Optimal Transportation for Data Assimilation
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**DATA ASSIMILATION**

Given
- a physical system and its state \( x(t, x, y) \);
- partial observations of the system \( \{y_i\} \); and
- a (numerical) model \( M \) simulating the evolution of \( x \).

Can we estimate the initial condition \( x_0 \) of the system?

\[ h \text{ set } u = \begin{cases} \frac{\partial h}{\partial t} + \text{div}(u) = 0, \\ \frac{\partial h}{\partial x} + u \cdot \nabla u = -g \nabla h. \end{cases} \]

Variational data assimilation consists in retrieving \( x_0 \) by minimizing

\[ J(x_0) := \sum_i d\left( H(x_0, x_i), y_i \right)^2 + \omega d\left( x_0, \tilde{x} \right)^2. \tag{1} \]

It is common for the distance \( d \) to be a weighted \( L^2 \) distance. Our main goal is to use the Wasserstein distance \( W_2 \) instead, which seems very interesting when dealing with dense data (see right panel). The Wasserstein cost function writes

\[ J_W(x_0) := \sum_i W_2\left( H(x_0, x_i), y_i \right)^2 + \omega W_2\left( x_0, \tilde{x} \right)^2. \tag{2} \]

**RÉSULTATS SUR UNE ÉQUATION DE L'EAU MINCE**

Let the model \( M \) be a Shallow-Water equation, with initial condition \( (h_0, u_0) \),

\[ M : \begin{cases} \frac{\partial h}{\partial t} + \text{div}(hu) = 0, \\ \frac{\partial u}{\partial t} + u \cdot \nabla u = -g \nabla h. \end{cases} \]

We control the initial condition \( h_0 \) only, thanks to the Wasserstein cost function \( J_W \). We set \( u_0 = 0 \).

The observations \( h^{\text{true}} \) at times \( t_i \)

| \( t_1 = 8h \) | \( t_2 = 6h \) | \( t_3 = 4h \) | \( t_4 = 2h \) |

True and background initial conditions

**OPTIMAL TRANSPORTATION AND THE WASSERSTEIN DISTANCE**

For two functions \( \rho_1(x) \) and \( \rho_2(x) \), the square of the Wasserstein distance \( W_2(\rho_1, \rho_2) \) is defined as the minimal kinetic energy necessary to transport \( \rho_1 \) to \( \rho_2 \),

\[ W_2(\rho_1, \rho_2)^2 = \inf_{\rho \in \Pi(\rho_1, \rho_2)} \int_{\Omega} \rho v \cdot d\nu. \]

For the Wasserstein distance to be well-defined, one needs \( \rho_0 \geq 0 \) and \( \int \rho_0 = 1 \).

Average w.r.t the Wasserstein distance

The average, or barycenter, minimizes \( W_2(\rho_0, \rho_{\text{bar}})^2 + W_2(\rho_{\text{bar}}, \rho_1)^2 \). It is also the optimal \( \rho \) in the definition of \( W_2(\rho_0, \rho_1)^2 \) at time \( t = 1/2 \)

**SPECIFICITIES ON USING THE WASSERSTEIN DISTANCE**

- The Wasserstein distance is only defined for probability measures, i.e. \( \rho \) s.t.

\[ \rho \geq 0 \quad \text{and} \quad \int \rho = 1. \]

- The Wasserstein distance (real valued) interpolates well if \( \rho_0 \) and \( \rho_1 \) are of distinct support;

- When \( J(\rho) \to \min_{\rho_0} J(\rho_0) \), then there is only weak convergence of \( \rho_n \) to \( \rho_0^{\text{opt}} \) : oscillations or drizas can occur!

- Computing the Wasserstein distance is expensive [Peyré, Papadakis, Oudet, 2013].

- The minimization of \( J_W \) is performed through a gradient descent, using the Wasserstein gradient, arising from the use of the following Wasserstein scalar product depending on \( \rho_0 \),

\[ \eta \cdot \eta'(t) = \int_{\Omega} \rho_0 \nabla \eta \cdot \nabla \eta' \, dx. \]

**Results :**

Analysis \( h_0 \) of the assimilation when using the Euclidean \( L^2 \) or the Wasserstein \( W_2 \) distance.

Values of \( h \) and \( u \) for the background and true states, as well as analysis for Euclidean and Wasserstein distances, at time \( t = t_b \).

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