Mathematics & physics: an innermost relationship. Didactical implications for their teaching & learning

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MATHEMATICS & PHYSICS: AN INNERMOST RELATIONSHIP

Didactical implications for their teaching & learning

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ABSTRACT

The interplay and mutual influence between mathematics and physics all along their history and their deep epistemological affinity are explored and summarized in three *theses* with a basic didactical moral: In teaching and learning either of them, neither history should be ignored, nor the close interrelation of the two disciplines should be circumvented or bypassed. Moreover, the *key issues of the integration of history* in mathematics or physics education are addressed and a common framework is outlined based on work done in the past. These ideas are illustrated by means of *three examples*: (a) measuring the distance of inaccessible objects; (b) rotations, space-time and special relativity; (c) differential equations, functional analysis and quantum mechanics.

1 Introduction

At all levels of education, teaching and learning mathematics is usually kept separated from physics, and vice versa, corresponding to a distinction between the two sciences at the research level: mathematicians are supposed to stay in a universe of ideal logical rigor, while physicists are simply users of (possibly very sophisticated) mathematics (Tzanakis 2000, §2). This is reflected in physics education (PE), where mathematics is merely a tool to describe and calculate, whereas, in mathematics education (ME), physics is only a possible context for applying mathematics previously conceived abstractly (Tzanakis & Thomaidis 2000, p.49). Overcoming this dichotomy (which creates significant learning problems for the students of both disciplines) demands systematic research in different fields, especially when aiming at informing educational practices by reflecting on historical, philosophical and sociological aspects of scientific knowledge (Karam 2015).

However, the historical development of the two sciences, does not verify this separation. On the contrary, it is a relatively recent fact, not older than 100 years (or, even less; Arnold 1998, p.229), characterizing their development up to the 1960-70s at the peak of the effort to extremely formalize mathematics (reflected in ME in the "New Math" reform). No such clear-cut separation existed before, whereas, in the last decades there is a strong tendency to overcome it, reflected in the often deeply interwoven research in both sciences[1]. I believe this historical fact is due to the deep epistemological affinity of the two sciences, concisely expressed by numerous famous mathematicians and physicists; Galileo's phrase that "*this grand book, the Universe, … is written in the language of Mathematics*"[2]; Wigner's (1960) view of the "*unreasonable effectiveness of mathematics in the natural sciences*"; or Hilbert's

[1] E.g. Casacuberta & Castellet 1992, where 5 of the 7 (Fields medallists) authors think that developments in mathematics will be in areas related to problems in physics.
[2] *The Assayer* (1623); in Drake 1957.

6th problem in his famous 1900 list of problems[3].

On the other hand, over the last 40 years, there has been an increasing awareness of the educational community that integrating history of mathematics in ME could be a promising possible way to teach and learn mathematics because, in principle, it provides the opportunity to appreciate the evolutionary nature of mathematical knowledge, hence, to go beyond its conventional understanding as a corpus of finished deductively structured intellectual products. Especially in the context of the international HPM Study Group, this has evolved into a worldwide intensively studied area of new pedagogical practices and specific research activities (Fasanelli & Fauvel 2006 for an account up to 2000; Furinghetti 2012, Barbin 2013, Barbin & Tzanakis 2014 for later developments). However, the educational value of history outlined above is not restricted to mathematics. It is also valid for physics, because physics - just like mathematics - is not only a deductively structured corpus of knowledge (additionally constrained by its compatibility with experiment), but has been ever-changing and evolving.

These comments will be further detailed in section 2 as **three** main theses on the **ontological status** of mathematics and physics, their **historical interrelation**, and their **epistemological affinity** as scientific disciplines, constituting a historical-epistemological framework with a basic didactical moral: *In teaching and learning mathematics or physics, neither history can be ignored, nor their close interrelation can be circumvented or bypassed*. In section 3, this general point of view is elaborated by addressing the main issues related to a *history-pedagogy-mathematics/physics* (*HPM/Ph*) perspective (§2.1): *Which* history for didactical purposes, with *which role(s)* and in *which way(s)* to be realized in practice? Here much work has been done in recent years, to be briefly reviewed mainly in the light of and aiming to support an interactive interplay in the teaching and learning of mathematics **with/for** physics, and **vice versa**. In section 4 the framework of sections 2, 3 is illustrated by three examples of increasing sophistication - from junior high school to advanced undergraduate or graduate level - each one pinpointing equally well the main issues raised, presented and advocated in sections 2 & 3. The paper's main points are summarized in section 5.

2 Historical-epistemological framework

2.1 What is mathematics? What is physics?

It is customary to think of mathematics as a deductively structured, logically self-consistent corpus of knowledge; a point of view deeply influencing education and getting stronger at its higher levels. It is adopted by many mathematicians and mathematics teachers either because it leads to the quickest way of presentation (lectures, books etc), or/and because the **logical** clarity thus obtained, is often thought **identical** with the complete **understanding** of the subject. Similar comments hold for physics, additionally requiring this knowledge to be consistent with known empirical facts.

But because mathematics and physics are intellectual enterprises with a long history and a

[3] "*Mathematical treatment of the axioms of physics: To treat in the same manner, by means of axioms, those physical sciences in which mathematics plays an important part; in the first rank are the theory of probabilities and mechanics*" (Hilbert 1902, p.454).

vivid present, knowledge gained in their context is determined not only by the circumstances in which this knowledge becomes a deductively structured theory and/or supported by experiments, but also by the procedures that originally led or may lead to it. Thus learning mathematics or physics includes not only the "polished products" of the associated intellectual activity, but also the understanding of implicit motivations, the sense-making actions and the reflective processes of scientists, which aim to the construction of meaning. Although the "polished products" of both disciplines form the part of this knowledge that is communicated and criticized, and forms the basis for new work, didactically the **processes** producing this knowledge are equally important. Perceiving mathematics or physics **both** as a *logically structured collection of intellectual products* **and** as *knowledge-producing endeavours*[4] should be the core of their teaching and central to their image communicated to the outside world. Along these lines, putting emphasis on integrating historical and epistemological issues in the teaching and learning of mathematics and physics constitutes a possible natural way for exposing them in the making that may lead to a better understanding of their specific parts, and to a deeper awareness of what they are as disciplines. This is very important from an educational viewpoint: It helps to realize that they have been undergoing changes over time as a result of contributions from many different cultures, in uninterrupted dialogue with other scientific disciplines, philosophy, the arts and technology; a constant force for stimulating and supporting scientific, technical, artistic and social development.

The last paragraph frames the *HPM/Ph perspective* (Clark *et al.* 2016, §1.1), summarized in the following thesis on the **ontological** status of both disciplines and its educational implications (cf. Tzanakis & Thomaidis 2000, p.45).

Thesis A: Mathematics and physics should be conceived (hence, taught and learnt) both as the *result* of intellectual enterprises and as the *procedures* leading to these results. Knowledge gained in their context has an evolutionary character; by its very nature, *historicity*[5] is a deeply-rooted characteristic.

2.2 The interrelated historical development of mathematics and physics

Numerous important historical examples attest the following

Thesis B: Throughout their historical development from antiquity to the present, mathematics and physics have been evolving in a close, continuous, uninterrupted, bidirectional, multifaceted and fruitful way.

It goes from Hero's geometrical proof of the *law of reflection*, Eratosthenes' estimation of the *earth's circumference*, and Archimedes' "mechanical arguments" to compute areas and volumes in his *Method*, to Poincaré's derivation of the *Lorentz transformations* in special relativity using group-theoretic arguments, Hilbert's deduction of *General Relativity's field equations* from a variational principle[6], von Neumann's rigorous formulation of quantum

[4] Kjeldsen & Lützen 2015, p.552.
[5] Barbin & Bénard 2007; Barbin 2010.
[6] Both Poincaré and Hilbert got their results before Einstein's **physically-oriented** work, by following **mathematically-oriented** and totally different approaches from his. Though curious at first sight, this is due to the fact that these great mathematicians followed closely every development in physics of their time, which they knew deeply (Tzanakis 1999a, pp.114-115; Mehra 1973); just one striking example supporting *Theses* B & C of this paper.

mechanics, to more recent examples, like Penrose's *singularity theorems* in general relativity, Feynman's *path-integrals* in quantum mechanics and *functional integration*, Thom's *Catastrophe Theory*, or Connes' *Non-commutative Geometry* and its relation to quantum field theory (Tzanakis 1999b, §§4, 5; 2002, §3).

A simplified scheme of 3 scenarios highlights this relation (Tzanakis 2000, §2; 2002 §1.2):

(S_1) *Parallel development*: The physical problems asking for solution and the formulation of appropriate mathematics (concepts, methods, or theories) evolve in parallel.

Typical examples: Infinitesimal calculus and classical mechanics in the 17th century; the interrelated development of vector analysis, electromagnetic theory, and fluid mechanics in the 19th century (Crowe 1967); the multifaceted development in the 19th century of statistical concepts and methods, error theory in celestial mechanics, kinetic theory, and social statistics (Kourkoulos & Tzanakis 2010); Hamilton's unified treatment of geometrical optics and classical mechanics and the solution of 1st order partial differential equations (§ 4.3 below).

(S_2) *Mathematical concepts, methods or theories precede their integration into physics*: The corresponding physical problems naturally stress the need for the appropriate mathematics.

Typical examples: Riemannian geometry and tensor calculus as the indispensable framework for Einstein's formulation of General Relativity (Tzanakis 1999b, §4.3); matrix algebra and Heisenberg's experimentally-induced matrix mechanics (§4.3 below).

(S_3) *Physical problems precede the formulation of mathematics appropriate to tackle them*: Partially intuitive, formal or experimentally-induced models, and logically incomplete or ill-defined concepts, motivate and/or guide the development of new mathematics.

Typical examples: Langevin's equation modelling Brownian motion in statistical physics and the development of stochastic differential equations (Tzanakis 1999b, §5.3); Dirac's introduction and use of his δ-function in quantum mechanics as a key initial step for the development of distribution theory (Lützen 1982); Feynman's path-integral approach to quantum mechanics and the development of functional integration (Gelfand & Yaglom 1960).

A **word of caution**: This **simplified** picture aims just to emphasize the historically rich and bidirectional relation of the two disciplines. In fact, none of the three "pure" scenarios appears isolated. If one appears as the original step in the historical development, it is usually followed by fruitful feedback in the context of the others. This is supported by the examples in section 4.

2.3 The epistemological affinity of mathematics and physics

The conventional view of the epistemological relation between mathematics and physics, shared by many mathematicians, physicists and teachers of these disciplines, is that mathematics is simply the **language** of physics, and physics is an **exterior** to mathematics **source of problems** to be solved mathematically and/or a **domain to apply** already **available** mathematical **tools**. At least indirectly, this means that mathematics is simply a toolkit for handling physical problems, but otherwise is **extrinsic** to it, whereas physics is a huge, but **external** reservoir of a priori non-mathematical problems, which are nonetheless capable of mathematical formulation. If this were so, it would be difficult to understand the interrelation of the two sciences (§2.2). In fact, this conventional viewpoint has been challenged by distinguished mathematicians and physicists.

Maxwell stated his "… *opinion as to the necessity of mathematics for the study of Natural Philosophy*…" as "*Natural philosophy is, and ought to be, Mathematics… the greatest advances in mathematics have been due to enquirers into physical laws*" (in Harman 1990).

Hilbert wrote that *"... while the creative power of pure reason is at work, the outer world ... comes into play, forces upon us new questions from actual experience, opens up new branches of mathematics, and while we seek to conquer these new fields of knowledge for the realm of pure thought, we often find the answers to old unsolved problems and thus ... advance most successfully the old theories. ... the numerous and surprising analogies and that* ***apparently prearranged harmony*** *which the mathematician so often perceives in the questions, methods and ideas of the various branches of his science,* ***have their origin*** *in this* ***ever-recurring interplay between thought and experience***" (Hilbert 1902, p.440; my emphasis).

Weyl declared that "*...gaz*[ing] *up… towards the stars is… to strengthen faith in reason, to realise the "harmonia mundi" that transfuses all phenomena... it was my wish to present this great subject* [the general theory of relativity] *as an illustration of the intermingling of philosophical, mathematical, and physical thought...*" (Weyl 1952, pp.x, ix) and that "*We shall see more and more clearly… that Geometry, Mechanics, and Physics form an* ***inseparable theoretical whole***" (*ibid*, p.67, my emphasis; cf. Pyenson 1982).

Einstein was convinced that "*…pure mathematical construction enables us to discover the concepts and the laws ... which give us the key to the understanding of the phenomena of Nature. Experience can… guide us in our choice of serviceable mathematical concepts…* [and] *remains the sole criterion of the serviceablility of a mathematical construction for physics, but the truly creative principle resides in mathematics*" (Einstein 1934) and stressed the criterion of "*naturalness*" (or "*inner perfection*") for trusting a physical theory (Einstein 1969).

Wigner argued that the unreasonable effectiveness of mathematics in the natural sciences "*…shows that it is in a* ***very real sense****, the correct language*…" and its predictions, often in amazing agreement with experimental data, indicate that "[s]*urely… we 'got something out' of the equations that we* ***did not put in***" (Wigner 1960, pp.8, 9; my emphasis)[7], whereas, Dirac (referring to General Relativity) argued that "*Anyone who appreciates the fundamental* ***harmony*** *connecting the way* ***nature*** *runs,* ***and*** *general* ***mathematical*** *principles, must feel that a theory with… beauty and elegance…* ***has to be*** *substantially correct*" (Dirac 1979, my emphasis).

These quotations suggest that mathematics is the language of physics in the deepest sense, often determining the content and meaning of physical concepts and theories (Tzanakis 1999b, §3), or even instigating revolutions in physics (Brush 2015, pp.495, 511). Conversely, physics furnishes and provides to mathematics, not only problems, but also ideas, methods

[7] In Tzanakis 1997 §5, this is called "*naturality*", meaning "… *that the explanation of known unintelligible facts, or the prediction of new ones, is not originally in the intentions of the founders of the theory, whose starting point may be quite independent of these facts*" (*ibid*, p.2101).

and concepts, crucial for many mathematical innovations (Tzanakis 1999b, §3; Kragh 2015, particularly §8). These comments are summarized in the first part of the following

***Thesis C*(a):** Mathematics and physics have always been closely interwoven in the sense of a bidirectional[8] process:

***From* mathematics *to* physics**: Mathematics is the language of physics, not only as a tool for expressing, handling and developing logically physical concepts, methods and theories, but also as an indispensable, formative characteristic that shapes them, by deepening, sharpening, and extending their meaning, or even endowing them with meaning.

***From* physics *to* mathematics**: Physics constitutes a (or maybe, *the*) natural framework for testing, applying and elaborating mathematical theories, methods and concepts, or even motivating, stimulating, instigating and creating all kinds of mathematical innovations.[9]

This historically-evidenced mutual process of dialectical interplay between mathematics and physics helps to moderate Wigner's puzzlement (Kjeldsen & Lützen 2015, p.544) and seems to be based on a deeper epistemological affinity of the two disciplines. Arnold (1998, p.231) has claimed that "…*the scheme of construction of a mathematical theory is exactly the same as that in any other natural science*". Detailed analysis suggests that both disciplines use the same procedures as invention/discovery patterns, or for (partial, in general) justification of results: logical reasoning (by deduction, induction, or analogy), algorithmic procedures, and experimental investigations (Tzanakis & Kourkoulos 2000, §2; Tzanakis 1998, §1). In fact, the supposedly key difference between the two disciplines, namely, what constitutes a "true" statement (logical self-consistency and consistency with available empirical data, respectively) is less sharp than conventionally claimed (Tzanakis & Kourkoulos 2000, §3; Tzanakis & Thomaidis 2000, §2), a point of view (at least indirectly) adopted by important physicists and mathematicians. Dirac claimed that "*A theory with mathematical beauty is more likely to be correct than an ugly one that fits some experimental data*" (Dirac 1970; see also Brush 2015, §9; Kragh 1990) and Weyl declared that "[his] *work always tried to unite the true with the beautiful; but when* [he] *had to choose one or the other* [he] *usually chose the beautiful*" (quoted in Chandrasekhar 1987). Arnold went further arguing that "*Mathematics is a part of physics… the part of physics where experiments are cheap*" (Arnold 1998, p.229). Thus, we are led to the second part of

***Thesis C*(b):** Mathematics and physics as embodiments of general attitudes in regard to the description, exploration, and understanding of empirically and/or mentally conceived objects, are so closely interwoven, that any distinction between them is related more to the point of view adopted while studying particular aspects of an object, than to the object itself.[10].

Theses **A-C** will be further illustrated and supported by examples in section 4. A basic educational conclusion to be drawn from this analysis is summarized as follows:

By *Thesis* **A**, history cannot be ignored in the teaching and learning either of mathematics, or

[8] Kragh has called it "*dialectical, or loop-like*" reflecting the historical interaction between the two disciplines which "… *is multifarious and cannot be encapsulated by a single formula*" (Kragh 2015, p.525).

[9] In Tzanakis 1999b, these two aspects are called "*mathematical physics*" & "*physical mathematics*", respectively.

[10] Tzanakis & Thomaidis 2000, §4. This is close to Weyl's view of an "*inseparable, theoretical whole*" above.

physics; By *Theses* **B** & **C** the teaching and learning one of the two should take into account, be supported, or include aspects of the other[11]; *Thesis* **C** gives general orientation to motivate, stimulate, support, deepen and widen the teaching and learning of either discipline specialized for particular examples into precise guidelines with the aid and/or in the light of Thesis B.

For the actual implementation of this conclusion, a sufficiently clear framework for the *HPM/Ph* perspective is needed, which is the subject of the following section.

3 The *HPM/Ph* issues and framework

To proceed further having adopted *Theses* **A-C**, some fundamental issues should be faced related to the integration of history in ME and/or PE. Although much of what follows originated in the context of mathematics, it is **equally valid** for physics, which is reasonable in view of the close interrelation of the two disciplines. The presentation is based on Clark *et al* 2016, §2.3.

3.1 *Which history is suitable, pertinent, and relevant for didactical purposes?*

This has been a permanent issue of debate among historians and educators with an interest in the HPM/Ph perspective. Implicit to some objections against the introduction of history in ME or PE is the idea that the term "history" is used in the same sense by historians, mathematicians, physicists, or teachers. That this is not so was stressed early by Grattan-Guinness, in relation to the history of mathematics (Grattan-Guinness 1973)[12]. On the other hand, it is undeniable that often history followed a complicated zig-zag path, led to dead ends, included notions, methods and problems no longer used in mathematics or physics today etc. Thus, its integration in education is nontrivial, posing the question why it must be done at all, since in this way history may be forced "*…to serve aims not only foreign to its own but even antithetical to them*" (Fried 2011). In other words, there is real danger of either unacceptably simplify or/and distort history to serve education as still another of its tools by adopting a "Whig" (approach to) history, in which "*…the present is the measure of the past. Hence, what one considers significant in history is precisely what leads to something deemed significant today*" (Fried 2001).

An important step to clarify existing conflicts and tensions between a mathematician's and a historian's approach to mathematical knowledge, with due attention to the relevance of history to ME, was Grattan-Guiness' distinction between *History* and *Heritage* (Grattan-Guiness 2004a, b): The ***History*** of a particular mathematical subject *N* refers to

> "*… the development of N during a particular period: its launch and early forms, its impact* [in the following years and decades], *and applications in and/or outside mathematics. It addresses the question 'What happened in the past?' by offering descriptions. Maybe some kinds of explanation will also be attempted to answer the companion question 'Why did it happen?'*"... "[It] *should also address the dual questions 'what did not happen in the past?' and 'why not?'; false starts, missed*

[11] Arnold argues that "[a]*ttempts to create 'pure' deductive-axiomatic mathematics have led to the rejection of the scheme used in physics (observation, model, investigation of the model, conclusions, testing by observations) and its replacement by the scheme definition, theorem, proof*", which "*can do nothing but harm to the teaching and practical work*" (Arnold 1998, pp.233).

[12] See Kjeldsen 2011a; 2011b; Kjeldsen & Blomhøj 2012, for a different approach.

opportunities…, sleepers, and repeats are noted and maybe explained… ***differences*** *between N and seemingly similar more modern notions are likely to be emphasized*" (Grattan-Guiness, 2004b, p.1; 2004a, p.164).

The ***Heritage*** of a particular mathematical subject *N* refers

"*…. to the impact of N upon later work, both at the time and afterward, especially the forms which it may take, or be embodied, in later contexts. Some modern form of N is usually the main focus, with attention paid to the course of its development. ... the mathematical relationships will be noted, but historical ones… will hold much less interest.* [It] *addresses the question 'how did we get here?' ... The modern notions are inserted into N when appropriate, and thereby N is unveiled...* ***similarities*** *between N and its more modern notions are likely to be emphasized; the present is* ***photocopied*** *onto the past*" (Grattan-Guiness, 2004a, p.165).

Though "***both kinds of activity are quite legitimate***, and… *important in their own right...*", they are **incompatible** in the sense "…*that both history and heritage are legitimate ways of handling the mathematics of the past; but muddling the two together, or asserting that one is subordinate to the other, is not*." (Grattan-Guinness 2004a, p.165; 2004b, p.1).

This distinction - clearly valid for physics as well - is close to similar ones between pairs of methodological approaches: for mathematics Tzanakis, Arcavi *et al* 2000, pp.209-210; for physics Tzanakis 1998, Tzanakis & Coutsomitros 1988. Hence, this distinction is potentially of great relevance to education (Rogers 2009; Tzanakis & Thomaidis 2012), contributing to answering the recurrent question "*Why and which history is appropriate to be used for educational purposes?*" (Barbin 1997).

3.2 *Which role can history of mathematics and/or physics play in their teaching and learning?*

This question has been extensively discussed from various angles, especially in relation to the appropriateness and pertinence of original historical sources in ME. In this context, history can play three **mutually complementary roles** (Barbin 1997; Jahnke *et al* 2000; Furinghetti *et al* 2006; Furinghetti 2012, §5; Jankvist 2013) valid for physics as well:

(I) *Replacement*: Replacing mathematical and/or physical knowledge as usually understood (a corpus of knowledge consisting of final results; an externally given set of techniques to solve problems; school units useful for exams etc) by something different (not only final results, but also mental processes leading to them; hence perception of this knowledge, not only as a collection of well-defined deductively organized results, but also as a vivid intellectual activity).

(II) *Reorientation*: Changing what is (supposed to be) familiar, to something unfamiliar; thus modifying the learners and teachers' conventional perception of mathematical and/or physical knowledge as something that has always been existing in its current established form, into a deeper awareness that this knowledge was an invention based on a dialectical interplay between human mind's creativity and careful intelligent experimentation; an evolving intellectual activity.

(III) A *cultural* role: Making possible the awareness that mathematics and physics develop in a specific scientific, technological or societal context at a given time and place; thus appreciating knowledge gained in the context of these disciplines as an integral part of human intellectual history in the development of society; hence, perceiving mathematics and/or physics from perspectives beyond their currently established boundaries as (separate) disciplines.

Considered from the viewpoint of the *objective* of integrating history in ME and/or PE, there are five main areas where this could be valuable (Tzanakis, Arcavi *et al*, 2000, §7.2; Tzanakis & Thomaidis 2012, §3):

1. *Learning specific pieces of mathematics and/or physics*: Historical development vs. polished final results; history as a resource; history as a bridge between different domains and disciplines; history's general educational value in the development of personal growth and skills.

2. *Views on the nature of mathematics, physics and the associated activities*: About their content (to get insights into concepts, conjectures & proofs by looking from a different viewpoint; to appreciate "failure" as part of mathematics and physics in the making; to make visible the evolutionary nature of meta-concepts); about their form (to compare old and modern; to motivate learning by stressing clarity, conciseness and logical completeness).

3. *The didactical background of teachers and their pedagogical repertoire*: Identifying motivations; becoming aware of difficulties & obstacles; getting involved and/or becoming aware of the creative process of "doing mathematics and/or physics"; enriching the didactical repertoire; deciphering & understanding idiosyncratic or non-conventional approaches.

4. *The affective predisposition towards mathematics and physics*: Understanding mathematics and physics as human endeavours; persisting with ideas, attempting lines of inquiry, posing questions; not getting discouraged by failures, mistakes, uncertainties, misunderstandings.

5. *The appreciation of mathematics and physics as a cultural-human endeavour*: They form part of local cultures; they evolve under the influence of factors intrinsic and/or extrinsic to them.

From the point of view of the *way* history is accommodated into education, Jankvist's (2009a) distinction in relation to mathematics is pertinent:

(i) *History as a tool*: That is "*… the use of history as an assisting means, or an* ***aid****, in the learning* [or teaching] *of mathematics* [or physics]*... in this sense, history may be an aid…*" "*as a motivational or affective tool, and… as a cognitive tool…*" (Jankvist 2009b, p.69; 2009c, p.8).

(ii) *History as a goal*: That is, history is "*…an* ***aim*** *in itself... posing and suggesting answers to questions about the evolution and development of mathematics,* [or physics]... *about the inner and outer driving forces of this evolution, or the cultural and societal aspects of mathematics* [or physics] *and its history*" (Jankvist 2009b, p.69).

3.3 *In which way(s) history's role can be realized in educational practice?*

There are three broad ways to integrate history in ME and PE, each one emphasizing a different **aim**. They **complement** each other in the sense that each one, if taken alone, is insufficient to exhaust the multifarious constructive influence history can have on education (Tzanakis, Arcavi *et al* 2000; §7.3; Tzanakis & Thomaidis 2000, §3):

(I) *To provide direct historical information*, aiming to learn history;

(II) *To implement a teaching approach inspired by history* (explicitly or implicitly), aiming to learn mathematics and/or physics;

(III) *To focus on mathematics and/or physics as disciplines and the cultural & social context in which they have been evolving*, aiming to develop deeper awareness of their evolutionary character, epistemological characteristics, and relation to other disciplines, and the influence exerted by factors both intrinsic and extrinsic to them.

From a **methodological** point of view, Jankvist (2009a, §6) classified the teaching & learning approaches in three categories:

1. *Illumination approaches*: teaching and learning is supplemented by historical information of varying size and emphasis.

2. *Module approaches*: instructional units devoted to history, often based on specific cases.

3. *History-based approaches*: history shapes the order and the way of presentation, often without history appearing explicitly, but rather being integrated into teaching.

Approaches may vary in size and scope, according to the specific didactical aim, the subject matter, the level and orientation of the learners, the available didactical time, and external constraints (curriculum regulations, number of learners in a classroom etc).

4 Examples

In sections 2 & 3 a general framework for integrating history in ME and PE was presented, based on the epistemological similarities between the two disciplines and their interrelated historical evolution, and on a unified approach to the basic issues involved in any such integration. Below, this is illustrated by three examples, explicitly indicating their placement within the general framework.

4.1 Measuring the distance of inaccessible objects: *Mathematics & Physics in their wider cultural context*

The general theme is the determination of the size or distance of completely inaccessible objects like the celestial bodies,[13] some important cases being:

(a) *Eratosthenes' measurement of the Earth's circumference*.
(b) *Aristarchus' method of measuring the earth-sun-moon relative distances*.
(c) *Copernicus' method of measuring inner planets' relative distances from the sun*.
(d) *Trigonometric parallax* for measuring: (i) The *earth-sun distance* using inner planets' transits across the sun's disk; (ii) *Star distances* using *stellar parallax*.

This is a **rich** example that can be extended in several different directions, depending on the course orientation and the available didactical time. It could constitute a sufficiently self-contained *teaching module* (§3.3.2) aiming to reveal the wider strong cultural significance of mathematical thinking and its applications to important real problems in the course of history, seen and evaluated from our modern viewpoint (that is, adopting a *heritage-like* perspective;

[13] Along the lines of this example, see Siu *et al* 2000, for the distances of objects on earth.

§3.1)[14]. It reveals the fruitful, far-reaching connections among elementary Euclidean geometry, modelling of physical situations, astronomical observations, the significance of technically accurate instrumentation, and the crucial role of approximate computations. Or, parts of such a module could be used with the same objectives, as *illuminating* examples (§3.3.1) in high-school or university courses on Euclidean geometry, trigonometry, geography, or history of science and mathematics.

From a **mathematical** point of view, examples (a)-(d) are **elementary**. However, the **emphasis** is on how **elementary geometrical ideas** and **reasoning** led historically to **astronomically** and **physically non-trivial** consequences with **far-reaching cultural implications** of the highest importance that can be posed didactically (§3.3.III). Therefore, in this example, *history* appears mainly as a *goal* (§3.2(ii)), playing a *cultural role* (§3.2.III) by helping the learners to appreciate the significance of mathematics and natural sciences in the development of human civilization. That is, the focus is on the highly innovative and revolutionary boldness of **non**-mathematical hypotheses **implicit** to this example (tacitly taken for granted as self-evident from our **modern** perspective), which, **once** formulated and **accepted**, lead to **consequences** of **far-reaching** cultural importance by means of **elementary** high-school mathematics! Besides bridging mathematics with other subjects (history, physics, astronomy, geography, philosophy) as a vast resource of specific information and meaningful problems (§3.2.1), this example greatly enriches and widens teachers' didactical repertoire (§3.2.3), and develops students' awareness that mathematics and the natural sciences may evolve in a constant dialogue with societal needs and philosophical queries that has a strong mutual influence on both the sciences and the society as a whole (§3.2.5). Lack of space allows only a few comments on each case. The *italicized text* points to possible stimulating elaborations.

4.1.1 ***Eratosthenes' measurement of the earth's circumference***: This is a well know example; see figure 1 (for history e.g. van Helden 1985, pp.4-6; original texts in Thomas 1941, §XVIII(d); Heath 1991, pp.109-112; for a didactical implementation, de Hosson 2015). The elementary geometry involved lies on three non-mathematical assumptions: (i) *Earth is spherical*; (ii) Alexandria and Syene *lie on the same meridian;* (iii) The *sun is so far away* that its light rays are *practically* parallel.

Assertions (i), (iii) are **bold** hypotheses of far-reaching implications. Considerable didactical elaboration can be designed by raising culturally and physically important questions e.g.:

(1) *How do we know that the earth is spherical?* This could be approached in various ways, including *Aristotle's reasoning* in *De Cælo* (Book II, 14, 297b, 25ff) based on the earth's circular shadow during lunar eclipses, and the appearance of new constellations while moving southward (Berry 1961, §29; Crowe 1990, ch.2, p.27).

(2) *How do we know that two places lie on the same meridian?* A problem central during the geographical expeditions in Renaissance until the 18th century, for the determination of *geographical longitude in the sea*, closely related to *the accurate measurement of time* (Berry 1961, §§127, 226; Whitrow 1988; Dörrie 1965, problem 78); and the much easier: *How do we*

[14] A *history-like perspective* is also possible by putting emphasis on original sources and their proper contextualization.

determine the geographical latitude of a place? (Dörrie 1965).

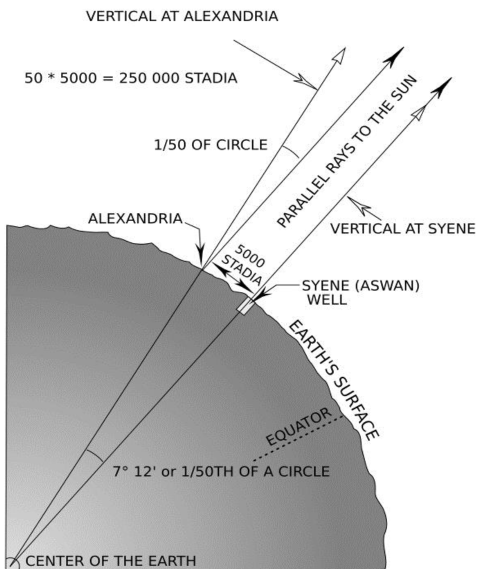

Figure 1: Eratosthenes' measurement of the earth's circumference[15]

(3) *How can we check that the sun is really so far away*? (cf. 4.1.2) etc.

Any discussion of these questions will reveal their interrelations (evidence for one can be used, or has been used to illuminate the others), their importance in the emergence and establishment of our modern view of the Cosmos, their appearance and treatment within different cultures etc.

4.1.2 ***Aristarchus' "lunar dichotomy" to measure* the *earth-sun-moon relative distances***[16]: Inherent to drawing figure 2 are *two hypotheses* about the moon: That it is (i) *spherical*; and (ii) *illuminated by the sun*, so that when the three bodies form a right-angled triangle, half of the moon disc is seen from the earth. By measuring φ, the relative distances of the sun and moon are obtained: $\cos\varphi = a_M/a$.

Remarks: (1) *How do we know that (i), (ii) hold?* Assertion (ii) is explicitly hypothesized by Aristarchus and as suggested by Pappus, the existence of *lunar eclipses strongly supports it* (Heath 1991, p.102).

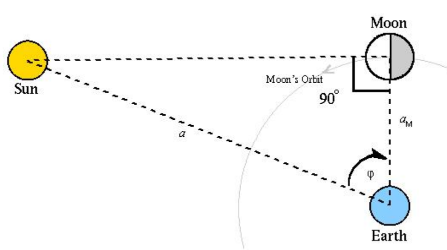

Figure 2: Aristarchus' *lunar dichotomy* method for the earth-sun-moon distances

(2) Aristarchus measured φ^o as 29/30 of a right angle (87^o), deducing that $18 < a/a_M < 20$. The modern value $a/a_M \cong 390$ amounts to $\varphi^o \cong 89^o52'$; hence, Aristarchus' was a great underestimation. This can be discussed in relation to (i) the *instruments* available in antiquity (*dioptra, quadrant, astrolabe* etc), their limited accuracy, the subsequent development of more

[15] https://en.wikipedia.org/wiki/Eratosthenes

[16] Original in Thomas 1941, §XVI(b); see also Heath 1981, chs. II.II, II.III; van Helden pp.6-7.

accurate ones (*telescope, sextant, theodolite*) etc (King 1979) - cf. §4.1.4; (ii) the *sensitive dependence of computations on the data used*: for many problems 2^o-3^o is a negligible error, but not for secφ near $\varphi = 90^o$ (to be discussed in various contexts; trigonometry, calculus etc).

(3) *Aristarchus' work* (and its further elaboration by Hipparchus) on *measuring the solar and lunar radii* using, the accidental empirical fact that the solar and lunar apparent diameters almost coincide (about 32′ on the average), *solar eclipses*, and a_M/a, provides a more complicated – though still elementary – geometrical problem in the context of this example (see e.g. van Helden 1985, pp.7-14; Crowe 1990 pp.27-30).

4.1.3 ***Copernicus' method for measuring inner planets' relative distances from the sun***: At greatest angular elongation from the sun (*existing only for the inner planets*; Mercury and Venus) the planet's distance $a_P = a \sin\theta$ (a being the earth-sun distance). Already by drawing figure 3, *it is* ***assumed*** *that the inner planets* ***revolve*** *around the sun* in ***circular*** orbits (for a more accurate description of Copernicus model see van Helden 1985, pp.42-44).

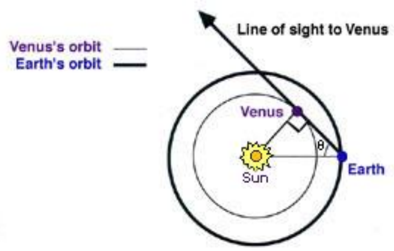

Figure 3: Copernicus' measurement of the inner planets' distances at their greatest elongation

Remarks: (1) Asking for *Copernicus' motivation of such a bold, counter-intuitive assumption* opens a fruitful discussion both on whether the earth moves and whether the planets' orbits are circular, with far-reaching implications on historical and philosophical issues (Kuhn 1973, ch.5; for a recent outline see Brush 2015, §3).

(2) The ***observed*** *greatest elongation of the inner planets*, fitted *naturally* in Copernicus' system (but not in Ptolemy's where extra hypotheses were needed), hence constituting a *strong supporting argument* (Crowe 1990, p.92).

(3) *Tycho Brahe's semi-heliocentric system*[17](and the crucial role of *his* careful astronomical *observations*) can be further explored as an intermediate step towards *Kepler's heliocentric system of elliptic orbits* (Berry 1961, §105; Crowe 1990, pp.140-143; King 1979, pp.16-23).

(4) A similar, though more elaborated *method was invented by Copernicus* for *the distances of the outer planets* (Crowe 1990, pp.95-96) that could be discussed in this context.

4.1.4 ***Trigonometric parallax*****: (i)** The *earth-sun distance using Mercury or Venus' transits across the sun's disk.*

In this method, originally suggested by Halley (van Helden 1985, p.144; Berry 1961, §§202, 227), the planet's projections *A′*, *B′* onto the solar disk are seen by observers *A*, *B* at two distant places on earth and their angular separation is measured simultaneously (figure 4).

[17] The planets revolve around the sun but the latter revolves around the immovable earth.

Remarks: (1) *It is **assumed*** that the *planet is very far away*, so that *PA*, *PB*>>*AB* the *linear distance* of *A*, *B* (cf. example 4.1.1)

(2) $\varphi^o < 0^o.5$ (sun's angular diameter), hence by (1), *approximately* $\sin\varphi \cong \varphi \cong AB/PA$ (φ expressed in radians; $\sin 0^o.5 \cong 2\pi/720$ to 6 decimals).

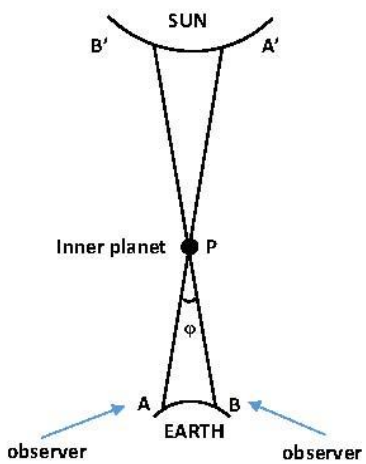

Figure 4: The earth-sun distance using inner planets' transitions across the sun

(3) $PA \cong PB$ is known either from example 4.1.3, or – more accurately – from *Kepler's 3rd law of planetary motions* (e.g. for Venus $PA = 0.72AA'$).

(4) The *earth-sun distance* gives *meaning* to all celestial *relative* distances obtained by *other methods* (examples 4.1.2, 4.1.3, 4.1.4(ii)).

(5) A pair of *Venus' transits* with an 8-year time difference occurs *rarely*, every 110 years at least. *Mercury's transits* are more frequent (every few years) but provide a *less favorable method*, because Mercury is considerably fainter than Venus.

(6) Today we use *radar methods* to accurately get AA' directly (*conceptually simple*, but *technically sophisticated*).

(ii) *Stellar (annual) parallax*

This is a method to measure the *distance of "nearby" stars*, applicable to distances less than about 100 *light years*[18] (*l.y.*; 1 light year = 9.46×10^{12}km) by observing them from two diametrically opposite places on the earth's orbit; that is, in a time span of 6 months (figure 5; AU ≡ astronomical unit, the semi-major axis of the earth's orbit $a = 1.49\times10^8$km). Even for the nearest star, $p < 1''$ ($0''.7687$), hence $a \cong d \sin p$ and $\sin p \cong \tan p \cong p$ to 11 decimals. Given that $p = (2\pi/360\times3600)p'' = p''/206{,}265$, one gets $d = (206{,}265/p'')$AU, where p'' is p expressed in seconds of arc[19].

Remarks: (1) There are two *crucial assumptions* inherent in this method: (i) *the earth revolves around the sun*; (ii) *the faint stars are (statistically) very far away* compared to the star whose parallax is sought, so that they constitute a sufficiently *immovable background* on which the star's two projections are formed and their angular separation is measured.

(2) The *conceptually* simple, but *technically* sophisticated idea of *parallax* was used by Copernicus' and Galileo's *Aristotelian opponents* and critics (including Tycho Brahe) *against* the *earth's motion*, given that no parallax had been observed at that time (Berry 1961, §129, Crowe 1990, pp.99-100, 141)[20].

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[18] Earth-based measurments. This limit has been greatly extended by using modern sophisticated photographic techniques in space telescopes.

[19] Or, $d = (1/p'')$ parsec; one parsec being the distance of an object of parallax $p=1''$, i.e. 206,265AU=3.25*l.y.*

[20] Galileo argued that this was due to the smallness of *p*, because of the huge stellar distances (Galilei 1967).

(3) *Technically possible measurement of parallax* required the telescope and became possible as late as 1838 (originally by Bessel), thus giving a *final definite experimental test of the earth's revolution* (Berry 1961, §§278-279). *Other such tests can be discussed*.

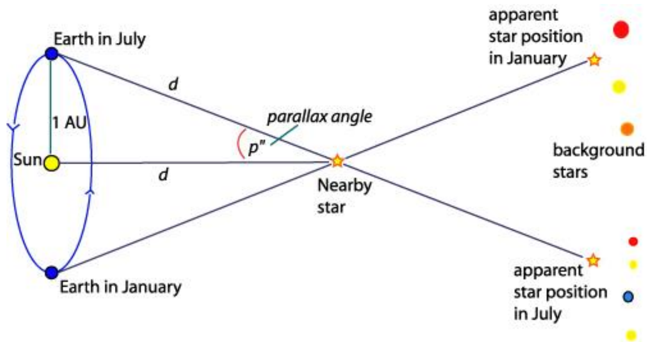

Figure 5: Trigonometric parallax for measuring stellar distances[21].

4.2 Rotations, Space-Time and the Special Theory of Relativity: *How did we get here?*

The *Theory of Special Relativity* (SR) is a standard subject for physics (but not mathematics) undergraduates. On the other hand, *matrices* and *linear algebra* are introduced early in undergraduate mathematics and physics curricula, or even last-year high-school math (as 3D analytic geometry and matrix algebra). Since the power of algebra lies in the unified-through-abstraction approach to otherwise distinct concrete problems, early undergraduates or last-year high-school students meet grave difficulties in the study of abstract algebraic concepts (vector spaces, matrices, linear transformations, groups etc) because of their limited mathematical maturity. Therefore, if such concepts are taught at this level, this ought to be done by giving as many concrete examples as possible.

A nice historical example, illustrating the framework of sections 2 & 3, is provided by an elementary**,** but fairly complete account of the foundations of the theory of *SR* in the spirit of Minkowski's original ideas about space-time and using simple matrix algebra. Below, an outline of a possible *illumination approach* (§3.3.1) is given, strongly *inspired by and based on history* (§3.3(II)), with history playing a *re-orientation role* (§3.2(II)) and serving mainly *as a tool* (§3.2(i)) for *learning mathematics & physics* (§3.2.1; by unfolding the interplay between these disciplines) and for *enriching teachers' didactical repertoire* (§3.2.3). For a detailed presentation, see Tzanakis 1999a, §3; 2000, §3.2. The didactical steps do not necessarily respect historical order; the approach is mainly *heritage-oriented* (§3.1) aiming to provide insights into the development and establishment of basic modern mathematical and physical results, and - whenever necessary - using notions and methods not available or used at the time[22]. This helps to link basic innovations in physics and their mathematical formulation, to their modern counterparts, thus illuminating better "*how did we get here?*"

Key historical elements[23]

(a) *SR* is based on *Lorentz transformations* (LT), the coordinate transformations between

[21] http://www.atnf.csiro.au/outreach/education/senior/astrophysics/astrometry1.html
[22] Vector methods and matrix algebra; a posterior custom in physics, after quantum mechanics in the 1920s.
[23] For details see Tzanakis 1999a, pp.114-115.

inertial systems (IS), i.e. systems moving with constant velocity relative to each other. Einstein derived LT in 1905 introducing and using the basic principles of *SR*: the *Special Relativity Principle* (*SRP*: the invariance of physical laws under IS coordinate transformations); and the *Principle of the constancy of the light speed* (the vacuum light speed *c* is constant in all IS, whether or not the source is moving). His derivation is based on the epistemological analysis of "simultaneity" (what does it mean that "two events are simultaneous"?), is mathematically elementary and appeals much on physical intuition and some "common-sense" assumptions about the homogeneity of space (Sommerfeld 1952, paper III).

(b) Others before him (especially Lorentz in 1899 and 1904) had obtained the *LT* while searching for the coordinate transformations that leave unaltered Maxwell's equations, without however attributing to them any **general** physical significance. They were simply the transformations between IS moving relative to the *ether* (by its definition in *absolute rest*), necessary to explain the null result of experiments that aimed to detect motion relative to the ether (especially the famous Michelson-Morley experiment).

(c) Poincaré in 1904 was the first to realize the deep general significance of the *SRP* and to derive the *LT* by a mathematically-oriented approach: He explicitly used the **group structure** of the sought transformations implied by the *SRP* and got their general form and fundamental consequences of *SR* (e.g. the relativistic velocity addition law); see Pais 1982, pp.128-130.

(d) In a seminal lecture in 1908 (Sommerfeld 1952, paper V), Minkowski introduced the *space-time* concept, unfolding the rich geometrical content implicit to Einstein's 1905 paper[24].

physically-oriented approach

Lorentz (1904)
transformations leaving invariant *Maxwell's equations*
Einstein (1905)
→ - analysis of *simultaneity*
- *Special Relativity Principle*
- Constancy *of the light speed*

mathematically-oriented approach

Poincaré (1904)
→ *group structure* of transformations between *inertial systems*

Minkowski (1908)
→ - the *space-time* concept
- transformations leaving invariant the *light cone*:
$$x^2 + y^2 + z^2 - c^2 t^2 = 0$$

Figure 6: The initial key steps in the historical development of the *Theory of Special Relativity*

Brief sketch of a possible didactical implementation

These basic historical facts are schematically shown in figure 6. An outline of their didactical reconstruction follows:

(1) The LT in 2D (one spatial *x*, one temporal *t*) result by following Minkowski's key ideas: (i) The introduction of *space-time* as a natural concept inherent to Einstein's 1905 analysis of the relative character of the *simultaneity* of events; and (ii) The constancy of *c* trivially implies that the sought transformations leave invariant the *light cone* (i.e. the surface

[24] Though *SR* was conceived **without** the *space-time* concept, Einstein's formulation of *General Relativity* ten years later would have been impossible without it (Tzanakis 1999b, §§4.3, 4.4).

on which light signals lay). This derivation uses elementary *matrix algebra* and proceeds in **close analogy** with the determination of **plane rotations** in 2D-analytic geometry: Rotations in the *xy*-plane conserve the Euclidean distance x^2+y^2; LT in the *xt*-plane conserve the Minkowski (pseudo)distance $x^2-c^2t^2$ that vanishes on the light cone (Tzanakis 1999a, §3). Proceeding along these lines, Euclidean rotations are represented by orthogonal transformations with matrix parametrized by the rotation angle φ,

$$R_\varphi = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$$

whereas the LT are represented by pseudo-rotations with matrix parametrized by φ, with $\tanh\varphi = v/c$; v being the relative speed of the two IS

$$L_\varphi = \begin{pmatrix} \cosh\varphi & \sinh\varphi \\ \sinh\varphi & \cosh\varphi \end{pmatrix}$$

(2) Strictly speaking, conservation of the *light cone* implies only that the sought transformations are *conformal*, a fact whose significance was appreciated first by Weyl in 1918 (Sommerfeld 1952, paper XI). It is more advanced - especially in 4D - to show that in the context of *SR* these conformal transformations are indeed isometries of the Minkowski (pseudo)-distance if it is assumed that they map straight lines to straight lines, as a consequence of *Newton's law of inertia*. This derivation may constitute a student project along the lines of Poincaré, using explicitly the **group structure** of the sought transformations. Fundamental results of *SR* are obtained in this way (velocity addition, length contraction etc), at the same time illustrating important abstract concepts, like *group*, *pseudo-Euclidean structure*, *conformal transformations* etc. For instance, the geometrically clear fact that successive plane rotations by angles φ and φ' coincide with a single rotation by an angle $\varphi+\varphi'$, is equivalent to the orthogonal matrices R_φ forming a group under multiplication, $R_\varphi R_{\varphi'} = R_{\varphi+\varphi'}$; a fact verified by direct calculation and using standard trigonometric identities. **Similarly**, by direct calculation and using the corresponding identities for the hyperbolic functions, the LT, hence the matrices L_φ, form a multiplication group, $L_\varphi L_{\varphi'} = L_{\varphi+\varphi'}$. But now, in view of $\tanh\varphi = v/c$, the identity

$$\tanh(\phi+\phi') = \frac{\tanh\phi + \tanh\phi'}{1+\tanh\phi\tanh\phi'}$$

becomes the important, **highly nontrivial** and counter-intuitive **relativistic law of velocity addition** for three inertial systems with parameter φ, φ', $\varphi''=\varphi+\varphi'$, moving respectively with velocities relative to each other v, v', v''; i.e. $v''=\dfrac{v+v'}{1+\frac{1}{c^2}vv'}$ Since $|v/c|<1$[25], this composition law if seen algebraically, is a new "addition" $\bullet$ in (-1,1], such that this interval with the operation $x\bullet x'=\dfrac{x+x'}{1+xx'}$ becomes a commutative group (isomorphic to the 2D Lorentz group),

[25] Negative values of v in one dimension express motion to the left of the axis.

with $x \bullet 1 = 1$ for any x in (-1,1][26] (Tzanakis 1999a for details and original references).

(3) Conformal transformations in the special case of *similarities* can be introduced naturally by looking for the *symmetry group* of *Maxwell's equations*. It is a nice example – dual to (1) above - to consider this problem for both the 2D *Laplace* and the *wave equation* and to arrive at the **orthogonal** and the **Lorentz** group of transformations, respectively (cf. Heras 2016). In fact, this is the idea behind the pre-relativistic derivations of the *LT* by Larmor and Lorentz (Schaffner 1972; Sommerfeld 1952, paper II; Whittaker 1951).

(4) At a more advanced level, one can pursue further Weyl's original idea that the two basic principles of *SR* alone, imply **only** that the transformations between IS are *conformal*[27], hence that the basic geometrical structure of space-time as a manifold is **not** its (pseudo)-metric, but vector *parallelism*. Actually, Weyl was trying to formulate a unified theory of gravitation **and** electromagnetism. In this way one has a natural path to introduce the first example of what - from today's perspective - was the first *gauge field theory* (Weyl 1952, §35, 36; Sommerfeld 1952, paper XI; see also Tzanakis 2002, §3.1).

4.3 Differential Equations, (Functional) Analysis and Quantum Mechanics: *What did (or did not happen) in the past and why (or why not)?*

For most mathematics and physics undergraduates (at least), the following subjects, if ever taught, are taught separately so as heterogeneous; thus learnt as unrelated, and finally conceived as completely alien to each other:

- *Jacobi's general method* to solve 1st order partial differential equations (PDEs) by solving an equivalent system of 1st order ordinary differential equations (ODEs).
- The *canonical (Hamiltonian) formulation of classical mechanics* and its associated *Hamilton-Jacobi theory* for solving mechanical problems.
- The formal analogy as variational principles of Fermat's *Principle of Least Time* in Geometrical Optics (GO) and Maupertuis' *Principle of Least Action* in Classical Mechanics (CM).
- *Schrödinger's equation* as the cornerstone of Quantum Mechanics (QM).
- Heisenberg's *Matrix Mechanics* and infinite dimensional matrices.
- Infinite dimensional linear spaces; in particular (separable) *Hilbert spaces*.
- Observable quantities in QM as (*hermitian/self adjoint*) operators in linear spaces and their non-commutative algebraic structure.
- Fourier analysis, Lebesgue integration and square-integrable functions.

This is related to the fact that key issues are introduced unmotivated e.g. (i) *Schrödinger's equation* is introduced ad hoc as a basic axiom, whose significance is to be evaluated **a posteriori** by its compatibility with experiment (see Tzanakis & Coutsomitros 1988, §3); (ii) *Separable Hilbert spaces* are introduced rather mysteriously as linear spaces with a countable, dense subset (e.g. Richtmyer 1978).

However, historically these subjects have strong interconnections that motivated,

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[26] Physically this means that in any IS a light signal travels at speed *c*, no matter what the speed of this IS is.
[27] They become isometries only if they map straight lines to straight lines (see (2) above); an extra assumption abandoned after the formulation of general relativity in 1915.

stimulated, and guided the course of development to their current form, which can be beneficial for their teaching and learning today. This example has been presented from a variety of perspectives elsewhere[28]. Here, a possible didactical illustration is given within the framework of sections 2 & 3: Briefly outlining a possible *history-based approach* (§3.3.3) *inspired by history* (§3.3(II)), in which history has a *replacement role* (§3.2(I)), serving mainly *as a tool* (§3.2(i)) for *learning mathematics & physics* (§3.2.1; contrasting the historical development vs. its polished final results, and unfolding the strong bonds between mathematics and physics), *enriching teachers' didactical repertoire* (§3.2.3; furnishing ample insight to the motivation for introducing new concepts and theories) and possibly amending students' *affective predisposition* towards learning abstract and difficult concepts (§3.2.4; by stressing the constructive role of posing questions and of attempting lines of inquiry despite any existing vagueness and uncertainties). The emphasis is on a *history-oriented* approach (§3.1) by enlightening "*what and why did/did not happen?*", but a *heritage-oriented* approach (§3.2) is also possible if the didactical sequence is structured differently to fit better dealing with the question "*how did we get here?*"

Key historical elements

(a) In the 18th century the formulation of Maupertuis' *Principle of Least Action* in CM was motivated by and in analogy with Fermat's *Principle of Least Time* in GO (Dugas 1988, part III, §§V.4-V.7, V.11, V.12). For a mechanical system with finite degrees of freedom and constant total energy, Maupertuis' principle as formulated by Hamilton (in the mid 1830s) was a main motivation for Hamilton's **mathematically unified** development of the two theories, which (together with Jacobi's important work since 1837; Nakane & Fraser 2002) provided a general method for solving 1st order PDEs and constituted another formulation of CM that became central to the solution of mechanical problems as well, the *Hamilton-Jacobi method* (Dugas 1988, part IV, §§VI.2-VI.5; Klein 1979, pp.179-203; Lanczos 1970; Yourgrau & Maldestam 1968; Goldstein 1980, ch.10; Tzanakis 2000, §3.3).

(b) Hamilton's ideas stimulated de Broglie to see this similarity as an indication of a deep relation between mechanical and optical phenomena in his 1924 doctoral thesis[29]. By relativistic arguments, he extended to matter the Planck-Einstein quantization relations for radiation, thus predicting the wave nature of atomic particles (Tzanakis 1999b, §4.2). In 1926, elaborating on this idea, Schrödinger arrived at the formulation of *Wave Mechanics*[30].

(c) In the 1920s, atomic physics was a complicated mixture of CM, electrodynamics, semi-empirical rules, and heuristic arguments. Physicists were trying hard to develop models of atomic phenomena in a rather vague and confusing landscape. Heisenberg's breakthrough in 1925 (van der Waerden 1967, paper 12) was the development of an algebraic manipulation of

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[28] Tzanakis, 1998, 1999b, 2000, 2002, Tzanakis & Coutsomitros 1988; Tzanakis & Thomaidis 2000.

[29] "*Guidé par l'idée d'une identité profonde du principe de moindre action et de celui de Fermat, j'ai été conduit… à* ***admettre*** [que] *les trajectoires dynamiquement possible de l'un coïncidaient avec les rayons de l'autre*" (de Broglie 1925).

[30] "*…our classical mechanics is the* ***complete*** *analogy of geometrical optics…Then it becomes a question of searching for an undulatory mechanics…* [by] *working out… the Hamiltonian analogy on the lines of undulatory optics*" (Schrödinger 1982, paper II, p.18); see also Dugas 1988, p.401; Goldstein 1980, §10-8.

atomic quantities, **in analogy** with *Fourier series operations*, with the crucial difference that the Fourier-like frequencies and coefficients were **doubly** indexed as a consequence of *Ritz' combination principle* in atomic spectroscopy (see figure 8). The novel fact was the non-commutative multiplication of atomic quantities, originally unintelligible to Heisenberg. However, it was immediately realized by the mathematically educated Born and Jordan, that Heisenberg's calculus was just the algebra of (generally, infinite-dimensional) matrices (Born 1969, §V.3; Heisenberg 1949, Appendix §1). This led to *Matrix Mechanics*, the first formulation of QM (van der Waerden 1967, paper 15; Mehra & Rechenberg 1982).

(d) After the formulation of *Wave Mechanics*, physicists were puzzled by the existence of two **conceptually** and **mathematically** totally **different** theories of atomic phenomena (*Matrix Mechanics* and *Wave Mechanics*), which nevertheless gave **identical results** compatible with the empirical data. Schrödinger (in 1926) and von Neumann (from 1927) showed that the two theories were mathematically equivalent. However they proceeded in different ways:

Schrödinger provided a **formal** proof: Functions in his *Wave Mechanics* were elements of the linear space $L^2(\boldsymbol{R})$ of Lebesgue quadratically integrable complex-valued functions, equipped with the scalar product $< f, g >= \int_{-\infty}^{\infty} f^{*} g d\mu$, f^{*} being the complex conjugate of *f*. The matrices in *Matrix Mechanics* could be seen as acting on the infinite-dimensional linear space l^2 of complex sequences $\alpha=(\alpha_1, \alpha_2, ...)$, such that $\sum_k |\alpha_k|^2<+\infty$; a straightforward generalization of the familiar *n*-dimensional Euclidean space. By expanding the wave functions in an orthonormal (ON) basis of $L^2(\boldsymbol{R})$, solving his PDE becomes the solution of the eigenvalue problem for the Hamiltonian matrix $\hat{H}$ of *Matrix Mechanics* and vice versa; i.e. Schrödinger's equation reduces to a system of linear equations, whose matrix was $\hat{H}$ (Schrödinger 1982, paper 4).

von Neumann's approach was **mathematically-oriented** (von Neumann 1947, ch.I). He identified the basic properties of the objects with which the two theories were dealing, emphasizing the **linear structure** of the **function spaces** (that is $L^2(\boldsymbol{R})$ and l^2) underlying them. Thus he was led to introduce axiomatically what became known as a *separable Hilbert space* (von Neumann 1947, ch.II). Then he resolved the puzzle by proving that all these (normed) spaces are isometric: The two **conceptually different** theories were **different representations** of the **same abstract mathematical structure** that underlies the formalism of QM (von Neumann 1947, ch.II, theorem 9).

Outline of a possible didactical sequence

In view of these historical facts, an approach to this vast and rich subject is outlined below:

(1) The *Least Action Principle* and the *Principle of Least Time* constitute **important** examples of variational principles, leading to mathematically interesting equations that are central in CM (the *Hamilton-Jacobi equation*) and (Geometrical) Optics (the *eikonal equation*)[31]; Pauli 1973;

[31] In Schrödinger's concise wording: "*The **inner** connection between Hamilton's theory and the process of wave propagation is anything but a new idea... Hamilton's variational principle… correspond*[s] *to Fermat's **Principle** for a wave propagation in configuration space… and the Hamilton-Jacobi equation expresses Huygen's **Principle** for the wave propagation… we must regard* [this] *analogy as one between mechanics and **geometrical** optics and not… **undulatory** optics*" (Schrödinger 1982, pp.13, 17; see also next paragraph).

Courant & Hilbert 1962 §II.9.2. Both are **generic** examples, making possible the establishment of a general result in the theory of PDEs: the solution of a large class of 1st order PDEs (the Hamilton-Jacobi equation being an important special case) is *equivalent* to the solution of a system of 1st order ODEs; the associated *canonical* (or *Hamilton*'s) equations (cf. Nakane & Fraser 2002, pp.49, 56). This result is of central importance both in the theory of differential equations and the calculus of variations (e.g. Courant & Hilbert 1962 §§II.8-II.10; Sneddon 1957; Gelfand & Fomin 1963) and in CM (e.g. Goldstein 1980, §§10.1, 10.3; Lanczos 1970). In fact, one can proceed close to Hamilton's and Jacobi's approaches to illuminate the subject from two different, but equally important angles (Dugas 1988, part IV, ch.VI; Klein 1979, pp.182-196).

(2) Schrödinger's elaboration of Hamilton's mathematically unified treatment of CM and GO, was based on **arguing by analogy**: If *CM* is mathematically similar to *GO*, and since the latter is **only** an approximation to *Wave Optics*, *CM* may be **only** an **approximation** to a *Wave Mechanics*, which is similar to Wave Optics in the same way that CM is similar to GO. Thus, entirely **within** the conceptual and mathematical framework of **CM** an equation results as the **mechanical equivalent** of the *wave equation*, which is **formally** identical with *Schrödinger's* up to an undetermined constant σ. This is schematically shown in figure 7.

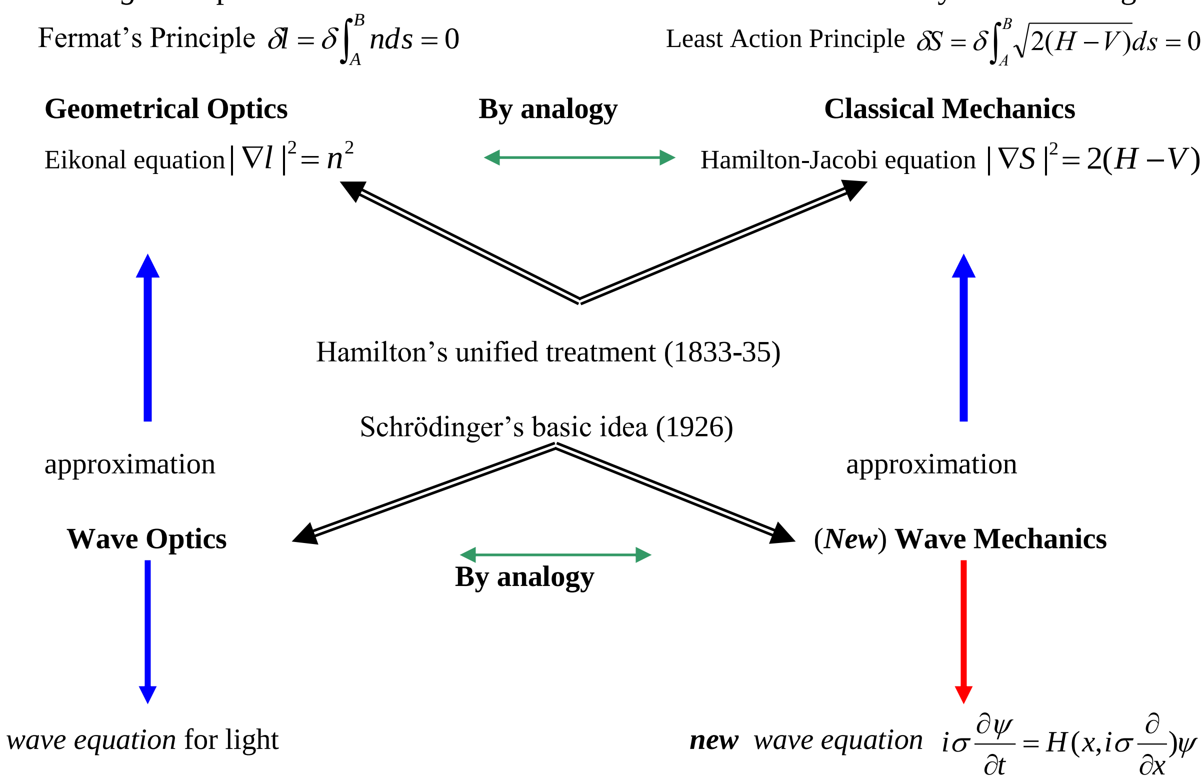

Figure 7: A schematic representation of Schrödinger's reasoning by analogy

Here *l*, *n* are the *optical length* and the *index of refraction* $n=c/v$ (c, v the speed of light in vacuum and in the optical medium); *S* the *action* of a mechanical system with potential energy *V* and total (mechanical) energy $H(x, p)$, *x* and *p* the coordinates in configuration space and the corresponding generalized momenta, and *ds* the element of arc length in real and configuration space respectively (Tzanakis 1998, §2; cf. Schrödinger 1982, pp.160-163, 189-192).

But then, why Hamilton did **not** formulate *Wave Mechanics*? To this end an extra condition was needed, supplied by de Broglie, who - based on *SR* theory - postulated the wave nature of **matter** by "symmetrising" the Planck-Einstein conception of the **corpuscular** nature of **radiation** with the aid of the same formal relations: as for the *light quanta* (photons), the energy E and the momentum $\boldsymbol{p}$ of a particle are proportional to the frequency ν and the wave number $\boldsymbol{k}$ of the associated wave; $E=h\nu$, $\boldsymbol{p}=h\boldsymbol{k}$, h being Planck's universal constant (Tzanakis 1998). In this case, σ necessarily coincides with h. This was the crucial **physical** idea that led from Hamilton's **mathematically** unified treatment of the **conceptually different** theories of *GO* and *CM*, to a deep, fruitful **physical** theory of atomic phenomena. Lack of this crucial physical idea prevented Hamilton from inventing wave mechanics.[32]

(3) Heisenberg formulated *matrix mechanics* (see (c) above) reasoning by analogy, as schematically shown in figure 8 (n, l, m, k being integers and $\boldsymbol{I}$ the identity matrix).

Classical positions & momenta $\boldsymbol{q, p}$		**Quantum** positions & momenta $\boldsymbol{q, p}$
Fourier representation		(looking for a) ***new*** representation (because of)
Singly indexed frequencies		***Doubly*** indexed spectral frequencies
$\nu_k = k\omega$		ν_{nm}
		obeying ***Ritz principle***
$\nu_k + \nu_l = \nu_{k+l}$	**by analogy**	$\nu_{nl} + \nu_{lm} = \nu_{nm}$
$q(t) = \sum q_k \exp(i\, \nu_k t)$	$\longleftrightarrow$	$q(t) \sim q_{nm} \exp(i\, \nu_{nm} t)$
	Operations **by analogy**	
$p+q \longrightarrow (p_k+q_k) \exp(i\, \nu_k t)$	$\longleftrightarrow$	$p+q \longrightarrow (p_{nm}+q_{nm}) \exp(i\, \nu_{nm} t)$
$pq \longrightarrow (\sum_l p_l\, q_{k-l}) \exp(i\, \nu_k t)$	$\longleftrightarrow$ **by analogy + Ritz Principle**	$pq \longrightarrow (\sum_l p_{nl}\, q_{lm}) \exp(i\, \nu_{nm} t)$
		$qp - pq = \frac{h}{2\pi i} I$
		(hence, $pq \neq qp$ leading to Heisenberg's uncertainty relations)

Figure 8: Schematic representation of Heisenberg's formulation of *Matrix Mechanics*

It is important to note that (i) the non-commutativity of quantum quantities was **imposed** by the doubly-indexed spectral frequencies so that the **experimentally obtained** *Ritz principle* is satisfied; (ii) the famous *Heisenberg uncertainty relations* are a necessary consequence of this non-commutativity (Heisenberg 1949, §II.1; Born 1969, Appendices XII, XXVI; for the

[32] In Hamilton's time, there was no **physical** reason to consider the **formal** similarity between *GO* and *CM* nothing more than a **mathematical** analogy. Hamilton never attributed to it deeper physical meaning and it never attracted enough attention after Hamilton, despite Klein's use of this optical analogy to develop the Hamilton-Jacobi theory as a kind of optics in a multidimensional configuration space (Tzanakis 1998, p.73; Goldstein 1980, pp.491-492; Jammer 1966, pp.237-238). At that time neither the wave, nor the corpuscular theory of light were generally accepted. Hamilton's motivation was the desire to formulate GO in a way capable of interpretation in the context of either of the two theories (Dugas 1988, pp.390-391; cf. Goldstein 1980, p.489).

history see Jammer 1966, §7.1).

(4) One can proceed along these lines to motivate the unforced introduction of important concepts and to prove basic results in *Functional Analysis* and *Fourier Analysis*. For instance:

(i) To present in a simple form the basic **mathematical problem** of the Heisenber-Born-Jordan *Matrix Mechanics* and Schrödinger's *Wave Mechanics*: the diagonalization of the hamiltonian matrix $\widehat{H}$ in l^2, and the solution of Schrödinger's equation in $L^2(\boldsymbol{R})$, respectively.

(ii) Given that these physically and mathematically a priori different theories yield identical, experimentally correct predictions, the question of their relation naturally arises. Hence Schrödinger's heuristic and non-rigorous arguments can be used to show formally that Schrödinger's equation reduces to a matrix eigenvalue problem, once an ON basis of $L^2(\boldsymbol{R})$ has been chosen (Schrödinger 1982, paper 4).

(iii) Motivated by this approach, a rigorous proof that l^2 and $L^2(\boldsymbol{R})$ are isometric Hilbert spaces can be given, a fact already included in the works of Riesz and Fisher in 1907 (Bourbaki 1974; Dorier 1996).

(iv) In view of the heuristic approach in (ii), it is reasonable to reverse the argument and to consider linear spaces with a scalar product spanned by a countable ON basis, thus arriving at their isomorphism and their various equivalent characterizations (spaces having either an ON sequence spanning the space, or an ON sequence not perpendicular to any element, or obeying a generalized Parseval identity with respect to an ON sequence, or having a countable dense subset). This is essentially von Neumann's approach (1947, chs.I, II; see also Dieudonné 1981).

(5) Continuing along these lines, to introduce many other important concepts and results of functional analysis: *bounded* vs *unbounded operators* and the associated concept of a *closed operator*; the distinction between *hermitian* and *self-adjoint operators*; the *extension of an operator* etc; all of which are both basic in functional analysis and indispensable to QM (and modern theoretical physics, in general); see Tzanakis 2000, §3.4.

5 Final comments

In this paper the innermost relationship of mathematics and physics considered both from the point of view of their epistemological characteristics and their historical development has been explored and arguments have been presented to support the three main *theses* formulated in section 2. From an educational point of view, they imply that in mathematics and physics education this relationship should be taken into account explicitly. The main issues to be faced in any such attempt have been addressed in section 3 and a framework for integrating history into teaching and learning these disciplines has been outlined on a common ground. The general ideas presented in sections 2, 3 have been illustrated by analyzing three examples of quite different content and orientation. Hopefully, enough evidence has been presented to support that (a) it is impossible to deeply understand either mathematics or physics without being sufficiently aware of their interconnections and mutual influence; (b) on the contrary, taking into account their interrelation is beneficial for teaching and learning either discipline.

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