Optimal Transportation for Data Assimilation
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**Data assimilation**

Gained
- a physical system and its state \(x(t, x, y)\);
- partial observations of the system \(y(t)\);
- a (numerical) model \(M\) simulating the evolution of \(x\).

Can we estimate the initial condition \(x_0\) of the system?

**Variational data assimilation consists in retrieving** \(x_0\) **by minimizing**

\[
\mathcal{J}(x_0) := \sum_i d(H(M(x_0), y_i)^2 + \omega d(x_0, x_0)^2).
\]

(1)

For the Wasserstein distance to be well-defined, one needs \(\rho \geq 0\) and \(\int \rho = 1\).

**Results on a Shallow-water equation**

Let the model \(M\) be a Shallow-Water equation, with initial condition \((h_0, u_0)\).

\[
M: \begin{align*}
\frac{\partial h}{\partial t} + \text{div}(\rho u) &= 0, \\
\frac{\partial u}{\partial t} + u \cdot \text{grad}u &= -gVh,
\end{align*}
\]

We control the initial condition \(h_0\) only, thanks to the Wasserstein cost function \(\mathcal{J}_W\). We set \(u_0 = 0\).

**Results**

**Analysis** \(h_0\) of the assimilation when using the Euclidean (\(L^2\)) or the Wasserstein (\(W_2\)) distance.

**Example of use of the Wasserstein distance**

\[
\begin{align*}
W_2(\rho_{\text{true}}, \rho_{\text{bg}}) &= \int \rho_1 \sqrt{x - y} \, dx, \\
W_2(\rho_{\text{true}}, \rho_{\text{opt}}) &= \int \rho \sqrt{x - \omega} \, dx.
\end{align*}
\]

\(\rho_{\text{opt}}\) is the optimal \(\rho\) in the definition of \(W_2(\rho_{\text{true}}, \rho_{\text{opt}})^2\) at time \(t = 1/2\).

**Optimal Transportation and the Wasserstein distance**

For two functions \(\rho_i(x)\) and \(\rho_1(x)\), the square of the Wasserstein distance \(W_2(\rho_0, \rho_1)^2\) is defined as the minimal kinetic energy necessary to transport \(\rho_0\) to \(\rho_1\),

\[
W_2(\rho_0, \rho_1)^2 = \inf_{\Phi} \frac{1}{2} \int_{\mathbb{R}^2} \rho \sqrt{x - y} \, dx.
\]

(2)

\(\Phi\) being a Shallow-Water equation, with initial condition \(h_0\) and \(u_0\), \(\rho_0\) and \(\rho_1\) are of distinct support.

Variational data assimilation consists in retrieving \(x_0\) by minimizing

\[
\mathcal{J}(x_0) := \sum_i d(H(M(x_0), y_i)^2 + \omega d(x_0, x_0)^2).
\]

It is common for the distance \(d\) to be a weighted \(L^2\) distance. Our main goal to use the Wasserstein distance \(W_2\) instead, which seems very interesting when dealing with dense data (see right panel). The Wasserstein cost function writes

\[
\mathcal{J}_W(\rho_0) := \sum_i W_2(\rho_0, \rho_i)^2 + \omega W_2(x_0, x_0)^2.
\]

(2)

**Spectrality on using the Wasserstein distance**

- The Wasserstein distance is only defined for probability measures, i.e. \(\rho\) s.t.
  \[\rho \geq 0\] and \(\int \rho = 1\).
  - Relaxes of the latter constraint are possible, however complex;
    - the \(W_2\) interpolation works well if \(\rho_0 \) and \(\rho_1\) are of distinct support;
    - when \(\mathcal{J}(\rho_0) \rightarrow \min_{\rho_0} \mathcal{J}(\rho_0)\), then there is only weak convergence of \(\rho_0\) to \(\rho^{opt}\) : oscillations or diracs can occur!
  - Computing the Wasserstein distance is expensive [Peyré, Papadakis, Oudet, 2013].

\[\text{Results }\]

- The minimization of \(\mathcal{J}_W\) is performed through a gradient descent, using the Wasserstein gradient, arising from the use of the following Wasserstein scalar product depending on \(\rho_0\),

\[
\int \rho \sqrt{x - \omega} \, dx.
\]

\(\eta, \eta'\) s.t. \(\int \eta = \int \eta' = 0\),

\[
\begin{align*}
\text{Let } \Phi, \Phi' &\text{ s.t. } \int \rho_0 \nabla \Phi = \eta \text{ (with Neumann BC)} \\
\text{and } \int \rho_0 \nabla \Phi' &= \eta'.
\end{align*}
\]

\[
\begin{align*}
\text{Then } \eta, \eta' &\text{ s.t. } \int \rho \nabla \Phi \cdot \nabla \Phi' \, dx.
\end{align*}
\]

\[\text{Values of } h \text{ and } u \text{ for the background and true states, as well as analysis for Euclidean and Wasserstein distances, at time } t = t_0\]