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BIFURCATIONS AT COMBINATION RESONANCE AND QUASIPERIODIC VIBRATIONS OF FLEXIBLE BEAMS

K. V. Avramov

The nonlinear dynamic behavior of flexible beams is described by nonlinear partial differential equations. The beam model accounts for the tension of the neutral axis under vibrations. The Bubnov–Galerkin method is used to derive a system of ordinary differential equations. The system is solved by the multiple-scale method. A system of modulation equations is analyzed.

Keywords: vibrations, flexible beam, Bubnov–Galerkin method, multiple-scale method, bifurcations, quasiperiodic vibrations

Introduction. Nonlinear dynamics of continuum systems is still one of the most complicated divisions of mechanics [2, 4, 5, 7, 11, 12]. The nonlinear vibrations of flexible beams were analyzed in many publications, which propose various formulations of problems and various methods for their solution. How the tension of the neutral axis of beams affects their free plane vibrations is studied in [15]. Evensen [10] used the perturbation method to solve the partial differential equation of nonlinear free vibrations. Forced vibrations of a beam in single-mode approximation for the first buckling mode were addressed in [13]. Bennet and Easley [9] applied double-mode approximation to represent the plane flexural vibrations of a beam. Note that the formulation of the problems to be considered below was discussed in [1].

1. Problem Formulation. Consider a vibrating hinged flexible rod whose neutral axis is stretched under bending. The vibration equation has the form [1, 11]:

$$\rho A W_{tt} + \beta W_t + E J W_{xxxx} = \frac{EA}{2l} \int_0^l W_x^2 dx + F(x, t), \quad (1)$$

where $F(x, t) = F_0 \delta(x - l/3) \cos(\Omega t)$ is the periodic transverse force, ρ is the density of the material, E is the elastic modulus, A and J are the area and moment of inertia of the cross section, and $\delta(\cdot)$ is the delta function.

Introduce the following dimensionless parameters and variables:

$$\begin{aligned} \sqrt{\epsilon} W^* &= \frac{W}{r}, \quad x^* = \frac{x}{l}, \quad t^* = \sqrt{\frac{EJ}{\rho A l^4}} t, \quad \frac{\beta l^2}{A r \sqrt{E \rho}} = 2\mu\epsilon, \\ \sqrt{\epsilon} f_0 &= \frac{l^3 F_0}{E J r}, \quad \delta(x^* - \frac{1}{3}) = l \delta(x - \frac{l}{3}), \end{aligned} \quad (2)$$

where $\epsilon \ll 1$ and r is the radius of inertia of the cross section.

Omitting the asterisks in (2), we rearrange (1) in terms of the new variables:

$$W_{tt} + W_{xxxx} = \varepsilon \left(\frac{1}{2} W_{xx} \int_0^1 W_x^2 dx - 2\mu W_t \right) + f(x, t), \quad (3)$$

where $f(x, t) = f_0 \delta(x - 1/3) \cos(\Omega t)$.

Represent the vibrations in the form $W = \sqrt{2} \sum_n \eta_n(t) \sin(n\pi x)$. Applying the Bubnov–Galerkin method to (3), we obtain the following system of ordinary differential equations:

$$\begin{aligned} \ddot{\eta}_k + \omega_k^2 \eta_k &= \varepsilon \left(-\frac{\pi^4}{2} k^2 \eta_k \sum_i i^2 \eta_i^2 - 2\mu \dot{\eta}_k \right) + 2h_k \cos \Omega t, \\ \omega_k &= k^2 \pi^2, \quad h_k = \frac{f_0}{\sqrt{2}} \sin\left(\frac{k\pi}{3}\right), \quad k = 1, 2, \dots \end{aligned} \quad (4)$$

Consider a combination resonance: $2\Omega = \omega_1 + \omega_2 + \varepsilon\sigma$. To analyze Eqs. (4), we will apply the multiple-scale method [6] based on the following change of variables:

$$\eta_k = a_k(t) \cos(k^2 \pi^2 t + \beta_k(t)) + 2\Lambda_k \cos \Omega t + O(\varepsilon), \quad (5)$$

where $\Lambda_k = \frac{4h_k}{\pi^4(4k^4 - 25)}$.

As a result, we obtain the system of modulation equations

$$\begin{aligned} \gamma' - \sigma + \frac{5}{16} \pi^2 a_1^2 + \frac{5}{4} \pi^2 a_2^2 + \frac{6f_0^2 \chi}{\pi^6} - \frac{f_0^2 \chi}{\pi^6} \left(\frac{4a_2}{a_1} + \frac{a_1}{a_2} \right) \cos \gamma &= 0, \\ a_1' + \mu a_1 - 4 \frac{f_0^2 \chi_1}{\pi^6} a_2 \sin \gamma &= 0, \quad a_2' + \mu a_2 - \frac{f_0^2 \chi_1}{\pi^6} a_1 \sin \gamma = 0, \end{aligned} \quad (6)$$

where

$$\begin{aligned} \gamma &= \sigma T_1 - \beta_1 - \beta_2, \quad \chi_1 = 3.7 \cdot 10^{-3}, \\ \chi &= \frac{3285}{670761} + \sum_{m=0}^{\infty} \frac{(3m+1)^2}{[4(3m+1)^4 - 25]^2} + \sum_{m=0}^{\infty} \frac{(3m+2)^2}{[4(3m+2)^4 - 25]^2} = 9.82 \cdot 10^{-3}. \end{aligned} \quad (7)$$

The transition from (4) to (6) through the change of variables (5) is based on rigorous mathematical considerations. This procedure is fully described in [6]. The stability of the periodic motions of (6) corresponds to the stability of the periodic motions of (4). System (6) can be written in terms of the variables $(x, y, z) = (a_2 \cos \gamma, a_2 \sin \gamma, a_1)$:

$$\begin{aligned} x' &= -\mu x - \left(\sigma - \frac{6f_0^2 \chi}{\pi^6} \right) y + \frac{5}{16} \pi^2 (z^2 + 4x^2 + 4y^2) y - \frac{4f_0^2 \chi_1}{\pi^6} \frac{xy}{z}, \\ y' &= \left(\sigma - \frac{6f_0^2 \chi}{\pi^6} \right) x - \mu y + \frac{f_0^2 \chi_1}{\pi^6} z - \frac{5}{16} \pi^2 (z^2 + 4x^2 + 4y^2) x + \frac{4f_0^2 \chi_1}{\pi^6} \frac{x^2}{z}, \\ z' &= -\mu z + \frac{4f_0^2 \chi_1}{\pi^6} y. \end{aligned} \quad (8)$$

To analyze the nonlinear vibrations of flexible beams, we will represent the solution of Eq. (3) in the form

Fig. 1

$$W = \sqrt{2}a_1 \cos(\pi^2 t + \beta_1) \sin(\pi x) + \sqrt{2}a_2 \cos(4\pi^2 t + \beta_2) \sin(2\pi x) + \psi(x, t) + O(\varepsilon),$$

$$\psi(x, t) = \frac{4\sqrt{3}f_0}{\pi^4} \cos \Omega t \left[\sum_{m=0}^{\infty} \frac{(-1)^m \sin[(3m+1)\pi x]}{4(3m+1)^4 - 25} + \sum_{m=0}^{\infty} \frac{(-1)^m \sin[(3m+2)\pi x]}{4(3m+2)^4 - 25} \right]. \quad (9)$$

2. Bifurcations of Fixed Points. It is easy to verify that the dynamic system (6) has a fixed point $(a_1, a_2) = (0, 0)$ and fixed points that satisfy the system of equations

$$a_1 = 2a_2, \quad 4\mu^2 + \left(\frac{6f_0^2\chi}{\pi^6} - \sigma + \frac{5}{8}\pi^2 a_1^2 \right)^2 = 16 \frac{f_0^4\chi_1^2}{\pi^{12}}. \quad (10)$$

The solutions of Eqs. (10) have the form

$$a_1^{(A,B)} = \frac{2\sqrt{2}}{\sqrt{5}\pi} \sqrt{\sigma_1 \pm p}, \quad (11)$$

where $p^2 = 16 \frac{f_0^4\chi_1^2}{\pi^{12}} - 4\mu^2$.

Equality (11) defines two groups of fixed points, denoted by the superscripts A and B . These points correspond to two groups of equilibrium states of the dynamic system (8):

$$r_A = (x_A, y_A, z_A), \quad r_B = (x_B, y_B, z_B),$$

where

$$\delta = \sigma - \frac{6f_0^2\chi}{\pi^6}, \quad x_{A,B} = \pm \frac{\pi^5 p}{2\sqrt{10}f_0^2\chi_1} \sqrt{\delta \pm p}, \quad y_{A,B} = \frac{\pi^5 \mu}{\sqrt{10}f_0^2\chi_1} \sqrt{\delta \pm p}, \quad z_{A,B} = \frac{4}{\sqrt{10}\pi} \sqrt{\delta \pm p}. \quad (12)$$

Let the above fixed points be associated with an amplitude surface (see Fig. 1) that represents the dependence $a_1 = a_1(\sigma, f_0^2/\pi^6)$ in the three-dimensional space $(a_1, \sigma, f_0^2/\pi^6) \in R^3$. This surface consists of joined sheets marked by A , B , and C in Fig. 1. The sheet C corresponds to the fixed point $(a_1, a_2) = (0, 0)$; and the sheets A and B represent fixed points marked by the same letters.

To analyze the fixed points for stability, let us determine the eigenvalues λ_i of the Jacobian matrix of the vector field (8). The stability of the fixed points $r_{A,B}$ is defined by quantities $\lambda_i^{(A,B)}$ as follows:

$$\lambda_1^{(A,B)} = -2\mu, \quad \lambda_{2,3}^{(A)} = -\mu \pm R_1, \quad \lambda_{2,3}^{(B)} = -\mu \pm R_2, \quad (13)$$

where $R_{1,2} = \sqrt{\mu^2 \mp \frac{5}{8}\pi^2 a_1^2 p}$.

To describe the stability of the fixed point $(a_1, a_2) = (0, 0)$, we will use the quantities

$$\lambda_1 = -2\mu, \quad \lambda_{2,3} = -\mu \pm \sqrt{4 \frac{f_0^4 \chi_1^2}{\pi^{12}} - \frac{1}{4} \left(\sigma - \frac{6f_0^2 \chi}{\pi^6} \right)^2}. \quad (14)$$

Note that the nonhyperbolic fixed points of the sheet C satisfy the equation $\lambda_2 = 0$. Therefore, we have

$$\left(\sigma - \frac{6f_0^2 \chi}{\pi^6} \right)^2 = 16 \frac{f_0^4 \chi_1^2}{\pi^{12}} - 4\mu^2. \quad (15)$$

To analyze Eq. (15), we introduce new variables $(\tilde{\sigma}, \tilde{f}_0^2 / \pi^6)$. Then this equation rearranges to

$$-\frac{\tilde{\sigma}^2}{4\mu^2} + \left(\frac{\tilde{f}_0^2}{\pi^6} \right)^2 \frac{4\chi_1^2}{\mu^2} = 1. \quad (16)$$

Equation (16) describes the hyperbole L in Fig. 1. The domains of stable and unstable states of the dynamic system (8) on the amplitude surface are labeled by S and U , respectively. The line L represents supercritical pitchfork bifurcation on the section GC_1 and subcritical bifurcation on the section C_1B . The codimension-two bifurcation point C_1 separates the lines of supercritical and subcritical bifurcation. The amplitude surface contains a saddle-node bifurcation line L_1 , which connects the fixed points of the sheets A and B .

To analyze the dynamics of system (8) at C_1 , we will take advantage of the central-manifold method [3, 14], which allows reducing the dimension of a dynamic system. Note that it is central manifolds on which bifurcation phenomena occur. In the neighborhood of the fixed point $x = y = z = 0$, Eqs. (10) can be represented as

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} -2\mu & 0 & 0 \\ 0 & -\mu & v_1 \chi_1 + \frac{\mu}{2} \\ 0 & 4\chi_1 v_1 + 2\mu & -\mu \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} -\frac{\delta}{4} \left(2y + \frac{\mu}{2\chi_1 v_1 + \mu} z \right) + \frac{5\pi^2}{16} (z^2 + 4x^2 + 4y^2) y \\ -\frac{\delta^2 z}{8(2\chi_1 v_1 + \mu)} - \frac{5\pi^2}{16} (z^2 + 4x^2 + 4y^2) x \\ 0 \end{pmatrix}, \quad (17)$$

where $v_1 = \frac{f_0^2}{\pi^6} - \frac{\mu}{2\chi_1}$.

Note that the bifurcation point C_1 is observed when $v_1 = \delta = 0$. Write system (17) in terms of the new variables $(u, v, w) = (x, y + z/2, z/2 - y)$:

$$u' = -2\mu u + r_1(u, v, w, \delta), \quad v' = 2v_1 \chi_1 v + r_2(u, v, w, \delta), \quad w' = -2(\mu + \chi_1 v_1)w - r_2(u, v, w, \delta), \quad (18)$$

where

$$\begin{aligned} r_1 &= \frac{\delta [\chi_1 v_1 (w - v) - \mu v_1]}{2(2\chi_1 v_1 + \mu)} + \frac{5\pi^2}{16} (v - w)(v^2 + w^2 + 2u^2), \\ r_2 &= -\frac{\delta^2 (v_1 + w)}{8(2\chi_1 v_1 + \mu)} - \frac{5\pi^2}{8} u(v^2 + w^2 + 2u^2). \end{aligned} \quad (19)$$

The central manifold of the dynamic system (18) can be represented as power series with undetermined coefficients. After determining these coefficients by the method described in [14], we represent the central manifold in the form

$$u = -\frac{v_1 \delta}{4(2\chi_1 v_1 + \mu)} + \frac{5\pi^2 v^3}{32(3\chi_1 v_1 + \mu)} + O(4), \quad w = \frac{v_1 \delta^2}{16(\mu + 2\chi_1 v_1)^2} + O(4), \quad (20)$$

where $O(4)$ denotes terms of orders $v^4, \delta v^3, \delta^2 v^2, \delta^3 v$, and δ^4 .

The motions of the dynamic system (8) on the central manifold are described by the equation

$$v' = a_1 v + a_2 v^3 + a_3 v^5 + O(v^7), \quad (21)$$

where $a_1 = 2\chi_1 v_1 - \frac{\delta^2}{8\mu} + O(\delta^4)$, $a_2 = \frac{5\pi^2 \delta}{32\mu} \left[1 + \frac{\delta^2}{8\mu^2} + O(\delta^4) \right]$, and $a_3 = -\frac{25\pi^4}{256\mu} \left[1 + \frac{3\delta^2}{4\mu^2} + O(\delta^4) \right]$.

System (21) has the fixed point $v=0$ and those defined by

$$v = \frac{8}{5\pi^2} \sqrt{\delta \pm 4\sqrt{\chi_1 v_1 \mu}}. \quad (22)$$

Since fixed points are described by the dependence $v = v(\delta, v_1)$, the point $v=0$ represents the sheet C (Fig. 1); and Eq. (22) with plus (minus) sign represents the sheet $A(B)$ at C_1 . The point $v=0$ is unstable in the domain $16\mu\chi_1 v > \delta^2$. Note that the bifurcation line of the dynamic system (21) is defined by the equation $16\mu\chi_1 v = \delta^2$. This equation approximates the curve L at C_1 (Fig. 1). The equation

$$v' = \frac{5\pi^2 \delta}{32\mu} v^3 + O(v^5) \quad (23)$$

describes the motions of the dynamic system (8) on the central manifolds of the fixed points of the line L . This manifold is stable when $\delta < 0$ and unstable when $\delta > 0$.

The motion of system (8) on the central manifold of the point C_1 is described by the equation

$$v' = -\frac{25\pi^4}{256\mu} v^5 + O(v^7). \quad (24)$$

It follows from this equation that the point C_1 is stable.

Consider saddle-node bifurcation L_1 (Fig. 1) at C_1 . To this end, let us introduce the variables

$$x_1 = x, \quad y_1 = y - 2\sqrt{\delta_1}, \quad z_1 = z - 4\sqrt{\delta_1}, \quad (25)$$

where $10\pi^2 \delta_1 = \delta$.

Rearranged in terms of the new variables, system (8) has the form

$$\begin{pmatrix} x'_1 \\ y'_1 \\ z'_1 \end{pmatrix} = \begin{pmatrix} -2\mu & \delta & \frac{\delta}{2} \\ 0 & -\mu & \frac{\mu}{2} \\ 0 & 2\mu & -\mu \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} f_1(x_1, y_1, z_1) \\ f_2(x_1, y_1, z_1) \\ f_3(x_1, y_1, z_1) \end{pmatrix}. \quad (26)$$

We do not present the expressions of the functions $f_1(x_1, y_1, z_1)$, $f_2(x_1, y_1, z_1)$, and $f_3(x_1, y_1, z_1)$ because of their awkwardness. Rearranged in terms of the new variables, Eqs. (26) become

$$(u, v, w) = \left(x_1 - \frac{\delta}{2\mu} y_1 - \frac{\delta}{4\mu} z_1, \frac{\delta}{4\mu} (2y_1 + z_1), \frac{z_1}{2} - y_1 \right),$$

Fig. 2

$$u' = -2\mu u + F_1(u, v, w), \quad v' = F_2(u, v, w), \quad w' = -2\mu w + F_3(u, v, w). \quad (27)$$

The functions F_1 , F_2 , and F_3 are also unwieldy and, thus, are not presented. The central manifold is

$$u = a_1 v^2 + b_1 v w + c_1 v^2 + O(3), \quad w = a_2 v^2 + b_2 v w + c_2 v^2 + O(3), \quad (28)$$

where $O(3)$ denotes terms of orders v^3 , $v^2 w$, $v w^2$, and w^3 .

The motion of system (8) on the central manifold is described by the differential equation

$$v' = -v^3 \left(\frac{5\pi^2 \mu}{2\delta} + \frac{5\delta \pi^2}{8\mu} \right) - \frac{\pi \sqrt{10\delta}}{4} v^2 + 2\chi_1 v w + \frac{4\chi_1 \delta^{3/2} v}{\sqrt{10\mu\pi}} + O(v^4). \quad (29)$$

Equation (29) describes saddle-node bifurcation in system (8). The fixed points of this equation form two branches v_1 and v_2 :

$$v_{1,2} = \mp \frac{2}{\pi} \sqrt{\frac{2\chi_1 \delta v}{5\mu}} + O(v). \quad (30)$$

The stability of the fixed points $v_{1,2}$ is determined by the characteristic numbers $\lambda_{1,2}$

$$\lambda_{1,2} = \pm 2\delta \sqrt{\frac{\chi_1 v}{\mu}} + O(v). \quad (31)$$

The equation $v = 0$ describes a saddle-node bifurcation line. The behavior of trajectories near the bifurcation point C_1 is shown in Fig. 2 as a bifurcational diagram [11].

3. Quasiperiodic Vibrations of a Flexible Beam. The steady vibrations of a flexible beam that correspond to the fixed points of the dynamic system (8) are defined by

$$W = \sqrt{2} a_1 \cos(v_1 t - C_*) \sin(\pi x) + \frac{a_1}{\sqrt{2}} \cos(v_2 t + C_* - \gamma^0) \sin(2\pi x) + \Psi(x, t) + O(\varepsilon), \quad (32)$$

where $\gamma^0 = \arcsin(\mu \pi^6 / (2f_0^2 \chi_1))$.

Thus, the vibrations of the beam are represented by the sum of three functions of time with frequencies v_1 , v_2 , and Ω ,

$$v_j = j^2 \pi^2 + \epsilon \Omega_j + O(\epsilon^2), \quad \Omega_j = \frac{3}{16} \pi^2 a_1^2 + \frac{6f_0^2}{\pi^6} \rho_j + \frac{\pi^2}{2} a_2^2 + 2F \cos \gamma^0, \quad j=1, 2, \dots, \quad (33)$$

where $\rho_1 + \rho_2 = \chi$, $\rho_1 = 4.73 \cdot 10^{-3}$, and $\rho_2 = 5.09 \cdot 10^{-3}$.

Using (6), we obtain

$$v_1 + v_2 = 2\Omega + O(\epsilon^2). \quad (34)$$

As follows from (34), vibrations (32) occur at two frequencies v_1 and v_2 and their combination Ω . It is well known that if the frequencies v_1 and v_2 are not related as integers, then the vibrations are quasiperiodic. Minor changes in v_1 or v_2 may turn quasiperiodic vibrations into periodic. This phenomenon is called synchronization. Note that based on the results presented in this paper we may demonstrate that vibrations (32) are always quasiperiodic.

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