Optimal well-posedness for the inhomogeneous incompressible Navier-Stokes system with general viscosity

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July 26, 2016

Abstract

In this paper we obtain new well-possedness results concerning a linear inhomogenous Stokes-like system. These results are used to establish local well-posedness in the critical spaces for initial density ρ_0 and velocity u_0 such that $\rho_0-\rho\in\dot{B}^{\frac{3}{p}}_{p,1}(\mathbb{R}^3)$, $u_0\in\dot{B}^{\frac{3}{p}-1}_{p,1}(\mathbb{R}^3)$, $p\in\left(\frac{6}{5},4\right)$, for the inhomogeneous incompressible Navier-Stokes system with variable viscosity. To the best of our knowledge, regarding the $3D$ case, this is the first result in a truly critical framework for which one does not assume any smallness condition on the density.

Keywords Inhomogeneous Navier-Stokes system; critical regularity; Lagrangian coordinates;

MSC: 35Q30, 76D05

1 Introduction

In this paper we deal with the well-posedness of the inhomogeneous, incompressible Navier-Stokes system:

$$\begin{cases} \partial_t\rho+\operatorname{div}(\rho u)=0,\\ \partial_t(\rho u)+\operatorname{div}(\rho u\otimes u)-\operatorname{div}(\mu(\rho)D(u))+\nabla P=0,\\ \operatorname{div}u=0,\\ u_{|t=0}=u_0. \end{cases}\tag{1.1}$$

In the above, $\rho>0$ stands for the density of the fluid, $u\in\mathbb{R}^n$ is the fluid's velocity field while P is the pressure. The viscosity coefficient μ is assumed to be a smooth, strictly positive function of the density while

$$D(u)=\nabla u+Du.$$

is the deformation tensor. This system is used to study fluids obtained as a mixture of two (or more) incompressible fluids that have different densities: fluids containing a melted substance, polluted air/water etc.

There is a very rich literature devoted to the study of the well-posedness of (1.1) which we will review in the following lines. Briefly, the question of existence of weak solutions with finite energy was first considered by Kazhikov in [23] (see also [5]) in the case of constant viscosity. The case with a general viscosity law was treated in [26]. Weak solutions for more regular data were considered in [18]. Recently, weak solutions were investigated by Huang, Paicu and Zhang in [22].

The unique solvability of (1.1) was first addressed in the seminal work of Ladyženskaja and Solonnikov in [25]. More precisely, considering $u_0\in W^{2-\frac{2}{p},p}(\Omega)$, with $p>2$, a divergence free vector field that vanishes on $\partial\Omega$ and $\rho_0\in C^1(\Omega)$ bounded away from zero, they construct a global strong solution in the $2D$ case respectively a local solution in the $3D$ case. Moreover, if u_0 is small in $W^{2-\frac{2}{p},p}(\Omega)$ then global well-posedness holds true.

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†This work was partially supported by a grant of the Romanian National Authority for Scientific Research and Innovation, CNCS - UEFISCDI, project number PN-II-RU-TE-2014-4-0320

The question of weak-strong uniqueness was addressed in [8] for the case of sufficiently smooth data with vanishing viscosity.

Over the last thirteen years, efforts were made to obtain well-posedness results in the so called critical spaces i.e. the spaces which have the same invariance with respect to time and space dilation as the system itself, namely

$$\begin{cases} (\rho_0(x), u_0(x)) \to (\rho_0(lx), u_0(lx)), \\ (\rho(t,x), u(t,x)) \to \left(\rho\left(l^2t, lx\right), lu\left(l^2t, lx\right), l^2P\left(l^2t, lx\right)\right). \end{cases}$$

For more details and explanations for nowadays a classical approach we refer to [9] or [17]. In the Besov space context, which includes in particular the more classical Sobolev spaces, these are

$$\rho_0 - \bar{\rho} \in \dot{B}^{\frac{n}{p_1}}_{p_1,r_1} \text{ and } u_0 \in \dot{B}^{\frac{n}{p_2}-1}_{p_2,r_2}. \tag{1.2}$$

where $\bar{\rho}$ is some constant density state and n is the space dimension. Working with densities close (in some appropriate norm) to a constant has led to a rich literature. In [9] local and global existence results are obtained for the case of constant viscosity and by taking the initial data

$$\rho_0 - \bar{\rho} \in L^\infty \cap \dot{B}^{\frac{n}{2}}_{2,\infty},\ u_0 \in \dot{B}^{\frac{n}{2}-1}_{2,1}$$

and under the assumption that $\|\rho_0 - \bar{\rho}\|_{L^\infty \cap \dot{B}^{\frac{n}{2}}_{2,\infty}}$ is sufficiently small. The case with variable viscosity and for initial data

$$\rho_0 - \bar{\rho} \in \dot{B}^{\frac{n}{p}}_{p,1} \text{ and } u_0 \in \dot{B}^{\frac{n}{p}-1}_{p,1},$$

$p \in [1, 2n)$, is treated in [1]. However, uniqueness is guaranteed once $p \in [1, n)$. These results where further extended by H. Abidi and M. Paicu in [4] by noticing that $\rho_0 - \bar{\rho}$ can be taken in a larger Besov space. In [19], B. Haspot established results in the same spirit as those mentioned above (however, the results are obtained in the nonhomogeneous framework and thus do not fall into the critical framework) in the case where the velocity field is not Lipschitz. In [15], using the Lagrangian formulation, R. Danchin and P.B. Mucha establish local and global results for (1.1) with constant viscosity when $\rho_0 - \bar{\rho} \in \mathcal{M}(\dot{B}^{\frac{n}{p}-1}_{p,1})$, $u_0 \in \dot{B}^{\frac{n}{p}-1}_{p,1}$ and under the smallness condition:

$$\|\rho_0 - \bar{\rho}\|_{\mathcal{M}(\dot{B}^{\frac{n}{p}-1}_{p,1})} \ll 1,$$

where $\mathcal{M}(\dot{B}^{\frac{n}{p}-1}_{p,1})$ stands for the multiplier space of $\dot{B}^{\frac{n}{p}-1}_{p,1}$. In particular, functions with small jumps enter this framework. Moreover, as a consequence of their approach, the range of Lebesgue exponents for which uniqueness of solutions holds is extended to $p \in [1, 2n)$. In [27], [21], [20], [22] the authors improve the smallness assumptions used in order to obtain global existence. To summarize, all the previous well-posedness results in critical spaces were established assuming that the density is close in some sense to a constant state.

When the later assumption is removed, one must impose more regularity on the data. For the case of constant viscosity, in [10], R. Danchin obtains local well posedness respectively global well posedness in dimension $n = 2$ for data drawn from the nonhomogeneous Sobolev spaces: $(\rho_0 - \bar{\rho}, u_0) \in H^{\frac{n}{2}+\alpha} \times H^{\frac{n}{2}-1+\beta}$ with $\alpha, \beta > 0$. The same result for the case of general viscosity law is established in [1]. For data with non Lipschitz velocity results were established in [19]. Concerning rougher densities, in [16], considering $\rho_0 \in L^\infty(\mathbb{R}^d)$ bounded from below and $u_0 \in H^2\left(\mathbb{R}^d\right)$ Danchin and Mucha construct a unique local solution. Again, supposing that the density is close to some constant state they prove global well-posedness. These results are generalized in [28]. Taking the density as above the authors construct: a global unique solution provided that $u_0 \in H^s\left(\mathbb{R}^2\right)$ for any $s > 0$ in the $2D$ case respectively a local unique solution in the $3D$ case considering $u_0 \in H^1\left(\mathbb{R}^3\right)$. Moreover, assuming that u_0 is suitably small the solution constructed is global even in the three dimensional case.

In critical spaces of the Navier-Stokes system i.e. (1.2) there are few well posedness results. Very recently, in the $2D$ case and allowing variable viscosity, H. Xu, Y. Li and X. Zhai, [30] constructed a unique local solution to (1.1) provided that the initial data satisfies $\rho_0 - \bar{\rho} \in \dot{B}^{\frac{2}{p}}_{p,1}\left(\mathbb{R}^2\right)$ and $u_0 \in \dot{B}^{\frac{2}{p}-1}_{p,1}\left(\mathbb{R}^2\right)$. Moreover, if $\rho_0 - \bar{\rho} \in L^p \cap \dot{B}^{\frac{2}{p}}_{p,1}\left(\mathbb{R}^2\right)$ and the viscosity is supposed constant, their solution becomes global. In the $3D$ situation, to the best of our knowledge, the results that are closest to the critical regularity are those presented in [2] and [3] (for a similar result in the periodic case one can consult [29]). More precisely, in $3D$, assuming that

$$\rho_0 - \bar{\rho} \in L^2 \cap \dot{B}^{\frac{3}{2}}_{2,1} \text{ and } u_0 \in \dot{B}^{\frac{1}{2}}_{2,1}$$

and taking constant viscosity, H. Abidi, G. Gui and P.Zhang, [2], show the local well-posedness of system (1.1). Moreover, if the initial velocity is small then global well-posedness holds true. In [3] they establish the same kind of result for initial data

$$\rho_0 - \bar{\rho} \in L^\lambda \cap \dot{B}^{\frac{3}{\lambda}}_{\lambda,1} \text{ and } u_0 \in \dot{B}^{\frac{3}{p}-1}_{p,1}$$

where $\lambda \in [1,2]$, $p \in [3,4]$ are such that $\frac{1}{\lambda} + \frac{1}{p} > \frac{5}{6}$ and $\frac{1}{\lambda} - \frac{1}{p} \leq \frac{1}{3}$.

One of the goals of the present paper is to establish local well-posedness in the critical spaces:

$$\rho_0 - \bar{\rho} \in \dot{B}^{\frac{3}{p}}_{p,1}\left(\mathbb{R}^3\right),\ u_0 \in \dot{B}^{\frac{3}{p}-1}_{p,1}\left(\mathbb{R}^3\right),\ p \in \left(\frac{6}{5}, 4\right)$$

for System (1.1)

- with general smooth variable viscosity law,
- without any smallness assumption on the density,
- without the extra low frequencies assumption. In particular, we generalize the local existence and uniqueness result of H. Abidi, G. Gui and P. Zhang from [2] thus achieving the critical regularity.

As in [15] we will not work directly with system (1.1) instead we will rather use its Lagrangian formulation. By proceeding so, we are naturally led to consider the following Stokes problem with time independent, nonconstant coefficients:

$$\begin{cases} \partial_t u - a \operatorname{div}(bD(u)) + a\nabla P = f, \\ \operatorname{div} u = \operatorname{div} R, \\ u_{|t=0} = u_0. \end{cases} \tag{1.3}$$

We establish global well-posedness results for System (1.3). This can be viewed as a first step towards generalizing the results of Danchin and Mucha obtained in [17], Chapter 4, for the case of general viscosity and without assuming that the density is close to a constant state. Let us mention that the estimates that we obtain for System (1.3) have a wider range of applications: in a forthcoming paper we will investigate the well-posedness issue of the Navier-Stokes-Korteweg system under optimal regularity assumptions.

To summarize all the above, our main result reads:

Theorem 1.1. *Let us consider $p \in \left(\frac{6}{5}, 4\right)$. Assume that there exists positive constants $(\bar{\rho}, \rho_\star, \rho^\star)$ such that $\rho_0 - \bar{\rho} \in \dot{B}^{\frac{3}{p}}_{p,1}\left(\mathbb{R}^3\right)$ and $0 < \rho_\star < \rho_0 < \rho^\star$. Furthermore, consider u_0 a divergence free vector field with coefficients in $\dot{B}^{\frac{3}{p}-1}_{p,1}\left(\mathbb{R}^3\right)$. Then, there exists a time $T > 0$ and a unique solution $(\rho, u, \nabla P)$ of system (1.1) with*

$$\rho - \bar{\rho} \in \mathcal{C}_T(\dot{B}^{\frac{3}{p}}_{p,1}(\mathbb{R}^3)) \cap L^\infty_T(\dot{B}^{\frac{3}{p}}_{p,1}(\mathbb{R}^3)),\ u \in \mathcal{C}_T(\dot{B}^{\frac{3}{p}-1}_{p,1}(\mathbb{R}^3)) \text{ and } \left(\partial_t u, \nabla^2 u, \nabla P\right) \in L^1_T(\dot{B}^{\frac{3}{p}-1}_{p,1}(\mathbb{R}^3)).$$

One salutary feature of the Lagrangian formulation is that the density becomes independent of time. More precisely, considering $(\rho, u, \nabla P)$ a solution of (1.1) and denoting by X the flow associated to the vector field u:

$$X(t,y) = y + \int_0^t u(\tau, X(\tau, y))\, dy$$

we introduce the new Lagrangian variables:

$$\bar{\rho}(t,y) = \rho(t, X(t,y)),\ \bar{u}(t,y) = u(t, X(t,y)) \text{ and } \bar{P}(t,y) = P(t, X(t,y)).$$

Then, using the chain rule and Proposition 4.18 we gather that $\bar{\rho}(t,\cdot) = \rho_0$ and

$$\begin{cases} \rho_0 \partial_t \bar{u} - \operatorname{div}(\mu(\rho_0) A_{\bar{u}} D_{A_{\bar{u}}}(\bar{u})) + A^T_{\bar{u}} \nabla \bar{P} = 0, \\ \operatorname{div}(A_{\bar{u}} \bar{u}) = 0, \\ \bar{u}_{|t=0} = u_0. \end{cases} \tag{1.4}$$

where $A_{\bar{u}}$ is the inverse of the differential of X, and

$$D_A(\bar{u}) = D\bar{u} A_{\bar{u}} + A^T_{\bar{u}} \nabla \bar{u}.$$

Note that we can give a meaning to (1.4) independently of the Eulerian formulation by stating:

$$X(t,y) = y + \int_0^t \bar{u}(\tau, y)\, d\tau.$$

Theorem 1.1 will be a consequence of the following result:

Theorem 1.2. *Let us consider $p \in \left(\frac{6}{5}, 4\right)$. Assume that there exists positive $(\bar{\rho}, \rho_\star, \rho^\star)$ such that $\rho_0 - \bar{\rho} \in \dot{B}^{\frac{3}{p}}_{p,1}(\mathbb{R}^3)$ and $0 < \rho_\star < \rho_0 < \rho^\star$. Furthermore, consider u_0 a divergence free vector field with coefficients in $\dot{B}^{\frac{3}{p}-1}_{p,1}(\mathbb{R}^3)$. Then, there exists a time $T > 0$ and a unique solution $\left(\bar{u}, \nabla \bar{P}\right)$ of system (1.4) with*

$$\bar{u} \in \mathcal{C}_T(\dot{B}^{\frac{3}{p}-1}_{p,1}(\mathbb{R}^3)) \text{ and } \left(\partial_t \bar{u}, \nabla^2 \bar{u}, \nabla \bar{P}\right) \in L^1_T(\dot{B}^{\frac{3}{p}-1}_{p,1}(\mathbb{R}^3)).$$

Moreover, there exists a positive constant $C = C(\rho_0)$ such that:

$$\|u\|_{L^\infty_T(\dot{B}^{\frac{3}{p}-1}_{p,1})} + \left\|\left(\nabla^2 u, \nabla P\right)\right\|_{L^1_T(\dot{B}^{\frac{3}{p}-1}_{p,1})} \leq \|u_0\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}} \exp(CT).$$

The study of system (1.4) naturally leads to the Stokes-like system (1.3). In Section 2 we establish the global well-posedness of System (1.3). More precisely, we prove:

Theorem 1.3. *Let us consider $n \in \{2,3\}$ and $p \in (1,4)$ if $n = 2$ or $p \in \left(\frac{6}{5}, 4\right)$ if $n = 3$. Assume there exist positive constants $\left(a_\star, b_\star, a^\star, b^\star, \bar{a}, \bar{b}\right)$ such that $a - \bar{a} \in \dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n)$, $b - \bar{b} \in \dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n)$ and*

$$0 < a_\star \leq a \leq a^\star,$$
$$0 < b_\star \leq b \leq b^\star.$$

Furthermore, consider the vector fields u_0 and f with coefficients in $\dot{B}^{\frac{n}{p}-1}_{p,1}(\mathbb{R}^n)$ respectively in $L^1_{loc}(\dot{B}^{\frac{n}{p}-1}_{p,1}(\mathbb{R}^n))$. Also, let us consider the vector field $R \in (\mathcal{S}'(\mathbb{R}^n))^n$ with[1]

$$\mathcal{Q}R \in \mathcal{C}([0,\infty); \dot{B}^{\frac{n}{p}-1}_{p,1}(\mathbb{R}^n)) \text{ and } (\partial_t R, \nabla \operatorname{div} R) \in L^1_{loc}(\dot{B}^{\frac{n}{p}-1}_{p,1}(\mathbb{R}^n))$$

such that:

$$\operatorname{div} u_0 = \operatorname{div} R(0,\cdot).$$

Then, system (1.3) has a unique solution $(u, \nabla P)$ with:

$$u \in C([0,\infty), \dot{B}^{\frac{n}{p}-1}_{p,1}(\mathbb{R}^n)) \text{ and } \partial_t u, \nabla^2 u, \nabla P \in L^1_{loc}(\dot{B}^{\frac{n}{p}-1}_{p,1}(\mathbb{R}^n)).$$

Moreover, there exists a constant $C = C(a,b)$ such that:

$$\begin{aligned} &\|u\|_{L^\infty_t(\dot{B}^{\frac{n}{p}-1}_{p,1})} + \left\|\left(\partial_t u, \nabla^2 u, \nabla P\right)\right\|_{L^1_t(\dot{B}^{\frac{n}{p}-1}_{p,1})} \\ &\leq \left(\|u_0\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}} + \|(f, \partial_t R, \nabla \operatorname{div} R)\|_{L^1_t(\dot{B}^{\frac{n}{p}-1}_{p,1})}\right) \exp(C(t+1)), \end{aligned} \tag{1.5}$$

for all $t \in [0,\infty)$.

The difficulty in establishing such a result comes from the fact that the pressure and velocity are "strongly" coupled as opposed to the case where ρ is close to a constant see Remark 2.3 below. The key idea is to use the high-low frequency splitting technique first used in [11] combined with the special structure of the "incompressible" part of $a\nabla P$ i.e.

$$\begin{aligned} \mathcal{P}(a\nabla P) &= \mathcal{P}((a-\bar{a})\nabla P) = \mathcal{P}((a-\bar{a})\nabla P) - (a-\bar{a})\mathcal{P}(\nabla P) \\ &:= [\mathcal{P}, a-\bar{a}]\nabla P. \end{aligned}$$

[1] $\mathcal{P}$ is the Leray projector over divergence free vector fields, $\mathcal{Q} = Id - \mathcal{P}$

which is, loosely speaking, more regular than ∇P. Let us mention that a similar principle holds for u which is divergence free[2]: whenever we estimate some term of the form $\mathcal{Q}\left(bM\left(D\right)u\right)$ where b lies in an appropriate Besov space and $M\left(D\right)$ is some pseudo-differential operator then we may write it

$$\mathcal{Q}\left(bM\left(D\right)u\right)=[\mathcal{Q},b]M\left(D\right)u$$

and use the fact that the later expression in more regular than $M\left(D\right)u$, see Proposition 4.16.

The proof of Theorem 1.3 in the 3-dimensional case is more subtle: first we prove a more restrictive result by demanding an extra low-frequency information on the initial data. Then, using a perturbative version of Danchin and Mucha's results of [17] we arrive at constructing a solution with the optimal regularity. The uniqueness is obtained by a duality method.

Once the estimates of Theorem 1.3 are established, we proceed with the proof of Theorem 1.2 which is the object of Section 3. Finally, we show the equivalence between system (1.4) and system (1.1) thus achieving the proof of Theorem 1.1. We end this paper with an Appendix where results of Littlewood-Paley theory used through the text are gathered.

2 The Stokes system with nonconstant coefficients

2.1 Pressure estimates

Before handling System (1.3) we shall study the following elliptic equation:

$$\operatorname{div}\left(a\nabla P\right)=\operatorname{div}f. \tag{2.1}$$

For the reader's convenience let us cite the following classical result, a proof of which can be found, for instance in [12]:

Proposition 2.1. *For all vector field f with coefficients in $L^2\left(\mathbb{R}^n\right)$, there exists a tempered distribution P unique up to constant functions such that $\nabla P\in L^2\left(\mathbb{R}^n\right)$ and Equation (2.1) is satisfied. In addition, we have:*

$$a_\star\left\|\nabla P\right\|_{L^2}\leq\left\|\mathcal{Q}f\right\|_{L^2}.$$

Recently, in [30], in the $2D$ case, H. Xu, Y. Li and X. Zhai studied the eliptic equation (2.1) with the data $(a-\bar{a},f)$ in Besov spaces. Using a different approach, we obtain estimates in both two dimensional and three dimensional situations. Let us also mention that our method allows to obtain a wider range of indices than the one of Proposition 3.1. i) of [30].

We choose to focus on the $3D$ case. We aim at establishing the following result:

Proposition 2.2. *Let us consider $p\in\left(\frac{6}{5},2\right)$ and $q\in[1,\infty)$ such that $\frac{1}{p}-\frac{1}{q}\leq\frac{1}{2}$. Assume that there exists positive constants $(\bar{a},a_\star,a^\star)$ such that $a-\bar{a}\in\dot{B}^{\frac{3}{q}}_{q,1}\left(\mathbb{R}^3\right)$ and $0<a_\star\leq a\leq a^\star$. Furthermore, consider $f\in\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2}\left(\mathbb{R}^3\right)$. Then there exists a tempered distribution P unique up to constant functions such that $\nabla P\in\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2}\left(\mathbb{R}^3\right)$ and Equation (2.1) is satisfied. Moreover, the following estimate holds true:*

$$\left\|\nabla P\right\|_{\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2}}\lesssim\left(\frac{1}{\bar{a}}+\left\|\frac{1}{a}-\frac{1}{\bar{a}}\right\|_{\dot{B}^{\frac{3}{q}}_{q,1}}\right)\left(1+\frac{1}{a_\star}\left\|a-\bar{a}\right\|_{\dot{B}^{\frac{3}{q}}_{q,1}}\right)\left\|\mathcal{Q}f\right\|_{\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2}}. \tag{2.2}$$

Remark 2.1. *Working in Besov spaces with third index $r=2$ is enough in view of the applications that we have in mind. However similar estimates do hold true when the third index is choosen in the interval $[1,2]$.*

Proof. Because $p<2$, Proposition 4.5 ensures that $\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2}\hookrightarrow L^2$ and owing to Proposition 2.1, we get the existence of $P\in\mathcal{S}'\left(\mathbb{R}^3\right)$ with $\nabla P\in L^2$ and

$$a_\star\left\|\nabla P\right\|_{L^2}\leq\left\|\mathcal{Q}f\right\|_{L^2}. \tag{2.3}$$

[2] and thus $\mathcal{Q}u=0$.

Moreover, as $\mathcal{Q}$ is a continuous operator on L^2 we deduce from (2.1) that

$$\mathcal{Q}\left(a\nabla P\right)=\mathcal{Q}f\ . \tag{2.4}$$

Using the Bony decomposition (see Definition 4.3 and the remark that follows) and the fact that $\mathcal{P}\left(\nabla P\right)=0$ we write that:

$$\mathcal{P}\left(a\nabla P\right)=\mathcal{P}\left(\dot{T}'_{\nabla P}(a-\bar{a})\right)+\left[\mathcal{P},\dot{T}_{a-\bar{a}}\right]\nabla P.$$

Using Proposition 4.11 along with Proposition 4.5 and relation (2.3), we get that

$$\left\|\mathcal{P}\left(\dot{T}'_{\nabla P}(a-\bar{a})\right)\right\|_{\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2}}\lesssim\left\|\nabla P\right\|_{L^2}\left\|a-\bar{a}\right\|_{\dot{B}^{\frac{3}{p^\star}-\frac{3}{2}}_{p^\star,2}}\lesssim\frac{1}{a_\star}\left\|\mathcal{Q}f\right\|_{L^2}\left\|a-\bar{a}\right\|_{\dot{B}^{\frac{3}{q}}_{q,1}} \tag{2.5}$$

where

$$\frac{1}{p}=\frac{1}{2}+\frac{1}{p^\star}.$$

Next, proceeding as in Proposition 4.15 we get that:

$$\left\|\left[\mathcal{P},\dot{T}_{a-\bar{a}}\right]\nabla P\right\|_{\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2}}\lesssim\left\|\nabla a\right\|_{\dot{B}^{\frac{3}{p}-\frac{5}{2}}_{p^\star,2}}\left\|\nabla P\right\|_{L^2}\lesssim\frac{1}{a_\star}\left\|\mathcal{Q}f\right\|_{L^2}\left\|a-\bar{a}\right\|_{\dot{B}^{\frac{3}{q}}_{q,1}}\ . \tag{2.6}$$

Putting together relations (2.5) and (2.6) we get

$$\left\|\mathcal{P}\left(a\nabla P\right)\right\|_{\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2}}\lesssim\frac{1}{a_\star}\left\|\mathcal{Q}f\right\|_{L^2}\left\|a-\bar{a}\right\|_{\dot{B}^{\frac{3}{q}}_{q,1}}\ .$$

Combining this with (2.4) and Proposition 4.5, we find that:

$$\left\|a\nabla P\right\|_{\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2}}\lesssim\left(1+\frac{1}{a_\star}\left\|a-\bar{a}\right\|_{\dot{B}^{\frac{3}{q}}_{q,1}}\right)\left\|\mathcal{Q}f\right\|_{\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2}}.$$

Of course, writing

$$\nabla P=\frac{1}{a}a\nabla P$$

using product rules one gets that:

$$\left\|\nabla P\right\|_{\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2}}\lesssim\left(\frac{1}{\bar{a}}+\left\|\frac{1}{a}-\frac{1}{\bar{a}}\right\|_{\dot{B}^{\frac{3}{q}}_{q,1}}\right)\left(1+\frac{1}{a_\star}\left\|a-\bar{a}\right\|_{\dot{B}^{\frac{3}{q}}_{q,1}}\right)\left\|\mathcal{Q}f\right\|_{\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2}} \tag{2.7}$$

□

Applying the same technique as above leads to the 2 dimensional estimate:

Proposition 2.3. *Let us consider $p\in(1,2)$ and $q\in[1,\infty)$ such that $\frac{1}{p}-\frac{1}{q}\leq\frac{1}{2}$. Assume that there exists positive constants $(\bar{a},a_\star,a^\star)$ such that $a-\bar{a}\in\dot{B}^{\frac{2}{q}}_{q,1}\left(\mathbb{R}^2\right)$ and $0<a_\star\leq a\leq a^\star$. Furthermore, consider $f\in\dot{B}^{\frac{2}{p}-1}_{p,2}\left(\mathbb{R}^2\right)$. Then there exists a tempered distribution P unique up to constant functions such that $\nabla P\in\dot{B}^{\frac{2}{p}-1}_{p,2}\left(\mathbb{R}^2\right)$ and Equation (2.1) is satisfied. Moreover, the following estimate holds true:*

$$\left\|\nabla P\right\|_{\dot{B}^{\frac{2}{p}-1}_{p,2}}\lesssim\left(\frac{1}{\bar{a}}+\left\|\frac{1}{a}-\frac{1}{\bar{a}}\right\|_{\dot{B}^{\frac{2}{q}}_{q,1}}\right)\left(1+\frac{1}{a_\star}\left\|a-\bar{a}\right\|_{\dot{B}^{\frac{2}{q}}_{q,1}}\right)\left\|\mathcal{Q}f\right\|_{\dot{B}^{\frac{2}{p}-1}_{p,2}}. \tag{2.8}$$

Let us point out that the restriction $p>\frac{6}{5}$ comes from the fact that we need $\frac{3}{p}-\frac{5}{2}<0$ in relation (2.6). In $2D$ instead of $\frac{3}{p}-\frac{5}{2}$ we will have $\frac{2}{p}-2$ which is negative provided that $p>1$.

The next result covers the range of integrability indices larger than 2 :

Proposition 2.4. *Let us consider $p \in (2,6)$ and $q \in [1,\infty)$ such that $\frac{1}{p}+\frac{1}{q} \geq \frac{1}{2}$. Assume that there exists positive constants $(\bar{a}, a_\star, a^\star)$ such that $a-\bar{a} \in \dot{B}^{\frac{3}{q}}_{q,1}(\mathbb{R}^3)$ and $0<a_\star \leq a \leq a^\star$. Furthermore, consider $f \in \dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2}(\mathbb{R}^3)$ and a tempered distribution P with $\nabla P \in \dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2}(\mathbb{R}^3)$ such that Equation (2.1) is satisfied. Then, the following estimate holds true:*

$$\|\nabla P\|_{\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2}} \lesssim \left(\frac{1}{\bar{a}}+\left\|\frac{1}{a}-\frac{1}{\bar{a}}\right\|_{\dot{B}^{\frac{3}{q}}_{q,1}}\right)\left(1+\frac{1}{a_\star}\|a-\bar{a}\|_{\dot{B}^{\frac{3}{q}}_{q,1}}\right)\|\mathcal{Q}f\|_{\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2}}. \tag{2.9}$$

Proof. Let us notice that p' the conjugate Lebesgue exponent of p satisfies $p' \in \left(\frac{6}{5},2\right)$ and $\frac{1}{p'}-\frac{1}{q} \leq \frac{1}{2}$. Thus, according to Proposition 2.2, for any g belonging to the unit ball of $\mathcal{S}\cap\dot{B}^{\frac{3}{p'}-\frac{3}{2}}_{p',2}$ there exists a $P_g \in \mathcal{S}'(\mathbb{R}^3)$ with $\nabla P_g \in \mathcal{S}\cap \dot{B}^{\frac{3}{p'}-\frac{3}{2}}_{p',2}$ such that

$$\operatorname{div}(a\nabla P_g) = \operatorname{div} g$$

and

$$\|\nabla P_g\|_{\dot{B}^{\frac{3}{p'}-\frac{3}{2}}_{p',2}} \lesssim \left(\frac{1}{\bar{a}}+\left\|\frac{1}{a}-\frac{1}{\bar{a}}\right\|_{\dot{B}^{\frac{3}{q}}_{q,1}}\right)\left(1+\frac{1}{a_\star}\|a-\bar{a}\|_{\dot{B}^{\frac{3}{q}}_{q,1}}\right).$$

We write that

$$\begin{aligned}\langle \nabla P, g\rangle &= \langle P, \operatorname{div} g\rangle = \langle P, \operatorname{div}(a\nabla P_g)\rangle \\ &= \langle \operatorname{div}\mathcal{Q}f, P_g\rangle = \langle \mathcal{Q}f, \nabla P_g\rangle,\end{aligned}$$

and consequently

$$\begin{aligned}|\langle \nabla P, g\rangle| &\lesssim \|\mathcal{Q}f\|_{\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2}}\|\nabla P_g\|_{\dot{B}^{\frac{3}{p'}-\frac{3}{2}}_{p',2}} \\ &\lesssim \left(\frac{1}{\bar{a}}+\left\|\frac{1}{a}-\frac{1}{\bar{a}}\right\|_{\dot{B}^{\frac{3}{q}}_{q,1}}\right)\left(1+\frac{1}{a_\star}\|a-\bar{a}\|_{\dot{B}^{\frac{3}{q}}_{q,1}}\right)\|\mathcal{Q}f\|_{\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2}}.\end{aligned}$$

Using Proposition 4.6 we get that relation (2.9) holds true. □

As in the previous situation, by applying the same technique we get a similar result in $2D$:

Proposition 2.5. *Let us consider $p \in (2,\infty)$ and $q \in [1,\infty)$ such that $\frac{1}{p}+\frac{1}{q} \geq \frac{1}{2}$. Assume that there exists positive constants $(\bar{a}, a_\star, a^\star)$ such that $a-\bar{a} \in \dot{B}^{\frac{2}{q}}_{q,1}(\mathbb{R}^2)$ and $0<a_\star \leq a \leq a^\star$. Furthermore, consider $f \in \dot{B}^{\frac{2}{p}-1}_{p,2}(\mathbb{R}^2)$ and a tempered distribution P with $\nabla P \in \dot{B}^{\frac{2}{p}-1}_{p,2}(\mathbb{R}^2)$ such that Equation (2.1) is satisfied. Then, following estimate holds true:*

$$\|\nabla P\|_{\dot{B}^{\frac{2}{p}-1}_{p,2}} \lesssim \left(\frac{1}{\bar{a}}+\left\|\frac{1}{a}-\frac{1}{\bar{a}}\right\|_{\dot{B}^{\frac{2}{q}}_{q,1}}\right)\left(1+\frac{1}{a_\star}\|a-\bar{a}\|_{\dot{B}^{\frac{2}{q}}_{q,1}}\right)\|\mathcal{Q}f\|_{\dot{B}^{\frac{2}{p}-1}_{p,2}}. \tag{2.10}$$

2.2 Some preliminary results

In this section we derive estimates for a Stokes-like problem with time independent, nonconstant coefficients. Before proceeding to the actual proof, for the reader's convenience, let us cite the following results pertaining to the case $a=\bar{a}$, $b=\bar{b}$ constants:

Proposition 2.6. *Let us consider $u_0 \in \dot{B}^{\frac{n}{p}-1}_{p,1}$ and $(f, \partial_t R, \nabla \operatorname{div} R) \in L^1_T(\dot{B}^{\frac{n}{p}-1}_{p,1})$ with $\mathcal{Q}R \in C_T(\dot{B}^{\frac{n}{p}-1}_{p,1})$ such that*

$$\operatorname{div} u_0 = \operatorname{div} R(0,\cdot).$$

Then, system

$$\begin{cases} \partial_t u - \bar{a}\bar{b}\Delta u + \bar{a}\nabla P = f, \\ \operatorname{div} u = \operatorname{div} R, \\ u_{|t=0} = u_0, \end{cases}$$

has a unique solution $(u,\nabla P)$ *with:*

$$u\in\mathcal{C}([0,T);\dot{B}^{\frac{n}{p}-1}_{p,1})\text{ and }\partial_t u,\nabla^2 u,\nabla P\in L^1_T(\dot{B}^{\frac{n}{p}-1}_{p,1})$$

and the following estimate is valid:

$$\|u\|_{L^\infty_T(\dot{B}^{\frac{n}{p}-1}_{p,1})}+\left\|\left(\partial_t u,\bar{a}\bar{b}\nabla^2 u,\bar{a}\nabla P\right)\right\|_{L^1_T(\dot{B}^{\frac{n}{p}-1}_{p,1})}\lesssim\|u_0\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}+\left\|\left(f,\partial_t R,\bar{a}\bar{b}\nabla\operatorname{div}R\right)\right\|_{L^1_T(\dot{B}^{\frac{n}{p}-1}_{p,1})}.$$

As a consequence of the previous result, one can establish via a perturbation argument:

Proposition 2.7. *Let us consider* $u_0\in\dot{B}^{\frac{n}{p}-1}_{p,1}$ *and* $(f,\partial_t R,\nabla\operatorname{div}R)\in L^1_T(\dot{B}^{\frac{n}{p}-1}_{p,1})$ *with* $\mathcal{Q}R\in C_T(\dot{B}^{\frac{n}{p}-1}_{p,1})$ *such that*

$$\operatorname{div}u_0=\operatorname{div}R(0,\cdot).$$

Then, there exists a $\eta=\eta(\bar{a})$ *small enough such that for all* $c\in\dot{B}^{\frac{n}{p}}_{p,1}$ *with*

$$\|c\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\leq\eta\ ,$$

the system

$$\begin{cases}\partial_t u-\bar{a}\bar{b}\Delta u+(\bar{a}+c)\nabla P=f,\\ \operatorname{div}u=\operatorname{div}R,\\ u_{|t=0}=u_0,\end{cases}$$

has a unique solution $(u,\nabla P)$ *with:*

$$u\in\mathcal{C}([0,T);\dot{B}^{\frac{n}{p}-1}_{p,1})\text{ and }\partial_t u,\nabla^2 u,\nabla P\in L^1_T(\dot{B}^{\frac{n}{p}-1}_{p,1})$$

and the following estimate is valid:

$$\|u\|_{L^\infty_T(\dot{B}^{\frac{n}{p}-1}_{p,1})}+\left\|\left(\partial_t u,\bar{a}\bar{b}\nabla^2 u,\bar{a}\nabla P\right)\right\|_{L^1_T(\dot{B}^{\frac{n}{p}-1}_{p,1})}\lesssim\|u_0\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}+\left\|\left(f,\partial_t R,\bar{a}\bar{b}\nabla\operatorname{div}R\right)\right\|_{L^1_T(\dot{B}^{\frac{n}{p}-1}_{p,1})}.$$

The above results were established by Danchin and Mucha in [14] and [17].

In all what follows we denote by E_{loc} the space of $(u,\nabla P)$ such that:

$$u\in\mathcal{C}([0,\infty);\dot{B}^{\frac{n}{p}-1}_{p,1})\text{ and }\left(\nabla^2 u,\nabla P\right)\in L^1_{loc}(\dot{B}^{\frac{n}{p}-1}_{p,1})\times L^1_{loc}(\dot{B}^{\frac{n}{p}-\frac{n}{2}}_{p,2}\cap\dot{B}^{\frac{n}{p}-1}_{p,1}).$$

Also, let us introduce the space E_T of $u\in\mathcal{C}_T(\dot{B}^{\frac{n}{p}-1}_{p,1})$ with $\nabla^2 u\in L^1_T(\dot{B}^{\frac{n}{p}-1}_{p,1})$ and $\nabla P\in L^1_T(\dot{B}^{\frac{n}{p}-\frac{n}{2}}_{p,2}\cap\dot{B}^{\frac{n}{p}-1}_{p,1})$ such that:

$$\|(u,\nabla P)\|_{E_T}=\|u\|_{L^\infty_T(\dot{B}^{\frac{n}{p}-1}_{p,1})}+\left\|\nabla^2 u\right\|_{L^1_T(\dot{B}^{\frac{n}{p}-1}_{p,1})}+\|\nabla P\|_{L^1_T(\dot{B}^{\frac{n}{p}-\frac{n}{2}}_{p,2}\cap\dot{B}^{\frac{n}{p}-1}_{p,1})}<\infty.$$

The first ingredient in proving Theorem 1.3 is the following:

Proposition 2.8. *Let us consider* $n\in\{2,3\}$ *and* $p\in(1,4)$ *if* $n=2$ *or* $p\in\left(\frac{6}{5},4\right)$ *if* $n=3$. *Assume there exists positive constants* $\left(a_\star,b_\star,a^\star,b^\star,\bar{a},\bar{b}\right)$ *such that* $a-\bar{a}\in\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n)$, $b-\bar{b}\in\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n)$ *and*

$$0<a_\star\leq a\leq a^\star,$$
$$0<b_\star\leq b\leq b^\star.$$

Furthermore, let us consider u_0,f *vector fields with coefficients in* $\dot{B}^{\frac{n}{p}-1}_{p,1}(\mathbb{R}^n)$ *respectively in* $L^1_{loc}(\dot{B}^{\frac{n}{p}-\frac{n}{2}}_{p,2}(\mathbb{R}^n)\cap\dot{B}^{\frac{n}{p}-1}_{p,1}(\mathbb{R}^n))$ *and a vector field* $R\in(\mathcal{S}'(\mathbb{R}^n))^n$ *with* $\mathcal{Q}R\in\mathcal{C}([0,\infty);\dot{B}^{\frac{n}{p}-1}_{p,1}(\mathbb{R}^n))$ *and* $(\partial_t R,\nabla\operatorname{div}R)\in L^1_{loc}(\dot{B}^{\frac{n}{p}-\frac{n}{2}}_{p,2}(\mathbb{R}^n)\cap\dot{B}^{\frac{n}{p}-1}_{p,1}(\mathbb{R}^n))$ *such that*

$$\operatorname{div}u_0=\operatorname{div}R(0,\cdot).$$

Then, there exists a constant C_{ab} depending on a and b such that any solution $(u, \nabla P) \in E_T$ of the Stokes system (1.3) will satisfy:

$$\begin{aligned}&\|u\|_{L^\infty_t(\dot B^{\frac{n}{p}-1}_{p,1})} + \left\|\nabla^2 u\right\|_{L^1_t(\dot B^{\frac{n}{p}-1}_{p,1})} + \|\nabla P\|_{L^1_t(\dot B^{\frac{n}{p}-\frac{n}{2}}_{p,2}\cap \dot B^{\frac{n}{p}-1}_{p,1})}\\ &\leq \left(\|u_0\|_{\dot B^{\frac{n}{p}-1}_{p,1}} + \|(f, \partial_t R, \nabla \operatorname{div} R)\|_{L^1_t(\dot B^{\frac{n}{p}-\frac{n}{2}}_{p,2}\cap \dot B^{\frac{n}{p}-1}_{p,1})}\right)\exp\left(C_{ab}(t+1)\right)\end{aligned} \tag{2.11}$$

for all $t \in (0, T]$.

Before proceeding with the proof, a few remarks are in order:

Remark 2.2. *Proposition 2.8 is different from Theorem 1.3 when $n = 3$. Indeed, in the 3 dimensional case the theory is more subtle and thus, as a first step we construct a unique solution for the case of more regular initial data.*

Remark 2.3. *The difficulty when dealing with the Stokes system with non constant coefficients lies in the fact that the pressure and the velocity u are coupled. Indeed, in the constant coefficients case, in view of*

$$\operatorname{div} u = \operatorname{div} R,$$

one can apply the divergence operator in the first equation of (1.3) in order to obtain the following elliptic equation verified by the pressure:

$$a\Delta P = \operatorname{div}\left(f - \partial_t R + 2ab\nabla \operatorname{div} R\right). \tag{2.12}$$

From (2.12) we can construct the pressure. Having built the pressure, the velocity satisfies a classical heat equation. In the non constant coefficient case, proceeding as above we find that:

$$\operatorname{div}(a\nabla P) = \operatorname{div}\left(f - \partial_t R + a\operatorname{div}(bD(u))\right). \tag{2.13}$$

such that the strategy used in the previous case is not well-adapted. We will establish a priori estimate and use a continuity argument like in [13]. In order to be able to close the estimates on u, we have to bound $\|a\nabla P\|_{L^1_t(\dot B^{\frac{n}{p}-1}_{p,1})}$ in terms of $\|u\|^{\beta}_{L^\infty_t(\dot B^{\frac{n}{p}-1}_{p,1})}\left\|\nabla^2 u\right\|^{1-\beta}_{L^1_t(\dot B^{\frac{n}{p}-1}_{p,1})}$ for some $\beta \in (0,1)$. Thus, the difficulty is to find estimates for the pressure which do not feature the time derivative of the velocity.

In view of Proposition 2.6, let us consider $(u_L, \nabla P_L)$ the unique solution of the system

$$\begin{cases} \partial_t u - \bar a \operatorname{div}(\bar b D(u)) + \bar a \nabla P = f,\\ \operatorname{div} u = \operatorname{div} R,\\ u_{|t=0} = u_0,\end{cases} \tag{2.14}$$

with

$$u_L \in \mathcal{C}([0,\infty); \dot B^{\frac{n}{p}-1}_{p,1}) \text{ and } (\partial_t u_L, \nabla^2 u_L, \nabla P_L) \in L^1_{loc}(\dot B^{\frac{n}{p}-1}_{p,1}).$$

Recall that for any $t \in [0,\infty)$ we have

$$\|u_L\|_{L^\infty_t(\dot B^{\frac{n}{p}-1}_{p,1})} + \left\|\left(\partial_t u_L, \bar a\bar b\nabla^2 u_L, \bar a \nabla P_L\right)\right\|_{L^1_t(\dot B^{\frac{n}{p}-1}_{p,1})} \leq C(\|u_0\|_{\dot B^{\frac{n}{p}-1}_{p,1}} + \left\|\left(f, \partial_t R, \bar a\bar b\nabla \operatorname{div} R\right)\right\|_{L^1_t(\dot B^{\frac{n}{p}-1}_{p,1})}). \tag{2.15}$$

In what follows, we will use the notation:

$$\tilde u = u - u_L,\ \nabla \tilde P = \nabla P - \nabla P_L. \tag{2.16}$$

Obviously, we have

$$\operatorname{div} \tilde u = 0. \tag{2.17}$$

Thus, the system (1.3) is recasted into

$$\begin{cases} \partial_t \tilde u - a\operatorname{div}(bD(\tilde u)) + a\nabla \tilde P = \tilde f,\\ \operatorname{div} \tilde u = 0,\\ \tilde u_{|t=0} = 0,\end{cases} \tag{2.18}$$

where

$$\tilde{f}=a\,\mathrm{div}(bD(u_L))-\bar{a}\,\mathrm{div}(\bar{b}D(u_L))-(a-\bar{a})\nabla P_L.$$

Using the last equality along with Proposition 4.12, we infer that:

$$\begin{aligned}\left\|\tilde{f}\right\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}&\leq\left\|a\,\mathrm{div}(bD(u_L))-\bar{a}\,\mathrm{div}(\bar{b}D(u_L))\right\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}+\|(a-\bar{a})\nabla P_L\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}\\&\lesssim(\bar{a}+\|a-\bar{a}\|_{\dot{B}^{\frac{n}{p}}_{p,1}})(\bar{b}+\left\|b-\bar{b}\right\|_{\dot{B}^{\frac{n}{p}}_{p,1}})\|\nabla u_L\|_{\dot{B}^{\frac{n}{p}}_{p,1}}+\|a-\bar{a}\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\|\nabla P_L\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}.\end{aligned}\tag{2.19}$$

Let us estimate the pressure $a\nabla\tilde{P}$. First, we write that

$$\left\|a\nabla\tilde{P}\right\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}\leq\left\|\mathcal{Q}\left(a\nabla\tilde{P}\right)\right\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}+\left\|\mathcal{P}\left(a\nabla\tilde{P}\right)\right\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}.$$

Applying the $\mathcal{Q}$ operator in the first equation of (2.18) we get that

$$\mathcal{Q}\left(a\nabla\tilde{P}\right)=\mathcal{Q}\tilde{f}+\mathcal{Q}(a\,\mathrm{div}\,(bD(\tilde{u}))).$$

Thus, we get that:

$$\left\|\mathcal{Q}\left(a\nabla\tilde{P}\right)\right\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}\leq\left\|\mathcal{Q}\tilde{f}\right\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}+\|\mathcal{Q}\,(a\,\mathrm{div}\,(bD(\tilde{u})))\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}.\tag{2.20}$$

Let write that:

$$\mathcal{Q}\,(a\,\mathrm{div}\,(bD(\tilde{u})))=\mathcal{Q}\left(D(\tilde{u})\dot{S}_m\,(a\nabla b)\right)+\mathcal{Q}\left(\dot{S}_m\left(ab-\bar{a}\bar{b}\right)\Delta\tilde{u}\right)\tag{2.21}$$

$$+\mathcal{Q}\left(D(\tilde{u})\left(Id-\dot{S}_m\right)(a\nabla b)\right)\tag{2.22}$$

$$+\mathcal{Q}\left(\left(Id-\dot{S}_m\right)\left(ab-\bar{a}\bar{b}\right)\Delta\tilde{u}\right).\tag{2.23}$$

According to Proposition 4.12 we have:

$$\left\|\mathcal{Q}\left(D\,(\tilde{u})\,\dot{S}_m\,(a\nabla b)\right)\right\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}\lesssim\left\|\dot{S}_m\,(a\nabla b)\right\|_{\dot{B}^{\frac{n}{p}-\frac{1}{2}}_{p,1}}\|\nabla\tilde{u}\|_{\dot{B}^{\frac{n}{p}-\frac{1}{2}}_{p,1}}.\tag{2.24}$$

Owing to the fact that $\tilde{u}$ is divergence free we can write that

$$\mathcal{Q}\left(\dot{S}_m\left(ab-\bar{a}\bar{b}\right)\Delta\tilde{u}\right)=[\mathcal{Q},\dot{S}_m\left(ab-\bar{a}\bar{b}\right)]\Delta\tilde{u},\tag{2.25}$$

such that applying Proposition 4.16 we get that

$$\begin{aligned}\left\|\mathcal{Q}\left(\dot{S}_m\left(ab-\bar{a}\bar{b}\right)\Delta\tilde{u}\right)\right\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}&\lesssim\left\|\left(\dot{S}_m\,(a\nabla b),\dot{S}_m\,(b\nabla a)\right)\right\|_{\dot{B}^{\frac{n}{p}-\frac{1}{2}}_{p,1}}\|\Delta\tilde{u}\|_{\dot{B}^{\frac{n}{p}-\frac{3}{2}}_{p,1}}\\&\lesssim\left\|\left(\dot{S}_m\,(a\nabla b),\dot{S}_m\,(b\nabla a)\right)\right\|_{\dot{B}^{\frac{n}{p}-\frac{1}{2}}_{p,1}}\|\nabla\tilde{u}\|_{\dot{B}^{\frac{n}{p}-\frac{1}{2}}_{p,1}}.\end{aligned}\tag{2.26}$$

The last two terms of (2.21)-(2.23) are estimated as follows:

$$\left\|\mathcal{Q}\left(\left(Id-\dot{S}_m\right)(a\nabla b)\,D\,(\tilde{u})\right)+\mathcal{Q}\left(\left(Id-\dot{S}_m\right)\left(ab-\bar{a}\bar{b}\right)\Delta\tilde{u}\right)\right\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}\tag{2.27}$$

$$\lesssim\left(\left\|\left(Id-\dot{S}_m\right)(a\nabla b)\right\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}+\left\|\left(Id-\dot{S}_m\right)\left(ab-\bar{a}\bar{b}\right)\right\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\right)\|\nabla\tilde{u}\|_{\dot{B}^{\frac{n}{p}}_{p,1}}.\tag{2.28}$$

Thus, putting together relations (2.20)-(2.28) we get that:

$$\begin{aligned}\left\|\mathcal{Q}\left(a\nabla\tilde{P}\right)\right\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}&\lesssim\left\|\mathcal{Q}\tilde{f}\right\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}+\left\|\left(\dot{S}_m\,(a\nabla b),\dot{S}_m\,(b\nabla a)\right)\right\|_{\dot{B}^{\frac{n}{p}-\frac{1}{2}}_{p,1}}\|\nabla\tilde{u}\|_{\dot{B}^{\frac{n}{p}-\frac{1}{2}}_{p,1}}\\&+\|\nabla\tilde{u}\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\left(\left\|\left(Id-\dot{S}_m\right)(a\nabla b,b\nabla a)\right\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}+\left\|\left(Id-\dot{S}_m\right)\left(ab-\bar{a}\bar{b}\right)\right\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\right).\end{aligned}\tag{2.29}$$

Next, we turn our attention towards $\mathcal{P}\left(a\nabla\tilde{P}\right)$. The $2D$ case respectively the $3D$ case have to be treated differently.

2.2.1 The $3D$ case

Noticing that

$$\mathcal{P}\left(a\nabla\tilde{P}\right)=\mathcal{P}\left(\left(Id-\dot{S}_m\right)(a-\bar{a})\nabla\tilde{P}\right)+[\mathcal{P},\dot{S}_m(a-\bar{a})]\nabla\tilde{P},$$

and using again Proposition 4.16 combined with Proposition 2.2 and Proposition 2.4 we get that:

$$\left\|\nabla\tilde{P}\right\|_{\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2}}+\left\|\mathcal{P}\left(a\nabla\tilde{P}\right)\right\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}}\lesssim\left\|\nabla\tilde{P}\right\|_{\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2}}+\left\|\mathcal{P}\left(\left(Id-\dot{S}_m\right)(a-\bar{a})\nabla\tilde{P}\right)\right\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}}+\left\|[\mathcal{P},\dot{S}_m(a-\bar{a})]\nabla\tilde{P}\right\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}} \tag{2.30}$$

$$\lesssim\left\|\left(Id-\dot{S}_m\right)(a-\bar{a})\right\|_{\dot{B}^{\frac{3}{p}}_{p,1}}\left\|\nabla\tilde{P}\right\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}}+\left(1+\left\|\dot{S}_m\nabla a\right\|_{\dot{B}^{\frac{3}{p}-\frac{1}{2}}_{p,2}}\right)\left\|\nabla\tilde{P}\right\|_{\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2}} \tag{2.31}$$

$$\lesssim\left\|\left(Id-\dot{S}_m\right)(a-\bar{a})\right\|_{\dot{B}^{\frac{3}{p}}_{p,1}}\left(\frac{1}{\bar{a}}+\left\|\frac{1}{a}-\frac{1}{\bar{a}}\right\|_{\dot{B}^{\frac{3}{p}}_{p,1}}\right)\left\|a\nabla\tilde{P}\right\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}} \tag{2.32}$$

$$+\tilde{C}(a)\left(1+\left\|\dot{S}_m\nabla a\right\|_{\dot{B}^{\frac{3}{p}-\frac{1}{2}}_{p,2}}\right)\left(\left\|\tilde{f}\right\|_{\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2}}+\|a\operatorname{div}(bD(\tilde{u}))\|_{\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2}}\right), \tag{2.33}$$

where

$$\tilde{C}(a)=\left(\frac{1}{\bar{a}}+\left\|\frac{1}{a}-\frac{1}{\bar{a}}\right\|_{\dot{B}^{\frac{3}{p}}_{p,1}}\right)\left(1+\frac{1}{a_\star}\|a-\bar{a}\|_{\dot{B}^{\frac{3}{p}}_{p,1}}\right).$$

We observe that

$$\|a\operatorname{div}(bD(\tilde{u}))\|_{\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2}}\lesssim\left(\bar{a}+\|a-\bar{a}\|_{\dot{B}^{\frac{3}{p}}_{p,1}}\right)\left(\bar{b}+\left\|b-\bar{b}\right\|_{\dot{B}^{\frac{3}{p}}_{p,1}}\right)\|\nabla\tilde{u}\|_{\dot{B}^{\frac{3}{p}-\frac{1}{2}}_{p,1}}. \tag{2.34}$$

Putting together (2.30)-(2.33) along with (2.34) we get that

$$\left\|\nabla\tilde{P}\right\|_{\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2}}+\left\|\mathcal{P}\left(a\nabla\tilde{P}\right)\right\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}}\lesssim\left\|\left(Id-\dot{S}_m\right)(a-\bar{a})\right\|_{\dot{B}^{\frac{3}{p}}_{p,1}}\left(\frac{1}{\bar{a}}+\left\|\frac{1}{a}-\frac{1}{\bar{a}}\right\|_{\dot{B}^{\frac{3}{p}}_{p,1}}\right)\left\|a\nabla\tilde{P}\right\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}}$$
$$+\tilde{C}(a)\left(1+\left\|\dot{S}_m\nabla a\right\|_{\dot{B}^{\frac{3}{p}-\frac{1}{2}}_{p,2}}\right)\left(\left\|\tilde{f}\right\|_{\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2}}+\left(\bar{a}+\|a-\bar{a}\|_{\dot{B}^{\frac{3}{p}}_{p,1}}\right)\left(\bar{b}+\left\|b-\bar{b}\right\|_{\dot{B}^{\frac{3}{p}}_{p,1}}\right)\|\nabla\tilde{u}\|_{\dot{B}^{\frac{3}{p}-\frac{1}{2}}_{p,1}}\right). \tag{2.35}$$

Combining (2.29) with (2.35) yields:

$$\left\|\nabla\tilde{P}\right\|_{\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2}}+\left\|a\nabla\tilde{P}\right\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}}\lesssim T^1_m(a,b)\left\|a\nabla\tilde{P}\right\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}}$$
$$+T^2_m(a,b)\left\|\tilde{f}\right\|_{\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2}\cap\dot{B}^{\frac{3}{p}-1}_{p,1}}+T^3_m(a,b)\|\nabla\tilde{u}\|_{\dot{B}^{\frac{3}{p}-\frac{1}{2}}_{p,1}}+T^4_m(a,b)\|\nabla\tilde{u}\|_{\dot{B}^{\frac{3}{p}}_{p,1}}$$

where

$$T^1_m(a,b)=\left\|\left(Id-\dot{S}_m\right)(a-\bar{a})\right\|_{\dot{B}^{\frac{3}{p}}_{p,1}}\left(\frac{1}{\bar{a}}+\left\|\frac{1}{a}-\frac{1}{\bar{a}}\right\|_{\dot{B}^{\frac{3}{p}}_{p,1}}\right),$$

$$T^2_m(a,b)=\tilde{C}(a)\left(1+\left\|\dot{S}_m\nabla a\right\|_{\dot{B}^{\frac{3}{p}-\frac{1}{2}}_{p,2}}\right),$$

$$T^3_m(a,b)=\left\|\left(\dot{S}_m(a\nabla b),\dot{S}_m(b\nabla a)\right)\right\|_{\dot{B}^{\frac{3}{p}-\frac{1}{2}}_{p,1}}$$
$$+\tilde{C}(a)\left(1+\left\|\dot{S}_m\nabla a\right\|_{\dot{B}^{\frac{3}{p}-\frac{1}{2}}_{p,2}}\right)\left(\bar{a}+\|a-\bar{a}\|_{\dot{B}^{\frac{3}{p}}_{p,1}}\right)\left(\bar{b}+\left\|b-\bar{b}\right\|_{\dot{B}^{\frac{3}{p}}_{p,1}}\right),$$

$$T_m^4(a,b)=\left\|\left(Id-\dot{S}_m\right)(a\nabla b,b\nabla a)\right\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}}+\left\|\left(Id-\dot{S}_m\right)(ab-\bar{a}\bar{b})\right\|_{\dot{B}^{\frac{3}{p}}_{p,1}}.$$

Observe that m could be chosen large enough such that $T_m^1(a,b)$ and $T_m^4(a,b)$ can be made arbitrarily small. Thus, there exists a constant C_{ab} depending on a and b such that:

$$\left\|\nabla\tilde{P}\right\|_{\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2}\cap\dot{B}^{\frac{3}{p}-1}_{p,1}}\leq C_{ab}\left(\left\|\tilde{f}\right\|_{\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2}\cap\dot{B}^{\frac{3}{p}-1}_{p,1}}+\|\nabla\tilde{u}\|_{\dot{B}^{\frac{3}{p}-\frac{1}{2}}_{p,1}}\right)+\eta\|\nabla\tilde{u}\|_{\dot{B}^{\frac{3}{p}}_{p,1}},\tag{2.36}$$

where η can be made arbitrarily small (of course, with the price of increasing the constant C_{ab}). Let us take a look at the $\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2}$-norm of $\tilde{f}$; we get that:

$$\begin{aligned}\left\|\tilde{f}\right\|_{\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2}}&\leq\left\|a\,\mathrm{div}(bD(u_L))-\bar{a}\,\mathrm{div}(\bar{b}D(u_L))\right\|_{\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2}}+\|(a-\bar{a})\nabla P_L\|_{\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2}}\\&\lesssim\left(\bar{a}+\|a-\bar{a}\|_{\dot{B}^{\frac{3}{p}}_{p,1}}\right)\left(\bar{b}+\left\|b-\bar{b}\right\|_{\dot{B}^{\frac{3}{p}}_{p,1}}\right)\|\nabla u_L\|_{\dot{B}^{\frac{3}{p}-\frac{1}{2}}_{p,1}}+\|a-\bar{a}\|_{\dot{B}^{\frac{3}{p}}_{p,1}}\|\nabla P_L\|_{\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2}}.\end{aligned}\tag{2.37}$$

As $u_L\in C([0,\infty),\dot{B}^{\frac{3}{p}-1}_{p,1})\cap L^1([0,\infty),\dot{B}^{\frac{3}{p}+1}_{p,1})$ and $\mathcal{Q}$ is continuous operator on homogeneous Besov spaces from

$$\mathrm{div}\,(u_L-R)=0,$$

we deduce that

$$\mathcal{P}(u_L-R)=u_L-R,$$

which implies

$$\mathcal{Q}u_L=\mathcal{Q}R.$$

By applying the operator $\mathcal{Q}$ in the first equation of System (2.14) we get that:

$$\begin{aligned}\bar{a}\nabla P_L&=\mathcal{Q}f-\mathcal{Q}\partial_t u_L+\bar{a}\bar{b}\mathcal{Q}\Delta u_L+\bar{a}\bar{b}\nabla\,\mathrm{div}\,R\\&=\mathcal{Q}f-\mathcal{Q}\partial_t R+2\bar{a}\bar{b}\nabla\,\mathrm{div}\,R\end{aligned}$$

and thus

$$\|\nabla P_L\|_{\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2}}\leq\frac{1}{\bar{a}}\|\mathcal{Q}f\|_{\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2}}+\frac{1}{\bar{a}}\|\partial_t\mathcal{Q}R\|_{\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2}}+2\bar{b}\|\nabla\,\mathrm{div}\,R\|_{\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2}}.$$

In view of (2.36), (2.19), (2.37) and interpolation we gather that there exists a constant C_{ab} such that:

$$\left\|\nabla\tilde{P}\right\|_{\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2}\cap\dot{B}^{\frac{3}{p}-1}_{p,1}}\leq C_{ab}\left(\|\nabla u_L\|_{\dot{B}^{\frac{3}{p}-\frac{1}{2}}_{p,1}}+\|\nabla P_L\|_{\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2}}+\left\|(\nabla^2u_L,\nabla P_L)\right\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}}+\|\nabla\tilde{u}\|_{\dot{B}^{\frac{3}{p}-\frac{1}{2}}_{p,1}}\right)+\eta\|\nabla\tilde{u}\|_{\dot{B}^{\frac{3}{p}}_{p,1}}.\tag{2.38}$$

$$\leq C_{ab}\|(\mathcal{Q}f,\partial_t\mathcal{Q}R,\nabla\,\mathrm{div}\,R)\|_{\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2}}+C_{ab}\|u_L\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}}\tag{2.39}$$

$$+C_{ab}\left\|(\nabla^2u_L,\nabla P_L)\right\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}}+C_{ab}\|\tilde{u}\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}}+2\eta\|\nabla\tilde{u}\|_{\dot{B}^{\frac{3}{p}}_{p,1}}.\tag{2.40}$$

where, again, at the price of increasing C_{ab}, η can be made arbitrarily small.

2.2.2 The $2D$ case

In this case, using again Proposition 4.16 combined with Proposition with Proposition 2.3 and Proposition 2.5 we get that:

$$\begin{aligned}\left\|\nabla\tilde{P}\right\|_{\dot{B}^{\frac{2}{p}-1}_{p,2}}+\left\|\mathcal{P}(a\nabla\tilde{P})\right\|_{\dot{B}^{\frac{2}{p}-1}_{p,1}}&\lesssim\left\|\nabla\tilde{P}\right\|_{\dot{B}^{\frac{2}{p}-1}_{p,2}}+\left\|\left(Id-\dot{S}_m\right)(a-\bar{a})\right\|_{\dot{B}^{\frac{2}{p}}_{p,1}}\left\|\nabla\tilde{P}\right\|_{\dot{B}^{\frac{2}{p}-1}_{p,1}}+\left\|[\mathcal{P},\dot{S}_m(a-\bar{a})]\nabla\tilde{P}\right\|_{\dot{B}^{\frac{2}{p}-1}_{p,1}}\\&\lesssim\left\|\left(Id-\dot{S}_m\right)(a-\bar{a})\right\|_{\dot{B}^{\frac{2}{p}}_{p,1}}\left\|\nabla\tilde{P}\right\|_{\dot{B}^{\frac{2}{p}-1}_{p,1}}+\left(1+\left\|\nabla\dot{S}_ma\right\|_{\dot{B}^{\frac{2}{p}}_{p,2}}\right)\left\|\nabla\tilde{P}\right\|_{\dot{B}^{\frac{2}{p}-1}_{p,2}}\end{aligned}$$

$$\lesssim \left\|\left(Id-\dot{S}_m\right)(a-\bar{a})\right\|_{\dot{B}^{\frac{2}{p}}_{p,1}} \left\|\nabla\tilde{P}\right\|_{\dot{B}^{\frac{2}{p}-1}_{p,1}}$$
$$+\tilde{C}(a)\left(1+\left\|\nabla\dot{S}_m a\right\|_{\dot{B}^{\frac{2}{p}}_{p,2}}\right)\left(\left\|\tilde{f}\right\|_{\dot{B}^{\frac{2}{p}-1}_{2,2}}+\left\|\mathcal{Q}(a\,\mathrm{div}\,(bD(\tilde{u})))\right\|_{\dot{B}^{\frac{2}{p}-1}_{p,2}}\right)$$

where, as before

$$\tilde{C}(a)=\left(\frac{1}{\bar{a}}+\left\|\frac{1}{a}-\frac{1}{\bar{a}}\right\|_{\dot{B}^{\frac{2}{p}}_{p,1}}\right)\left(1+\frac{1}{a_\star}\|a-\bar{a}\|_{\dot{B}^{\frac{2}{p}}_{p,1}}\right).$$

As we have already estimated $\|\mathcal{Q}(a\,\mathrm{div}\,(bD(\tilde{u})))\|_{\dot{B}^{\frac{2}{p}-1}_{p,2}}$ in (2.29), we gather that:

$$\left\|\nabla\tilde{P}\right\|_{\dot{B}^{\frac{2}{p}-1}_{p,2}}+\left\|a\nabla\tilde{P}\right\|_{\dot{B}^{\frac{2}{p}}_{p,1}} \lesssim T^1_m(a,b)\left\|a\nabla\tilde{P}\right\|_{\dot{B}^{\frac{2}{p}-1}_{p,1}}+T^2_m(a,b)\left\|\tilde{f}\right\|_{\dot{B}^{\frac{2}{p}-1}_{p,1}}$$
$$+T^3_{m,M}(a,b)\|\nabla\tilde{u}\|_{\dot{B}^{\frac{2}{p}-\frac{1}{2}}_{p,1}}+T^4_{m,M}(a,b)\|\nabla\tilde{u}\|_{\dot{B}^{\frac{2}{p}}_{p,1}}, \tag{2.41}$$

where

$$T^1_m(a,b)=\left\|\left(Id-\dot{S}_m\right)(a-\bar{a})\right\|_{\dot{B}^{\frac{2}{p}}_{p,1}}\left(\frac{1}{\bar{a}}+\left\|\frac{1}{a}-\frac{1}{\bar{a}}\right\|_{\dot{B}^{\frac{2}{p}}_{p,1}}\right),$$
$$T^2_m(a,b)=\tilde{C}(a)\left(1+\left\|\nabla\dot{S}_m a\right\|_{\dot{B}^{\frac{2}{p}}_{p,2}}\right),$$
$$T^3_{m,M}(a,b)=\left\|\left(\dot{S}_m(a\nabla b),\dot{S}_m(b\nabla a)\right)\right\|_{\dot{B}^{\frac{2}{p}}_{p,1}}$$
$$+\tilde{C}(a)\left(1+\left\|\nabla\dot{S}_m a\right\|_{\dot{B}^{\frac{2}{p}}_{p,2}}\right)\left\|\left(\dot{S}_M(a\nabla b),\dot{S}_M(b\nabla a)\right)\right\|_{\dot{B}^{\frac{2}{p}}_{p,1}},$$
$$T^4_{m,M}(a,b)=\left\|(Id-\dot{S}_m)(a\nabla b)\right\|_{\dot{B}^{\frac{2}{p}-1}_{p,1}}+\left\|(Id-\dot{S}_m)\left(ab-\bar{a}\bar{b}\right)\right\|_{\dot{B}^{\frac{2}{p}}_{p,1}}$$
$$+\tilde{C}(a)\left(1+\left\|\nabla\dot{S}_m a\right\|_{\dot{B}^{\frac{2}{p}}_{p,2}}\right)\left(\left\|(Id-\dot{S}_M)(a\nabla b)\right\|_{\dot{B}^{\frac{2}{p}-1}_{p,1}}+\left\|(Id-\dot{S}_M)\left(ab-\bar{a}\bar{b}\right)\right\|_{\dot{B}^{\frac{2}{p}-1}_{p,1}}\right).$$

First, we fix an $\eta>0$. Let us fix an $m\in\mathbb{N}$ such that $T^1_m(a,b)\left\|a\nabla\tilde{P}\right\|_{\dot{B}^{\frac{2}{p}-1}_{p,1}}$ can be "absorbed" by the LHS of (2.41) and that

$$\left\|(Id-\dot{S}_m)(a\nabla b)\right\|_{\dot{B}^{\frac{2}{p}-1}_{p,1}}+\left\|(Id-\dot{S}_m)\left(ab-\bar{a}\bar{b}\right)\right\|_{\dot{B}^{\frac{2}{p}}_{p,1}}\leq\eta/2.$$

Next, we see that by choosing M large enough we have

$$T^4_{m,M}(a,b)\leq\eta.$$

Thus, using interpolation we can write that:

$$\left\|\nabla\tilde{P}\right\|_{\dot{B}^{\frac{2}{p}-1}_{p,2}}+\left\|a\nabla\tilde{P}\right\|_{\dot{B}^{\frac{2}{p}-1}_{p,1}}\leq C_{ab}\left(\left\|(\nabla^2u_L,\nabla P_L)\right\|_{\dot{B}^{\frac{2}{p}-1}_{2,1}}+\|\tilde{u}\|_{\dot{B}^{\frac{2}{p}-1}_{p,1}}\right)+2\eta\left\|\nabla^2\tilde{u}\right\|_{\dot{B}^{\frac{2}{p}-1}_{p,1}}. \tag{2.42}$$

2.2.3 End of the proof of Proposition 2.8

Obviously, combining the two estimates (2.38)-(2.40) and (2.42) we can continue in a unified manner the rest of the proof of Proposition 2.8. First, choose $m\in\mathbb{N}$ large enough such that

$$\bar{a}\bar{b}+\dot{S}_m(ab-\bar{a}\bar{b})\geq\frac{a_\star b_\star}{2}.$$

We apply $\dot{\Delta}_j$ to (2.18) and we write that:

$$\begin{aligned}\partial_t\tilde{u}_j-\operatorname{div}\left((\bar{a}\bar{b}+\dot{S}_m(ab-\bar{a}\bar{b}))\nabla\tilde{u}_j\right)&=\tilde{f}_j-\dot{\Delta}_j\left(a\nabla\tilde{P}\right)\\&\quad+\dot{\Delta}_j\operatorname{div}\left((Id-\dot{S}_m)(ab-\bar{a}\bar{b})\nabla\tilde{u}\right)+\operatorname{div}\left[\dot{\Delta}_j,\dot{S}_m(ab-\bar{a}\bar{b})\right]\nabla\tilde{u}\\&\quad+\dot{\Delta}_j\left(D\tilde{u}\dot{S}_m(b\nabla a)\right)+\dot{\Delta}_j\left(D\tilde{u}(Id-\dot{S}_m)(b\nabla a)\right)\\&\quad+\dot{\Delta}_j\left(\nabla\tilde{u}\dot{S}_m(a\nabla b)\right)+\dot{\Delta}_j\left(\nabla\tilde{u}(Id-\dot{S}_m)(a\nabla b)\right).\end{aligned}$$

Multiplying the last relation by $|\tilde{u}_j|^{p-1}\operatorname{sgn}\tilde{u}_j$, integrating and using Lemma 8 from the Appendix B of [12], we get that:

$$\begin{aligned}&\|\tilde{u}_j\|_{L^p}+a_\star b_\star 2^{2j}C\int_0^t\|\tilde{u}_j\|_{L^p}\\&\lesssim\int_0^t\left\|\tilde{f}_j\right\|_{L^p}+\int_0^t\left\|\dot{\Delta}_j\left(a\nabla\tilde{P}\right)\right\|_{L^p}\\&+\int_0^t\left\|\operatorname{div}\left[\dot{\Delta}_j,\dot{S}_m(ab-\bar{a}\bar{b})\right]\nabla\tilde{u}\right\|_{L^p}+\int_0^t\left\|\dot{\Delta}_j\operatorname{div}\left((Id-\dot{S}_m)(ab-\bar{a}\bar{b})\nabla\tilde{u}\right)\right\|_{L^p}\\&+\int_0^t\left\|\dot{\Delta}_j\left(D\tilde{u}\dot{S}_m(b\nabla a)\right)\right\|_{L^p}+\int_0^t\left\|\dot{\Delta}_j\left(D\tilde{u}(Id-\dot{S}_m)(b\nabla a)\right)\right\|_{L^p}\\&+\int_0^t\left\|\dot{\Delta}_j\left(\nabla\tilde{u}\dot{S}_m(a\nabla b)\right)\right\|_{L^p}+\int_0^t\left\|\dot{\Delta}_j\left(\nabla\tilde{u}(Id-\dot{S}_m)(a\nabla b)\right)\right\|_{L^p}.\end{aligned}$$

Multiplying the last relation by $2^{j(\frac{n}{p}-1)}$, performing an $\ell^1(\mathbb{Z})$-summation and using Proposition 4.14 to deal with $\left\|\operatorname{div}\left[\dot{\Delta}_j,\dot{S}_m(ab-\bar{a}\bar{b})\right]\nabla\tilde{u}\right\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}$ along with (2.38)-(2.40) and (2.41) to deal with the pressure, we get that:

$$\begin{aligned}&\|\tilde{u}\|_{L^\infty_t(\dot{B}^{\frac{n}{p}-1}_{p,1})}+a_\star b_\star C\left\|\nabla^2\tilde{u}\right\|_{L^1_t(\dot{B}^{\frac{n}{p}-1}_{p,1})}\\&\lesssim\left\|\tilde{f}\right\|_{L^1_t(\dot{B}^{\frac{n}{p}-1}_{p,1})}+C\int_0^t\left\|a\nabla\tilde{P}\right\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}\\&+\int_0^t\left\|\left(\dot{S}_m(b\nabla a),\dot{S}_m(a\nabla b)\right)\right\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\|\nabla\tilde{u}\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}+T_m(a,b)\left\|\nabla^2\tilde{u}\right\|_{L^1_t(\dot{B}^{\frac{n}{p}-1}_{p,1})}\\&\leq C_{ab}(1+t)\left(\|u_0\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}+\|(f,\partial_tR,\nabla\operatorname{div}R)\|_{L^1_t(\dot{B}^{\frac{n}{p}-\frac{n}{2}}_{p,2}\cap\dot{B}^{\frac{n}{p}-1}_{p,1})}\right)\qquad(2.43)\\&\quad+C_{ab}\int_0^t\|\tilde{u}\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}+(T_m(a,b)+\eta)\left\|\nabla^2u\right\|_{L^1_t(\dot{B}^{\frac{n}{p}-1}_{p,1})}.\qquad(2.44)\end{aligned}$$

where

$$\begin{aligned}T_m(a,b)&=\left\|(Id-\dot{S}_m)(b\nabla a)\right\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}+\left\|(Id-\dot{S}_m)(a\nabla b)\right\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}\\&\quad+\left\|(Id-\dot{S}_m)(ab-\bar{a}\bar{b})\right\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}.\end{aligned}$$

Assuming that m is large enough respectively η is small enough, we can "absorb" $(T_m(a,b)+\eta)\left\|\nabla^2u\right\|_{L^1_t(\dot{B}^{\frac{n}{p}-1}_{p,1})}$ in the LHS of (2.43). Thus, we end up with

$$\begin{aligned}\|\tilde{u}\|_{L^\infty_t(\dot{B}^{\frac{n}{p}-1}_{p,1})}+a_\star b_\star\frac{C}{2}\left\|\nabla^2\tilde{u}\right\|_{L^1_t(\dot{B}^{\frac{n}{p}-1}_{p,1})}&\leq C_{ab}(1+t)\left(\|u_0\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}+\|(f,\partial_tR,\nabla\operatorname{div}R)\|_{L^1_t(\dot{B}^{\frac{n}{p}-\frac{n}{2}}_{p,2}\cap\dot{B}^{\frac{n}{p}-1}_{p,1})}\right)\\&\quad+C_{ab}\int_0^t\|\tilde{u}\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}\end{aligned}$$

such that using Gronwall's lemma, (2.15) and the classical inequality

$$1+t^{\alpha}\leq C_{\alpha}\exp(t)$$

yields:

$$\|\tilde{u}\|_{L^{\infty}_{t}(\dot{B}^{\frac{n}{p}-1}_{p,1})}+a_{\star}b_{\star}\frac{C}{2}\left\|\nabla^{2}\tilde{u}\right\|_{L^{1}_{t}(\dot{B}^{\frac{n}{p}-1}_{p,1})}\leq C_{ab}\left(\|u_{0}\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}+\|(f,\partial_{t}R,\nabla\operatorname{div}R)\|_{L^{1}_{t}(\dot{B}^{\frac{n}{p}-\frac{n}{2}}_{p,2}\cap\dot{B}^{\frac{n}{p}-1}_{p,1})}\right)\exp\left(C_{ab}t\right). \tag{2.45}$$

Using the fact that $u=u_{L}+\tilde{u}$ along with (2.15) and (2.45) gives us:

$$\|u\|_{L^{\infty}_{t}(\dot{B}^{\frac{n}{p}-1}_{p,1})}+a_{\star}b_{\star}\frac{C}{2}\left\|\nabla^{2}u\right\|_{L^{1}_{t}(\dot{B}^{\frac{n}{p}-1}_{p,1})}\leq C_{ab}\left(\|u_{0}\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}+\|(f,\partial_{t}R,\nabla\operatorname{div}R)\|_{L^{1}_{t}(\dot{B}^{\frac{n}{p}-\frac{n}{2}}_{p,2}\cap\dot{B}^{\frac{n}{p}-1}_{p,1})}\right)\exp\left(C_{ab}t\right). \tag{2.46}$$

Next, using (2.38)-(2.40) and (2.41) combined with (2.15) and interpolation, we infer that:

$$\|\nabla P\|_{L^{1}_{t}((\dot{B}^{\frac{n}{p}-\frac{n}{2}}_{p,2}\cap\dot{B}^{\frac{n}{p}-1}_{p,1})}\leq C_{a}\|a\nabla P_{L}\|_{L^{1}_{t}(\dot{B}^{\frac{n}{p}-\frac{n}{2}}_{p,2}\cap\dot{B}^{\frac{n}{p}-1}_{p,1})}+C_{a}\left\|a\nabla\tilde{P}\right\|_{L^{1}_{t}(\dot{B}^{\frac{n}{p}-\frac{n}{2}}_{p,2}\cap\dot{B}^{\frac{n}{p}-1}_{p,1})} \tag{2.47}$$

$$\leq C_{ab}\left(\|u_{0}\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}+\|(f,\partial_{t}R,\nabla\operatorname{div}R)\|_{L^{1}_{t}(\dot{B}^{\frac{n}{p}-\frac{n}{2}}_{p,2}\cap\dot{B}^{\frac{n}{p}-1}_{p,1})}\right)\exp\left(C_{ab}t\right). \tag{2.48}$$

Combing (2.48) with (2.46) we finally get that:

$$\begin{aligned}&\|u\|_{L^{\infty}_{t}(\dot{B}^{\frac{n}{p}-1}_{p,1})}+\left\|\nabla^{2}u\right\|_{L^{1}_{t}(\dot{B}^{\frac{n}{p}-1}_{p,1})}+\|\nabla P\|_{L^{1}_{t}((\dot{B}^{\frac{n}{p}-\frac{n}{2}}_{p,2}\cap\dot{B}^{\frac{n}{p}-1}_{p,1})}\\&\leq\left(\|u_{0}\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}+\|(f,\partial_{t}R,\nabla\operatorname{div}R)\|_{L^{1}_{t}(\dot{B}^{\frac{n}{p}-\frac{n}{2}}_{p,2}\cap\dot{B}^{\frac{n}{p}-1}_{p,1})}\right)\exp\left(C_{ab}\left(t+1\right)\right).\end{aligned} \tag{2.49}$$

Obviously, by obtaining the last estimate we conclude the proof of Proposition 2.8.

Next, let us deal with the existence part of the Stokes problem with the coefficients having regularity as in Proposition 2.8. More precisely, we have:

Proposition 2.9. *Let us consider (a,b,u_{0},f,R) as in the statement of Proposition 2.8. Then, there exists a unique solution $(u,\nabla P)\in E_{loc}$ of the Stokes system (1.3). Furthermore, there exists a constant C_{ab} depending on a and b such that:*

$$\begin{aligned}&\|u\|_{L^{\infty}_{t}(\dot{B}^{\frac{n}{p}-1}_{p,1})}+\left\|\nabla^{2}u\right\|_{L^{1}_{t}(\dot{B}^{\frac{n}{p}-1}_{p,1})}+\|\nabla P\|_{L^{1}_{t}((\dot{B}^{\frac{n}{p}-\frac{n}{2}}_{p,2}\cap\dot{B}^{\frac{n}{p}-1}_{p,1})}\\&\leq\left(\|u_{0}\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}+\|(f,\partial_{t}R,\nabla\operatorname{div}R)\|_{L^{1}_{t}(\dot{B}^{\frac{n}{p}-\frac{n}{2}}_{p,2}\cap\dot{B}^{\frac{n}{p}-1}_{p,1})}\right)\exp\left(C_{ab}\left(t+1\right)\right).\end{aligned} \tag{2.50}$$

for all $t>0$.

The uniqueness property is a direct consequence of the estimates of Proposition 2.8. The proof of existence relies on Proposition 2.8 combined with a continuity argument as used in [13], see also [24]. Let us introduce

$$(a_{\theta},b_{\theta})=(1-\theta)\left(\bar{a},\bar{b}\right)+\theta\left(a,b\right)$$

and let us consider the following Stokes systems

$$\left\{\begin{array}{r}\partial_{t}u-a_{\theta}(\operatorname{div}\left(b_{\theta}D(u)\right)-\nabla P)=f,\\ \operatorname{div}u=\operatorname{div}R,\\ u_{|t=0}=u_{0}.\end{array}\right. \tag{$\mathcal{S}_{\theta}$}$$

First of all, a more detailed analysis of the estimates established in Proposition 2.8 enables us to conclude that the constant $C_{a_{\theta}b_{\theta}}$ appearing in (2.49) is uniformly bounded with respect to $\theta\in[0,1]$ by a constant $c=c_{ab}$. Indeed, when repeating the estimation process carried out in Proposition 2.8 with (a_{θ},b_{θ}) instead of (a,b) amounts

in replacing $(a-\bar{a})$ and $\left(b-\bar{b}\right)$ with $\theta\,(a-\bar{a})$ and $\theta\left(b-\bar{b}\right)$. Taking in account Proposition 4.10 and the remark that follows we get that there exists

$$c:=\sup_{\theta\in[0,1]}C_{a_\theta b_\theta}<\infty.$$

Let us take $T>0$ and let us consider $\mathcal{E}_T$ the set of those $\theta\in[0,1]$ such that for any (u_0,f,R) as in the statement of Proposition 2.8 Problem $(\mathcal{S}_\theta)$ admits a unique solution $(u,\nabla P)\in E_T$ which satisfies

$$\begin{aligned}&\|u\|_{L^\infty_t(\dot{B}^{\frac{n}{p}-1}_{p,1})}+\left\|\nabla^2u\right\|_{L^1_t(\dot{B}^{\frac{n}{p}-1}_{p,1})}+\|\nabla P\|_{L^1_t((\dot{B}^{\frac{n}{p}-\frac{n}{2}}_{p,2}\cap\dot{B}^{\frac{n}{p}-1}_{p,1})}\\&\leq\left(\|u_0\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}+\|(f,\partial_tR,\nabla\operatorname{div}R)\|_{L^1_t(\dot{B}^{\frac{n}{p}-\frac{n}{2}}_{p,2}\cap\dot{B}^{\frac{n}{p}-1}_{p,1})}\right)\exp\left(c\,(t+1)\right),\end{aligned}\tag{2.51}$$

for all $t\in[0,T]$. According to Proposition 2.6, $0\in\mathcal{E}_T$.

Let us suppose that $\theta\in\mathcal{E}_T$. First of all, we denote by $(u_\theta,\nabla P_\theta)\in E_T$ the unique solution of $(\mathcal{S}_\theta)$. We consider the space

$$E_{T,\mathrm{div}}=\{\left(\tilde{w},\nabla\tilde{Q}\right)\in E_T:\operatorname{div}\tilde{w}=0\}$$

and let $S_{\theta\theta'}$ be the operator which associates to $\left(\tilde{w},\nabla\tilde{Q}\right)\in E_{T,\mathrm{div}}$, $\left(\tilde{u},\nabla\tilde{P}\right)$ the unique solution of

$$\left\{\begin{array}{r}\partial_t\tilde{u}-a_\theta(\operatorname{div}\left(b_\theta D(\tilde{u})\right)-\nabla\tilde{P})=g_{\theta\theta'}\left(u_\theta,\nabla P_\theta\right)+g_{\theta\theta'}\left(\tilde{w},\nabla\tilde{Q}\right),\\ \operatorname{div}u=0,\\ u_{|t=0}=0.\end{array}\right.\tag{2.52}$$

where

$$g_{\theta\theta'}\left(u,\nabla P\right)=\left(a_\theta-a_{\theta'}\right)\nabla P+a_{\theta'}\operatorname{div}\left(b_{\theta'}D\left(u\right)\right)-a_\theta\operatorname{div}\left(b_\theta D\left(u\right)\right).\tag{2.53}$$

Obviously, $S_{\theta\theta'}$ maps $E_{T,\mathrm{div}}$ into $E_{T,\mathrm{div}}$. We claim that there exists a positive quantity $\varepsilon=\varepsilon\left(T\right)>0$ such that if $|\theta-\theta'|\leq\varepsilon\left(T\right)$ then $S_{\theta\theta'}$ has a fixed point $\left(\tilde{u}^\star,\nabla\tilde{P}^\star\right)$ in a suitable ball centered at the origin of the space $E_{T,\mathrm{div}}$. Obviously,

$$\left(\tilde{u}^\star+u_\theta,\nabla\tilde{P}^\star+\nabla P_\theta\right)$$

will solve $(\mathcal{S}_{\theta'})$ in E_T.

First, we note that, as a consequence of Proposition 2.8, we have that:

$$\left\|\left(\tilde{u},\nabla\tilde{P}\right)\right\|_{E_T}\leq\left(\|g_{\theta\theta'}\left(u_\theta,\nabla P_\theta\right)\|_{L^1_T(\dot{B}^{\frac{n}{p}-\frac{n}{2}}_{p,2}\cap\dot{B}^{\frac{n}{p}-1}_{p,1})}+\left\|g_{\theta\theta'}\left(\tilde{w},\nabla\tilde{Q}\right)\right\|_{L^1_T(\dot{B}^{\frac{n}{p}-\frac{n}{2}}_{p,2}\cap\dot{B}^{\frac{n}{p}-1}_{p,1})}\right)\exp\left(c(T+1)\right).\tag{2.54}$$

Let us observe that

$$\|\left(a_\theta-a_{\theta'}\right)\nabla P\|_{L^1_T(\dot{B}^{\frac{n}{p}-\frac{n}{2}}_{p,2}\cap\dot{B}^{\frac{n}{p}-1}_{p,1})}\leq|\theta-\theta'|\,\|a-\bar{a}\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\,\|\nabla P\|_{L^1_T(\dot{B}^{\frac{n}{p}-\frac{n}{2}}_{p,2}\cap\dot{B}^{\frac{n}{p}-1}_{p,1})}.\tag{2.55}$$

Next, we write that:

$$a_{\theta'}\operatorname{div}\left(b_{\theta'}D\left(u\right)\right)-a_\theta\operatorname{div}\left(b_\theta D\left(u\right)\right)=\left(a_{\theta'}-a_\theta\right)\operatorname{div}\left(b_{\theta'}D\left(u\right)\right)+a_\theta\operatorname{div}\left(\left(b_{\theta'}-b_\theta\right)D\left(u\right)\right).$$

The first term of the last identity is estimated as follows:

$$\|\left(a_{\theta'}-a_\theta\right)\operatorname{div}\left(b_{\theta'}D\left(u\right)\right)\|_{L^1_T(\dot{B}^{\frac{n}{p}-1}_{p,1})}\leq|\theta-\theta'|\,\|a-\bar{a}\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\left(\bar{b}+\left\|b-\bar{b}\right\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\right)\|D(u)\|_{L^1_T(\dot{B}^{\frac{n}{p}}_{p,1})}.$$

Regarding the second term, we have that:

$$\|a_\theta\operatorname{div}\left(\left(b_{\theta'}-b_\theta\right)D\left(u\right)\right)\|_{L^1_T(\dot{B}^{\frac{n}{p}-1}_{p,1})}\leq|\theta-\theta'|\left\|b-\bar{b}\right\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\left(\bar{a}+\|a-\bar{a}\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\right)\|D(u)\|_{L^1_T(\dot{B}^{\frac{n}{p}}_{p,1})}$$

and thus:

$$\begin{aligned}&\|a_{\theta'}\operatorname{div}(b_{\theta'}D(u))-a_\theta\operatorname{div}(b_\theta D(u))\|_{L^1_T(\dot B^{\frac{n}{p}-1}_{p,1})}\\&\leq|\theta-\theta'|\left(\bar a+\|a-\bar a\|_{\dot B^{\frac{n}{p}}_{p,1}}\right)\left(\bar b+\left\|b-\bar b\right\|_{\dot B^{\frac{n}{p}}_{p,1}}\right)\|Du\|_{L^1_T(\dot B^{\frac{n}{p}}_{p,1})}.\end{aligned}\tag{2.56}$$

The only thing left is to treat the $L^1_t(\dot B^{\frac{3}{p}-\frac{3}{2}}_{p,2})$-norm of $a_{\theta'}\operatorname{div}(b_{\theta'}D(u))-a_\theta\operatorname{div}(b_\theta D(u))$ in the case where $n=3$. Using the fact that $\nabla u\in L^4_T(\dot B^{\frac{3}{p}-\frac{1}{2}}_{2,1})$ write that:

$$\begin{aligned}\|(a_{\theta'}-a_\theta)\operatorname{div}(b_{\theta'}D(u))\|_{L^1_T(\dot B^{\frac{3}{p}-\frac{3}{2}}_{p,2})}&\leq|\theta-\theta'|\,\|a-\bar a\|_{\dot B^{\frac{3}{p}}_{p,1}}\,\|\operatorname{div}(b_{\theta'}D(u))\|_{L^1_T(\dot B^{\frac{3}{p}-\frac{3}{2}}_{p,1})}\\&\leq|\theta-\theta'|\,\|a-\bar a\|_{\dot B^{\frac{3}{p}}_{p,1}}\left(\bar b+\left\|b-\bar b\right\|_{\dot B^{\frac{3}{p}}_{p,1}}\right)\|Du\|_{L^1_T(\dot B^{\frac{3}{p}-\frac{1}{2}}_{p,1})}\qquad(2.57)\\&\leq|\theta-\theta'|\,\|a-\bar a\|_{\dot B^{\frac{3}{p}}_{p,1}}\left(\bar b+\left\|b-\bar b\right\|_{\dot B^{\frac{3}{p}}_{p,1}}\right)T^{\frac34}\|u\|^{\frac14}_{L^\infty_T(\dot B^{\frac{3}{p}-1}_{p,1})}\|u\|^{\frac34}_{L^1_T(\dot B^{\frac{3}{p}+1}_{p,1})}\qquad(2.58)\\&\leq|\theta-\theta'|\,C(T,a,b)\left(\|u\|_{L^\infty_T(\dot B^{\frac{3}{p}-1}_{p,1})}+\left\|\nabla^2u\right\|_{L^1_T(\dot B^{\frac{3}{p}+1}_{p,1})}\right)\qquad(2.59)\end{aligned}$$

and, proceeding in a similar manner we can estimate $\|a_\theta\operatorname{div}((b_{\theta'}-b_\theta)D(u))\|_{L^1_T(\dot B^{\frac{3}{p}-\frac{3}{2}}_{p,2})}$.

Combining (2.55), (2.56) along with (2.59) we get that:

$$\|g_{\theta\theta'}(u,\nabla P)\|_{L^1_T(\dot B^{\frac{3}{p}-\frac{3}{2}}_{p,2}\cap\dot B^{\frac{3}{p}-1}_{p,1})}\leq|\theta-\theta'|\,C(T,a,b)\left(\|u\|_{L^\infty_T(\dot B^{\frac{3}{p}-1}_{p,1})}+\left\|\nabla^2u\right\|_{L^1_T(\dot B^{\frac{3}{p}+1}_{p,1})}+\|\nabla P\|_{L^1_T(\dot B^{\frac{3}{p}-\frac{3}{2}}_{p,2}\cap\dot B^{\frac{3}{p}-1}_{p,1})}\right).\tag{2.60}$$

Let us replace this into (2.54) to get that

$$\left\|\left(\tilde u,\nabla\tilde P\right)\right\|_{E_T}\leq|\theta-\theta'|\,C(T,a,b)\left(\|(u_\theta,\nabla P_\theta)\|_{E_T}+\left\|\left(\tilde w,\nabla\tilde Q\right)\right\|_{E_T}\right)$$

and by linearity

$$\left\|\left(\tilde u^1-\tilde u^2,\nabla\tilde P^1-\nabla\tilde P^2\right)\right\|_{E_T}\leq|\theta-\theta'|\,C(T,a,b)\left\|\left(\tilde w^1-\tilde w^2,\nabla\tilde Q^1-\nabla\tilde Q^2\right)\right\|_{E_T}$$

where for $k=1,2$:

$$\left(\tilde u^i,\nabla\tilde P^i\right)=S_{\theta\theta'}\left(\left(\tilde w^i,\nabla\tilde Q^i\right)\right)$$

Thus one can choose $\varepsilon(T)$ small enough such that $|\theta-\theta'|\leq\varepsilon(T)$ gives us a fixed point of the solution operator $S_{\theta\theta'}$ in $B_{E_{T,\operatorname{div}}}\left(0,2\|(u_\theta,\nabla P_\theta)\|_{E_T}\right)$.

Thus, for all $T>0$, $E_T=[0,1]$ and owing to the uniqueness property and to Proposition 2.8, we can construct a unique solution $(u,\nabla P)\in E_{loc}$ to (1.3) such that for all $t>0$ the estimate (2.11) is valid. This ends the proof of Proposition 2.9.

2.3 The proof of Theorem 1.3 in the case $n=3$

As it was discussed earlier, in dimension $n=3$, Proposition 2.8 is weaker than Theorem 1.3 as one requires additional low frequency informations on the data $(f,\partial_tR,\nabla\operatorname{div}R)\in L^1_t(\dot B^{\frac{3}{p}-\frac{3}{2}}_{p,2})$. Thus, we have to bring an extra argument in order to conclude the validity of Theorem 1.3. This is the object of interest of this section.

2.3.1 The existence part

We begin by taking $m \in \mathbb{N}$ large enough and owing to Proposition 2.7 we can consider $\left(u^1, \nabla P^1\right)$ the unique solution with $u^1 \in \mathcal{C}(\mathbb{R}^+; \dot{B}^{\frac{3}{p}-1}_{p,1})$ and $\left(\partial_t u^1, \nabla^2 u^1, \nabla P^1\right) \in L^1_{loc}(\dot{B}^{\frac{3}{p}-1}_{p,1})$ of the system

$$\begin{cases} \partial_t u - \bar{a}\bar{b}\,\mathrm{div}\, D(u) + \left(\bar{a} + \dot{S}_{-m}\left(a-\bar{a}\right)\right)\nabla P = f, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \mathrm{div}\, u = \mathrm{div}\, R, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad u_{|t=0} = u_0, \end{cases}$$

which also satisfies:

$$\left\|u^1\right\|_{L^\infty_T(\dot{B}^{\frac{3}{p}-1}_{p,1})} + \left\|\left(\partial_t u^1, \bar{a}\bar{b}\nabla^2 u^1, \bar{a}\nabla P^1\right)\right\|_{L^1_T(\dot{B}^{\frac{3}{p}-1}_{p,1})} \leq C(\|u_0\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}} + \left\|\left(f, \partial_t R, \bar{a}\bar{b}\nabla\,\mathrm{div}\, R\right)\right\|_{L^1_T(\dot{B}^{\frac{3}{p}-1}_{p,1})}),$$

for all $T > 0$. Let us consider

$$G\left(u^1, \nabla P^1\right) = a\,\mathrm{div}\left(bD(u^1)\right) - \bar{a}\,\mathrm{div}(\bar{b}D(u^1)) - \left(\left(Id - \dot{S}_{-m}\right)\left(a-\bar{a}\right)\right)\nabla P^1.$$

We claim that $G\left(u^1, \nabla P^1\right) \in L^1_{loc}(\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2} \cap \dot{B}^{\frac{3}{p}-1}_{p,1})$. Indeed

$$a\,\mathrm{div}\left(bD(u^1)\right) - \bar{a}\,\mathrm{div}(\bar{b}D(u^1)) = (a-\bar{a})\,\mathrm{div}\left(bD(u^1)\right) + \bar{a}\,\mathrm{div}\left(\left(b-\bar{b}\right)D\left(u^1\right)\right)$$

and proceeding as in (2.56) and (2.58) we get that

$$\begin{aligned} \left\|a\,\mathrm{div}\left(bD(u^1)\right) - \bar{a}\,\mathrm{div}(\bar{b}D(u^1))\right\|_{L^1_t(\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2}\cap\dot{B}^{\frac{3}{p}-1}_{p,1})} &\leq C_{ab}\left(1+t^{\frac{3}{4}}\right)\left(\left\|u^1\right\|_{L^\infty_t(\dot{B}^{\frac{3}{p}-1}_{p,1})} + \left\|u^1\right\|_{L^1_t(\dot{B}^{\frac{3}{p}+1}_{p,1})}\right) \\ &\leq \exp\left(C_{ab}\left(t+1\right)\right)\left(\|u_0\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}} + \|(f, \partial_t R, \nabla\,\mathrm{div}\, R)\|_{L^1_T(\dot{B}^{\frac{3}{p}-1}_{p,1})}\right). \end{aligned} \tag{2.61}$$

Next, we obviously have

$$\left\|\left(\left(Id - \dot{S}_{-m}\right)\left(a-\bar{a}\right)\right)\nabla P^1\right\|_{L^1_t(\dot{B}^{\frac{3}{p}-1}_{p,1})} \leq C\left\|(a-\bar{a})\right\|_{\dot{B}^{\frac{3}{p}}_{p,1}}\left\|\nabla P^1\right\|_{L^1_t(\dot{B}^{\frac{3}{p}-1}_{p,1})}. \tag{2.62}$$

Using the fact that the product maps $\dot{B}^{\frac{3}{p}-\frac{1}{2}}_{p,1} \times \dot{B}^{\frac{3}{p}-1}_{p,1} \to \dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2}$ we get that:

$$\left\|\left(\left(Id - \dot{S}_{-m}\right)\left(a-\bar{a}\right)\right)\nabla P^1\right\|_{L^1_t(\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2})} \leq C\left\|\left(Id - \dot{S}_{-m}\right)\left(a-\bar{a}\right)\right\|_{\dot{B}^{\frac{3}{p}-\frac{1}{2}}_{p,1}}\left\|\nabla P^1\right\|_{L^1_t(\dot{B}^{\frac{3}{p}-1}_{p,1})}. \tag{2.63}$$

Of course

$$\begin{aligned} \left\|\left(Id - \dot{S}_{-m}\right)\left(a-\bar{a}\right)\right\|_{\dot{B}^{\frac{3}{p}-\frac{1}{2}}_{p,1}} &\leq C\sum_{j\geq -m} 2^{j\left(\frac{3}{p}-\frac{1}{2}\right)}\left\|\dot{\Delta}_j\left(a-\bar{a}\right)\right\|_{L^2} \leq C2^{\frac{m}{2}}\sum_{j\geq -m} 2^{\frac{3}{p}j}\left\|\dot{\Delta}_j\left(a-\bar{a}\right)\right\|_{L^2}. \\ &\leq C2^{\frac{m}{2}}\left\|a-\bar{a}\right\|_{\dot{B}^{\frac{3}{p}}_{p,1}} \end{aligned}$$

so that the first term in the RHS of (2.63) is finite. We thus gather from (2.61), (2.62) and (2.63) that $G\left(u^1, \nabla P^1\right) \in L^1_{loc}(\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2} \cap \dot{B}^{\frac{3}{p}-1}_{p,1})$ and that for all $t > 0$ there exists a constant C_{ab} such that

$$\left\|G\left(u^1, \nabla P^1\right)\right\|_{L^1_t(\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2}\cap\dot{B}^{\frac{3}{p}-1}_{p,1})} \leq (\|u_0\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}} + \|(f, \partial_t R, \nabla\,\mathrm{div}\, R)\|_{L^1_t(\dot{B}^{\frac{3}{p}-1}_{p,1})})\exp\left(C_{ab}\left(t+1\right)\right).$$

According to Proposition 2.9, there exists a unique solution $\left(u^2, \nabla P^2\right) \in E_{loc}$ of the system:

$$\begin{cases} \partial_t u - a\,\mathrm{div}(bD(u)) + a\nabla P = G\left(u^1, \nabla P^1\right), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \mathrm{div}\, u = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad u_{|t=0} = 0, \end{cases}$$

which satisfies the following estimate

$$\begin{aligned}\left\|u^2\right\|_{L^\infty_t(\dot B^{\frac3p-1}_{p,1})}+\left\|\left(\nabla^2u^2,\nabla P^2\right)\right\|_{L^1_t(\dot B^{\frac3p-1}_{p,1})}&\le\left\|G\left(u^1,\nabla P^1\right)\right\|_{L^1_t(\dot B^{\frac3p-\frac32}_{p,2}\cap\dot B^{\frac3p-1}_{p,1})}\exp\left(C_{ab}\left(t+1\right)\right)\\&\le\left(\|u_0\|_{\dot B^{\frac3p-1}_{p,1}}+\|(f,\partial_tR,\nabla\operatorname{div}R)\|_{L^1_t(\dot B^{\frac3p-1}_{p,1})}\right)\exp\left(C_{ab}\left(t+1\right)\right).\end{aligned}$$

We observe that

$$(u,\nabla P):=\left(u^1+u^2,\nabla P^1+\nabla P^2\right)$$

is a solution of (1.3) which satisfies

$$\|u\|_{L^\infty_t(\dot B^{\frac3p-1}_{p,1})}+\left\|\left(\nabla^2u,\nabla P\right)\right\|_{L^1_t(\dot B^{\frac3p-1}_{p,1})}\le\left(\|u_0\|_{\dot B^{\frac3p-1}_{p,1}}+\|(f,\partial_tR,\nabla\operatorname{div}R)\|_{L^1_t(\dot B^{\frac3p-1}_{p,1})}\right)\exp\left(C_{ab}\left(t+1\right)\right).\tag{2.64}$$

Of course, using again the first equation of (1.3) we get that

$$\|\partial_tu\|_{L^1_t(\dot B^{\frac3p-1}_{p,1})}\le C_{ab}\left\|\left(f,\nabla^2u,\nabla P\right)\right\|_{L^1_t(\dot B^{\frac3p-1}_{p,1})}$$

and thus, we get the estimate

$$\|u\|_{L^\infty_t(\dot B^{\frac3p-1}_{p,1})}+\left\|\left(\partial_tu,\nabla^2u,\nabla P\right)\right\|_{L^1_t(\dot B^{\frac3p-1}_{p,1})}\le\left(\|u_0\|_{\dot B^{\frac3p-1}_{p,1}}+\|(f,\partial_tR,\nabla\operatorname{div}R)\|_{L^1_t(\dot B^{\frac3p-1}_{p,1})}\right)\exp\left(C_{ab}\left(t+1\right)\right).\tag{2.65}$$

2.3.2 Uniqueness

Next, let us prove the uniqueness property. Let us suppose that there exists a $T>0$ and a pair $(u,\nabla P)$ that solves

$$\begin{cases}\partial_tu-a\operatorname{div}(bD(u))+a\nabla P=0,\\ \operatorname{div}u=0,\\ u_{|t=0}=0,\end{cases}\tag{2.66}$$

with

$$u\in C_T(\dot B^{\frac3p-1}_{p,1})\text{ and }\left(\partial_tu,\nabla^2u,\nabla P\right)\in L^1_T(\dot B^{\frac3p-1}_{p,1}).$$

Observe that we cannot directly conclude to the uniqueness property by appealing to Proposition 2.9 because the pressure does not belong (a priori) to $L^1_T(\dot B^{\frac3p-\frac32}_{p,2})$. Recovering this low frequency information is done in the following lines. Let us suppose that $3<p<4$. Applying the operator $\mathcal{Q}$ in the first equation of (2.66) we write that:

$$\mathcal{Q}((\bar a+\dot S_{-m}(a-\bar a))\nabla P)=Q(a\operatorname{div}(bD(u)))-\mathcal{Q}((Id-\dot S_{-m})\left(a-\bar a\right)\nabla P)$$

where $m\in\mathbb{N}$ will be fixed later. We observe that:

$$\begin{aligned}\left\|\mathcal{Q}(\left(\bar a+\dot S_{-m}\left(a-\bar a\right)\right)\nabla P)\right\|_{L^1_T(\dot B^{\frac3p-\frac32}_{p,1})}&\lesssim\|Q(a\operatorname{div}(bD(u)))\|_{L^1_T(\dot B^{\frac3p-\frac32}_{p,1})}+\left\|\mathcal{Q}((Id-\dot S_{-m})\left(a-\bar a\right)\nabla P)\right\|_{L^1_T(\dot B^{\frac3p-\frac32}_{p,1})}\\&\lesssim T^{\frac14}\left(\bar a+\|a-\bar a\|_{\dot B^{\frac3p}_{p,1}}\right)\left(\bar b+\left\|b-\bar b\right\|_{\dot B^{\frac3p}_{p,1}}\right)\|\nabla u\|_{L^{\frac43}_T(\dot B^{\frac3p-\frac12}_{p,1})}\\&\quad+\left\|\left(Id-\dot S_{-m}\right)\left(a-\bar a\right)\right\|_{\dot B^{\frac3p-\frac12}_{p,1}}\|\nabla P\|_{L^1_T(\dot B^{\frac3p-1}_{p,1})}.\end{aligned}$$

Consequently, we get that

$$\mathcal{Q}((\bar a+\dot S_{-m}(a-\bar a))\nabla P)\in L^1_T(\dot B^{\frac3p-\frac32}_{p,1}).\tag{2.67}$$

Let us observe that the condition $p\in(3,4)$ ensures that $\dot B^{\frac3p}_{p,1}$ is contained in the multiplier space of $\dot B^{-\frac3p+1}_{p',2}=\dot B^{\frac3{p'}-2}_{p',2}$. More precisely, we get

Proposition 2.10. *Let us consider $(u,v)\in\dot B^{\frac3p}_{p,1}\times\dot B^{-\frac3p+1}_{p',2}$. Then $uv\in\dot B^{-\frac3p+1}_{p',2}$ and*

$$\|uv\|_{\dot B^{-\frac3p+1}_{p',2}}\lesssim\|u\|_{\dot B^{\frac3p}_{p,1}}\|v\|_{\dot B^{-\frac3p+1}_{p',2}}.$$

Proof. Indeed, considering $(u,v) \in \dot{B}^{\frac{3}{p}}_{p,1} \times \dot{B}^{-\frac{3}{p}+1}_{p',2}$ and using the Bony decomposition we get that

$$\left\|\dot{T}_u v\right\|_{\dot{B}^{-\frac{3}{p}+1}_{p',2}} \lesssim \|u\|_{L^\infty} \|v\|_{\dot{B}^{-\frac{3}{p}+1}_{p',2}}.$$

Next, considering

$$\frac{1}{p'} = \frac{1}{2} + \frac{1}{p^\star}.$$

we see that

$$\begin{aligned} 2^{j\left(-\frac{3}{p}+1\right)} \left\|\dot{\Delta}_j \dot{T}'_v u\right\|_{L^{p'}} &\lesssim \sum_{\ell \geq j-3} 2^{\left(-\frac{3}{p}+1\right)(j-\ell)} 2^{\left(-\frac{3}{p}+1\right)\ell} \|S_{\ell+1} v\|_{L^2} \left\|\dot{\Delta}_\ell u\right\|_{L^{p^\star}} \\ &= \sum_{\ell \geq j-3} 2^{\left(-\frac{3}{p}+1\right)(j-\ell)} 2^{-\frac{1}{2}\ell} \|S_{\ell+1} v\|_{L^2} 2^{\frac{3}{p^\star}\ell} \left\|\dot{\Delta}_\ell u\right\|_{L^{p^\star}}, \end{aligned}$$

and consequently, we get

$$\left\|\dot{T}'_v u\right\|_{\dot{B}^{-\frac{3}{p}+1}_{p',2}} \lesssim \|v\|_{\dot{H}^{-\frac{1}{2}}} \|u\|_{\dot{B}^{\frac{3}{p^\star}}_{p^\star,1}} \lesssim \|v\|_{\dot{B}^{-\frac{3}{p}+1}_{p',2}} \|u\|_{\dot{B}^{\frac{3}{p}}_{p,1}}.$$

□

Proposition 2.11. *Let us consider $p \in (3,4)$. Furthermore, consider a constant $\bar{c} > 0$ and $c \in \dot{B}^{\frac{3}{p}}_{p,1}$. Then there exists an universal constant $\eta > 0$ such that if*

$$\|c\|_{\dot{B}^{\frac{3}{p}}_{p,1}} \leq \eta,$$

then for any $\psi \in \dot{B}^{\frac{3}{p'}-\frac{3}{2}}_{p',2} \cap \dot{B}^{\frac{3}{p'}-2}_{p',2}$ there exists a unique solution $\nabla P \in \dot{B}^{\frac{3}{p'}-\frac{3}{2}}_{p',2} \cap \dot{B}^{\frac{3}{p'}-2}_{p',2}$ of the eliptic equation

$$\operatorname{div}\left((\bar{c}+c)\nabla P\right) = \operatorname{div}\psi.$$

Moreover, the following estimate holds true

$$\|\nabla P\|_{\dot{B}^{\frac{3}{p'}-\sigma}_{p',2}} \lesssim \|\mathcal{Q}\psi\|_{\dot{B}^{\frac{3}{p'}-\sigma}_{p',2}},$$

where $\sigma \in \left\{\frac{3}{2}, 2\right\}$.

Proof. The proof is standard. Under some smallness condition on $c \in \dot{B}^{\frac{3}{p}}_{p,1}$ the operator

$$\nabla R \to \nabla P = \frac{1}{\bar{c}} \mathcal{Q}\left(\psi - c\nabla R\right)$$

has a fixed point in a suitable chosen ball of the space $\dot{B}^{\frac{3}{p'}-\frac{3}{2}}_{p',2} \cap \dot{B}^{\frac{3}{p'}-2}_{p',2}$. □

We choose $m \in \mathbb{N}$ such that $\left\|\dot{S}_{-m}(a-\bar{a})\right\|_{\dot{B}^{\frac{3}{p}}_{p,1}}$ is small enough such that we can apply Proposition 2.11 with $\bar{a}$ and $\dot{S}_{-m}(a-\bar{a})$ instead of $\bar{c}$ and c. Let us consider ψ a vector field with coefficients in $\mathcal{S}$. As the Schwartz class is included in $\dot{B}^{\frac{3}{p'}-\frac{3}{2}}_{p',2} \cap \dot{B}^{\frac{3}{p'}-2}_{p',2}$, let us consider $\nabla P_\psi \in \dot{B}^{\frac{3}{p'}-\frac{3}{2}}_{p',2} \cap \dot{B}^{\frac{3}{p'}-2}_{p',2}$ the solution of the equation

$$\operatorname{div}\left(\left(\bar{a} + \dot{S}_{-m}(a-\bar{a})\right)\nabla P_\psi\right) = \operatorname{div}\psi,$$

the existence of which is granted by Proposition 2.11. Then, using Proposition 4.6 and Proposition 4.7, we write that[3]:

$$\langle \nabla P, \psi \rangle_{\mathcal{S}' \times \mathcal{S}} = \sum_j \left\langle \dot{\Delta}_j \nabla P, \tilde{\Delta}_j \psi \right\rangle = \sum_j - \left\langle \dot{\Delta}_j P, \tilde{\Delta}_j \operatorname{div}\psi \right\rangle \tag{2.68}$$

[3]We denote $\tilde{\Delta}_j := \dot{\Delta}_{j-1} + \dot{\Delta}_j + \dot{\Delta}_{j+1}$.

$$= \sum_j - \left\langle \dot{\Delta}_j P, \tilde{\Delta}_j \operatorname{div}\left(\left(\bar{a} + \dot{S}_{-m}(a-\bar{a})\right)\nabla P_\psi\right)\right\rangle = \sum_j \left\langle \dot{\Delta}_j \nabla P, \tilde{\Delta}_j\left(\left(\bar{a} + \dot{S}_{-m}(a-\bar{a})\right)\nabla P_\psi\right)\right\rangle \tag{2.69}$$

$$= \sum_j \left\langle \dot{\Delta}_j\left(\bar{a} + \dot{S}_{-m}(a-\bar{a})\right)\nabla P, \tilde{\Delta}_j \nabla P_\psi\right\rangle = \sum_j \left\langle \dot{\Delta}_j \mathcal{Q}\left(\left(\bar{a} + \dot{S}_{-m}(a-\bar{a})\right)\nabla P\right), \tilde{\Delta}_j \nabla P_\psi\right\rangle \tag{2.70}$$

$$\lesssim \left\|\mathcal{Q}\left(\left(\bar{a} + \dot{S}_{-m}(a-\bar{a})\right)\nabla P\right)\right\|_{\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2}} \|\nabla P_\psi\|_{\dot{B}^{\frac{3}{p'}-\frac{3}{2}}_{p',1}} \tag{2.71}$$

$$\lesssim \left\|\mathcal{Q}\left(\left(\bar{a} + \dot{S}_{-m}(a-\bar{a})\right)\nabla P\right)\right\|_{\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2}} \|\psi\|_{\dot{B}^{\frac{3}{p'}-\frac{3}{2}}_{p',1}}. \tag{2.72}$$

Taking the supremum over all $\psi \in \mathcal{S}$ with $\|\psi\|_{\dot{B}^{\frac{3}{p'}-\frac{3}{2}}_{p',2}} \leq 1$, owing to (2.67) and Proposition 4.6, it follows that $\nabla P \in L^1_T(\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2})$ and that

$$\|\nabla P\|_{L^1_T(\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2})} \lesssim \left\|\mathcal{Q}\left(\left(\bar{a} + \dot{S}_{-m}(a-\bar{a})\right)\nabla P\right)\right\|_{L^1_T(\dot{B}^{\frac{3}{p}-\frac{3}{2}}_{p,2})}.$$

According to the uniqueness property of Proposition 2.9 we conclude that $(u, \nabla P) = (0,0)$.

Let us observe that in the case $p \in \left(\frac{6}{5}, 3\right]$, owing to the fact that $\dot{B}^{\frac{3}{p}-1}_{p,1} \hookrightarrow \dot{B}^{\frac{3}{q}-1}_{q,1}$ for any $q \in (3,4)$ and $u \in C_T(\dot{B}^{\frac{3}{p}-1}_{p,1})$ along with $\left(\partial_t u, \nabla^2 u, \nabla P\right) \in L^1_T(\dot{B}^{\frac{3}{p}-1}_{p,1})$ we get that $u \in C_T(\dot{B}^{\frac{3}{q}-1}_{q,1})$ along with $\left(\partial_t u, \nabla^2 u, \nabla P\right) \in L^1_T(\dot{B}^{\frac{3}{q}-1}_{q,1})$. Thus, owing to the uniqueness property for the case $q \in (3,4)$ we conclude that $(u, \nabla P)$ is identically null for $p \in \left(\frac{6}{5}, 3\right]$.

3 Proof of Theorem 1.2

In the rest of the paper we aim at proving Theorem 1.2. Thus, from now on we will work in a 3 dimensional framework.

3.1 The linear theory

Let us introduce the space F_T of $\left(\tilde{w}, \nabla\tilde{Q}\right)$ with $\tilde{w} \in \mathcal{C}_T(\dot{B}^{\frac{3}{p}-1}_{p,1})$ and $\left(\partial_t \tilde{w}, \nabla^2\tilde{w}, \nabla\tilde{Q}\right) \in L^1_T(\dot{B}^{\frac{3}{p}-1}_{p,1})$ with the norm

$$\left\|\left(\tilde{w}, \nabla\tilde{Q}\right)\right\|_{F_T} = \|\tilde{w}\|_{L^\infty_T(\dot{B}^{\frac{3}{p}-1}_{p,1})} + \left\|\left(\partial_t\tilde{w}, \nabla^2\tilde{w}, \nabla\tilde{Q}\right)\right\|_{L^1_T(\dot{B}^{\frac{3}{p}-1}_{p,1})}.$$

Before attacking the well-posedness of (1.4), we first have to solve the following linear system:

$$\begin{cases} \rho_0\partial_t\bar{u} - \operatorname{div}\left(\mu(\rho_0) A_{\bar{v}} D_{A_{\bar{v}}}(\bar{u})\right) + A^T_{\bar{v}}\nabla\bar{P} = 0, \\ \operatorname{div}\left(A_{\bar{v}}\bar{u}\right) = 0, \\ \bar{u}_{|t=0} = u_0. \end{cases} \tag{3.1}$$

where $\bar{v} \in \mathcal{C}_T(\dot{B}^{\frac{3}{p}-1}_{p,1})$ with $\nabla\bar{v} \in L^1_T(\dot{B}^{\frac{3}{p}}_{p,1}) \cap L^2_T(\dot{B}^{\frac{3}{p}-1}_{p,1})$,

$$X_{\bar{v}}(t,y) = y + \int_0^t \bar{v}(\tau, y)\, d\tau,$$

with $\det DX_{\bar{v}} = 1$ and $A_{\bar{v}} = (DX_{\bar{v}})^{-1}$. Moreover, we suppose that:

$$\|\nabla\bar{v}\|_{L^2_T(\dot{B}^{\frac{3}{p}-1}_{p,1})} + \|\nabla\bar{v}\|_{L^1_T(\dot{B}^{\frac{3}{p}}_{p,1})} \leq 2\alpha \tag{3.2}$$

for a suitably small α. Obviously, this will be achieved using the estimates of the Stokes system established in the previous section, see Theorem 1.3. Let us write (3.1) in the form

$$\begin{cases} \partial_t \bar{u} - \frac{1}{\rho_0} \operatorname{div}\left(\mu\left(\rho_0\right) D\left(\bar{u}\right)\right) + \frac{1}{\rho_0}\nabla \bar{P} = \frac{1}{\rho_0} F_{\bar{v}}\left(\bar{u}, \nabla \bar{P}\right), \\ \operatorname{div} \bar{u} = \operatorname{div}\left(\left(Id - A_{\bar{v}}\right)\bar{u}\right), \\ \bar{u}_{t=0} = u_0. \end{cases}$$

with

$$F_{\bar{v}}\left(\bar{w}, \nabla \bar{Q}\right) := \operatorname{div}\left(\mu\left(\rho_0\right) A_{\bar{v}} D_{A_{\bar{v}}}\left(\bar{w}\right) - \mu\left(\rho_0\right) D\left(\bar{w}\right)\right) + \left(Id - A_{\bar{v}}^T\right)\nabla \bar{Q}.$$

Let us consider $(u_L, \nabla P_L)$ with $u_L \in \mathcal{C}(\mathbb{R}^+, \dot{B}^{\frac{3}{p}-1}_{p,1})$ and $\left(\partial_t u_L, \nabla^2 u_L, \nabla P_L\right) \in L^1_{loc}(\dot{B}^{\frac{3}{p}-1}_{p,1})$ the unique solution of

$$\begin{cases} \partial_t u_L - \frac{1}{\rho_0} \operatorname{div}\left(\mu\left(\rho_0\right) D\left(u_L\right)\right) + \frac{1}{\rho_0}\nabla P_L = 0, \\ \operatorname{div} u_L = 0, \\ u_{L|t=0} = u_0, \end{cases} \tag{3.3}$$

for which we know that:

$$\left\|(u_L, \nabla P_L)\right\|_{E_T} \leq \|u_0\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}} \exp\left(C_{\rho_0}\left(T+1\right)\right).$$

Moreover, T can be chosen small enough such that

$$\|\nabla u_L\|_{L^2_T(\dot{B}^{\frac{3}{p}-1}_{p,1})} + \left\|\left(\partial_t u_L, \nabla^2 u_L, \nabla P_L\right)\right\|_{L^1_T(\dot{B}^{\frac{3}{p}-1}_{p,1})} \leq \alpha \tag{3.4}$$

Following the idea in [15], and owing to Theorem 1.3, we consider the operator Φ which associates to $\left(\tilde{w}, \nabla \tilde{Q}\right) \in F_T$, the unique solution $\left(\tilde{u}, \nabla \tilde{P}\right) \in F_T$ of:

$$\begin{cases} \partial_t \tilde{u} - \frac{1}{\rho_0} \operatorname{div}\left(\mu\left(\rho_0\right) D\left(\tilde{u}\right)\right) + \frac{1}{\rho_0}\nabla \tilde{P} = \frac{1}{\rho_0} F_{\bar{v}}\left(u_L + \tilde{w}, \nabla P_L + \nabla \tilde{Q}\right), \\ \operatorname{div} \tilde{u} = \operatorname{div}\left(\left(Id - A_{\bar{v}}\right)\left(u_L + \tilde{w}\right)\right), \\ \tilde{u}_{|t=0} = 0. \end{cases}$$

We will show in the following that for a sufficiently small $T > 0$, there exists a fixed point for Φ in the unit ball centered at the origin of F_T. More precisely, according to Theorem 1.3 we get that

$$\begin{aligned} \left\|\Phi\left(\tilde{w}, \nabla \tilde{Q}\right)\right\|_{F_T} \leq & \left\|\frac{1}{\rho_0} F_{\bar{v}}\left(u_L + \tilde{w}, \nabla P_L + \nabla \tilde{Q}\right)\right\|_{L^1_T(\dot{B}^{\frac{3}{p}-1}_{p,1})} + \left\|\partial_t\left(Id - A_{\bar{v}}\right)\left(u_L + \tilde{w}\right)\right\|_{L^1_T(\dot{B}^{\frac{3}{p}-1}_{p,1})} \\ & + \left\|\nabla \operatorname{div}\left(\left(Id - A_{\bar{v}}\right)\left(u_L + \tilde{w}\right)\right)\right\|_{L^1_T(\dot{B}^{\frac{3}{p}-1}_{p,1})}. \end{aligned} \tag{3.5}$$

We begin by treating the first term:

$$\left\|\frac{1}{\rho_0} F_{\bar{v}}\left(u_L + \tilde{w}, \nabla P_L + \nabla \tilde{Q}\right)\right\|_{L^1_T(\dot{B}^{\frac{3}{p}-1}_{p,1})} \lesssim \left(\frac{1}{\bar{\rho}} + \left\|\frac{1}{\rho_0} - \frac{1}{\bar{\rho}}\right\|_{\dot{B}^{\frac{3}{p}}_{p,1}}\right)\left\|F_{\bar{v}}\left(u_L + \tilde{w}, \nabla P_L + \nabla \tilde{Q}\right)\right\|_{L^1_T(\dot{B}^{\frac{3}{p}-1}_{p,1})}. \tag{3.6}$$

We write that

$$\begin{aligned} T_1 &= \operatorname{div}\left(\mu\left(\rho_0\right) A_{\bar{v}} D_{A_{\bar{v}}}\left(u_L + \tilde{w}\right)\right) - \operatorname{div}\left(\mu\left(\rho_0\right) D\left(u_L + \tilde{w}\right)\right) \\ &= \operatorname{div}\left(\mu\left(\rho_0\right)\left(A_{\bar{v}} - Id\right) D_{A_{\bar{v}}}\left(u_L + \tilde{w}\right)\right) + \operatorname{div}\left(\mu\left(\rho_0\right) D_{A_{\bar{v}} - Id}\left(u_L + \tilde{w}\right)\right) \\ &= \operatorname{div}\left(\mu\left(\rho_0\right)\left(A_{\bar{v}} - Id\right) D_{A_{\bar{v}} - Id}\left(u_L + \tilde{w}\right)\right) + \operatorname{div}\left(\mu\left(\rho_0\right)\left(A_{\bar{v}} - Id\right) D\left(u_L + \tilde{w}\right)\right) \\ &\quad + \operatorname{div}\left(\mu\left(\rho_0\right) D_{A_{\bar{v}} - Id}\left(u_L + \tilde{w}\right)\right). \end{aligned}$$

Thus, using (4.15) we get the following bound for T_1:

$$\|T_1\|_{L^1_T(\dot{B}^{\frac{3}{p}-1}_{p,1})} \lesssim C_{\rho_0}\|A_{\bar{v}} - Id\|_{L^\infty_T(\dot{B}^{\frac{3}{p}}_{p,1})}\left(1 + \|A_{\bar{v}} - Id\|_{L^\infty_T(\dot{B}^{\frac{3}{p}}_{p,1})}\right)\left(\|\nabla u_L\|_{L^1_T(\dot{B}^{\frac{3}{p}}_{p,1})} + \|\nabla \tilde{w}\|_{L^1_T(\dot{B}^{\frac{3}{p}}_{p,1})}\right)$$

$$\lesssim C_{\rho_0}\|\nabla\bar{v}\|_{L^1_T(\dot{B}^{\frac{3}{p}}_{p,1})}\left(1+\|\nabla\bar{v}\|_{L^1_T(\dot{B}^{\frac{3}{p}}_{p,1})}\right)\left(\|\nabla u_L\|_{L^1_T(\dot{B}^{\frac{3}{p}}_{p,1})}+\|\nabla\tilde{w}\|_{L^1_T(\dot{B}^{\frac{3}{p}}_{p,1})}\right)$$
$$\lesssim C_{\rho_0}\alpha\left(\alpha+\left\|(\tilde{w},\nabla\tilde{Q})\right\|_{F_T}\right). \tag{3.7}$$

The second term is estimated as follows:

$$\left\|\left(Id-A_{\bar{v}}^T\right)(\nabla P_L+\nabla\tilde{Q})\right\|_{L^1_T(\dot{B}^{\frac{3}{p}-1}_{p,1})}\lesssim\|\nabla\bar{v}\|_{L^1_T(\dot{B}^{\frac{3}{p}}_{p,1})}\left(\|\nabla P_L\|_{L^1_T(\dot{B}^{\frac{3}{p}-1}_{p,1})}+\left\|\nabla\tilde{Q}\right\|_{L^1_T(\dot{B}^{\frac{3}{p}-1}_{p,1})}\right)$$
$$\lesssim\alpha\left(\alpha+\left\|(\tilde{w},\nabla\tilde{Q})\right\|_{F_T}\right) \tag{3.8}$$

such that combining (3.6), (3.7) and (3.8) we get that:

$$\left\|\frac{1}{\rho_0}F_{\bar{v}}\left(u_L+\tilde{w},\nabla P_L+\nabla\tilde{Q}\right)\right\|_{L^1_T(\dot{B}^{\frac{3}{p}-1}_{p,1})}\lesssim C_{\rho_0}\alpha\left(\alpha+\left\|(\tilde{w},\nabla\tilde{Q})\right\|_{F_T}\right). \tag{3.9}$$

In order to treat the second term of (3.5) we use relations (4.15), (4.16) along with interpolation in order to obtain:

$$\|\partial_t\left(Id-A_{\bar{v}}\right)(u_L+\tilde{w})\|_{L^1_T(\dot{B}^{\frac{3}{p}-1}_{p,1})}\lesssim\|\partial_t A_{\bar{v}}(u_L+\tilde{w})\|_{L^1_T(\dot{B}^{\frac{3}{p}-1}_{p,1})}+\|\left(Id-A_{\bar{v}}\right)\left(\partial_t u_L+\partial_t\tilde{w}\right)\|_{L^1_T(\dot{B}^{\frac{3}{p}-1}_{p,1})}$$
$$\lesssim\|\partial_t A_{\bar{v}}\|_{L^2_T(\dot{B}^{\frac{3}{p}-1}_{p,1})}\|u_L+\tilde{w}\|_{L^2_T(\dot{B}^{\frac{3}{p}}_{p,1})}+\|Id-A_{\bar{v}}\|_{L^\infty_T(\dot{B}^{\frac{3}{p}}_{p,1})}\|\partial_t u_L+\partial_t\tilde{w}\|_{L^1_T(\dot{B}^{\frac{3}{p}-1}_{p,1})}$$
$$\lesssim\|\nabla\bar{v}\|_{L^2_T(\dot{B}^{\frac{3}{p}-1}_{p,1})}\left(\alpha+\left\|(\tilde{w},\nabla\tilde{Q})\right\|_{F_T}\right)+\alpha\left(\alpha+\left\|(\tilde{w},\nabla\tilde{Q})\right\|_{F_T}\right)$$
$$\lesssim\alpha\left(\alpha+\left\|(\tilde{w},\nabla\tilde{Q})\right\|_{F_T}\right). \tag{3.10}$$

Treating the last term of (3.5) is done using the following formula:

$$\operatorname{div}\left(\left(Id-A_{\bar{v}}\right)(u_L+\tilde{w})\right)=\left(Du_L+D\tilde{w}\right):\left(Id-A_{\bar{v}}\right)$$

which is a consequence of the fact that $\det DX_{\bar{v}}=1$ and Proposition 4.19. Thus, we may write:

$$\|\nabla\operatorname{div}\left(\left(Id-A_{\bar{v}}\right)(u_L+\tilde{w})\right)\|_{L^1_T(\dot{B}^{\frac{3}{p}-1}_{p,1})}\lesssim\|\left(Du_L+D\tilde{w}\right):\left(Id-A_{\bar{v}}\right)\|_{L^1_T(\dot{B}^{\frac{3}{p}}_{p,1})}$$
$$\lesssim\|Id-A_{\bar{v}}\|_{L^1_T(\dot{B}^{\frac{3}{p}}_{p,1})}\|Du_L+D\tilde{w}\|_{L^1_T(\dot{B}^{\frac{3}{p}}_{p,1})}$$
$$\lesssim\alpha\left(\alpha+\left\|(\tilde{w},\nabla\tilde{Q})\right\|_{F_T}\right). \tag{3.11}$$

Combining the estimates (3.9), (3.10) and (3.11) we get that:

$$\left\|\Phi\left(\tilde{w},\nabla\tilde{Q}\right)\right\|_{E_T}\lesssim\alpha\left(\alpha+\left\|(\tilde{w},\nabla\tilde{Q})\right\|_{F_T}\right).$$

Thus, for a suitably small α the operator Φ maps the unit ball centered at the origin of F_T into itself. Due to the linearity of Φ one can repeat the above arguments in order to show that for small values of α, Φ is a contraction. This concludes the existence of a fixed point of Φ, say $\left(\tilde{u}^\star,\nabla\tilde{P}^\star\right)\in F_T$. Of course,

$$\left(\bar{u},\nabla\bar{P}\right)=\left(\tilde{u}^\star,\nabla\tilde{P}^\star\right)+\left(u_L,\nabla P_L\right)$$

is a solution of (3.1).

3.2 Proof of Theorem 1.2

Let us consider T small enough such that $(u_L, \nabla P_L)$ the solution of (3.3) satisfies

$$\|\nabla u_L\|_{L^2_T(\dot{B}^{\frac{3}{p}-1}_{p,1})} + \|\nabla u_L\|_{L^1_T(\dot{B}^{\frac{3}{p}}_{p,1})} \leq \alpha$$

and let us consider the closed set:

$$\tilde{F}_T(R) = \left\{ \left(\tilde{v}, \nabla\tilde{Q}\right) \in F_T : \tilde{v}_{|t=0} = 0, \det DX_{(u_L+\tilde{v})} = 1, \ \left\| \left(\tilde{v}, \nabla\tilde{Q}\right) \right\|_{F_T} \leq R \right\}$$

with $R \leq \alpha$ sufficiently small such that:

$$\|\nabla\tilde{v}\|_{L^2_T(\dot{B}^{\frac{3}{p}-1}_{p,1})} + \|\nabla\bar{v}\|_{L^1_T(\dot{B}^{\frac{3}{p}}_{p,1})} \leq \alpha. \tag{3.12}$$

Let us consider the operator S which associates to $\left(\tilde{v}, \nabla\tilde{Q}\right) \in \tilde{F}_T(R)$, the solution of:

$$\begin{cases} \partial_t\tilde{u} - \frac{1}{\rho_0}\operatorname{div}\left(\mu(\rho_0) D(\tilde{u})\right) + \frac{1}{\rho_0}\nabla\tilde{P} = \frac{1}{\rho_0}F_{(u_L+\tilde{v})}\left(u_L + \tilde{u}, \nabla P_L + \nabla\tilde{P}\right), \\ \qquad\qquad \operatorname{div}\left(A_{(u_L+\tilde{v})}(u_L+\tilde{u})\right) = 0, \\ \qquad\qquad \tilde{u}_{|t=0} = 0, \end{cases}$$

constructed in the previous section. We will show that that for a suitably small T, the operator S maps the closed set $\tilde{F}_T(R)$ into itself and that S is a contraction. First of all, recalling that $\left(\tilde{u}, \nabla\tilde{P}\right)$ is in fact the fixed point of the operator Φ defined above and using the estimates established in the last section we conclude that

$$\left\| S\left(\tilde{v}, \nabla\tilde{Q}\right) \right\|_{F_T} \leq R$$

for some small enough T. Moreover, because

$$\det DX_{(u_L+\tilde{v})} = 1 \text{ and } \operatorname{div}\left(A_{(u_L+\tilde{v})}(u_L+\tilde{u})\right) = 0$$

we invoke Proposition 4.19 in order to conclude that

$$\det DX_{(u_L+\tilde{u})} = 1$$

so that S maps $\tilde{F}_T(R)$ into itself.

Next, we will deal with the stability estimates. For $i = 1, 2$, let us consider $\left(\tilde{v}_i, \nabla\tilde{Q}_i\right) \in \tilde{F}_T(R)$ and $\left(\tilde{u}_i, \nabla\tilde{P}_i\right) = S\left(\tilde{v}_i, \nabla\tilde{Q}_i\right)$. Denoting by

$$\begin{aligned} \left(\delta\tilde{v}, \nabla\delta\tilde{Q}\right) &= \left(\tilde{v}_1 - \tilde{v}_2, \nabla\tilde{Q}_1 - \nabla\tilde{Q}_2\right), \\ \left(\delta\tilde{u}, \nabla\delta\tilde{P}\right) &= \left(\tilde{u}_1 - \tilde{u}_2, \nabla\tilde{P}_1 - \nabla\tilde{P}_2\right), \end{aligned}$$

we see that:

$$\begin{cases} \partial_t\delta\tilde{u} - \frac{1}{\rho_0}\operatorname{div}\left(\mu(\rho_0) D(\delta\tilde{u})\right) + \frac{1}{\rho_0}\nabla\delta\tilde{P} = \frac{1}{\rho_0}\tilde{F}, \\ \qquad\qquad \operatorname{div}\left(A_{(u_L+\tilde{v}_1)}\delta\tilde{u}\right) = \operatorname{div}\tilde{G}, \\ \qquad\qquad \delta\tilde{u}_{|t=0} = 0, \end{cases}$$

where

$$\begin{aligned} \tilde{F} &= F_1(\delta\tilde{v}, u_L + \tilde{u}_1) + F_1(u_L + \tilde{v}_2, \delta\tilde{u}) \\ &\quad + F_2\left(\delta\tilde{v}, \nabla P_L + \nabla\tilde{P}_1\right) + F_2(u_L + \tilde{v}_2, \nabla\delta\tilde{P}), \\ \tilde{G} &= -\left(A_{(u_L+\tilde{v}_1)} - A_{(u_L+\tilde{v}_2)}\right)(u_L + \tilde{u}_2), \end{aligned}$$

and

$$F_1(\bar{v},\bar{w}) = \operatorname{div}\left(\mu\left(\rho_0\right) A_{\bar{v}} D_{A_{\bar{v}}}\left(\bar{w}\right) - \mu\left(\rho_0\right) D\left(\bar{w}\right)\right),$$
$$F_2(\bar{v},\nabla\bar{Q}) = \left(Id - A_{\bar{v}}^T\right)\nabla\bar{Q}.$$

According to Theorem 1.3 we get that

$$\left\|\left(\delta\tilde{u},\nabla\delta\tilde{P}\right)\right\|_{F_T} \lesssim C_{\rho_0}\left(\left\|\tilde{F}\right\|_{L^1_T(\dot{B}^{\frac{3}{p}-1}_{p,1})} + \left\|\nabla\operatorname{div}\tilde{G}\right\|_{L^1_T(\dot{B}^{\frac{3}{p}-1}_{p,1})} + \left\|\partial_t\tilde{G}\right\|_{L^1_T(\dot{B}^{\frac{3}{p}-1}_{p,1})}\right). \tag{3.13}$$

Proceeding as in relations (3.6) and (3.7) we get that

$$\begin{aligned}\left\|\tilde{F}\right\|_{L^1_T(\dot{B}^{\frac{3}{p}-1}_{p,1})} &\lesssim \|\nabla\delta\tilde{v}\|_{L^1_T(\dot{B}^{\frac{3}{p}}_{p,1})}\|\nabla u_L + \nabla\tilde{u}_1\|_{L^1_T(\dot{B}^{\frac{3}{p}}_{p,1})} + \|\nabla u_L + \nabla\tilde{v}_2\|_{L^1_T(\dot{B}^{\frac{3}{p}}_{p,1})}\|\nabla\delta\tilde{u}\|_{L^1_T(\dot{B}^{\frac{3}{p}}_{p,1})}\\ &+ \|\nabla\delta\tilde{v}\|_{L^1_T(\dot{B}^{\frac{3}{p}}_{p,1})}\left\|\nabla P_L + \nabla\tilde{P}_1\right\|_{L^1_T(\dot{B}^{\frac{3}{p}}_{p,1})} + \|\nabla u_L + \nabla\tilde{v}_2\|_{L^1_T(\dot{B}^{\frac{3}{p}}_{p,1})}\left\|\nabla\delta\tilde{P}\right\|_{L^1_T(\dot{B}^{\frac{3}{p}}_{p,1})}\\ &\lesssim \alpha\left\|\left(\nabla\delta\tilde{v},\nabla\delta\tilde{Q}\right)\right\|_{L^1_T(\dot{B}^{\frac{3}{p}}_{p,1})} + \alpha\left\|\left(\nabla\delta\tilde{u},\nabla\delta\tilde{P}\right)\right\|_{L^1_T(\dot{B}^{\frac{3}{p}}_{p,1})}.\end{aligned} \tag{3.14}$$

Of course, we will use the smallness of α to absorb $\alpha\left\|\left(\nabla\delta\tilde{u},\nabla\delta\tilde{P}\right)\right\|_{L^1_T(\dot{B}^{\frac{3}{p}}_{p,1})}$ into the LHS of (3.13).

Next, we treat $\left\|\nabla\operatorname{div}\tilde{G}\right\|_{L^1_T(\dot{B}^{\frac{3}{p}-1}_{p,1})}$. Using Proposition 4.19, we write that

$$\operatorname{div}\tilde{G} = (Du_L + D\tilde{u}_2) : \left(A_{(u_L+\tilde{v}_1)} - A_{(u_L+\tilde{v}_2)}\right)$$

and thus, using (4.18)

$$\begin{aligned}\left\|\nabla\operatorname{div}\tilde{G}\right\|_{L^1_T(\dot{B}^{\frac{3}{p}-1}_{p,1})} &= \left\|(Du_L + D\tilde{u}_2) : \left(A_{(u_L+\tilde{v}_1)} - A_{(u_L+\tilde{v}_2)}\right)\right\|_{L^1_T(\dot{B}^{\frac{3}{p}-1}_{p,1})}\\ &\lesssim \left(\|Du_L\|_{L^1_T(\dot{B}^{\frac{3}{p}}_{p,1})} + \|D\tilde{u}_2\|_{L^1_T(\dot{B}^{\frac{3}{p}}_{p,1})}\right)\left\|A_{(u_L+\tilde{v}_1)} - A_{(u_L+\tilde{v}_2)}\right\|_{L^\infty_T(\dot{B}^{\frac{3}{p}}_{p,1})}\\ &\lesssim \alpha\|\nabla\delta\tilde{v}\|_{L^1_T(\dot{B}^{\frac{3}{p}}_{p,1})}.\end{aligned} \tag{3.15}$$

Finally, we write that

$$\begin{aligned}\left(A_{(u_L+\tilde{v}_1)} - A_{(u_L+\tilde{v}_2)}\right)(u_L+\tilde{u}_2) &= \left(\partial_t A_{(u_L+\tilde{v}_1)} - \partial_t A_{(u_L+\tilde{v}_2)}\right)(u_L+\tilde{u}_2)\\ &+ \left(A_{(u_L+\tilde{v}_1)} - A_{(u_L+\tilde{v}_2)}\right)(\partial_t u_L + \partial_t\tilde{u}_2).\end{aligned}$$

Using (4.18), (4.19) and (4.20) gives us

$$\begin{aligned}\left\|\left(\partial_t A_{(u_L+\tilde{v}_1)} - \partial_t A_{(u_L+\tilde{v}_2)}\right)(u_L+\tilde{u}_2)\right\|_{L^1_T(\dot{B}^{\frac{3}{p}-1}_{p,1})} &\lesssim \left\|\partial_t A_{(u_L+\tilde{v}_1)} - \partial_t A_{(u_L+\tilde{v}_2)}\right\|_{L^2_T(\dot{B}^{\frac{3}{p}-1}_{p,1})}\|u_L\|_{L^2_T(\dot{B}^{\frac{3}{p}}_{p,1})}\\ &+ \left\|\partial_t A_{(u_L+\tilde{v}_1)} - \partial_t A_{(u_L+\tilde{v}_2)}\right\|_{L^1_T(\dot{B}^{\frac{3}{p}}_{p,1})}\|\tilde{u}_2\|_{L^\infty_T(\dot{B}^{\frac{3}{p}-1}_{p,1})}\\ &\lesssim \|\nabla\delta\tilde{v}\|_{L^2_T(\dot{B}^{\frac{3}{p}-1}_{p,1})}\|u_L\|_{L^2_T(\dot{B}^{\frac{3}{p}}_{p,1})} + \|\nabla\delta\tilde{v}\|_{L^1_T(\dot{B}^{\frac{3}{p}}_{p,1})}\|\tilde{u}_2\|_{L^\infty_T(\dot{B}^{\frac{3}{p}-1}_{p,1})}\\ &\lesssim \alpha\|\delta\tilde{v}\|_{L^2_T(\dot{B}^{\frac{3}{p}}_{p,1})} + \alpha\|\nabla\delta\tilde{v}\|_{L^1_T(\dot{B}^{\frac{3}{p}}_{p,1})}.\end{aligned}$$

Also, using (4.18), we have that:

$$\begin{aligned}&\left\|\left(A_{(u_L+\tilde{v}_1)} - A_{(u_L+\tilde{v}_2)}\right)(\partial_t u_L + \partial_t\tilde{u}_2)\right\|_{L^1_T(\dot{B}^{\frac{3}{p}-1}_{p,1})}\\ &\lesssim \left\|A_{(u_L+\tilde{v}_1)} - A_{(u_L+\tilde{v}_2)}\right\|_{L^\infty_T(\dot{B}^{\frac{3}{p}}_{p,1})}\left(\|\partial_t u_L\|_{L^1_T(\dot{B}^{\frac{3}{p}-1}_{p,1})} + \|\partial_t u_2\|_{L^1_T(\dot{B}^{\frac{3}{p}-1}_{p,1})}\right)\end{aligned}$$

$$\lesssim \alpha \left\|\nabla \delta\tilde{v}\right\|_{L^1_T(\dot{B}^{\frac{3}{p}}_{p,1})}.$$

The conclusion is that

$$\left\|\nabla \operatorname{div} \tilde{G}\right\|_{L^1_T(\dot{B}^{\frac{3}{p}-1}_{p,1})} \lesssim \alpha \left\|\delta\tilde{v}\right\|_{L^2_T(\dot{B}^{\frac{3}{p}}_{p,1})} + \alpha \left\|\nabla\delta\tilde{v}\right\|_{L^1_T(\dot{B}^{\frac{3}{p}}_{p,1})}. \tag{3.16}$$

Gathering the information of (3.14), (3.15) and (3.16) we get that if α is chosen sufficiently small then

$$\left\|\left(\left(\delta\tilde{u}, \nabla\delta\tilde{P}\right)\right)\right\|_{F_T} \leq \frac{1}{2}\left\|\left(\left(\delta\tilde{v}, \nabla\delta\tilde{Q}\right)\right)\right\|_{F_T} \tag{3.17}$$

the operator S is also a contraction over $\tilde{F}_T(R)$. Thus, according to Banach's theorem there exists a fixed point $\left(\bar{u}^\star, \nabla\bar{P}^\star\right)$ of S. Obviously,

$$\left(\bar{u}, \nabla\bar{P}\right) = \left(u_L, \nabla P_L\right) + \left(\bar{u}^\star, \nabla\bar{P}^\star\right)$$

is a solution of

$$\begin{cases} \rho_0\partial_t\bar{u} - \operatorname{div}\left(\mu\left(\rho_0\right)A_{\bar{u}}D_{A_{\bar{u}}}\left(\bar{u}\right)\right) + A^T_{\bar{u}}\nabla\bar{P} = 0, \\ \operatorname{div}\left(A_{\bar{u}}\bar{u}\right) = 0, \\ \bar{u}_{|t=0} = u_0. \end{cases} \tag{3.18}$$

The only thing left to prove is the uniqueness property. Let us consider $\left(\bar{u}^1, \nabla\bar{P}^1\right)$, $\left(\bar{u}^2, \nabla\bar{P}^2\right) \in F_T$, two solutions of (3.18) with the same initial data $u_0 \in \dot{B}^{\frac{3}{p}-1}_{p,1}$. With $(u_L, \nabla P_L)$ defined above we let

$$\left(\tilde{u}^i, \nabla\tilde{P}^i\right) = \left(\bar{u}^i, \nabla\bar{P}^i\right) - \left(u_L, \nabla P_L\right) \text{ for } i = 1, 2$$

such that the system verified by $\left(\tilde{u}^i, \nabla\tilde{P}^i\right)$ is

$$\begin{cases} \partial_t\tilde{u}^i - \frac{1}{\rho_0}\operatorname{div}\left(\mu\left(\rho_0\right)D\left(\tilde{u}^i\right)\right) + \frac{1}{\rho_0}\nabla\tilde{P}^i = \frac{1}{\rho_0}F_{(u_L+\tilde{u}^i)}\left(u_L + \tilde{u}^i, \nabla P_L + \nabla\tilde{P}^i\right), \\ \operatorname{div}\left(A_{(u_L+\tilde{u}^i)}(u_L + \tilde{u}^i)\right) = 0, \\ \tilde{u}_{|t=0} = 0. \end{cases}$$

We are now in the position of performing exactly the same computations as above such that we obtain a time T' sufficiently small such that:

$$\left(\bar{u}^1, \nabla\bar{P}^1\right) = \left(\bar{u}^2, \nabla\bar{P}^2\right) \text{ on } [0, T'].$$

It is classical that the above local uniqueness property extends to all $[0, T]$.

3.2.1 Proof of Theorem 1.1

Finally, we are in the position of proving the result announced in Theorem 1.1. Considering $(\rho_0, u_0) \in \dot{B}^{\frac{3}{p}}_{p,1} \times \dot{B}^{\frac{3}{p}-1}_{p,1}$ and applying Theorem 1.2, there exists a positive $T > 0$ such that we may construct a solution $\left(\bar{u}, \nabla\bar{P}\right)$ to the system (1.4) in F_T. Then, considering $X_{\bar{u}}$, the "flow" of $\bar{u}$ defined by (4.12) and using Proposition 4.20 from the Appendix, one obtains that for all $t \in [0, T]$, $X_{\bar{u}}$ is a measure preserving C^1-diffeormorphism over $\mathbb{R}^n$. Thus we may introduce the Eulerian variable:

$$\rho(t,x) = \rho_0\left(X^{-1}_{\bar{u}}(t,x)\right), \; u(t,x) = \bar{u}\left(t, X^{-1}_{\bar{u}}(t,x)\right) \text{ and } P(t,x) = \bar{P}\left(t, X^{-1}_{\bar{u}}(t,x)\right).$$

Then, Proposition 4.18 assures that $(\rho, u, \nabla P)$ is a solution of (1.1). As $DX_{\bar{u}} - Id$ belongs to $\dot{B}^{\frac{3}{p}}_{p,1}$ we may conclude that $(\rho, u, \nabla P)$ has the announced regularity.

The uniqueness property comes from the fact that considering two solutions $\left(\rho^i, u^i, \nabla P^i\right)$ of (1.1), $i = 1, 2$, and considering Y_{u^i} the flow of u^i we find that $\left(u^i\left(t, Y_{u^i}(t,y)\right), \nabla P^i\left(t, Y_{u^i}(t,y)\right)\right)$ are solutions of the system (1.4) with the same data. Thus, they are equal according to the uniqueness property announced in Theorem 1.2. Thus, on some nontrivial interval $[0, T'] \subset [0, T]$, (chosen such as the condition (4.14) holds), the solutions $\left(\rho^i, u^i, \nabla P^i\right)$ are equal. This local uniqueness property obviously entails uniqueness on all $[0, T]$.

4 Appendix

We present here a few results of Fourier analysis used through the text. The full proofs along with other complementary results can be found in [6], Chapter 2.

Let us introduce the dyadic partition of the space:

Proposition 4.1. *Let $\mathcal{C}$ be the annulus $\{\xi \in \mathbb{R}^n : 3/4 \leq |\xi| \leq 8/3\}$. There exist a radial function $\varphi \in \mathcal{D}(\mathcal{C})$ valued in the interval $[0,1]$ and such that:*

$$\forall \xi \in \mathbb{R}^n \backslash \{0\}, \quad \sum_{j \in \mathbb{Z}} \varphi(2^{-j}\xi) = 1, \tag{4.1}$$

$$2 \leq |j - j'| \Rightarrow \mathrm{Supp}(\varphi(2^{-j}\cdot)) \cap \mathrm{Supp}(\varphi(2^{-j'}\cdot)) = \emptyset. \tag{4.2}$$

Also, the following inequality holds true:

$$\forall \xi \in \mathbb{R}^n \backslash \{0\}, \quad \frac{1}{2} \leq \sum_{j \in \mathbb{Z}} \varphi^2(2^{-j}\xi) \leq 1. \tag{4.3}$$

From now on we fix a functions χ and φ satisfying the assertions of the above proposition and let us denote by $\tilde{h}$ respectively h their Fourier inverses.

The homogeneous dyadic blocks $\dot{\Delta}_j$ and the homogeneous low frequency cutt-off operators $\dot{S}_j$ are defined below:

$$\dot{\Delta}_j u = \varphi\left(2^{-j}D\right) u = 2^{jn} \int_{\mathbb{R}^n} h\left(2^j y\right) u\left(x - y\right) dy$$

$$\dot{S}_j u = \chi\left(2^{-j}D\right) u = 2^{jn} \int_{\mathbb{R}^n} \tilde{h}\left(2^j y\right) u\left(x - y\right) dy$$

for all $j \in \mathbb{Z}$.

Definition 4.1. *We denote by $\mathcal{S}'_h$ the space of tempered distributions such that:*

$$\lim_{j \to -\infty} \left\| \dot{S}_j u \right\|_{L^\infty} = 0.$$

Let us now define the homogeneous Besov spaces:

Definition 4.2. *Let s be a real number and $(p,r) \in [1,\infty]$. The homogenous Besov space $\dot{B}^s_{p,r}$ is the subset of tempered distributions $u \in S'_h$ such that:*

$$\|u\|_{\dot{B}^s_{p,r}} := \left\| \left(2^{js} \left\| \dot{\Delta}_j u \right\|_{L^2} \right)_{j \in \mathbb{Z}} \right\|_{\ell^r(\mathbb{Z})} < \infty.$$

The next propositions gather some basic properties of Besov spaces.

Proposition 4.2. *Let us consider $s \in \mathbb{R}$ and $p, r \in [1,\infty]$ such that*

$$s < \frac{n}{p} \text{ or } s = \frac{n}{p} \text{ and } r = 1. \tag{4.4}$$

Then $\left(\dot{B}^s_{p,r}, \|\cdot\|_{\dot{B}^s_{p,r}}\right)$ is a Banach space.

Proposition 4.3. *A tempered distribution $u \in S'_h$ belongs to $\dot{B}^s_{p,r}(\mathbb{R}^n)$ if and only if there exists a sequence $(c_j)_j$ such that $\left(2^{js}c_j\right)_j \in \ell^r(\mathbb{Z})$ with norm 1 and a constant $C = C(u) > 0$ such that for any $j \in \mathbb{Z}$ we have*

$$\left\| \dot{\Delta}_j u \right\|_{L^p} \leq C c_j.$$

Proposition 4.4. *Let us consider s_1 and s_2 two real numbers such that $s_1 < s_2$ and $\theta \in (0,1)$. Then, there exists a constant $C > 0$ such that for all $r \in [1,\infty]$ we have:*

$$\|u\|_{\dot{B}^{\theta s_1+(1-\theta)s_2}_{p,r}} \leq \|u\|^{\theta}_{\dot{B}^{s_1}_{p,r}} \|u\|^{1-\theta}_{\dot{B}^{s_2}_{p,r}} \quad \text{and}$$
$$\|u\|_{\dot{B}^{\theta s_1+(1-\theta)s_2}_{p,1}} \leq \frac{C}{s_2-s_1}\left(\frac{1}{\theta}+\frac{1}{1-\theta}\right)\|u\|^{\theta}_{\dot{B}^{s_1}_{p,\infty}} \|u\|^{1-\theta}_{\dot{B}^{s_2}_{p,\infty}}$$

Proposition 4.5. *Let $1 \leq p_1 \leq p_2 \leq \infty$ and $1 \leq r_1 \leq r_2 \leq \infty$. Then, for any real number s, the space $\dot{B}^s_{p_1,r_1}$ is continuously embedded in $\dot{B}^{s-n\left(\frac{1}{p_1}-\frac{1}{p_2}\right)}_{p_2,r_2}$.*

Proposition 4.6. *For all $1 \leq p,r \leq \infty$ and $s \in \mathbb{R}$,*

$$\begin{cases} \dot{B}^s_{p,r} \times \dot{B}^{-s}_{p',r'} \to \mathbb{R}, \\ (u,v) \to \displaystyle\sum_j \left\langle \dot{\Delta}_j u, \tilde{\Delta}_j v \right\rangle, \end{cases} \tag{4.5}$$

where $\tilde{\Delta}_j := \dot{\Delta}_{j-1} + \dot{\Delta}_j + \dot{\Delta}_{j+1}$, defines a continuous bilinear functional on $\dot{B}^s_{p,r} \times \dot{B}^{-s}_{p',r'}$. Denote by $Q^{-s}_{p',r'}$ the set of functions $\phi \in \mathcal{S} \cap \dot{B}^{-s}_{p',r'}$ such that $\|\phi\|_{\dot{B}^{-s}_{p',r'}} \leq 1$. If $u \in \mathcal{S}'_h$, then we have

$$\|u\|_{\dot{B}^s_{p,r}} \lesssim \sup_{\phi \in Q^{-s}_{p',r'}} \langle u, \phi \rangle_{\mathcal{S}' \times \mathcal{S}}.$$

Proposition 4.7. *Let us consider $1 < p,r < \infty$ and $s \in \mathbb{R}$. Furthermore, let $u \in \dot{B}^s_{p,r}$, $v \in \dot{B}^{-s}_{p',r'}$ and $\rho \in L^\infty \cap \mathcal{M}\left(\dot{B}^s_{p,r}\right) \cap \mathcal{M}\left(\dot{B}^{-s}_{p',r'}\right)$. Then, we have that*

$$(\rho u, v) = \sum_j \left\langle \dot{\Delta}_j(\rho u), \tilde{\Delta}_j v \right\rangle = \sum_j \left\langle \dot{\Delta}_j u, \tilde{\Delta}_j(\rho v) \right\rangle = (u, \rho v). \tag{4.6}$$

The proof of Proposition 4.7 follows from a density argument. Relation (4.6) clearly holds for functions from the Schwartz class: then we may write

$$\int_{\mathbb{R}^n} \rho u v = (\rho u, v) = (u, \rho v).$$

The condition $1 < p,r < \infty$ and $s \in \mathbb{R}$ ensures that u and v may be approximated by Schwartz functions.

An important feature of Besov spaces with negative index of regularity is the following:

Proposition 4.8. *Let $s < 0$ and $1 \leq p,r \leq \infty$. Let u be a distribution in S'_h. Then, u belongs to $\dot{B}^s_{p,r}$ if and only if*

$$\left(2^{js}\left\|\dot{S}_j u\right\|_{L^p}\right)_{j\in\mathbb{Z}} \in \ell^r(\mathbb{Z}).$$

Moreover, there exists a constant C depending only on the dimension n such that:

$$C^{-|s|+1}\|u\|_{\dot{B}^s_{p,r}} \leq \left\|\left(2^{js}\left\|\dot{S}_j u\right\|_{L^p}\right)_{j\in\mathbb{Z}}\right\|_{\ell^r(\mathbb{Z})} \leq C\left(1+\frac{1}{|s|}\right)\|u\|_{\dot{B}^s_{p,r}}.$$

The next proposition tells us how certain multipliers act on Besov spaces.

Proposition 4.9. *Let us consider A a smooth function on $\mathbb{R}^n\backslash\{0\}$ which is homogeneous of degree m. Then, for any $(s,p,r) \in \mathbb{R}\times[1,\infty]^2$ such that*

$$s - m < \frac{n}{p} \text{ or } s - m = \frac{n}{p} \text{ and } r = 1$$

the operator[4] $A(D)$ maps $\dot{B}^s_{p,r}$ continuously into $\dot{B}^{s-m}_{p,r}$.

[4] $A(D)\,w = \mathcal{F}^{-1}(A\mathcal{F}w)$

The next proposition describes how smooth functions act on homogeneous Besov spaces.

Proposition 4.10. *Let f be a smooth function on $\mathbb{R}$ which vanishes at 0. Let us consider $(s,p,r)\in\mathbb{R}\times[1,\infty]^2$ such that*

$$0<s<\frac{n}{p}\ \text{or}\ s=\frac{n}{p}\ \text{and}\ r=1.$$

Then for any real-valued function $u\in\dot{B}^s_{p,r}\cap L^\infty$, the function $f\circ u\in\dot{B}^s_{p,r}\cap L^\infty$ and we have

$$\|f\circ u\|_{\dot{B}^s_{p,r}}\leq C\left(f',\|u\|_{L^\infty}\right)\|u\|_{\dot{B}^s_{p,r}}.$$

Remark 4.1. *The constant $C\left(f',\|u\|_{L^\infty}\right)$ appearing above can be taken to be*

$$\sup_{i\in\overline{1,[s]+1}}\left\|f^{(i)}\right\|_{L^\infty\left(|[-M\|u\|_{L^\infty},-M\|u\|_{L^\infty}]\right)}$$

where M is a constant depending only on the dimension n.

4.1 Commutator and product estimates

Next, we want to see how the product acts in Besov spaces. The Bony decomposition, introduced in [7] offers a mathematical framework to obtain estimates of the product of two distributions, when the later is defined.

Definition 4.3. *Given two tempered distributions $u,v\in S'_h$, the homogeneous paraproduct of v by u is defined as:*

$$\dot{T}_uv=\sum_{j\in\mathbb{Z}}\dot{S}_{j-1}u\dot{\Delta}_jv. \tag{4.7}$$

The homogeneous remainder of u and v is defined by:

$$\dot{R}\left(u,v\right)=\sum_{j\in\mathbb{Z}}\dot{\Delta}_ju\dot{\Delta}'_jv \tag{4.8}$$

where

$$\dot{\Delta}'_j=\dot{\Delta}_{j-1}+\dot{\Delta}_j+\dot{\Delta}_{j+1}.$$

Remark 4.2. *Notice that at a formal level, one has the following decomposition of the product of two (sufficiently well-behaved) distributions:*

$$uv=\dot{T}_uv+\dot{T}_vu+\dot{R}\left(u,v\right)=\dot{T}_uv+\dot{T}'_vu.$$

The next result describes how the paraproduct and remainder behave.

Proposition 4.11. *1) Assume that $(s,p,p_1,p_2,r)\in\mathbb{R}\times[1,\infty]^4$ such that:*

$$\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2},\ s<\frac{n}{p}\ \text{or}\ s=\frac{n}{p}\ \text{and}\ r=1.$$

Then, the paraproduct maps $L^{p_1}\times\dot{B}^s_{p_2,r}$ into $\dot{B}^s_{p,r}$ and the following estimates hold true:

$$\left\|\dot{T}_fg\right\|_{\dot{B}^s_{p,r}}\lesssim\|f\|_{L^{p_1}}\|g\|_{\dot{B}^s_{p_2,r}}.$$

2) Assume that $(s,p,p_1,p_2,r,r_1,r_2)\in\mathbb{R}\times[1,\infty]^6$ and $\nu>0$ such that

$$\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2},\ \frac{1}{r}=\frac{1}{r_1}+\frac{1}{r_2}$$

and

$$s<\frac{n}{p}-\nu\ \text{or}\ s=\frac{n}{p}-\nu\ \text{and}\ r=1.$$

Then, the paraproduct maps $\dot{B}^{-\nu}_{p_1,r_1} \times \dot{B}^{s+\nu}_{p_2,r_2}$ *into* $\dot{B}^{s}_{p,r}$ *and the following estimate holds true:*

$$\left\|\dot{T}_f g\right\|_{\dot{B}^{s}_{p,r}} \lesssim \|f\|_{\dot{B}^{-\nu}_{p_1,r_1}} \|g\|_{\dot{B}^{s+\nu}_{p_2,r_2}}.$$

3) Let us consider $(s_1, s_2, p, p_1, p_2, r, r_1, r_2) \in \mathbb{R}^2 \times [1,\infty]^6$ *such that*

$$0 < s_1 + s_2 < \frac{n}{p} \text{ or } s_1 + s_2 = \frac{n}{p} \text{ and } r = 1.$$

Then, the remainder maps $\dot{B}^{s_1}_{p_1,r_1} \times \dot{B}^{s_2}_{p_2,r_2}$ *into* $\dot{B}^{s_1+s_2}_{p,r}$ *and*

$$\left\|\dot{R}(f,g)\right\|_{\dot{B}^{s_1+s_2}_{p,r}} \leq \|f\|_{\dot{B}^{s_1}_{p_1,r_1}} \|g\|_{\dot{B}^{s_2}_{p_2,r_2}}.$$

As a consequence we obtain the following product rules in Besov space:

Proposition 4.12. *Consider* $p \in [1,\infty]$ *and the real numbers* $\nu_1 \geq 0$ *and* $\nu_2 \geq 0$

$$\nu_1 + \nu_2 < \frac{n}{p} + \min\left\{\frac{n}{p}, \frac{n}{p'}\right\}.$$

Then, the following estimate holds true:

$$\|fg\|_{\dot{B}^{\frac{n}{p}-\nu_1-\nu_2}_{p,1}} \lesssim \|f\|_{\dot{B}^{\frac{n}{p}-\nu_1}_{p,1}} \|g\|_{\dot{B}^{\frac{n}{p}-\nu_2}_{p,1}}.$$

Proposition 4.13. *Let us consider* θ *a* $\mathcal{C}^1$ *function on* $\mathbb{R}^n$ *such that* $(1+|\cdot|)\,\hat{\theta} \in L^1$. *Let us also consider* $p, q \in [1,\infty]$ *such that:*

$$\frac{1}{r} := \frac{1}{p} + \frac{1}{q} \leq 1.$$

Then, there exists a constant C *such that for any Lipschitz function* a *with gradient in* L^p, *any function* $b \in L^q$ *and any positive* λ*:*

$$\left\|\left[\theta\left(\lambda^{-1}D\right), a\right] b\right\|_{L^r} \leq C\lambda^{-1} \|\nabla a\|_{L^p} \|b\|_{L^q}.$$

In particular, when $\theta = \varphi$ *and* $\lambda = 2^j$ *we get that:*

$$\left\|\left[\dot{\Delta}_j, a\right] b\right\|_{L^r} \leq C 2^{-j} \|\nabla a\|_{L^p} \|b\|_{L^q}.$$

Proposition 4.14. *Assume that* s, ν *and* $p \in [1,\infty]$ *are such that*

$$0 \leq \nu \leq \frac{n}{p} \text{ and } -1 - \min\left\{\frac{n}{p}, \frac{n}{p'}\right\} < s \leq \frac{n}{p} - \nu.$$

Then, there exists a constant C *depending only on* s, ν, p *and* n *such that for all* $l \in \overline{1,n}$ *we have for some sequence* $(c_j)_{j\in\mathbb{Z}}$ *with* $\left\|(c_j)_{j\in\mathbb{Z}}\right\|_{\ell^1(\mathbb{Z})} = 1$*:*

$$\left\|\partial_l\left[a, \dot{\Delta}_j\right] w\right\|_{L^p} \leq C c_j 2^{-js} \|\nabla a\|_{\dot{B}^{\frac{n}{p}-\nu}_{p,1}} \|w\|_{\dot{B}^{s+\nu}_{p,1}}$$

for all $j \in \mathbb{Z}$.

For a proof of the above results we refer the reader to the Appendix of [13], Lemma $A.5$. and Lemma $A.6$.

Proposition 4.15. *Let us consider a homogeneous function* $A : \mathbb{R}^n \backslash \{0\} \to \mathbb{R}$ *of degree* 0. *Let us consider* $s \in \mathbb{R}$, $0 < \nu \leq 1$ *and* $p, r, r_1, r_2 \in [1,\infty]$ *such that*

$$\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$$

and

$$s < \frac{n}{p} - \nu \text{ or } s = \frac{n}{p} - \nu \text{ and } r_2 = 1. \tag{4.9}$$

Moreover, assume that $w \in \dot{B}^{s+\nu}_{p,r_2}$ *and that* $a \in L^\infty$ *with* $\nabla a \in \dot{B}^{-\nu}_{\infty,r_1}$. *Then, the following estimate holds true:*

$$\left\|[A(D), \dot{T}_a] w\right\|_{\dot{B}^{s+1}_{p,r}} \lesssim \|\nabla a\|_{\dot{B}^{-\nu}_{\infty,r_1}} \|w\|_{\dot{B}^{s+\nu}_{p,r_2}}. \tag{4.10}$$

As this result is of great importance in the analysis of the pressure term, we present a sketched proof below (see also [6], Chapter 2, Lemma 2.99)

Proof. The fact that $a \in L^\infty$ along with relation (4.9) guarantees that $A(D)\, w \in \dot{B}^{s+\nu}_{p,r}$ and that the paraproducts $\dot{T}_a w$ and $\dot{T}_a A(D)\, w$ are well-defined. We observe that there exists a function $\tilde{\varphi}$ supported in some annulus which equals 1 on the support of φ such that one may write (of course it is here that we use the homogeneity of A):

$$[A(D), \dot{T}_a]w = \sum_j \left[(A\tilde{\varphi})(2^{-j}D), \dot{S}_{j-1}a\right] \dot{\Delta}_j w.$$

But according to Lemma 4.13 we have

$$2^{j(s+1)} \left\| \left[(A\tilde{\varphi})(2^{-j}D), \dot{S}_{j-1}a\right] \dot{\Delta}_j w \right\|_{L^p} \lesssim 2^{-j\nu} \left\| \nabla \dot{S}_{j-1}a \right\|_{L^\infty} 2^{j(s+\nu)} \left\| \dot{\Delta}_j w \right\|_{L^p}.$$

The last relation obviously implies (4.10). □

As a consequence of the above proposition and Proposition 4.11 we get the following:

Proposition 4.16. *Let us consider a homogeneous function $A : \mathbb{R}^n \backslash \{0\} \to \mathbb{R}$ of degree 0, $s \in \mathbb{R}$, $0 < \nu \leq 1$ and $p, r, r_1, r_2 \in [1, \infty]$ such that*

$$\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$$

and

$$-1 - \min\left\{\frac{n}{p}, \frac{n}{p'}\right\} < s < \frac{n}{p} - \nu \text{ or } s = \frac{n}{p} - \nu \text{ and } r = r_2 = 1. \tag{4.11}$$

assume that $w \in \dot{B}^{s+\nu}_{p,r_2}$ and that $a \in L^\infty$ with $\nabla a \in \dot{B}^{-\nu}_{\infty,r_1}$. Then, the following estimate hold true:

$$\|[A(D), a]w\|_{\dot{B}^{s+1}_{p,r}} \lesssim \|\nabla a\|_{\dot{B}^{\frac{n}{p}-\nu}_{p,r_1}} \|w\|_{\dot{B}^{s+\nu}_{p,r_2}}.$$

4.2 Properties of Lagrangian coordinates

Proposition 4.17. *Let X be a globally defined bi-lipschitz diffeomorphism of $\mathbb{R}^3$ and $-\frac{3}{p'} < s \leq \frac{3}{p}$. Then $a \to a \circ X$ is a self map over $\dot{B}^s_{p,1}$ whenever*

$$1)\ s \in (0,1);$$

$$2)\ s \geq 1 \text{ and } (DX - Id) \in \dot{B}^{\frac{3}{p}}_{p,1}.$$

Proposition 4.18. *Let K be a C^1 scalar function over $\mathbb{R}^3$ and H a C^1 vector field. Let X be a C^1 diffeomorphism such that $\det(DX) = 1$. Then, the following relations hold true:*

$$(\nabla K) \circ X = \operatorname{div}\left(DX^{-1} K \circ X\right),$$
$$(\operatorname{div} H) \circ X = \operatorname{div}\left(DX^{-1} H \circ X\right).$$

Proposition 4.19. *Let us consider v and w two time-dependent vector fields with coefficients in $L^1_T(C^{0,1})$. Let us denote by Y_v and Y_w their corresponding flows. We denote by $A_v = (DY_v)^{-1}$ and $A_w = (DY_w)^{-1}$. Let us also assume that*

$$\det DY_v = 1 \text{ and } \operatorname{div}(A_v w \circ Y_v) = 0.$$

Then,

$$\det DY_w = 1$$

and for any C^1-vector field H one has

$$\operatorname{div} H = (D(H \circ Y_v) : A_v) \circ Y_v^{-1} = \operatorname{div}(A_w H \circ Y_w) \circ Y_w^{-1}.$$

This result interferes in a crucial manner in the proof of the well-posedness result for the inhomogeneous incompressible Navier-Stokes system. For a proof and other remarks see Corollary 2 from the Appendix of [15].

For any $\bar{v}$ a time dependent vector field we set:

$$X_{\bar{v}}(t,y) = y + \int_0^t \bar{v}(\tau,y)\,d\tau \tag{4.12}$$

and we denote

$$A_{\bar{v}} = (DX_{\bar{v}})^{-1}. \tag{4.13}$$

Proposition 4.20. *Let us consider $\bar{v} \in C_b([0,T], \dot{B}^{\frac{3}{p}-1}_{p,1})$ with $\partial_t \bar{v}$, $\nabla^2 \bar{v} \in L^1_T(\dot{B}^{\frac{3}{p}-1}_{p,1})$. Then, there exists a positive α such that if*

$$\|\nabla \bar{v}\|_{L^1_t(\dot{B}^{\frac{3}{p}}_{p,1})} \leq \alpha, \tag{4.14}$$

then, $X_{\bar{v}}$ introduced in (4.12) is a global C^1-diffeormorphism over $\mathbb{R}^3$. Moreover, if

$$\operatorname{div}(A_{\bar{v}}\bar{v}) = 0$$

then, $X_{\bar{v}}$ is measure preserving i.e.

$$\det DX_{\bar{v}} = 1.$$

Proposition 4.21. *Let us consider $\bar{v} \in E_T$ satisfying the smallness condition (3.2). Let X_v be defined by (4.12). Then for all $t \in [0,T]$:*

$$\|Id - A_{\bar{v}}(t)\|_{\dot{B}^{\frac{3}{p}}_{p,1}} \lesssim \|\nabla \bar{v}\|_{L^1_t(\dot{B}^{\frac{3}{p}}_{p,1})}, \tag{4.15}$$

$$\|\partial_t A_{\bar{v}}(t)\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}} \lesssim \|\nabla \bar{v}(t)\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}}, \tag{4.16}$$

$$\|\partial_t A_{\bar{v}}(t)\|_{\dot{B}^{\frac{3}{p}}_{p,1}} \lesssim \|\nabla \bar{v}(t)\|_{\dot{B}^{\frac{3}{p}}_{p,1}}. \tag{4.17}$$

In order to establish stability estimates we use the following

Proposition 4.22. *Let $\bar{v}_1, \bar{v}_2 \in E_T$ satisfying the smallness condition (4.14) and $\delta v = \bar{v}_2 - \bar{v}_1$. Then we have:*

$$\|A_{\bar{v}_1} - A_{\bar{v}_2}\|_{L^\infty_T(\dot{B}^{\frac{3}{p}}_{p,1})} \lesssim \|\nabla \delta v\|_{L^1_T(\dot{B}^{\frac{3}{p}}_{p,1})}, \tag{4.18}$$

$$\|\partial_t A_{\bar{v}_1} - \partial_t A_{\bar{v}_2}\|_{L^1_T(\dot{B}^{\frac{3}{p}}_{p,1})} \lesssim \|\nabla \delta v\|_{L^1_T(\dot{B}^{\frac{3}{p}}_{p,1})}, \tag{4.19}$$

$$\|\partial_t A_{\bar{v}_1} - \partial_t A_{\bar{v}_2}\|_{L^2_T(\dot{B}^{\frac{3}{p}-1}_{p,1})} \lesssim \|\nabla \delta v\|_{L^2_T(\dot{B}^{\frac{3}{p}-1}_{p,1})}. \tag{4.20}$$

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