are not roots of unity and $\lambda_{1,2} \neq \pm i$ (this property in fact holds for almost all ω and ϵ). Consequently, by Theorem 2, (2.2) in these cases is not integrable in the complex domain. We note that in the real domain this equation is completely integrable: it has an analytic integral $f(\dot{z}, z, t)$ that is 2π-periodic in t. The fact is that by a linear canonical change of variables that is 2π-periodic in t the equations (2.2) can be reduced to a linear autonomous Hamiltonian system with one degree of freedom. For f we can take the Hamiltonian function of the autonomous sytems.

We now consider the non-linear equation of the oscillations of a mathematical pendulum

$$\ddot{z} + (\omega^2 + \varepsilon f(t)) \sin z = 0.$$

We claim that it can have an analytic integral $f(\dot{z}, z, t)$ that is doubly-periodic in $t \in \mathbf{C}$ only for those values of ω and ϵ for which the linear equation (2.2) is integrable. To prove this we expand f in a convergent power series

$$\sum_{s\geqslant m} \sum_{k+l=s} f_{kl}(t)\, \dot{z}^k z^l, \tag{2.3}$$

whose coefficients f_{kl} are elliptic functions with periods 2π and $2\pi i$. The first form in (2.3) (when $s = m$) is obviously a single-valued integral of (2.2). Consequently, by hypothesis, it must be constant. But then the next form ($s = m + 1$) is an integral of (2.2) and therefore also constant, and so on.

4. The last remark can be generalized. Suppose that the non-linear Hamiltonian system

$$\dot{z} = \Im H', \ z \in C^{2n} \tag{2.4}$$

has a particular solution $z_0(t)$ that is single-valued on its Riemann surface X. We put $u = z - z_0(t)$. Then (2.4) can be rewritten as follows:

$$\dot{u} = \Im H''(z_0(t))u + \dots$$

The linear non-autonomous equation

$$\dot{u} = \Im H''(t)u \tag{2.5}$$

is an equation in variations for the solution $z_0(t)$. Of course, it is Hamiltonian with the Hamiltonian function

$$\frac{1}{2}\langle u, \ H''(t)\, u\rangle.$$

To the integral $H(z)$ of the autonomous system (2.4) there corresponds the linear integral of the equation in variations

$$\langle H'(z_0(t)), \ u\rangle.$$

With its help we can, for example, reduce the number of degrees of freedom of (2.5) by 1.

We assume that the non-linear equation (2.4) has several independent holomorphic integrals $F_s(z)$ ($1 \leqslant s \leqslant m$). Then (2.5) also has first integrals. They are the homogeneous forms of the expansion of F_s in a power series in u:

$$\langle F_s'(z_0(t)), \ u\rangle + \dots$$

These forms are holomorphic functions on the direct product $\mathbf{C}^{2n} \times X$. We have

Lemma. *If* (2.4) *has m independent integrals, then the equation in variations* (2.5) *has m independent polynomial integrals* [24].

Thus, the problem of the complete integrability of Hamiltonian systems in the complex domain reduces to an investigation of the integrability of linear canonical systems.

By this method Ziglin has proved the integrability of the Hamiltonian systems of Henon-Heile and Yang-Mills (see Ch. I). He has also applied it to the problem of the rotation of a heavy rigid body around a fixed point. It turned out that an additional holomorphic integral exists only in the three classical cases of Euler, Lagrange, and Kovalevskaya. If the area constant is fixed to be zero, then to these must be added the case of Goryachev-Chaplygin [24].

For the systems of Henon-Heile and Yang-Mills one can prove that there are no integrals even in a real domain. The question of the existence of an additional real analytic integral for an arbitrary mass distribution in a rigid body remains open.

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