HISTORY-BASED PROBLEMS IN THE TEACHING OF SENIOR HIGH SCHOOL

Mathematics in Mainland China

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ABSTRACT

This paper deals with the integration of history in mathematics teaching. Four possible approaches are considered: complementation, replication, accommodation, reconstruction. The main focus is on the relation between. mathematical problems posed on the basis of historical information (history-based problems) and mathematics teaching from the HPM perspective. In this paper 20 HPM lessons are analysed.

1 Introduction

Integrating the history of mathematics into mathematics teaching is one of the important research fields in HPM. As shown in table 1, four ways are identified in which historical information is used in classroom teaching in Mainland China.

Table 1. Four Approaches to Using History in Mathematics Teaching

Approaches	*Description*
Complementation	Pictures, stories, etc.
Replication	Original problems, methods, etc. in the history, copied without any adaption
Accommodation	Problems adapted from original ones or based on historical information
Reconstruction	The evolution of a subject (e.g. a concept or theory) inspired by the history of mathematics

In replication and accommodation approaches, most of historical materials used are mathematical problems. In the reconstruction approach, the reconstructed steps of evolution of a subject “are given as sequences of historically motivated problems of an increasing level of difficulty” (Tzanakis & Arcavi, 2000). Therefore, mathematical problem posing is involved in all approaches except the complementation one. In this paper, mathematical problems posed on the basis of historical information are called “history-based problems”. The relation between history-based problems and mathematics teaching from the HPM perspective is shown in Figure 1.

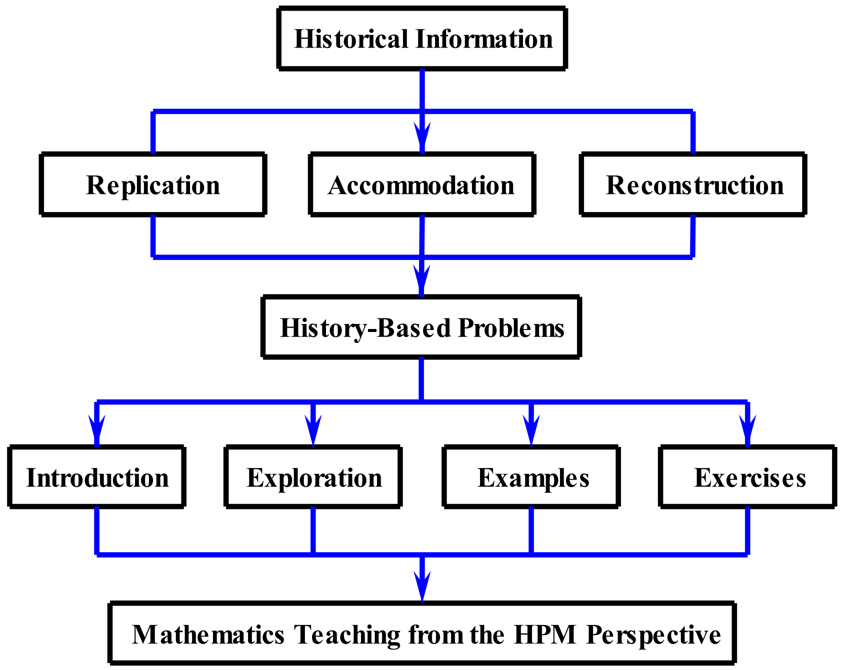

Figure 1. History-based problems and mathematics teaching from the HPM perspective

In mainland China, researches in HPM have focused on teaching practice and mathematics teachers' professional development and many lessons in which the history of mathematics is integrated (hereafter, abbreviated as **HPM lessons**) have been developed in recent years. Which history-based problems are used in these lessons? How are they posed? What roles do they play in these cases? How are they distributed at different stages of the lessons? In this paper, we will answer these research questions.

2 Selection of the HPM Lessons

A total of 20 HPM lessons (Chen, 2015; Chen, 2015; Chen & Wang, 2012; Fang & Wang, 2013; Guo & Wang, 2015; Jin & Wang, 2014; Li, 2015; Li & Wang, 2016; Li & Wang, 2016b; Shi, 2016; Wang & Wang, 2012); Wang & Wang, 2014; Wang & Zhang, 2006; Xu, 2015; Yang, 2016; Zhang, 2007; Zhang, 2012; Zhang & Wang, 2007; Zhang & Wang, 2015; Zhong & Wang, 2015; Zhong & Wang, 2016), published from 2007 to 2016, are selected and examined, as shown in Figure 2.

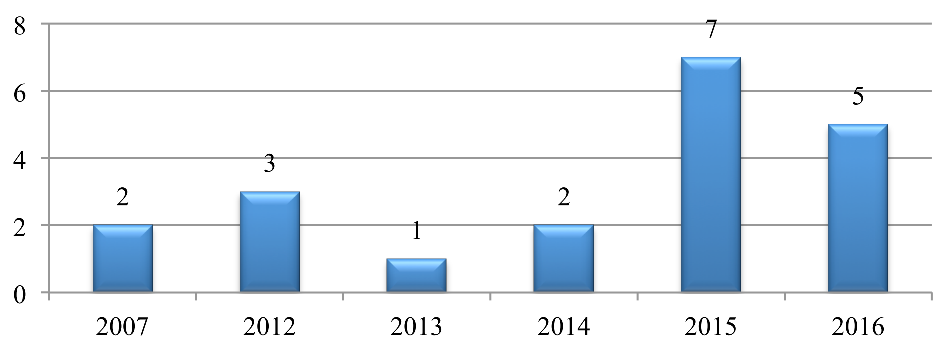

Figure 2. 20 HPM Lessons from 2007 to 2016

The subjects of the 20 instructional cases belong to algebra, geometry, trigonometry, analytical geometry and calculus respectively. Figure 3 shows their distribution in different areas.

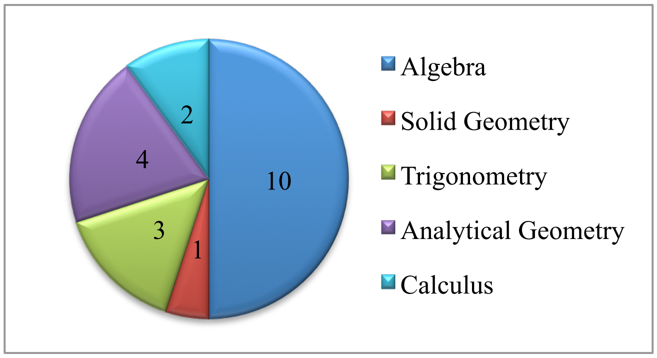

Figure 3. Subject Distribution of 20 HPM Lessons

All these HPM lessons are developed by senior high school mathematics teachers in cooperation with the university researchers, as shown in figure 4.

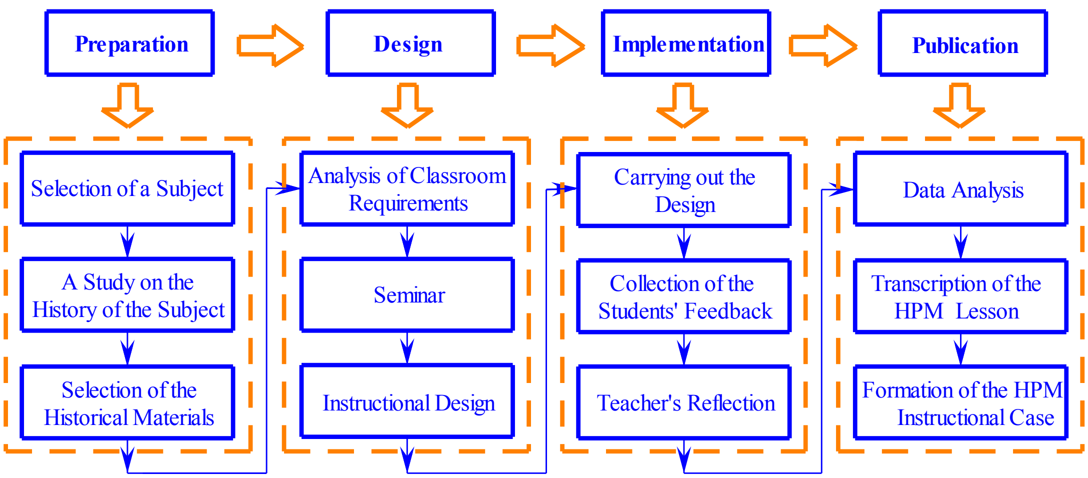

Figure 4. Development of the HPM Lessons

Figure 5 shows the frequency of each approach adopted in each lesson and Figure 6 shows the number of lessons in which different approaches are adopted.

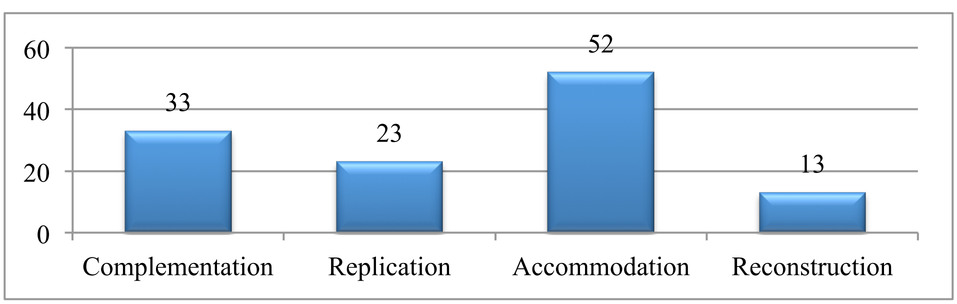

Figure 5. The Frequency of Each Approach Adopted in Each Lesson

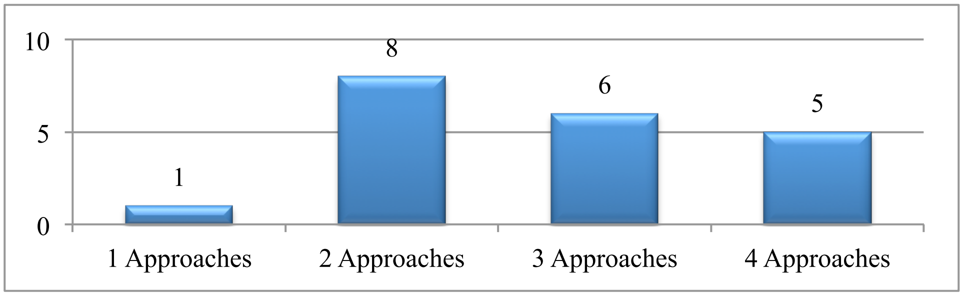

Figure 6. Different approaches adopted in the same lesson

The reconstruction approach is mainly adopted in lessons on mathematics concepts. While designing such a lesson, the teacher need have a basic knowledge of the historical evolution of the relevant concept. In one and the same lesson, this approach is used only one time. Thus we see that it appears less frequently. History-based problems, most of which are adapted from historical information, are indispensable for most HPM lessons and can be used at different stages of the lessons. Therefore, the accommodation approach appears most frequently. Historical resources are limited for mathematics teachers and most of historical materials available to them are unsuitable for direct use and need to be adapted. This is the reason why the replication approach appears less frequently than the accommodation one.

Figure 6 shows that two approaches are adopted in 8 lessons, three in 6 lessons, four in 5 lessons, and one in only one lesson.

3 History-based problem posing

Silver et al (1996) identified four strategies of posing new problems in terms of the given situations or problems: constraint manipulation (i.e., systematic manipulation of the task conditions or implicit assumptions), goal manipulation (i.e., manipulation of the goal of a given or previously posed problem where the assumptions of the problem are accepted with no change), symmetry (a symmetrical exchange between the existing problem's goal and conditions), chaining (i.e., expanding an existing problem so that a solution to the new problem would require solving the existing one first). According to these strategies, combined with the approaches to using history in teaching, six strategies of posing history-based problems are identified: copying, free posing, situation manipulation, constraint manipulation, goal manipulation, symmetry and chaining.

The copying strategy is that of directly using original problems without any adaption, corresponding to the replication approach to using history, the used historical materials being mathematics problems. Free posing strategy is that of posing problems according to some historical information (stories, problems, methods, etc.) in which the conditions and goals are selected according to classroom requirements. The situation manipulation strategy is that of adapting the situation of a problem in the history or adding real situation to the existing historical problem which is familiar to students, keeping the given conditions and goals unchanged. Manipulation of the constraints or goals of the problems in the history is called constraint or goal manipulation strategy. The case in which both situations and conditions are changed is classified among the constraint manipulation strategy and that in which both situations and goals are changed is classified among the goal manipulation strategy. Symmetrical exchange between conditions and goals of problems in the history is called symmetry strategy. Taking the goals of the original problems as the conditions to pose new problem is called the chaining strategy.

Figure 7 shows the relation between the strategies of posing history-based problems and approaches to using historical materials in teaching.

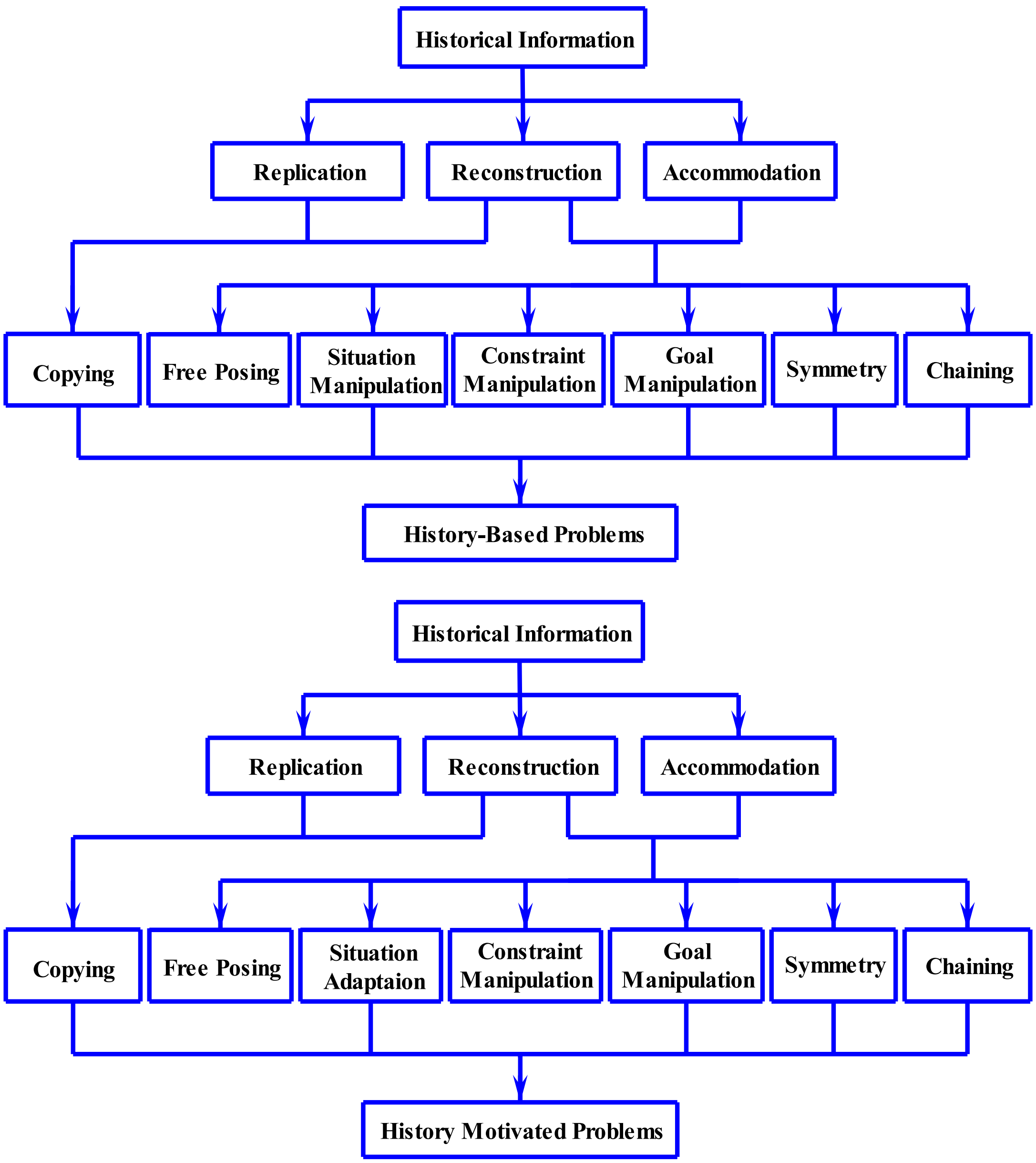

Figure 7. Strategies of posing problems based on historical information

4 History-based problems in 20 Lessons

In all 20 HPM lessons, a total of 63 history-based problems are found. Figure 8 is the frequency histogram of each type of problems. It is revealed from Figure 8 that the number of freely posed problems (FP problems) far exceed other types, followed by the copied original problems (CO problem). Only one problem posed by means of the goal manipulation strategy (GM problem) is found, and none of the problems posed by means of the symmetry or situation adaptation strategy appear. 9 problems posed by means of the constraint manipulation strategy (CM problem) appear only in four lessons, and 9 problems of the chaining strategy (CH problem) appear only in two lessons. On the one hand, few of original historical materials available to teachers are suitable for direct use in classroom teaching; on the other hand, it is easier for a teacher to pose a new problem freely on the basis of historical information than to pose a problem by means of constraint or goal manipulation.

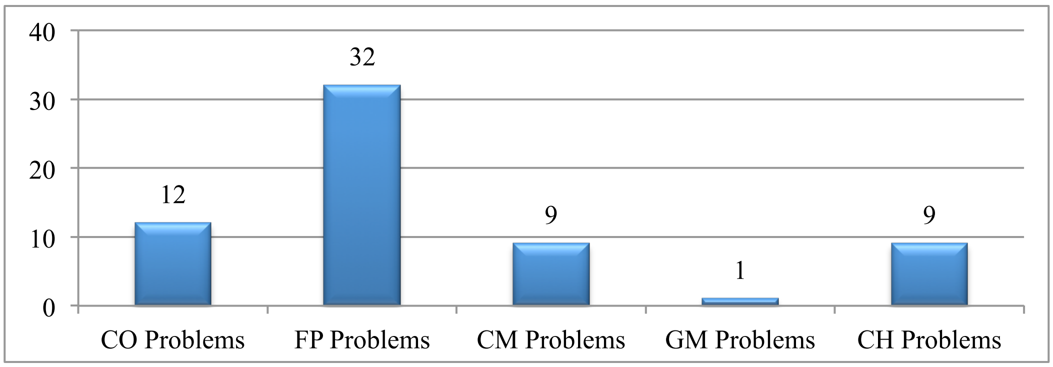

Figure 8. Various types of history-based problems

4.1 The CO Problems

Table 2 shows the detailed information of some CO problems.

The CO problems are used in 10 HPM lessons. In the lesson "The Roots of an Equation & the Zeroes of a Function", the problem of solving the cubic equation $x^3+2x^2+10x=20$, which was solved approximately by Leonardo Fibonacci, is used to introduce the topic (Chen,2015). In the lesson "The Mean Value Inequality", the mean value inequality is used to solve Regiomontanus' angle maximization problem: "Viewed from which point on the ground, a pole hanging vertically looks the longest?" At the beginning of the lesson "The Introduction of the Complex Numbers (I)", the students are asked to solve Leibniz's system of quadratic equations with two unknowns: $x^2+y^2=xy=2$,which creates students' motivation to learn the new type of numbers----the imaginary numbers. At the beginning of the lesson "The Introduction of the Complex Numbers (II), the teacher used Cardano's problem of dividing 10 to introduce the imaginary numbers: "To divide 10 into two parts so that their product is 40".

These problems are all original historical ones without any adaptation.

Table 2. Part of the CO problems

No.	CO Problems	HPM Lessons	Sources	Authors	Time
1	The Cats & Mice problem	The Concept of the Number Sequence	Rhind Papyrus	Ahmes	ca. 1650 B.C.
2	The Interest Problem	The Concept of Logarithm	Tablet	--	ca. 1700 B.C.
3	The Two-line Problem	The Curves & Ttheir Equations	Plane Loci	Apollonius of Perga	3rd Century B.C.
4	The Weaving Problem	The Sum of a Geometric Sequence	The Nine Chapters on the Mathematical Art	Zhang Cang, et al	ca. 100 A.D.
5	The Solution to the Cubic Equation (I)	The Roots of an Equation and Zeroes of a Function	*Flos*	Leonardo Fibonacci	1225
6	The Angle Maximization Problem	The Mean Value Inequality	Correspondence	Regiomontanus	1471
7	The Problem of Dividing 10 into Two Parts	The Introduction of the Complex Numbers (II)	*Ars Magna*	G. Cardano	1545
8	The Solution to the Cubic Equation (II）	The Introduction of the Complex Numbers (II)	*Algebra*	R. Bombelli	1572
9	Systems of quadratic Equations with Two Unknowns	The Introduction of the Complex Numbers（I）	Correspondence	G. W. Leibniz	1671
10	The derivation of the Law of Refraction	The Application of the Derivatives	Paper on the Calculus	G. W. Leibniz	1684

4.2 The FP Problems

Of 63 history-based problems, about half are FP ones. Table 2 gives their detailed information.

Table 3. The FP Problems

No.	FP Problems	HPM Lessons	Sources	Authors	Time
1	The Derivation of the Mean Value Inequality according to Euclid's Proposition	The Mean Value Inequality	*Elements*	Euclid	3^{rd} Century B.C.
2	The Counterexample of Euclid's Definition of the Prism	The Definition of the Prism	*Elements*	Euclid	3^{rd} Century B. C.
3	The Underlying Idea of the Cyclotomic Rule	The Geometric Mean of the Derivative	Commentary on *the Nine Chapters of the Mathematical Art*	Liu Hui	263 A.D.
4	Derivation of the Addition Formula for Sine Based on Pappus' Geometrical Model	Addition and Subtraction Formulas for Sine and Cosine	*Collection*	Pappus	3^{rd} Century A.D.
5	Derivation of the Addition Formula for Cosine Based on Pappus' Geometrical Model	Addition and Subtraction Formulas for Sine and Cosine	*Collection*	Pappus	3^{rd} Century A.D.
6	Derivation of the Subtraction Formula for Sine Based on Pappus' Geometrical Model	Addition and Subtraction Formulas for Sine and Cosine	*Collection*	Pappus	3^{rd} Century A.D.
7	Derivation of the Subtraction Formula for Cosine Based on Pappus' Geometrical Model	Addition and Subtraction Formulas for Sine and Cosine	*Collection*	Pappus	3^{rd} Century A.D.
8	The Survey of the Meteor	The Law of Sine	--	Al-Kuhi	10^{th} Century A.D.
9	A Problem of the Fibonacci's Sequence	The Recurrent Sequence	*Liber Abaci*	Leonardo Fibonacci	1202
10	The Product of the Sum of Two Square Numbers and That of Two Other Square Numbers	The Introduction of Complex Numbers（I）	*Liber Quadratorum*	Leonardo Fibonacci	1225
11	The Derivation of the	Addition and	*Canonen*	F. Viète	1571

	Product Formula Based on Viète's Method	Subtraction Formulas for Sine and Cosine	*Mathematicum*		
12	The Proof of the Mean Value Inequality Based on Wallis' Method	The Mean Value Inequality	--	J. Wallis	17th Century A.D.
13	The Proposition on the Minimum Perimeter of Rectangles with the Same Area	The Mean Value Inequality	--	J. Wallis	17th Century A.D.
14	Given the Product of Two Positive Numbers, to Find the Minimum Sum	The Mean Value Inequality	--	J. Wallis	17th Century A.D.
15	The Problem of the Dock Locating	The Application of the Derivatives	--	P. Fermat	17th Century A.D.
16	The Problem of the Communication Line Connecting	The Application of the Derivatives	--	P. Fermat	17th Century A.D.
17	The Problem of the Pipe Installing	The Application of the Derivatives	--	P. Fermat	17th Century A.D.
18	The Design of the Pop-Top Can	The Application of the Derivatives	*Nova Stereometria Doliorum Vinariorum*	J. Kepler	1615
19	A Problem on the Hanoi Tower	The Recurrent Sequence	*Mathematical Recreations & Essays*	W. W. R Ball	19th Century A.D.
20	The Proof the Mean Value Inequality Based on the Isoperimetrc Rectangle	The Mean Value Inequality	--	N. H. Abel	19th Century A.D.
21	A Problem on the Dirichlet's Function	The Concept of Function	--	P. G. L. Dirichlet	19th Century A.D.

The FP problems appear in 11 lessons. One of the propositions in Pappus' *Collection* acts as the geometrical model of the addition formulas (Heath, 1921), based on which the problems of deriving the addition formulas are posed in the lesson "The Addition & Difference Formulas".

In his *Nova Stereometria Doliorum Vinariorum*, John Kepler (1571-1630) studied the ratio of the height of the cylinder to the diameter of its base while the cylinder has maximum volume. In the lesson "The Application of the Derivative", this problem was changed into that of a Pop-Top can: "What is the ratio of the height to the radius of the base when the tin has the minimum surface?" It should be noticed that the two bases and the side surface has different thickness. The constraints, goals and situations of Kepler's original problem being changed, the new problem is a FP one.

4.3 The CM and GM problems

Table 4 gives the detailed information about some CM problems.

Table 4. The CM Problems

No.	Problems	Lessons	Sources	Authors	Time
1	The Three-line Problem	Curves & Their Equations	*Conics*	Apollonius of Perga	3rd Century B. C.
2	The Four-line problem	Curves & Their Equations	*Conics*	Apollonius of Perga	3rd Century B. C.
3	The Josephus Problem	The Concept of the Number Sequence	*Mathematical Recreations and Essays*	W. W. R. Ball	1892
4	The Problem of Rectangle Frame	The Introduction of Complex Numbers（II）	*Ars Magna*	G. Cardano	1545
5	A Cylinder with Maximum Volume (I)	The Application of the Derivatives	*Nova Stereometria Doliorum Vinariorum*	J. Kepler	1615
6	A Cylinder with Maximum Volume (II)	The Application of the Derivatives	*Nova Stereometria Doliorum Vinariorum*	J. Kepler	1615
7	A Prism with Maximum Volume	The Application of the Derivatives	*Nova Stereometria Doliorum Vinariorum*	J. Kepler	1615
8	A Cone with Maximum Volume	The Application of the Derivatives	*Nova Stereometria Doliorum Vinariorum*	J. Kepler	1615
9	A Triangular Pyramid with Maximum Volume	The Application of the Derivatives	*Nova Stereometria Doliorum Vinariorum*	J. Kepler	1615

The CM problems appear only in four lessons. In the lesson "Curves & Their Equations", special cases of the Greek three-line problem and four-line problem are adopted in which the lines are perpendicular or parallel to each other. At the beginning of the lesson “The Introduction of Complex Numbers”, the teacher first gave the problem of dividing 10 into two parts so that their product is 24, which is adapted from a problem in Cardano’s *Ars Magna*, then Cardano’s original problem is discussed.

In his *Nova Stereometria Doliorum Vinariorum*, John Kepler posed the following proposition: “Of all right cylinders with the fixed diagonal of the axial section, the one whose diameter is to the height as $\sqrt{2}$: 1 has maximum volume.” A problem solved in the lesson “The Application of the Derivatives” is adapted from this proposition, the cylinder being inscribed in a sphere.

Only one GM problem is found in all the 20 lessons, which is adapted from the chessboard problem in Fibonacci’s *Liber Abaci*:

If one grain of corn is put in the first place and from the second place on, any place in the sequence of places is proposed to be double the sum of all preceding places, how many grains of corn are there in total on the whole chessboard? (Stigler, 2002, 437).

In a problem of the lesson “The Recurrent Sequence”, the way of putting grains of corn remains the same but the goal is to find “the general term of the sequence of grains of corn in each place of the chessboard” instead of the sum of the sequence.

4.4 The CH Problems

Table 5 shows the detailed information about some of the CH problems.

Table 5. The CH Problems

No.	CH Problems	Lessons	Sources	Authors	时间
1	The Geometrical Representation of the Geometrical Mean	The Mean Value Inequality	*Elements*	Euclid	3rd Century B. C.
2	Application of the Addition Formulas for Sine and Cosine (I)	Addition and Subtraction Formulas for Sine and Cosine	*Collection*	Pappus	3rd Century A. D.
3	Application of the Addition Formulas for Sine and Cosine (II)	Addition and Subtraction Formulas for Sine and Cosine	*Collection*	Pappus	3rd Century A. D.
4	Application of the Subtraction Formulas for Sine and Cosine (I)	Addition and Subtraction Formulas for Sine and Cosine	*Collection*	Pappus	3rd Century A. D.
5	Application of the	Addition and	*Collection*	Pappus	3rd

	Subtraction Formulas for Sine and Cosine (II)	Subtraction Formulas for Sine and Cosine			Century A. D.
6	The Application of the Product Formula	Addition and Subtraction Formulas for Sine and Cosine	--	J. Werner	ca. 1510
7	The Geometrical Solution to the Regiomontanus' angle maximization problem	The Mean Value Inequality	Correspondence	Regiomontanus	1471

The CH problems appear only in the lessons "The Addition and Subtraction Formulas for Sine and Cosine" and "The Mean Value Inequality", in which the HPM Work Cards are used.

For Example, in the lesson "The Mean Value Inequality", after solving Regiomontanus' problem, a new problem is posed: How to solve this problem geometrically? Being based on the solution to the original problem, it is a CH problem.

5 Distribution of history-based problems at different stages of the lessons

Figure9 and Figure10 show the distribution of 12 CO problems and 51 Accommodation problems (FP, CM, GM & CH) at different stages of the lessons respectively.

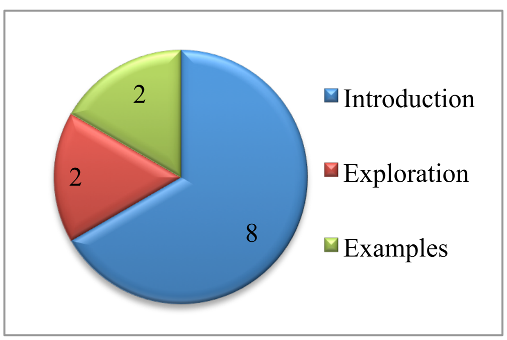

Figure 9. Distribution of the CO problems

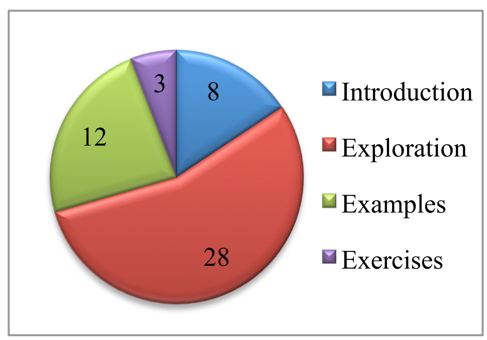

Figure10. Distribution of the Accommodation problems

Figure 9 shows that CO problems are mainly used at the stage of introduction and seldom appear at stages of exploration and example. Figure 10 shows that the accommodation problems are mainly used at the stage of exploration and seldom appear at stages of introduction and exercise. There are few lessons in which accommodation problems are used as examples. In most of lessons, history-based problems quit from the stage after playing relevant roles at stages of introduction and exploration.

On the on hand, history-based problems available to teachers which are suitable for classroom teaching are very few, on the other hand, teachers usually believe that they don't need such problems at stages of example and exercise. In recent years, history-based problems appear in college entrance examination, and perhaps this fact will change teachers' opinions.

6 Functions of History-based problems

In Mainland China, one of the general objectives of the mathematics curriculum is to develop students' core competencies, i.e., mathematical abstraction, logical reasoning, mathematical operation, intuitive imagination and data analysis. Except for data analysis, these competencies are embodied in all 63 history-based problems. Figure 12 shows the number of problems corresponding to each type of the competency.

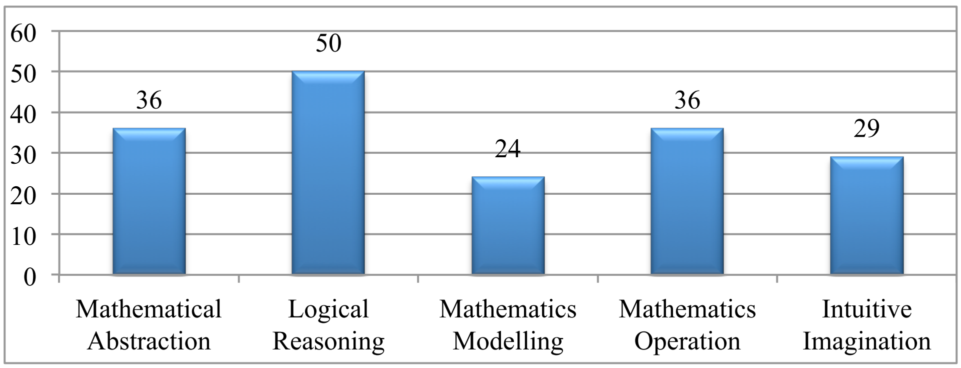

Figure 11. The number of history-based problems which reflect the key mathematics literacy

Take Regiomontanus' angle maximization problem as an example. Through intuitive imagination, the geometrical figure is constructed: In Figure 12, the segment AB designates the hanging pole, P is the view point on the ground. Let $OA = a$, $OB = b$. $OP = x$, $\angle OPA = \alpha$,

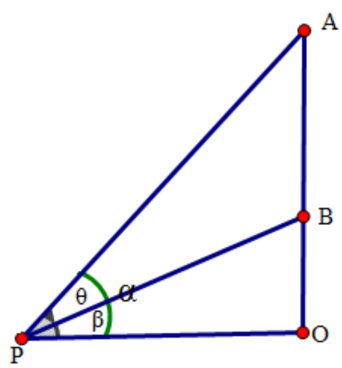

Figure 12. Regiomontanus' Angle maximization problem

$\angle OPB = \beta$, $\theta = \alpha - \beta$. Then

$$\tan\theta = \tan(\alpha-\beta) = \frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta} = \frac{\frac{a}{x}-\frac{b}{x}}{1+\frac{ab}{x^2}} = \frac{a-b}{x+\frac{ab}{x}}.$$

Using the mean value inequality $x+\frac{ab}{x}\geq 2\sqrt{ab}$, through operation and logical reasoning, we get $x=\sqrt{ab}$ while $\tan\theta$ or θ takes its maximum.

Figure 13 shows the distribution of problems relating to different types of core competencies, indicating that history-based problems are effective in developing students' core competencies.

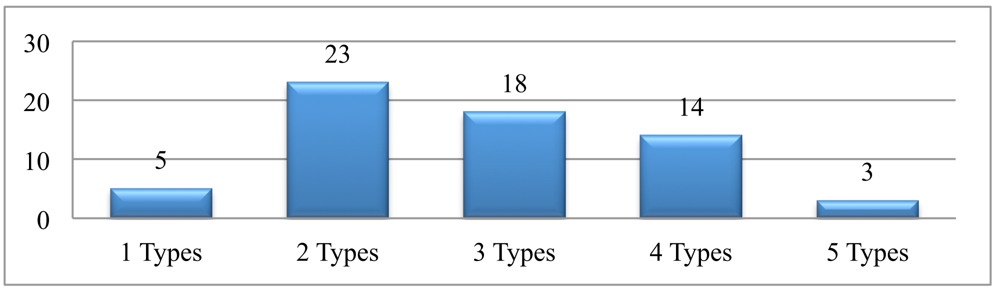

Figure 13. The distribution of history-based problems involving different competencies

Here arises a problem: are history-based problems irreplaceable in mathematics teaching? Undoubtedly, those problems unrelated to the history of mathematics can also develop students' core competencies. However, it is impossible for any mathematics problems to generate from the thin air. The history of mathematics offers us the origin of problems, from which endless new problems can be posed by means of various types of strategies. Moreover, history-based problems can easily form the chains of problems. For example, in the lesson "Curves and their Equations", the two-line problem, three-line problem and four-line problem form an inseparable whole, creating for students the opportunity to experience the birth process of analytical geometry. In the lesson "The Application of Derivatives", problems of cylinders, cones, prisms and pyramids with maximum volumes, a series of problems solved by mathematicians in the 17^{th} century, are reconstructed in the classroom.

On the other hand, history-based problems play unique roles in promoting positive affect and beliefs. In the lesson "The Concept of Number Sequence", the cat and rice problem of ancient Egypt and the Josephus' game aroused students' interest in number sequences. In the lesson "The Concept of Complex Numbers", Leibniz's system of equations with two unknowns stimulated students' learning motivation. In the lesson "The Concept of Prism", Euclid's incomplete definition of the prism told students that mathematicians might make mistakes and enabled them to recognize the nature of mathematics activity and reflect on their own mistakes and difficulties in learning mathematics. In the lesson "The Application of Derivatives", problems of can designing, dock locating, pipe installing and communication line connecting reveal the intimate links between mathematics and human real life, while the problem of deriving the refraction law shows that mathematics is closely related to other subjects.

Moreover, history-based problems can humanize the classroom. While solving such problems, students seem to traverse the time and space and make a conversation with a mathematician living in other culture and time, who seems to be a student in the classroom. The dialogue across time and space makes students fell close to mathematics.

7 Conclusions

Of the seven types of history-based problems, five are found in the total twenty HPM lessons, which, according to the frequency of being used, are FP, CO, CM, CH and GM problems respectively. No problems posed by means of symmetry strategy (SE problems) or situation manipulation strategy (SM problems) are found. Some problems are posed by means of situation manipulation, they are, however, classified among FP, CM or GM problems because the constraints or goals of original problems are also changed.

History-based problems in the twenty lessons mainly come from such sources as old Babylonian mathematics tablets, Egyptian papyrus, *Elements*, *Nine Chapters on the Mathematical Art*, Pappus' *Collection*, *Liber Abaci* and works of such mathematicians as Fermat, Kepler, Leibniz, etc., which are not limited to one culture or one time but, among the tremendous mathematics literatures, just a drop in the ocean.

As far as core mathematics competencies are concerned, history-based problems in 20 HPM lessons involves mathematics abstraction, mathematics modelling, logical reasoning, mathematical operations and intuitive imagination. In contrast with those non-historical problems, history-based problems had advantages in affective and cultural aspects of mathematics education.

In 20 HPM lessons, mathematics motivated problems are mainly used at the introduction and exploration stages and seldom appear at other stages.

History-based problems are indispensable for HPM lessons. Therefore, researchers should make deep and systematic historical researches on relevant subjects in the senior high school mathematics curriculum in an effort to amplify the repertoire of mathematical problems in mathematics curriculum, textbooks and teaching.

They should also cooperate with high school mathematics teachers, discussing and disseminating the history motivated problem posing strategies. Teachers should know more about the educational values of the history-based problems and using them at more stages of classroom teaching, completely exerting their educational functions.

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