Optimal Transportation for Data Assimilation
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To cite this version:
Nelson Feyeux, Maëlle Nodet, Arthur Vidard. Optimal Transportation for Data Assimilation. 5th International Symposium for Data Assimilation (ISDA 2016), Jul 2016, Reading, United Kingdom. 2016. hal-01349637

HAL Id: hal-01349637
https://hal.archives-ouvertes.fr/hal-01349637
Submitted on 28 Jul 2016

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DATA ASSIMILATION

Given
- a physical system and its state $x(t,x,y)$;
- partial observations of the system $(y_i)$;
- a (numerical) model $\mathcal{M}$ simulating the evolution of $x$;

Can we estimate the initial condition $x_0$ of the system?

We control the initial condition set $u_0 \cdot \cdot \cdot \cdot \cdot \cdot \cdot$

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2016

Can we estimate the initial condition $x_0$ of the system?

Variational data assimilation consists in retrieving $x_0$ by minimizing

$$J(x_0) := \sum_i \left( H(M(x_0), y_i) - \omega d(x_0, x_i)^2 \right)^2 + \omega d(x_0, x_i)^2. \quad (1)$$

It is common for the distance $d$ to be a weighted $L^2$ distance. Our main goal is to use the Wasserstein distance $W_2$ instead, which seems very interesting when dealing with dense data (see right panel). The Wasserstein cost function writes

$$J_W(x_0) = \sum_i W_2(H(M(x_0), y_i))^2 + \omega W_2(x_0, x_i)^2. \quad (2)$$

RESULTS ON A SHALLOW-WATER EQUATION

Let the model $\mathcal{M}$ be a Shallow-Water equation, with initial condition $(h_0, u_0)$.

$$\mathcal{M} : \begin{cases} \frac{\partial h}{\partial t} + \text{div}(mu) = 0 \\ \frac{\partial u}{\partial t} + u \cdot \nabla u = -g \nabla h. \end{cases}$$

We control the initial condition $h_0$ only, thanks to the Wasserstein cost function $J_W$. We set $u_0 = 0$.

The observations of $h^{\text{true}}$ at times $t_i$.

True and background initial conditions

SPECIFICITIES ON USING THE WASSERSTEIN DISTANCE

- The Wasserstein distance is only defined for probability measures, i.e. $\rho$ s.t.
  $$\rho \geq 0 \quad \text{and} \quad \int \rho = 1.$$

  Relaxations of the latter constraint are possible, however complex;
  - the $W_2$ interpolation works well if $\rho_0$ and $\rho_1$ are of distinct support;
  - when $J(\rho) \rightarrow \min_{\rho \in \mathcal{P}} J(\rho)$, then there is only weak convergence of $\rho^n_t$ to $\rho_0^\text{opt}$: oscillations or diracs can occur!
  - Computing the Wasserstein distance is expensive [Peyré, Papadakis, Oudet, 2013];

  - The minimization of $J_W$ is performed through a gradient descent, using the Wasserstein gradient, arising from the use of the following Wasserstein scalar product depending on $\rho_0$.

  For $\eta, \eta'$ s.t.
  $$\int \eta = \int \eta' = 0$$

  Let $\Phi, \Psi'$ s.t.
  $$-\text{div}(\rho_0 \nabla \Phi) = \eta \quad \text{(with Neumann BC)}$$
  $$-\text{div}(\rho_0 \nabla \Psi') = \eta'$$

  Then
  $$\eta(\eta') = \int \rho_0 \nabla \Phi \cdot \nabla \Psi' \, dx.$$

This work has been supported by the region Rhône-Alpes.