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Abstract: In this paper we study relationships between accelerations and central forces which lead to conics trajectories, that’s Darboux’s forces, with an especially look on their two limiting cases (the Hooke’s and the Newton’s). This work lead us to investigate the equivalence principle and, in a second time, to clarify mathematical form of our own attempt to build a theory of Gravitation. At end we present predictions about experiments which are in progress.

Keywords: Central force; conic; dark matter; force of gravitation; galaxies; modified gravitation; PACS Number: 04.70.Kd

1. Introduction

It is well known that classical theories of physics in the field of Gravitation fail to describe contemporary astronomical observations without add inside the Universe hypothetical matter and energy such “dark matter” or “dark energy”. Among these observations we can list for example the problem of the “flat” curve of rotation of the spiral galaxies [1,2,3]. We can also mention the problem of the expansion at an increasing rate of the Universe [4,5]. Consequently last decades several alternative theories have been built to solve these contradictions [6]. An important theoretical problem of a part of these theories is the question of equivalence principle. Indeed, this principle, which assume (in its weak form) that the ratio of gravitational and inertial mass is always equal, is the base of the classical theories of gravitation. It is the reason for what a part of alternative theories provide that this principle is false (see for example reference [7]). It is also one of the reason for what this principle regularly attracts attention of experimentalists. Indeed despite the extraordinary precision of the validation of this principle (at the part in $10^{13}$ level [8]) regularly other important experiments are a work in progress (for example actually with the spacecraft “Microscope” [9]).

This, then, is the reason why we considered it opportune to investigate relation between our own approach to build a theory of gravitation and this principle. We present in this paper results we obtained and predictions about experiments. Indeed it appears that our model lead to a mathematical relationship between inertial and gravitational masses and consequently becomes predictive.

2. A brief presentation of our model

We begin this paper with a brief look on our theory, which has been for a part previously published or pre-published [10,11,12].

This model is a non-relativistic theory and is based on two fundamental assumptions. First of them can be written: “if a point-particle interacts with a center of force without external disturbance its trajectory is conic (circle, ellipse, parabola or hyperbolae)”.

Second assumption of this model is the following: “force of gravitation is progressively modified when acceleration decreases”. It introduces a constant acceleration which is comparable to the acceleration used in MOND theory [13,14] and which is also a constant of Universe. Moreover model try to respect the corpus of classical physics and allows to conserve two fundamental quantities. These quantities are:

- The energy
The angular momentum

By a way of consequence model assume that force of gravitation is always central and, consistent with this hypothesis, is a part of the Darboux-Halphen’s forces [15]. Indeed these central forces which has been discovered in 1877 allow to obtain conic trajectories and admit two limiting cases, the Newton’s and the Hooke’s.

Moreover our force allows to solve the two body problem. The solving of it led us to investigate the question of equivalence principle.

At end naturally model try to describe astronomical observations we listed before. In particular if acceleration is very low model leads to:

- A flat curve of rotation
- Possibility that the force is repulsive in the cases of parabolic- hyperbolic motion

At the beginning of this paper we present thus accelerations and forces which lead to conic trajectories. In a second part we study the equivalence principle and we clarify the mathematical form of our model.

A Theory

A.1. Determination of the acceleration: generalization of the Binet’s equation

To determine accelerations we use Binet’s equation. Therefore we consider all cases and consequently we generalize it to the tangential component. Our polar system of coordinate is\((F;\mathbf{e}_R;\mathbf{e}_\theta)\) where \(F\) (foci of the conic) is the origin of this system, \(r\) is the radial distance to the origin with the relation

\[
FM = r\mathbf{e}_R
\]

And \(\theta\) the angle measured from the periapsis of the orbit. In this system of coordinate the acceleration is given by the classical relation

\[
\mathbf{a}_C = (\dot{r} - r\dot{\theta}^2)\mathbf{e}_R + (r\ddot{\theta} + 2r\dot{\theta})\mathbf{e}_\theta
\]

But the orbital shape is more concisely described by the reciprocal \(u = \frac{1}{r}\) as a function of \(\theta\).

And by using the relations

\[
\begin{align*}
\dot{r} &= \frac{d}{dt}r = \frac{d}{dt} \frac{1}{u} = -\frac{\dot{u}}{u^2}, \\
\dot{u} &= \frac{d}{dt}u = \frac{d\theta}{dt} \frac{du}{d\theta} = \dot{\theta}u', \\
\ddot{u} &= \frac{d}{dt}u' = \frac{d\theta}{dt} \frac{d}{dt} \frac{1}{u} = \dot{\theta}u''
\end{align*}
\]

We obtain a generalization of the Binet’s equation.

\[
\mathbf{a}_C = -\frac{u'u^3\dot{\theta}^2 - u^3\dot{\theta}^2 + 2u^2u\dot{\theta}^2 - u'u^2\dot{\theta}^2}{u^4}\mathbf{e}_R + \frac{2u'u^2\dot{\theta}^2 + u^3\dot{\theta}^2}{u^4}\mathbf{e}_\theta
\]

By noticing that this equation can be written

\[
\mathbf{a}_C = \left[ -\frac{u'u^2\dot{\theta}^2 - u^3\dot{\theta}^2}{u^4} + \frac{u' 2u'u^2\dot{\theta}^2 - u^3\dot{\theta}^2}{u^4}\right]e_R + \left[ -\frac{2u'u^2\dot{\theta}^2 + u^3\dot{\theta}^2}{u^4}\right]e_\theta
\]

We introduce two functions given by
\[ Y(u) = -\frac{u'' u^2 \dot{\theta}^2 - u^3 \ddot{\theta}^2}{u^4} \]

And
\[ Z(u) = -\frac{2u' u^2 \dot{\theta}^2 + u^3 \ddot{\theta}}{u^4} \]

And the acceleration becomes
\[ \ddot{a}_c = \left[ Y(u) - \frac{u'}{u} Z(u) \right] \vec{e}_r + Z(u) \vec{e}_\theta \]

We can now write the system of equation
\[
\begin{align*}
    \dot{a}_r &= Y(u) - \frac{u'}{u} Z(u) \\
    a_\theta &= Z(u)
\end{align*}
\]

We introduce a new function \( f(u) \) definite by
\[ Y(u) = -A f(u) \]

Where \( A \) is constant. To obtain \( r(\dot{\theta}) \) as a conic, we have to solve a differential equation as
\[ u'' + u = B \]

Where \( B \) is a second constant. Consequently we have now to introduce a relation between \( Y(u) \) and \( \dot{\theta} \).

This relation is
\[ \dot{\theta} = Cu \sqrt{f(u)} \quad (1) \]

Where \( C \) is a constant of the motion. Indeed with this relation we obtain
\[ u'' \dot{\theta}^2 - u \dot{\theta}^2 = -Au^2 f(u) \]

And
\[ u'' + u = B = \frac{A}{C^2} \]

This differential equation leads now to the classical solution
\[ r(\dot{\theta}) = \frac{p}{1 + e \cos \dot{\theta}} \]

The parameter \( p \) of the conic is
\[ p = a(1 - e^2) = \frac{C^2}{A} \]

Where \( e \) is the eccentricity and \( a \) the semi major axis. Thus we obtain
\[ C = \sqrt{A} \sqrt{a(1 - e^2)} \]

We have now to determine the tangential component of the acceleration and by using
\[ \ddot{\theta} = \frac{C \dot{r}}{r \sqrt{f}} \left[ \frac{1}{2} f' r - f \right] \]

Where
\[ f' = \frac{d}{dr} f(r) \]

We obtain
\[ a_\theta = r \ddot{\theta} + 2 \dot{r} \dot{\theta} = \frac{1}{2} \frac{C \ddot{r}}{\sqrt{f}} + C \frac{\dot{r}}{r} \sqrt{f} \]
And
\[
\ddot{a}_c = -Af \ddot{e}_r + \dot{r}^2 \left[ \frac{1}{2} f' + \frac{1}{r} \right] \dot{e}_r + Cr \sqrt{f} \left[ \frac{1}{2} f' + \frac{1}{r} \right] \dot{e}_\theta
\]

Or, more simply
\[
\ddot{a}_c = -Af \ddot{e}_r + \dot{r}^2 \left[ \frac{1}{2} f' + \frac{1}{r} \right] \dot{e}_r + r \frac{d}{d\theta} \left[ \frac{1}{2} f' + \frac{1}{r} \right] \dot{e}_\theta
\]

Where \( A \) and \( C \) are two constants. Their physical dimensions depend on the choice of \( f(r) \).

A.2. Central force

In previously part we determine a family of acceleration which lead to conic trajectories. In this one we link theses accelerations to forces by the Newton’s second law of motion given by
\[
\ddot{F} = \frac{d}{dt} m_V \ddot{V} \quad (3)
\]

Where \( m_V \) is the inertial mass and \( \ddot{V} \) the speed. If we consider the general case relation (3) becomes
\[
\ddot{F} = m_V \dddot{V} + m_V \ddot{a}_c \quad (4)
\]

Consequently we obtain a system of two equations
\[
\begin{align*}
F_r &= m_V \ddot{r} + m_V a_{CR} \\
F_\theta &= m_V \ddot{\theta} + m_V a_{C\theta}
\end{align*}
\]

By noticing that
\[
\ddot{m}_V = \frac{d}{dt} m_V = \frac{d}{dt} \frac{dm}{dt} = \dot{m} m'
\]

This system becomes
\[
\begin{align*}
F_r &= m'_V \dddot{r} + m_V a_{CR} \\
F_\theta &= m_V \dddot{\theta} + m_V a_{C\theta}
\end{align*} \quad (5)
\]

Where inertial mass is a function of \( r \) \( m_V = m_V(r) \). Note that acceleration given by equation (2) isn’t necessary central and consequently that vectors of accelerations and forces aren’t necessary parallel. Therefore in agreement with our initial assumption we are looking for a central force. To obtain it we introduce an unmovable point (we call it \( I \)) which is located between \( O \) and \( F \) (Figure 1).
We call \( \Delta \) the distance \( FI \). If the force is directed to \( I \) then the vector product

\[
\mathbf{IM} \times \mathbf{F} = \mathbf{0}
\]

By noting that, in our system of coordinate \( F e_{\rho} e_{\theta} \) the vector \( \mathbf{IM} \) is given by

\[
\mathbf{IM} = (\Delta \cos \theta + r \mathbf{e}_x - \Delta \sin \theta \mathbf{e}_y)
\]

We obtain

\[
\Delta \sin \theta \mathbf{F}_R + (\Delta \cos \theta + r) \mathbf{F}_\phi = 0 \tag{6}
\]

With the specific relationships to the conics

\[
r = \frac{a(1 - e^2)}{1 + e \cos \theta} \quad \text{and} \quad \dot{r} = \frac{e \sin \theta}{a(1 - e^2)} \sqrt{f} \tag{7}
\]

And by using relations (5) and (6) we obtain

\[
\Delta = \frac{ae(2f m_1 r' + m_1 f'' + 2m_1 f)}{f' m_1 (r - a) + 2f m' (r - a) + 2m_1 f}
\]

To obtain a central force we are looking for the family of functions \( f(r) \) and \( m_1(r) \) which leads to \( \Delta \) as a constant. Consequently we write the equation

\[
\frac{d}{dr} \Delta = 0
\]

This relation leads to

Figure 1. Representation of the central force
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Alternative theory of gravitation: implication on equivalence principle

\[ f(r) = \frac{1}{m_i^2(r)(C'_1 r + C'_2)^2} \] (8)

Where \( C'_1 \) and \( C'_2 \) are two constant (notations of X’ will be used in all the paper for accelerations, X for forces). It is really interesting to note that \( \Delta \) is independent from \( m_i(r) \) simply given by

\[ \Delta = \frac{eaC'_2}{aC'_1 + C'_2} \] (9)

We can now obtain the mathematical expression of our force. This one is given by introducing relation (8) in equations (2) and (5)

\[ \vec{F} = A' \frac{ar(C'_2 - rC'_1) - C'_2(a^2(1-e^2) + r^2)}{m_i(r)(C'_1 r + C'_2)^3} \hat{e}_r + A' \frac{eC'_2}{m_i(r)(C'_1 r + C'_2)^3} \sin \theta \hat{e}_\theta \]

Its magnitude is

\[ F = \frac{A' \sqrt{2ac'_1c'_2(-ar + r^2 + a^2(1-e^2)) + c'_2^2(2a^2 - e^2a^2 + r^2 - 2ar) + c'_1^2(a^2r^2)}}{a}\]

By noticing that the distance \( IM = R \) is given by

\[ R = \sqrt{\left(\Delta + r \cos \theta \right)^2 + \left( r \sin \theta \right)^2} = \sqrt{\Delta^2 + r^2 + 2r \cos \theta} \]

After simplification

\[ R = \frac{\sqrt{2ac'_1c'_2(-ar + r^2 + a^2(1-e^2)) + c'_2^2(2a^2 - e^2a^2 + r^2 - 2ar) + c'_1^2(a^2r^2)}}{(C'_1 a + C'_2)} \]

We obtain

\[ F = \frac{A' \frac{1}{a} \frac{c'_1 a + c'_2}{m_i(r)(C'_1 r + C'_2)^3} R} \] (10)

A.3. Darboux’s force and equivalence principle

As we presented it in introduction mathematician Halphen and Darboux published simultaneous in 1877 a famous paper where they presented a family of central force which lead to conics trajectories. At least two other demonstration has been published later [16,17]. The second part of this family can be written

\[ F = \frac{\mu}{R^2 \left( \frac{1}{R} - a_1 \cos w - b_1 \sin w \right)^3} \]

Where \( a_1, b_1, \mu \) are three constants and \( w \) the angle of revolution given by

\[ w = MIF \]

We showed elsewhere [12] that, when the center of force is located between the foci and the center of the conic this force is equal to the force given by equation

\[ F = \frac{A \frac{c'_1 a + c'_2}{(C'_1 r + C'_2)^2}}{a} R \] (11)

Where the center of force is given by

\[ \Delta = IF = \frac{eaC'_2}{aC'_1 + C'_2} \]

Where \( C'_1 \) and \( C'_2 \) are two constant (not necessary equal to \( C'_1 \) and \( C'_2 \)). We can thus write the equality of the expressions (10) and (11)
After simplification we obtain

\[
\frac{A'}{A} \frac{1}{m_f(r)} \frac{C'_1 a + C'_2}{(C'_1 r + C'_2)^3} R = \frac{A}{a} \frac{C_i a + C_2}{(C_i r + C_2)^3} R
\]

We can also write the equality of \( \Delta \)

\[
\frac{eaC_2}{aC_1 + C_2} = \frac{eaC'_2}{aC'_1 + C'_2}
\]

This equation becomes

\[
\frac{C_2}{C_1} = \frac{C'_2}{C'_1}
\]

We solve now the system of equations given by \{12,13\}. Results is

\[
m_f(r) = \frac{A'}{A} \frac{C'_2}{C'_1 R_2^2}
\]

Consequently \( m_f(r) \) is constant. We introduce now the ratio of mass \( \eta \) given by

\[
\eta = \frac{m_g}{m_f}
\]

\( (m_g \text{ is gravitational mass}) \). This ratio is constant for a given force of Darboux (for example the Newton’s or the Hooke’s) and on all a conic trajectory.

Moreover in our model this force is a force of gravitation. Consequently this one can be written

\[
F = \frac{GM_G m_f}{a} \frac{C'_1 a + C'_2}{(C'_1 r + C'_2)^3} R = \frac{GM_G m_g}{a} \frac{C_i a + C_2}{(C_i r + C_2)^3} R
\]

Where \( G \) is the constant of gravity and \( M_g \) the gravitational mass of the center of force.

**A.4. Coefficient of Darboux’s force**

Force is thus given by

\[
F = \frac{GM_G m_g}{a} \frac{C_i a + C_2}{(C_i r + C_2)^3} R
\]

Where \( C_i \) and \( C_2 \) are two constant. This force is directed to \( I \) definite by

\[
\Delta = IF = \frac{eaC_2}{aC_1 + C_2}
\]

We will use this force to build our theory of gravitation and consequently we will add drastic conditions on its coefficient, in agreement with a correct physical point of view. To do it as usual in physics we study the limiting cases. These cases are three:

Firstly If \( \Delta = 0 \) the force is naturally the Newton’s. The coefficients are given in this case by

\( C_i = 1 \) and \( C_2 = 0 \)
Secondly if $\Delta = ea$ the force is the Hooke’s and coefficient $C_1$ is given by

$$C_1 = 0$$

Thirsty if $\Delta = 2ea$ the center of force is located at the second foci of the conic. Consequently force is also the Newton’s. We write thus the equation

$$2ea = \frac{eaC_2}{aC_1 + C_2}$$

And obtain

$$C_2 = -2C_1a$$

With

$$C_1 = \pm 1$$

We can also have consider the possibility to solve the two body-problem. Indeed this solving add another drastic conditions on the forces. We write thus that, if two bodies $M_1$ and $M_2$ interacts only with each other’s the sum of these forces has to be equal to zero and the two ellipses have to be homothetic (Figure 2.).

![Figure 2. Representation of the two bodies-problem](image)

To respect laws of dynamics it appears that each terms of the expression of the force has to be linked to inertial mass. Classical relations are

$$a_1m_{11} = a_2m_{12}$$
$$r_1m_{11} = r_2m_{12}$$

And consequently

$$R_1m_{11} = R_2m_{12}$$

This indicates that we won’t solve two body problem if we don’t consider relationship
$C_{21}m_{i1} = C_{22}m_{i2}$

And we deduce

$$\frac{C_{21}}{a_i} = \frac{C_{22}}{a_2}$$

Consequently to these observations we write

$$C_{21} = a_1(1 - C_{11})$$
$$C_{22} = a_1(1 - C_{12})$$

With

$$C_{11} = C_{12}$$

We will use this property at the end of the paper. General relationship is consequently

$$C_2 = a(1 - C_1)$$  \hspace{1cm} (15)

We verify our initial conditions

If $C_1 = 1$ then $C_2 = 0$ and $\Delta = 0$
If $C_1 = 0$ then $C_2 = a$ and $\Delta = ea$
If $C_1 = -1$ then $C_2 = 2a$ and $\Delta = 2ea$

It is interesting to note that the force admit now a great simplification

$$F = \frac{G M_G m_G}{(C_1(r - a) + a)^3} R$$  \hspace{1cm} (16)

And distance $\Delta$ becomes simply

$$\Delta = ae(1 - C_1)$$  \hspace{1cm} (17)

B. Theory of gravitation

It is well known that the speed of the stars at the periphery of one spiral galaxy is constant and well described by the relation

$$V = \left[GM_G a_0 \right]^{1/4}$$  \hspace{1cm} (18)

Where $M_G$ is the visible mass of the galaxy and $a_0$ the Millgrom’s acceleration (around $1.2 \times 10^{-10}ms^{-2}$). Note that expression (17) has been often verified, see for example references [7,13,14] and is in agreement with Tully-Fisher law [18]. In this part we study if our model is compatible with this relation and we concluded about the principle of equivalence we expect.

To begin this study we introduce a constant $r_0$ given by

$$r_0 = \sqrt[3]{\frac{GM_G}{a_0}}$$  \hspace{1cm} (19)

We have two limiting cases: Firstly, when the force is the Newton’s we have relations

$$r_0 \gg a \text{ and } \eta \rightarrow 1$$

Secondly, when the force is the Hooke’s we have the conditions

$$a \gg r_0$$

We can obtain information about $C_2'$ in this case. To do it we study as usual the circular motion [7,13,14]. Indeed the acceleration is issue from equation (14)
\[ a_c = \frac{GM_G}{a} \left( \frac{C'_1 a + C'_2}{(C'_1 r + C'_2)^3} \right) R \]

We write in this case the equality between accelerations for circular motion (note that here \( C'_1 = 0 \) and \( r = R = a \)). Thus

\[ a_c = \frac{GM_G}{C'_2} = \frac{V^2}{a} \]

And by using (18) and (19) we obtain

\[ C'_2 = \sqrt{ar_0} \]

Consequently to this result we assume now a relationship between \( C'_1 \) and \( C'_2 \) comparable to (15)

\[ C'_2 = \sqrt{ar_0} (1 - C'_1) \]

We obtain here a problem because with this relation we can’t solve the two body problem. Indeed \( C'_2 \) is for a part linked to \( r_0 \) given by (19) and consequently isn’t entirely linked to the semi major axis (i.e. the inertial mass, as presented in part A4). This indicates that we have to investigate the equivalence principle.

Consequently we study now the general case. It appears that we can write at least two equations to build a system: the first of them is obtained by considering that the two expressions of the force are directed toward the same point \( I \). By using (16) and (9) we write thus

\[ \Delta = aC_1 = \frac{eaC'_2}{aC'_1 + C'_2} \]

And

\[ (1 - C_1) = \frac{r_0 (1 - C'_1)}{C'_1 (a - r_0) + r_0} \quad (20) \]

The second equation is obtained as previously by written the equality of the two expressions of the force

\[ \frac{GM_G m_i ((C'_1 a + C'_2) - a r + C'_2)^3}{a (C'_1 r + C'_2)^3} R = \frac{GM_G m_G a}{a (C'_1 (r - a) + a)^3} R \]

Consequently

\[ \frac{(C'_1 a + C'_2)}{\eta (C'_1 r + C'_2)^3} = \frac{1}{(C'_1 (r - a) + a)^3} \]

And

\[ \frac{1}{\eta} \frac{(a - \sqrt{ar_0}) + \sqrt{ar_0}}{a (C'_1 (r - \sqrt{ar_0}) + \sqrt{ar_0})^3} = \frac{1}{(C'_1 (r - a) + a)^3} \quad (21) \]

Solving of system given by equations (20) and (21) leads to the relations

\[ \eta = \left[ \frac{C'_1}{C'_1} \right]^2 \]

\[ C'_1 = \frac{C'_1 \sqrt{ar_0}}{C'_1 (\sqrt{r_0} - \sqrt{a}) + \sqrt{a}} \]
Note that we verify here that \( \eta \) is well constant on all the trajectory.

We suggest the solution given by

\[
C_1 = \frac{r_0^2}{a^2 + r_0^2}
\]

Indeed we deduce following relations

\[
\begin{align*}
C_1 &= \frac{r_0^2}{a \sqrt{ar_0 + r_0^2}} \\
C_1' &= \frac{r_0^2}{a^2 + r_0^2} \\
\eta &= \left[ \frac{a^2 + r_0^2}{a \sqrt{ar_0 + r_0^2}} \right]^2
\end{align*}
\]

Indeed it appears that these coefficients are now linked with simple relations and that all limiting conditions are respected. For example if \( r_0 \to \infty \) (case of solar system) \( C_1 \to 1 \) and the force is well the Newton’s. If \( C_1 \to 0 \) force becomes the Hooke’s and

\[
\eta = \frac{a}{r_0}
\]

Note that a proximate relation has been suggested [7] with

\[
\eta = \sqrt{g + g_0} = \sqrt{\frac{r_0^2 + a^2}{r_0^2}}
\]

We can give now several properties of the force we obtained.

**B.2. Properties of the force**

Magnitude of the force becomes now

\[
F = \frac{GM_c m_G}{r_0^2} R = \frac{GM_c m_G}{a \sqrt{ar_0 + r_0^2}} \left( \frac{r_0^2}{a \sqrt{ar_0 + r_0^2}} \right)^3 \left( r - a + a \right)
\]

This force is located at I definite by

\[
\Delta = IF = ea(1 - \frac{r_0^2}{a \sqrt{ar_0 + r_0^2}}) = ea - \frac{a \sqrt{ar_0 + r_0^2}}{a \sqrt{ar_0 + r_0^2}}
\]

We verify that our limiting cases are respected

\[0 \leq \Delta \leq ea\]

**B.2.1. Angular momentum**

The angular momentum is the cross product of the particle’s position vector and its momentum vector

\[
\vec{p} = m_i \vec{V} = m_i (\dot{r} \vec{e}_r + r \dot{\theta} \vec{e}_\theta)
\]

at \( F \), foci of the conic we obtain

\[
\vec{L}_F = r \vec{e}_r \times (\dot{r} \vec{e}_r + r \dot{\theta} \vec{e}_\theta) = r^2 \dot{\theta} \vec{e}_Z
\]

And by using

11
\[ \dot{\theta} = \frac{C}{r \sqrt{f(r)}} = \frac{1}{r C_1 r + C_2} \]  

we obtain

\[ \vec{L}_r = \frac{Cr}{C_1 r + C_2} \vec{e}_z \]

The angular momentum at \( I \) is given by

\[ \vec{L}_I = \vec{L}_r + I \vec{F} * (\vec{r} \vec{e}_r + r \dot{\theta} \vec{e}_\theta) \]

By noting that

\[ I \vec{F} = \Delta (\cos \theta \vec{e}_r - \sin \theta \vec{e}_\theta) \]

\( \vec{L}_I \) becomes

\[ \vec{L}_I = \vec{L}_r + \Delta [r \dot{\theta} \cos \theta + i \sin \theta] \vec{e}_z \]

And

\[ \vec{L}_I = \frac{Cr}{C_1 r + C_2} + \frac{aC_2 e}{C'_2 + aC'_1} \left[ r \dot{\theta} \cos \theta + i \sin \theta \right] \vec{e}_z \]

By using the specific relationships to the conics (7) we obtain

\[ \vec{L}_I = \frac{aC}{C'_2 + aC'_1} \vec{e}_z \text{ (m}^2 \text{s}^{-1}) \]

This vector is constant. Angular momentum becomes simply

\[ \vec{L}_I = \frac{aC}{r_0^2 (a^2 + r_0^2)} \left( a^2 + r_0^2 \right) \frac{\sqrt{aGM_G a(1 - e^2)}}{\sqrt{a r_0 a^2 + ar_0^2}} \]

\[ = \frac{C_1}{C'} \left( a^2 + r_0^2 \right) \frac{\sqrt{aGM_G (1 - e^2)}}{\sqrt{a^2 + r_0^2}} \vec{e}_z = \frac{C_1}{C'} \sqrt{aGM_G (1 - e^2)} \vec{e}_z \]

**B.2.2. Equation of time**

As usual, we write this equation by using the eccentric anomaly \( E \) (Figure 2). Indeed (in the case of an ellipse) we have the relation

\[ \dot{\theta} = \dot{E} \frac{\sqrt{1 - e^2}}{1 - e \cos E} \]

By using equation (23)

\[ \dot{E} \frac{\sqrt{1 - e^2}}{1 - e \cos E} = \frac{C}{r} \frac{1}{C_1 r + C_2} \]

And by using

\[ r = a(1 - e \cos E) \]

We obtain

\[ \dot{E} [C_1 a (1 - e \cos E) + C'_2] = \frac{C}{a \sqrt{1 - e^2}} = \frac{\sqrt{A}}{\sqrt{a}} \]

By an integration
Consequently if trajectory is bounded we obtain

\[
E(C_1' a + C_2') - eC_1' a SinE = \frac{\sqrt{A}}{\sqrt{a}}
\]

\[
(C_1' a + C_2') \left[ E - e \frac{C_1' a}{C_1' a + C_2'} SinE \right] = \frac{\sqrt{A}}{\sqrt{a}}
\]

And

\[
E - e \frac{C_1' a}{C_1' a + C_2'} SinE = \frac{1}{(C_1' a + C_2')} \frac{\sqrt{A}}{\sqrt{a}} = mt
\]

Thus the mean motion is given by

\[
m = \frac{1}{(C_1' a + C_2')} \frac{\sqrt{A}}{\sqrt{a}}
\]

If the force is the Newton’s we obtain the equation of time of Kepler

\[
E - eSinE = \frac{\sqrt{A}}{\sqrt{a}} t
\]

And if the force is the Hooke’s we obtain simply

\[
E = \frac{\sqrt{A}}{\sqrt{aC_2'^2}} t
\]

These results indicates that we have well a time invariance. In agreement with Noether’s theorem Energy is conserved.

**B.2.3. Can this force be conservative?**

In agreement with Bertrand’s theorem [19,20] we know that it exists only two central and conservative forces which lead to closed trajectories, the Newton’s and the Hooke’s. (Trajectories are naturally in the two cases conic).

Consequently the force we study here isn’t conservative except in this two limiting cases.

**B.3. Predictions**

We study in this part several predictions of our model.

**B.3.1 Curve of rotation of galaxies**

To obtain this curve we consider as usual the circular motion and we write in this case the fundamental equality of dynamics

\[
m_t V^2 = \frac{GM_G m_G}{(C_1 (r - a) + a)^3} R
\]

We have the relationships \( r = a = R \). Equation becomes thus

\[
V^2 = \frac{GM_G}{a} \frac{m_G}{m_t} = \frac{GM_G}{a} \left[ \frac{a^2 + r_0^2}{a \sqrt{ar_0 + r_0^2}} \right]^2
\]

And
We can distinguish two limiting cases:

If \( r_0 \gg a \) the speed becomes the Newton’s given by

\[
V = \sqrt{\frac{GM_G}{a}}
\]

If \( r_0 \ll a \) this speed becomes

\[
V = \sqrt{\frac{GM_G}{r_0}} = \left[ GM_G a_0 \right]^{1/4}
\]

And is in agreement with Tully-Fisher law.

A test for our model should to verify if the relation given by equation (24) allows to describe real curve of rotations. We have reasons to think that it could be the case. Indeed, for example, in his famous theory MOND M.Millgrom tested several functions which are proximate [13,14].

### B.3.2. Expansion of the Universe

A second test should be to see if our model could explain this expansion. Indeed the force we obtained is

\[
F = GM_G m_G \left[ \frac{a \sqrt{a r_0 + r_0^2}}{a^2 \sqrt{a r_0 + r_r^2}} \right]^3 R
\]

If trajectory is bounded this force is naturally always attractive. But if trajectory is parabolic or hyperbolic, this one can in certain cases becomes repulsive, especially when the force is proximate to the Hooke’s. This indicates that we can perhaps use this force to describe the expansion of the Universe at an increasing rate with a classical point of view.

In another paper we discuss this possibility by a comparison with Friedman’s equation [12]. Indeed in this equation Friedman suggested that a harmonic potential could be added to the Einstein’s equation.

### B.3.3. Tests around the principle of equivalence

In our model the ratio of gravitational/ inertial mass is constant on all a trajectory and given by

\[
\eta = \left[ \frac{a^2 + r_0^2}{a \sqrt{a r_0 + r_0^2}} \right]^2
\]

Moreover we assume previously that constant \( r_0 \) was given by relation

\[
r_0 = \sqrt{\frac{GM_G}{a_0}}
\]

It is right that this empirical relation (which is a consequence of Tully-Fisher law) seems correct for galaxies. But we don’t know if it is the case for small masses (as solar system). Moreover this constant is likely depending on the two masses which interact each other. Indeed this condition is necessary to solve the two body problem. For example relation could be

\[
r_0 = \sqrt{\frac{G(M_G + m_G)}{a_0}}
\]
By a way of consequence a difference should be measured between two bodies test which follow same trajectories. Therefore if we consider the ratio of mass between Earth and these test bodies this difference should be very small. For example mass of test bodies inside satellite Microscope [9] are around 0.5 and 1.5 Kg and Earth around $10^{23}$ Kg… Difference of $r_0$ and consequently of $\eta$ should be in these conditions really small and likely undetectable.

**B.3.4. A possible test inside solar system?**
In our model a more pertinent test should be an experiment around the parabolic or hyperbolic motion, for example with a spacecraft. Indeed we should detect (if our model is correct) a small variation around the Newton’s law, especially if the distance to the Sun is important.

This prediction is the consequence of the expression of acceleration obtained with

$$f(r) = \frac{1}{(C_1 + C_2 r)^2}$$

For example, in the case of parabolic motion, we have obtained with a series when $r_0 \rightarrow \infty$ the acceleration given by

$$\vec{a}_c = (-\frac{GM}{r^2} + \frac{GM}{r^3} \left[ \frac{a}{r_0} \right]^{3/2} [4a - r]) \vec{e}_r$$

A second test should be about trajectories of planes. Indeed model predicts that the center of Sun isn’t exactly located at the foci of the ellipse. Perhaps it is possible to detect a small difference.

**C. Discussion: a way to build a relativistic theory?**
We discuss in this part two weak points of our model:

Firstly, the lack of experiment. Therefore we can argue that it is the case of majorities of actual alternative theories of gravitation. Indeed experiments seem very difficult when accelerations are so small [21]. Moreover dominant theory (Dark matter/ Energy) seems have no more experimental proof [22].

Secondly this theory is classical and consequently can only be an approximation of reality. It is important that it becomes a day a limiting case of a relativistic theory. In our opinion, this theory (if it doesn’t exist today) should be based for a part on the expression of the principle of equivalence we presented in this paper.

**Conclusion**
In this paper I present a classical model as an alternative to dark matter and dark energy. I used the Darboux’s forces and I studied the principle of equivalence. Conclusion is that this principle is respected on all a conic trajectory but ratio of masses can be different between two trajectories. Moreover model respect conservation of Energy and of Angular Momentum, and seems in agreement with corpus of classical physics. It seems to describe correctly flat curve of rotation of galaxies and perhaps expansion of Universe.

**REFERENCES**

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