On the signed chromatic number of grids

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Abstract

The oriented (resp. signed) chromatic number $\chi_o(G)$ (resp. $\chi_s(G)$) of an undirected graph G, is defined as the maximum oriented (resp. signed) chromatic number of an orientation (resp. signature) of G. Although the difference between $\chi_o(G)$ and $\chi_s(G)$ can be arbitrarily large, there are, however, contexts in which these two parameters are quite comparable.

We here compare the behaviour of these two parameters in the context of (square) grids. While a series of works have been dedicated to the oriented chromatic number of grids, nothing was known about their signed chromatic number. We study this parameter throughout this paper. We show that the maximum signed chromatic number of a grid, lies in between 7 and 12. We also focus on 2-row and 3-row grids, and exhibit bounds on their signed chromatic number, some of which are tight. Although our results indicate that the oriented and signed chromatic numbers of grids are, in general, close, they also show that these parameters may differ, even for easy instances.

Keywords: signed chromatic number, oriented chromatic number, grids

1. Introduction

Colouring problems are among the most important problems of graph theory, as, notably, they can model many real-life problems under a graph-theoretical formalism. In its most common sense, a *colouring* of an undirected graph G refers to a *proper vertex-colouring*, which is a colouring of $V(G)$ such that every two adjacent vertices of G get assigned distinct colours. Many variants of this definition have been introduced and studied in the literature, including variants dedicated to augmented kinds of graphs, which are of interest in this paper.

Namely, our investigations are related to two kinds of augmented graphs, called *oriented graphs* and *signed graphs*. An *oriented graph* $\overrightarrow{G}$ is basically obtained from an undirected simple graph G, by orienting every edge uv either from u to v (resulting in an arc $\overrightarrow{uv}$), or conversely (yielding an arc $\overrightarrow{vu}$). We sometimes also call $\overrightarrow{G}$ an *orientation* of G. Now, from G, we can also get a *signed graph* (G, σ), by assigning a sign $\sigma(uv)$, being either $-$ (negative) or $+$ (positive), to every edge uv of G. We also call (G, σ) a *signature* of G. Signed graphs generally come along with a resigning operation, which we will not consider herein. So, though our signed graphs are thus nothing but 2-edge-coloured graphs, we here stick to the signed graphs terminology, which we find more convenient in our context.

As undirected graphs, oriented and signed graphs can as well be coloured in many different ways. We herein consider those colouring variants arising from the homomorphism definition of colourings. Indeed, a k-colouring ϕ of an undirected graph G can as well be regarded as a *homomorphism* from G to K_k, the complete graph on k vertices, that is, a mapping $\phi : V(G) \to V(K_k)$ preserving the edges (i.e. for every edge uv of G, we have that $\phi(u)\phi(v)$ is an edge of K_k). Graph homomorphisms can quite naturally be derived for oriented and signed graphs as well (i.e. we want the edge orientations and signs, respectively, to be preserved as well), yielding, in turn, colouring variants for oriented and signed graphs, which are precisely the variants we consider here. These variants,

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out of the homomorphism terminology, can be defined as follows. A colouring ϕ of an oriented graph is a proper colouring, such that, for any two arcs $\overrightarrow{u_1v_1}$ and $\overrightarrow{u_2v_2}$, if $\phi(u_1) = \phi(v_2)$, then $\phi(v_1) \neq \phi(u_2)$. Analogously, a colouring ϕ of a signed graph respects that, for any two edges u_1v_1 and u_2v_2 with the same sign, if $\phi(u_1) = \phi(v_2)$, then $\phi(v_1) \neq \phi(u_2)$.

Usually, the main objective, given a colouring notion, is to find a colouring of a graph using the least possible number of colours. For an undirected graph G, the least number of colours in a colouring is called the *chromatic number* of G, commonly denoted by $\chi(G)$. Concerning the aforementioned colouring notions for oriented and signed graphs, we call the associated chromatic parameters the *oriented chromatic number* and *signed chromatic number*, respectively, and denote them by $\chi_o(\overrightarrow{G})$ and $\chi_s((G,\sigma))$, respectively, for a given oriented graph $\overrightarrow{G}$ and a given signed graph (G,σ), respectively. The χ_o and χ_s parameters can also be derived for undirected graphs: for an undirected graph G, we set $\chi_o(G)$ as being the maximum value of χ_o for an orientation of G, while we set $\chi_s(G)$ as the maximum value of χ_s for a signature of G. In other words, $\chi_o(G)$ and $\chi_s(G)$ indicate whether G is the underlying graph of oriented or signed graphs needing many colours to be coloured. For more details on these two chromatic parameters, we refer the interested reader to the recent survey [5] by Sopena dedicated to the oriented chromatic number, and to the Ph.D. thesis [4] of Sen, which is dedicated, in particular, to both the oriented and signed chromatic numbers.

Our investigations, in this paper, are motivated by the general relation between $\chi_o(G)$ and $\chi_s(G)$ for a given undirected graph G. Intuitively, one could expect these two parameters to be close somehow, as oriented graphs and signed graphs are rather alike notions: in both an orientation and a signature of G, every edge basically has one of two possible "states" (being oriented in one way or the other, or being positive or negative). From a more local point of view, though, an oriented edge and a signed edge are perceived differently by their two ends. In light of these two points, it thus appears legitimate to wonder whether oriented and signed graphs, in some particular contexts, have comparable behaviours. This aspect was notably investigated by Sen in his Ph.D. thesis [4].

In general, it has to be known that, for a given undirected graph G, the difference between $\chi_o(G)$ and $\chi_s(G)$ can be arbitrarily large, as noted by Bensmail, Duffy and Sen in [1]. A natural arising question, is thus whether this behaviour is rather systematic, or can be observed for a restricted number of graph classes only. Towards this question, we here focus on the class of (square) *grids*, where the grid $G(n,m)$, with n rows and m columns, is defined as the graph being the Cartesian product of the path with order n and the path with order m. While, to the best of our knowledge, nothing is known about the signed chromatic number of grids, a series of works, namely [2, 3, 6], can be found in the literature on the oriented chromatic number of these graphs. In brief words, these works have (1) pointed out that the maximum oriented chromatic number of a grid lies in between 8 and 11, and have (2) exhibited the exact oriented chromatic number of grids with at most four rows. More details on these results will be given throughout this paper, as they connect to our investigations.

We thus initiate the study of the signed chromatic number of grids, our main objective being to compare how close the oriented and signed chromatic numbers of these graphs are. Prior to present our results, we first introduce, in Section 2, some definitions and terminology that are used throughout this paper. We then start, in Section 3, by providing a general constant upper bound on the signed chromatic number of grids. Namely, we prove that $\chi_s(G(n,m)) \leq 12$ holds for every $n, m \geq 1$. We then exhibit, in Sections 4 and 5, lower bounds on the signed chromatic number of grids, by focusing on signed grids with at most three rows. In particular, we point out that some signed 3-row grids cannot be coloured with less than 7 colours. We also provide refined bounds on the signed chromatic number of 2-row and 3-row grids, our bounds for 2-row grids being sharp. We finally conclude this paper by gathering, in Section 6, our results, and discussing how the oriented and signed chromatic numbers seem to behave for grids.

2. Definitions and terminology

Throughout this paper, we use σ to refer to the implicit signature function of any signed graph. Let G be a signed graph, and A be a signature of some graph. By an *A-colouring* of G, we refer to a homomorphism from G to A. We also say that G is *coloured by A*. To stick to the colouring point of view, the vertices of any colouring graph A are generally represented, in our proofs, by integers $0, ..., |V(A)|-1$, while, to avoid any misleading terminology, the edges of A are written under the form $\{\alpha, \beta\}$. In that spirit, we denote *k-paths* (i.e. paths of length k) of A under the form $(\alpha_1, ..., \alpha_{k+1})$, where $\alpha_1, ..., \alpha_{k+1}$ are the consecutive vertices of the path. Similarly we denote by $(\alpha_1, ..., \alpha_k, \alpha_1)$ any *k-cycle*, i.e. cycle of length k. Any signed path or cycle is said *alternating*, if no two of its consecutive edges have the same sign.

Signed graphs that can colour all signed graphs among a family, should, intuitively, have a rather "regular" and "symmetric" structure, with convenient properties. In the context of colouring of oriented graphs, examples of such nice oriented graphs include *circulant oriented graphs*, which are defined as follows. The *circulant oriented graph* $C(n, S)$, where $n \geq 1$ and $S \subseteq \{1, ..., n-1\}$, is the oriented graph on n vertices $0, ..., n-1$, in which, for every $j \in S$ and $i \in \{0, ..., n-1\}$, the arc $(i, (i+j) \pmod n)$ is present[1]. In some sense, set S "generates" the arcs of $C(n, S)$.

Our upper bounds in this paper, are established from colourings by signed graphs which are inspired from circulant oriented graphs, which we call *circulant signed graphs*. The definition is as follows. The *circulant signed graph* $C(n, S)$ is the signature of K_n obtained by denoting by $0, ..., n-1$ the vertices, then, for every $j \in S$ and $i \in \{0, ..., n-1\}$, letting $(i, (i+j) \pmod n)$ be a positive edge, and, eventually, letting all other edges being negative. This time, set S is used to generate the positive edges of $C(n, S)$.

3. A general upper bound

The only known upper bound on the oriented chromatic number of grids was exhibited by Fertin, Raspaud and Roychowdhury, who proved in [3] that $\chi_o(G(n, m)) \leq 11$ holds for every $n, m \geq 1$. In this section, we prove that, for every grid $G = G(n, m)$, we have $\chi_s(G) \leq 12$. We first prove the upper bound of 13 using a somewhat simpler method, so that the reader gets a first idea of our proof scheme. The upper bound of 12 will then be obtained by using more refined arguments.

Our upper bound of 13 follows from the following signed version of the method described in [3].

Proposition 3.1. *Suppose we have a signed graph A such that, for every two distinct vertices u, v of A, and for any every $\{s_1, s_2\}$ of $\{-, +\}^2$, there exist two distinct 2-paths uw_1v and uw_2v in A such that $\sigma(uw_1) = \sigma(uw_2) = s_1$, $\sigma(w_1v) = \sigma(w_2v) = s_2$. Then every signed grid is A-colourable.*

Proof. We describe how to obtain an A-colouring ϕ of any signed grid G, by extending ϕ from rows to rows. By the assumption on A, there is, in A, an alternating 4-cycle C. Since every signed path can clearly be C-coloured, there is a C-colouring, hence an A-colouring, ϕ of the first row of G.

Now assume, for some i, that ϕ is a partial A-colouring of all vertices from the first row to the $(i-1)^{\text{th}}$ row of G. Assuming the vertices of the i^{th} row are consecutively denoted by $b_1, ..., b_n$ (b_1 being in the first column), we extend ϕ to these vertices following the order given by their indexes.

Denote by $a_1, ..., a_n$ the vertices from the $(i-1)^{\text{th}}$ row of G, where a_jb_j is an edge for every $j = 1, ..., n$. When colouring a vertex b_j, we of course have to make sure that $\phi(b_j) \neq \phi(b_{j-1})$ (if this vertex exists), and that $\phi(b_j) \neq \phi(a_j)$. We also note that, if a_{j+1} exists, then b_{j+1} cannot be coloured if $\phi(b_j) = \phi(a_{j+1})$ and $\sigma(b_jb_{j+1}) \neq \sigma(a_{j+1}b_{j+1})$. One solution to avoid this technicality, is to always request that $\phi(b_j) \neq \phi(a_{j+1})$, which we will maintain at any step of the upcoming colouring procedure.

We are now ready to describe how to extend ϕ to $b_1, ..., b_n$. By the assumption on A, every vertex of A is incident to at least two positive edges, and incident to at least two negative edges. Hence, we can colour b_1 so that $\phi(b_1) \neq \phi(a_1), \phi(a_2)$.

We now go to the general case, i.e. we consider the case where all vertices $b_1, ..., b_{j-1}$ have been coloured, for some $j-1 \geq 1$, and consider b_j. Then we can easily extend ϕ to b_j, in the following way. Recall that $\phi(b_{j-1}) \neq \phi(a_j)$. By the property of A, applied to $u = \phi(b_{j-1})$, $v = \phi(a_j)$, $s_1 = \sigma(b_{j-1}b_j)$, $s_2 = \sigma(a_jb_j)$, there are at least two possible colours that can be assigned to b_j. By assigning to b_j a colour different from $\phi(a_{j+1})$ (if this vertex exists), we maintain our colouring

[1]In order to avoid digons, one should request S to not include two elements summing up to n.

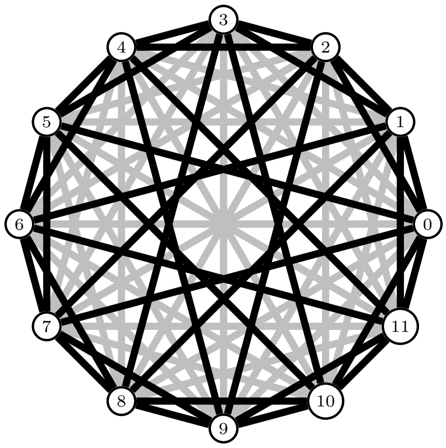

Figure 1: The circulant signed graph $C(12, \{1, 2, 5\})$. Black (resp. gray) edges are positive (resp. negative) edges.

condition, and hence make sure that the colouring process can be pursued. The i^{th} row of G can hence be coloured, thus all rows of G, concluding the proof. □

In order to get an upper bound on the signed chromatic number of all grids, we hence just have to exhibit signed graphs A, of lowest possible order, having the property described in Proposition 3.1. Using a computer program, we have generated all circulant signed graphs $C(n, S)$ with small order n, in order to find a smallest one having the desired property. Our algorithm is the following. For any fixed n, we thus want to generate every circulant signed graph on n vertices. To that aim, we consider every subset S of $\{1, 2, ..., n-1\}$, and build the graph $A = C(n, S)$. This can be done by starting from A being the empty graph on n vertices, then adding the positive edges of A (as defined by S), and finally adding all missing edges as negative edges. To then check whether A has the property described in Proposition 3.1, we just have to consider all distinct vertices $u, v \in \{0, ..., n-1\}$, all combinations of two signs $s_1, s_2 \in \{-, +\}$, and count, in A, the number of distinct vertices w such that $\sigma(uw) = s_1$, $\sigma(wv) = s_2$. If A has the desired property, this number should always be at least 2.

The main conclusion of our computation, is the following.

Proposition 3.2. *The smallest circulant signed graphs $C(n, S)$ having the property described in Proposition 3.1, have $n = 13$. An example of a such graph, is $C(13, \{1, 3, 4\})$.*

Combining Propositions 3.1 and 3.2, we directly get the following.

Theorem 3.3. *For every $n, m \geq 1$, we have $\chi_s(G(n, m)) \leq 13$.*

We now improve the bound in Theorem 3.3 down to 12, still by considering colourings by circulant signed graphs. Following Proposition 3.2, we unfortunately cannot find a circulant signed graph on 12 vertices having the nice property described in Proposition 3.1. We however show that such signed graphs having "almost" this property, can be used to colour any signed grid, provided we are a bit more careful when assigning the colours.

The circulant signed graph from which our bound is obtained, is $C(12, \{1, 2, 5\})$ (depicted in Figure 1). To show that every signed grid can be coloured by $C(12, \{1, 2, 5\})$, we essentially proceed as in the proof of Proposition 3.1, namely we colour the rows one after another, from the first vertex to the last vertex. As explained above, though, this time the colours should be cautiously chosen, as otherwise it may occur, during the course, that no colour is available for some vertex.

To overcome this issue, we proceed as follows. Assume all vertices from the first row up to the $(i-1)^{\text{th}}$ row of a signed grid G, have been successively coloured by $C(12, \{1, 2, 5\})$ in an arbitrary way (resulting in a partial colouring ϕ), and that we are now considering an extension of ϕ to the i^{th} row. Assume $a_1, ..., a_n$ and $b_1, ..., b_n$ denote the successive vertices of the $(i-1)^{\text{th}}$ and i^{th} rows, respectively. Prior to actually assigning a colour by ϕ to each b_j, we will first iteratively compute a set $\psi(b_j)$ of possible colours that can be assigned to b_j, where these possible colours depend on the set $\psi(b_{j-1})$ (if this vertex exists), and on some of the $\phi(a_j)$'s.

We call $\psi : V(G) \to \mathcal{P}(\{0,...,11\})$ a *choice function.* Formally, its definition is that, for every $j = 1,...,n$ and every $\alpha_j \in \psi(b_j) \neq \emptyset$, there are $\alpha_1 \in \psi(b_1),...,\alpha_{j-1} \in \psi(b_{j-1})$ such that, by setting $\phi(b_1) = \alpha_1,...,\phi(b_{j-1}) = \alpha_{j-1}$ and $\phi(b_j) = \alpha_j$, we obtain a correct extension of ϕ to the first j vertices of the i^{th} row of G. In other words, for every value α_j in $\psi(b_j)$, there are possible correct colours that can be assigned to $b_1,...,b_{j-1}$, i.e. colours indicated by ψ, which eventually allow us to correctly colour b_j with colour α_j. Note that this definition implies that if $\psi(b_n) \neq \emptyset$, then a colouring of the i^{th} row of G can be obtained.

It turns out that, assuming the $(i-1)^{\text{th}}$ row of G is arbitrarily coloured by $C(12, \{1,2,5\})$, a such choice function ψ for the i^{th} row can always be obtained. We state this result right away, so that we can then explain how ψ can lead to the desired extension of ϕ to the i^{th} row.

Proposition 3.4. *Let G be a signature of $G(2,n)$, whose first-row vertices are successively denoted by $a_1,...,a_n$, while its second-row vertices are denoted by $b_1,...,b_n$, so that a_i, b_i are the vertices of the i^{th} column for every $i = 1,...,n$. Then, for every colouring ϕ by $C(12, \{1,2,5\})$ of the a_i's, there exists a choice function ψ of the b_i's, such that every $\psi(b_i)$ is non-empty.*

With Proposition 3.4 in hand, we can finally prove that every signed grid is colourable by $C(12, \{1,2,5\})$, and, hence, that the signed chromatic number of every grid is at most 12.

Theorem 3.5. *For every $n, m \geq 1$, we have $\chi_s(G(n,m)) \leq 12$.*

Proof. We actually prove that every signed grid is colourable by $C(12, \{1,2,5\})$. The proof goes exactly the same way as that of Proposition 3.1. As a base case, we note that $C(12, \{1,2,5\})$ has alternating 4-cycles, such as $(0,1,4,3,0)$, so the first row of any given signed grid G can be coloured by $C(12, \{1,2,5\})$.

Assume now that all rows of G up to the $(i-1)^{\text{th}}$ one have been coloured by $C(12, \{1,2,5\})$, resulting in a partial colouring ϕ, and consider the i^{th} row, with consecutive vertices $b_1,...,b_n$. Denote by $a_1,...,a_n$ the consecutive vertices of the $(i-1)^{\text{th}}$ row of G, so that a_ib_i is an edge for every $i = 1,...,n$. According to Proposition 3.4, no matter how the a_j's are coloured, there is a non-trivial choice function ψ of the b_j's. In particular, every $\psi(b_j)$ is non-empty. By definition, there is hence a sequence $\alpha_1,...,\alpha_n \in \{0,1,...,11\}$ of colours, such that $\alpha_j \in \psi(b_j)$ for every $j = 1,...,n$, and, by setting $\phi(b_j) = \alpha_j$ for every $j = 1,...,n$, we get a correct extension of ϕ to the b_j's. This proves the inductive step, and hence concludes the proof. □

The remaining task, is thus to prove Proposition 3.4.

Proof of Proposition 3.4. We determine some of the values in the choice function $\psi(b_i)$ of every b_i, one after another, starting from b_1, and show, in particular, that none of the $\psi(b_i)$'s is empty. Throughout the proof, we call any value in a set assigned by ψ a *choice.* Furthermore, we say that a choice at a vertex b_i is *clean*, if this choice is different from $\phi(a_{i+1})$ (if this vertex exists). So, out of all choices in any $\psi(b_i)$, all but at most one of them are clean.

The proof goes as follows. Assuming we have determined part of the choice functions of all vertices up to, say, b_i, and $\psi(b_i)$ was shown to include at least two clean choices, we determine part of the choice functions of the vertices following b_i, until either 1) we determine part of ψ for all remaining vertices of the row, or 2) we reach a vertex b_j, for which $\psi(b_j)$ is shown to have at least two clean choices. In the latter case, we can then reapply the same procedure, but with b_j instead of b_i, and pursue the process until all $\psi(b_i)$'s are shown to be non-empty.

We start with b_1. Since all vertices of $C(12, \{1,2,5\})$ are incident to at least five negative edges and five positive edges, $\psi(b_1)$ includes at least five choices (deduced from $\phi(a_1)$), at least four of which are clean. It now suffices to prove the claim above, with $b_i = b_1$.

By the assumption, $\psi(b_1)$ includes at least two clean choices α_1, α_2. We note that if, by virtually assigning colour α_1 or α_2 to b_1, there are, regarding $\phi(a_2)$, at least three possible colours available for b_2, then we are done, as, then, $\psi(b_2)$ would include at least three choices, at least two of which are clean. We may thus assume that, by virtually colouring b_1 with α_1 or α_2, at most two choices are available at b_2. Actually, since $\alpha_1 \neq \alpha_2$, there cannot be less than two such choices, according to the following property of $C(12, \{1,2,5\})$, which we have checked using a computer.

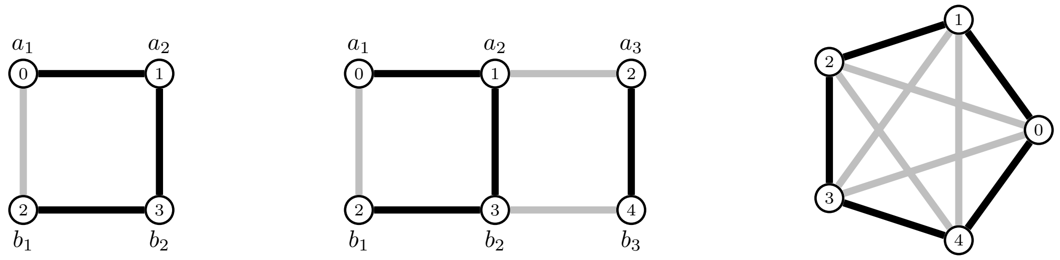

Figure 2: A 4-colouring of a signature of $G(2,2)$ (left), a 5-colouring of a signature of $G(2,3)$ (middle), and the circulant signed graph $C(5,\{1\})$ (right). Black (resp. gray) edges are positive (resp. negative) edges.

Observation 3.6. *The cases where a pair of vertices of $C(12,\{1,2,5\})$ is, for a fixed signature, not joined by at least two 2-paths with that signature, are when 1) the signature includes two negative edges, and 2) the pair is of the form $\{\alpha,\alpha+4\}$ or $\{\alpha,\alpha-4\}$, where the operations are understood modulo 12. Furthermore, for such signature and pair, the only joining 2-paths in $C(12,\{1,2,5\})$ are $(\alpha,\alpha+8,\alpha+4)$ and $(\alpha,\alpha-8,\alpha-4)$, respectively.*

Let thus β_1,β_2 be two choices at b_2. If these two choices are clean, then we are done. Assume thus that one of these two choices, say α_1, is not clean. Due to the symmetries of $C(12,\{1,2,5\})$, we may assume, without loss of generality, that $\phi(a_3)=0$, and hence that $\alpha_1=0$. We may, as well, assume that $\sigma(b_2b_3)\neq\sigma(a_3b_3)$, as, otherwise, by virtually colouring b_2 with colour 0, we would get at least five choices at b_3, and, hence, at least four clean choices.

According to Observation 3.6, since $\sigma(b_2b_3)\neq\sigma(b_3a_3)$, we get, by virtually colouring b_2 with colour α_2, at least two choices at b_3. If at least three choices are available, then we are done, according to the same reasons as above. So we may assume that there are only two choices γ_1,γ_2 at b_3, when b_2 is virtually coloured with colour α_2. This situation being exactly the same as earlier, it can be treated similarly. Namely, we can assume that one of γ_1,γ_2, say γ_1, is not clean, hence that $\phi(a_4)=\gamma_1$. Furthermore, we may assume that $\sigma(b_3b_4)\neq\sigma(a_4b_4)$. Under those assumptions, we again deduce that, by virtually colouring b_3 with colour γ_2, there are at least two choices available at b_4. And so on.

By repeatedly applying the above procedure, we either run into a b_i such that $\psi(b_i)$ includes at least two clean choices, or successively reveal, until we reach the last vertex of the row, that at least two choices are available, one of which is not clean, at every vertex. We hence end up with ψ being non-empty for every b_i, as claimed. □

4. Signed grids of the form $G(2,n)$

The oriented chromatic number of 2-row grids was fully determined by Fertin, Raspaud and Roychowdhury in [3], wherein it was proved that $\chi_o(G(2,n))=6$ for every $n\geq 4$, while $G(2,2)$ and $G(2,3)$ have oriented chromatic number 4 and 5, respectively. We here completely determine the signed chromatic number of 2-row grids, by mainly showing that $\chi_s(G(2,n))$ is bounded above by 5 for every $n\geq 3$. Hence, for this type of grids, the signed chromatic number is always smaller than the oriented chromatic number.

We start off by noting that $G(2,2)$, which is the cycle of length 4, admits a signature under which each of its vertices must be coloured with a unique colour.

Proposition 4.1. *We have $\chi_s(G(2,2))=4$.*

Proof. Consider the signature of $G(2,2)$ depicted in Figure 2 (left). Clearly, in this signed graph, every two non-adjacent vertices are joined by an alternating 2-path. It hence cannot be coloured with less than $|V(G(2,2))|$ colours, implying that $\chi_s(G(2,2))=4$. □

Since $G(2,2)$ is a subgraph of $G(2,n)$ for every $n\geq 2$, by Proposition 4.1 we get that $\chi_s(G(2,n))\geq 4$ for every $n\geq 2$. In the following, we prove that, actually, $\chi_s(G(2,n))\geq 5$ holds for every $n\geq 3$.

Proposition 4.2. *We have $\chi_s(G(2,3))\geq 5$.*

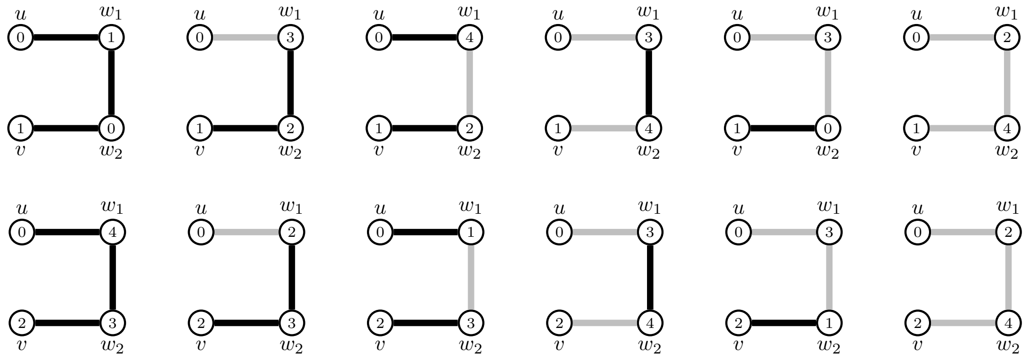

Figure 3: Examples of the 3-paths of $C(5, \{1\})$, claimed in the proof of Observation 4.4, for $(u, v) = (0, 1)$ (top), and $(u, v) = (0, 2)$ (bottom). Black (resp. gray) edges are positive (resp. negative) edges.

Proof. To be convinced of this statement, consider the signature of $G(2, 3)$ depicted in Figure 2 (middle), and assume, for contradiction, that it admits a 4-colouring ϕ. We note that the vertices a_1, a_2, b_1, b_2 form exactly the signature of $G(2, 2)$ described in the proof of Proposition 4.1. As explained earlier, these four vertices must be assigned different colours by ϕ. Assume $\phi(a_1) = 0$, $\phi(a_2) = 1$, $\phi(b_1) = 2$ and $\phi(b_2) = 3$ without loss of generality. Now, because a_3 is adjacent to a_2, and a_3 is joined by alternating 2-paths to both a_1 and b_2, clearly we must have $\phi(a_3) = 2$. But now, b_3 cannot be assigned any of colours 1, 2 or 3 for the same reasons, while it cannot be assigned colour 0, since a_1b_1 and a_3b_3 have different signs, and $\phi(a_3) = \phi(b_1) = 2$. So b_3 cannot be assigned a colour by ϕ, contradicting our initial hypothesis. □

Again, since $G(2, 3)$ is a subgraph of $G(2, n)$ for every $n \geq 3$, Proposition 4.2 implies that $\chi_s(G(2, n)) \geq 5$ holds for every $n \geq 3$. Actually, it turns out that five colours are sufficient to colour any signature of any 2-row grid.

Proposition 4.3. *For every $n \geq 1$, we have $\chi_s(G(2, n)) \leq 5$.*

Proof. We actually show that every signature of $G(2, n)$, with $n \geq 1$, can be coloured by the circulant signed graph $C(5, \{1\})$ (see Figure 2 (right)). To that aim, let us first point out the following property of $C(5, \{1\})$.

Observation 4.4. *For every two distinct vertices u, v of $C(5, \{1\})$, and for every set $\{s_1, s_2, s_3\}$ of $\{-, +\}^3$, there exists a 3-path uw_1w_2v in $C(5, \{1\})$ such that $\sigma(uw_1) = s_1$, $\sigma(w_1w_2) = s_2$, $\sigma(w_2v) = s_3$.*

Proof. Due to the signature-preserving automorphisms of $C(5, \{1\})$, it should be clear that we may restrict our attention to the cases $(u, v) = (0, 1)$ and $(u, v) = (0, 2)$. Furthermore, only six of the sets among $\{-, +\}^3$ have to be considered. To see that the claim holds, refer to Figure 3, which gathers examples of the claimed twelve 3-paths of $C(5, \{1\})$. □

Back to the proof of Proposition 4.3, we now describe how to get a colouring ϕ by $C(5, \{1\})$, of any signature G of $G(2, n)$ with $n \geq 1$. Let us denote by $a_1, ..., a_n$ and $b_1, ..., b_n$ the consecutive vertices of the first and second rows of G, respectively, where a_i, b_i are the vertices of the i^{th} column for every $i = 1, ..., n$. As a first step, we colour a_1 and b_1. For this purpose, we choose an edge $\{\alpha, \beta\}$ of $C(5, \{1\})$ having sign $\sigma(a_1b_1)$, and set $\phi(a_1) = \alpha$ and $\phi(b_1) = \beta$.

To complete the colouring by $C(5, \{1\})$, it now suffices to repeatedly apply the following procedure. Assuming vertices a_{i-1} and b_{i-1} have been coloured in the previous step, we extend ϕ to a_i and b_i. Let s_1, s_2, s_3 be the signs of $a_{i-1}a_i, a_ib_i, b_ib_{i-1}$, respectively. According to Observation 4.4 (applied to $u = \phi(a_{i-1})$, $v = \phi(b_{i-1})$ and s_1, s_2, s_3), there exists a 3-path $(\phi(a_{i-1}), \alpha, \beta, \phi(b_{i-1}))$ in $C(5, \{1\})$ whose edges have sign s_1, s_2, s_3, respectively. By hence setting $\phi(a_i) = \alpha$ and $\phi(b_i) = \beta$, we get an extension of ϕ to a_i and b_i. □

From all of the previous results, we end up with the following characterization of the signed chromatic number of 2-row grids.

Theorem 4.5. *We have:*

- $\chi_s(G(2,2)) = 4$,
- $\chi_s(G(2,n)) = 5$ *for every* $n \geq 3$.

5. Signed grids of the form $G(3, n)$

The investigations on the oriented chromatic number of 3-row grids were initiated by Fertin, Raspaud and Roychowdhury, who proved, in [3], that $\chi_o(G(3,3)) = \chi_o(G(3,4)) = \chi_o(G(3,5)) = 6$, while $\chi_o(G(3,n)) \in \{6,7\}$ for every $n \geq 6$. Later on, Szepietowski and Targan completely determined, in [6], the values of $\chi_o(G(3,n))$ for every $n \geq 6$, by proving that $\chi_o(G(3,6)) = 6$, while $\chi_o(G(3,n)) = 7$ for every $n \geq 7$.

Before presenting our results on 3-row grids, we first introduce some definitions and terminology that are used throughout this section, and raise some comments that are important to understand our investigations.

Whenever dealing with a (signed) grid $G = G(3,n)$, we assume that its vertices are labelled by $a_1, ..., a_n$, $b_1, ..., b_n$ and $c_1, ..., c_n$, where the a_i's are the consecutive vertices of the first row, the b_i's are the consecutive vertices of the second row, and the c_i's are the consecutive vertices of the third row. We also assume, for every $i = 1, ..., n$, that the vertices of the i^{th} column are a_i, b_i, c_i (see Figure 4 (left) for an illustration).

Let A be a fixed signed graph, and assume now that G is a signed grid. In the sequel, we will mainly colour G by extending an A-colouring ϕ from column to column, starting from the first column. In doing so, for each column i, we get a set of possible *triplets* of colours, which are 3-element sets $(\alpha, \beta, \gamma) \in \{0, 1, ..., |V(A)| - 1\}^3$ such that, when extending ϕ to the i^{th} column, we can set $\phi(a_i) = \alpha$, $\phi(b_i) = \beta$ and $\phi(c_i) = \gamma$. Note that every triplet (α, β, γ) verifies $\beta \neq \alpha, \gamma$.

When extending ϕ to the i^{th} column of G, it turns out that the possible colours for a_i, b_i, c_i, i.e. the possible triplets $(\alpha_i, \beta_i, \gamma_i)$ of colours that can be assigned to this column, are highly dependent of the triplet $(\alpha_{i-1}, \beta_{i-1}, \gamma_{i-1})$ of colours assigned to the $(i-1)^{\text{th}}$ column. Also, assuming $\phi(a_{i-1}) = \alpha_{i-1}$, $\phi(b_{i-1}) = \beta_{i-1}$, $\phi(c_{i-1}) = \gamma_{i-1}$, the possible triplets $(\alpha_i, \beta_i, \gamma_i)$ depend on the set of five edges $\{a_{i-1}a_i, b_{i-1}b_i, c_{i-1}c_i, a_ib_i, b_ic_i\}$, which form a signed subgraph that we call a *2-comb*. Formally, a 2-comb, refers to a graph obtained from a path $uw_1w_2w_3v$ of length 4, by joining w_2 to a new pendant vertex w. Under that labelling, we say that the 2-comb *joins* u, w, v, and call $w_1w_2w_3$ the *spine* of the 2-comb. We note that any (signed) 3-row grid can be obtained, starting from a (signed) 2-path, by repeatedly joining a new (signed) 2-comb onto a (signed) 2-path (first step) or the spine of a previous (signed) 2-comb (other steps).

Back to our context, the possible triplets $(\alpha_i, \beta_i, \gamma_i)$ for the i^{th} column are precisely those 3-element sets such that A has a 2-comb joining $\alpha_{i-1}, \beta_{i-1}, \gamma_{i-1}$, with spine $\alpha_i\beta_i\gamma_i$, and whose edge signs are precisely the signs of the 2-comb joining the $(i-1)^{\text{th}}$ and the i^{th} columns of G.

5.1. Lower bounds

We start off by investigating general lower bounds on the signed chromatic number of 3-row grids. As a starting point, we point out that, for some signatures of $G(3,3)$, at least six colours are needed.

Proposition 5.1. *We have* $\chi_s(G(3,3)) \geq 6$.

Proof. Let G be the signature of $G(3,3)$ depicted in Figure 4 (left), and assume, for contradiction, that there is a signature A of K_5 such that G admits an A-colouring ϕ.

We note that every two vertices of a_2, b_1, b_3, c_2 are joined by an alternating 2-path. For this reason, all colours $\phi(a_1), \phi(a_2), \phi(b_1), \phi(b_3), \phi(c_2)$ must be different. As in Figure 4 (left), let us assume, without loss of generality, that $\phi(b_2) = 0$, $\phi(a_2) = 1$, $\phi(b_3) = 2$, $\phi(c_2) = 3$ and $\phi(b_1) = 4$. This reveals that, in A, edges $\{0,1\}$ and $\{0,4\}$ are positive, while $\{0,2\}$ and $\{0,3\}$ are negative.

Now consider c_3. Since b_2 and c_3 are joined by an alternating 2-path, we have either $\phi(c_3) = 1$ or $\phi(c_3) = 4$. At this point of the proof, we may assume that $\phi(c_3) = 1$. This reveals that, in A, edge $\{1,2\}$ is negative, while $\{1,3\}$ is positive. Now consider c_1. Since c_1 is joined by an alternating

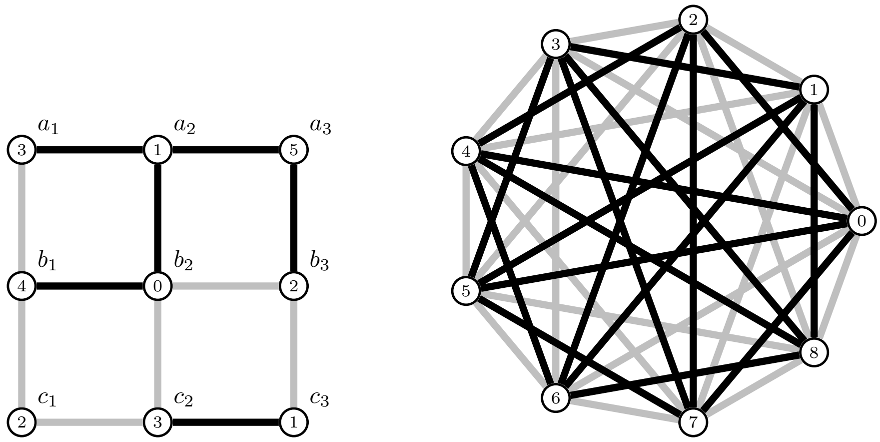

Figure 4: A 6-colouring of a signature of $G(3, 3)$ (left), and the circulant signed graph $C(9, \{2, 4\})$ (right). Black (resp. gray) edges are positive (resp. negative) edges.

2-path to both b_2 and c_3, we must have $\phi(c_1) = 2$. Hence, edges $\{2, 3\}$ and $\{2, 4\}$ are negative in A. For similar reasons, vertex a_1 must receive colour 2 or 3 by ϕ. Actually, we cannot have $\phi(a_1) = 2$ since edge $\{1, 2\}$ was shown to be negative in A. So, we have $\phi(a_1) = 3$.

We finally note that a_3 cannot be coloured with either of colours $0, 1, 2$, due to some edges or alternating 2-paths of G. Furthermore, we cannot have $\phi(a_3) = 3$ since edge $\{2, 3\}$ is negative in A, or $\phi(a_3) = 4$ since edge $\{2, 4\}$ is negative in A. Hence a_3 cannot be assigned a valid colour by ϕ, a contradiction. □

It turns out that some signed 3-row grids need at least seven colours to be coloured. The existence of such grids was attested by means of a computer, by employing the following arguments.

Roughly speaking, our method to show that a 7-chromatic signed grid exists, consisted in showing the existence, for every signature A of K_6, of a signed 3-row grid G_A that is not A-colourable. With such a signed grid in hand for every A, one can just imagine a big signed 3-row grid including every of the G_A's, so that it cannot be coloured by any signature of K_6. Our method for showing the existence of such grids G_A, was partly inspired from a computer-assisted proof described in [2], which was used to exhibit oriented grids with large oriented chromatic number.

So that our proof scheme could be used, a necessary ingredient was the explicit list $\mathcal{L}$ of all non-isomorphic signatures of K_6, out of the 2^{15} possible signatures. To obtain $\mathcal{L}$, we have proceeded as follows. Assuming the vertices of K_6 are denoted by $0, ..., 5$, while each edge (i, j) is denoted by $e_{i,j}$ (where $i < j$), we note that every signature A of K_6 is uniquely identified by an integer

$$r(A)_{10} = (b(e_{0,1})b(e_{0,2})...b(e_{0,5})b(e_{1,2})...b(e_{1,5})b(e_{2,3})...b(e_{2,5})b(e_{3,4})b(e_{3,5})b(e_{4,5}))_2 \ ,$$

where $b(e_{i,j}) = 1$ if $e_{i,j}$ is positive, and $b(e_{i,j}) = 0$ otherwise. Furthermore, $r(A)$ can easily be computed for every A, and, vice-versa, we can easily reconstruct A from $r(A)$.

Clearly, the signatures of K_6 that are isomorphic to A, are obtained by preserving the edge signs, but relabelling the vertices of A in every possible way. In other words, to every permutation π of $\{0, ..., 5\}$ corresponds a signature A' of K_6 which is isomorphic to A, and verifies $r(A') \neq r(A)$ (unless π is the identity function). Using a computer, we could hence generate the indexes of the non-isomorphic signatures of K_6, basically by iterating through all indexes one after another (from 0 to $2^{15} - 1$), next deducing, at each step, the indexes of the signatures that are isomorphic to the considered signature (by computing, for every permutation of the vertex set, the index of the resulting graph), and, during the course, "marking" those indexes which have not been deduced while treating a previous index. These marked indexes are basically our list $\mathcal{L}$, as, from every index $r(A)$ in $\mathcal{L}$, we can easily retrieve A by looking at the binary representation of $r(A)$.

To reduce the list $\mathcal{L}$, we also made use of the following observation. For a signature A of K_6, let A^{-1} be the signature being the *inverse* of A, namely the signature obtained by reversing all edge signs (all positive edges become negative, and vice-versa). Then we note that if a grid G_A cannot be A-coloured, then, by reversing all edge signs in G_A, we obtain a grid $G_{A^{-1}}$ which cannot be A^{-1}-coloured. Hence, in $\mathcal{L}$, we can as well keep only one of each $r(A), r(A^{-1})$.

Implementing the above algorithm, we have ended up with a list of 78 non-isomorphic signatures of K_6 to consider. For every A of these signatures, we have verified the existence of G_A, as follows.

Start from G_A being a 2-path $a_1b_1c_1$ with two positive edges, which will be the first column of G_A. Note that, since A is fixed, already there are some possible triplets $(\alpha_1, \beta_1, \gamma_1)$ of colours that can be assigned to a_1, b_1, c_1, respectively, by an A-colouring ϕ of G_A. We denote by $\mathcal{C}_1$ the set of all these possible triplets. Now add a second column with vertices a_2, b_2, c_2 to G_A, by just joining a_1, b_1, c_1 by a signed 2-comb with spine $a_2b_2c_2$, whose edges are signed in an arbitrary way. If G_A is A-colourable, then the set $\mathcal{C}_2$ of possible triplets $(\alpha_2, \beta_2, \gamma_2)$ of colours that can be assigned to a_2, b_2, c_2, when extending ϕ, should be non-empty. More precisely, for a fixed signature of the 2-comb, and for every $(\alpha_1, \beta_1, \gamma_1) \in \mathcal{C}_1$, set $\mathcal{C}_2$ contains those triplets $(\alpha_2, \beta_2, \gamma_2)$ such that A has a signed 2-comb joining $\alpha_1, \beta_1, \gamma_1$, with the corresponding signature, and with spine $\alpha_2\beta_2\gamma_2$. It is worth emphasizing the fact that the triplets in $\mathcal{C}_2$ are quite dependent of the signature of the 2-comb attached to the first column. By repeating this process (i.e. attaching a new signed 2-comb to the last column of G_A), we can get an arbitrarily long signed 3-row grid, and, for the specific signatures of the used 2-combs, we can iteratively (i.e. on the fly) deduce the triplets of colours in every $\mathcal{C}_i$.

In order to get a signed 3-row grid that is not A-colourable, we just need to show that, for specific signatures of the used 2-combs, there is an i such that $\mathcal{C}_i$ gets empty. The choice we made, is to always sign the joining 2-combs, so that next set $\mathcal{C}_i$ is of minimum size. Implementing this strategy through a computer program, we observed that, for every A of the 78 signatures of K_6 in $\mathcal{L}$, a non-colourable signed 3-row grid, with the edges from its first column being both positive, can always be obtained by using at most five joining 2-combs. Hence, we verified the following.

Theorem 5.2. *There exists a n_0, such that for every $n \geq n_0$, we have $\chi_{\mathrm{s}}(G(3, n)) \geq 7$.*

5.2. Upper bounds

Our upper bounds on the signed chromatic number of 3-row grids, rely on the existence of circulant signed graphs with properties analogous to that described in the statements of Proposition 3.1 and Observation 4.4 (but for 3-row grids).

We remind to the reader that, as described in the introduction of this section, we systematically colour any signed grid from column to column, by essentially extending triplets of colours from 2-comb to 2-comb. In that spirit, the following property directly yields upper bounds on the signed chromatic number of signed 3-row grids.

Proposition 5.3. *Suppose we have a signed graph A such that, for every three distinct vertices u, v, w of A, and for every set $\{s_1, s_2, s_3, s_4, s_5\}$ of $\{-, +\}^5$, there exists, in A, a 2-comb, joining u, w, v and with spine $w_1w_2w_3$, such that $\sigma(uw_1) = s_1$, $\sigma(ww_2) = s_2$, $\sigma(vw_3) = s_3$, $\sigma(w_1w_2) = s_4$, $\sigma(w_2w_3) = s_5$. Then every signed grid $G(3, n)$ is A-colourable.*

Proof. We prove by induction on n, the number of columns, that every signature G of $G(3, n)$ can be A-coloured, provided A has the desired property. In case $n = 1$, we note that G is actually a signed path on two edges. Since, by our assumption on A, signed graph A has both positive edges and negative edges, and has positive edges incident to negative edges, it should be clear that a_1, b_1, c_1 can be coloured.

Assume now that the claim is true for every n up to value $i-1$, and consider the case $n = i$. By the induction hypothesis, there exists an A-colouring ϕ of the $n-1$ first columns of G, which form a signature of $G(3, n-1)$. We now extend ϕ the i^{th} column, i.e. to the vertices a_i, b_i, c_i. To that aim, consider the signed 2-comb C of G joining $a_{i-1}, b_{i-1}, c_{i-1}$ with spine $a_ib_ic_i$. According to the initial assumption on A, no matter what the triplet $(\phi(a_{i-1}), \phi(b_{i-1}), \phi(c_{i-1}))$ is, and no matter what the signs on the edges of C are, we can find, in A, a 2-comb joining $\phi(a_{i-1}), \phi(b_{i-1}), \phi(c_{i-1})$, and with the same edge signs as C. Denote its spine by $\alpha_i\beta_i\gamma_i$. Then we can simply extend ϕ to a_i, b_i, c_i, by setting $\phi(a_i) = \alpha_i$, $\phi(b_i) = \beta_i$, $\phi(c_i) = \gamma_i$. This concludes the proof. $\square$

Hence, by showing that a signed graph A with small order has the property described in Proposition 5.3, we immediately get that every signed 3-row grid is A-colourable, thus that its signed chromatic number is at most $|V(A)|$. Using again a computer, we have determined that the smallest circulant signed graphs having that property, have order 10.

Proposition 5.4. *The smallest circulant signed graphs $C(n, S)$ having the property described in Proposition 5.3, have $n = 10$. An example of a such graph, is $C(10, \{2, 4\})$.*

From Propositions 5.3 and 5.4, we thus directly get the following.

Theorem 5.5. *For every $n \geq 1$, we have $\chi_s(G(3, n)) \leq 10$.*

We now improve the upper bound in Theorem 5.5 down to 9, by showing that every signed 3-row grid can be coloured by the circulant signed graph $C(9, \{2, 4\})$ (illustrated in Figure 4 (right)). The colouring strategy we use, is again the column-to-column one that we have used earlier. We however have to be more careful here, because, as indicated by Proposition 5.4, there are situations where a colouring of the $(i-1)^{\text{th}}$ column cannot be extended to the i^{th} one, namely because $C(9, \{2, 4\})$ does not admit all possible kinds of signed 2-combs.

Following Proposition 5.4, we know that $C(9, \{2, 4\})$ has *bad triplets*, namely triplets (α, β, γ) of colours such that $C(9, \{2, 4\})$ has no 2-comb, with a particular signature, joining α, β, γ. Hence, when colouring a new column of a signed 3-row grid, we should avoid getting a bad triplet, as it might then not be possible to extend the partial colouring to the next column.

Using a computer program to enumerate all 3-element sets of colours (α, β, γ) and, for every signature, all signed 2-combs joining α, β, γ in $C(9, \{2, 4\})$, we came up with the following characterization of the bad triplets in $C(9, \{2, 4\})$.

Observation 5.6. *A triplet (α, β, γ) of $C(9, \{2, 4\})$ is bad, if and only if $(\beta, \gamma) = (\alpha + 2, \alpha + 4)$, $(\beta, \gamma) = (\alpha - 2, \alpha - 4)$, $(\beta, \gamma) = (\alpha + 3, \alpha + 6)$ or $(\beta, \gamma) = (\alpha - 3, \alpha - 6)$, where the operations are understood modulo 9.*

When colouring a column, we should as well avoid using a non-bad triplet (α, β, γ) of colours such that, for a particular fixed signature, all signed 2-combs with that signature, joining α, β, γ in $C(9, \{2, 4\})$, have a bad spine, i.e. a spine $\alpha'\beta'\gamma'$ such that $(\alpha', \beta', \gamma')$ is bad. We call such a triplet *dangerous*. Once again, the dangerous triplets of $C(9, \{2, 4\})$ can easily be generated using a computer, and, hence, characterized.

Observation 5.7. *A non-bad triplet (α, β, γ) of $C(9, \{2, 4\})$ is dangerous, if and only if $(\beta, \gamma) = (\alpha + 2, \alpha + 5)$, $(\beta, \gamma) = (\alpha - 2, \alpha - 5)$, $(\beta, \gamma) = (\alpha + 2, \alpha + 6)$, $(\beta, \gamma) = (\alpha - 2, \alpha - 6)$, $(\beta, \gamma) = (\alpha + 3, \alpha + 5)$, $(\beta, \gamma) = (\alpha - 3, \alpha - 5)$, $(\beta, \gamma) = (\alpha + 4, \alpha + 6)$, or $(\beta, \gamma) = (\alpha - 4, \alpha - 6)$, where the operations are understood modulo 9.*

One should of course be cautious with non-bad and non-dangerous triplets (α, β, γ) of colours such that, for some signature, all signed 2-combs with that signature, joining α, β, γ in $C(9, \{2, 4\})$, have a bad or dangerous spine. However, we checked, using a computer, that every non-bad and non-dangerous triplet (α, β, γ) is *good*, in the sense that, in $C(9, \{2, 4\})$, for every signature there is a signed 2-comb with that signature, joining α, β, γ, and with a good spine, i.e. a spine $\alpha'\beta'\gamma'$ such that $(\alpha', \beta', \gamma')$ is good.

Observation 5.8. *Every non-bad and non-dangerous triplet is good.*

We are now ready to improve the bound in Theorem 5.5.

Theorem 5.9. *For every $n \geq 1$, we have $\chi_s(G(3, n)) \leq 9$.*

Proof. We actually prove, by induction on n, that every signature G of $G(3, n)$ can be coloured by $C(9, \{2, 4\})$, implying the result. The colouring strategy we use, is again the column-to-column strategy that we have been using so far, but restricted to good triplets of colours. More precisely, we show that the columns of G can be coloured one after another, in such a way that the triplets of colours, assigned by the colouring ϕ, are all good.

As a base case, assume $n = 1$. In case a_1b_1 and b_1c_1 are both positive, we can set e.g. $\phi(a_1) = 0$, $\phi(b_1) = 4$, $\phi(c_1) = 0$. If a_1b_1 and b_1c_1 are both negative, we can here set e.g. $\phi(a_1) = 0$, $\phi(b_1) = 1$,

$\phi(c_1) = 0$. Finally, if, say, a_1b_1 is positive while b_1c_1 is negative, we can set e.g. $\phi(a_1) = 0$, $\phi(b_1) = 2$, $\phi(c_1) = 1$. In every case, we get that $(\phi(a_1), \phi(b_1), \phi(c_1))$ is a good triplet, according to Observation 5.8, which concludes this case.

Assume now that the claim is true for every n up to some value $i-1$, and consider the next step $n = i$. By the induction hypothesis, we can colour the $i-1$ first columns of G, as they form a signature of $G(3, n-1)$, in such a way that all triplets of colours are good. Let ϕ be such a colouring. We now extend ϕ to the i^{th} column of G, namely to its vertices a_i, b_i, c_i, in a good way. To that aim, consider, in G, the signed comb C joining $a_{i-1}, b_{i-1}, c_{i-1}$ with spine $a_ib_ic_i$. According to the definition of a good triplet, and because $(\phi(a_{i-1}), \phi(b_{i-1}), \phi(c_{i-1}))$ is good, there has to be, in $C(9, \{2, 4\})$, a signed comb with the same edge signs as C, joining $(\phi(a_{i-1}), \phi(b_{i-1}), \phi(c_{i-1}))$, and with a good spine $\alpha_i\beta_i\gamma_i$, i.e. $(\alpha_i, \beta_i, \gamma_i)$ is a good triplet. So we can extend ϕ to a_i, b_i, c_i by just setting $\phi(a_i) = \alpha_i, \phi(b_i) = \beta_i, \phi(c_i) = \gamma_i$. This proves the inductive step, and, hence, the claim. $\square$

6. Conclusion

In this article, we have initiated the study of the signed chromatic number of grids, our main goal being to compare how the oriented and signed chromatic numbers behave in these graphs. As a conclusion, we first summarize and discuss our results, independently of our original motivation, before commenting on the connexion between the two chromatic parameters.

Concerning the signed chromatic number of grids, we have provided several bounds for both general grids and 2- or 3-row grids. We have notably shown that the maximum signed chromatic number of a grid lies in between 7 and 12. For 2-row grids, we managed to completely determine their signed chromatic number, while, for 3-row grids, we have obtained partial results.

In order to establish lower bounds on the signed chromatic number, it is necessary to prove that some signed grid cannot be A-coloured by any A being a signature of some complete graph. Due to the number of signatures and of possible colourings to consider, computers are useful tools in this context. Our experimentations, though, tend to show that, on small instances, determining, via a computer, the signed chromatic number of a grid is less tractable than computing its oriented chromatic number. Concerning 3-row grids, we did not manage to show that they have signed chromatic number at most 7, nor to disprove it. However, applying the exact same procedure from Section 5.1 on non-isomorphic signatures of K_7, we managed to reduce the list $\mathcal{L}$ to only 44 potential colouring signatures of K_7 to consider. It might be that one of these 44 candidates can colour every signed 3-row grid.

In order to establish upper bounds, we have decided to design colourings by circulant signed graphs only, as we thought the regular and symmetric structure of these graphs should grant convenient properties. It might be, though, that some of our upper bounds can easily be improved, by just considering colourings by other kinds of signed graphs. However, it is worth mentioning that, concerning the colouring strategies we have designed, we did our best to make sure that these strategies could not be easily applied with smaller circulant signed graphs.

Concerning the relation between the oriented and signed chromatic numbers, our results show that these two parameters are quite close for grids. This is mainly established by the lower and upper bounds we know on the maximum values of these parameters for grids: while it is known that the maximum oriented chromatic number of a grid lies in between 8 and 11, we have shown that the maximum signed chromatic number of a grid lies in between 7 and 12.

Some disparities, though, are worth mentioning. For 2-row grids, while the oriented chromatic number is 6 in general, the signed chromatic number is 5 in general. We still do not know whether 3-row grids with signed chromatic number 8 exist, but, if this is the case, that would be quite interesting, as these grids have oriented chromatic number at most 7. In that spirit, it could as well be interesting considering 4-row grids, which have oriented chromatic number at most 7, according to [6].

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