Moutard type transform for matrix generalized analytic functions and gauge transforms
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Considerable progress in the theory of Darboux-Moutard type transforms for two-dimensional linear differential systems with applications to geometry, spectral theory, and soliton equations has been achieved recently, see, e.g., [1, 2, 3, 4]. In the present note we derive such a transformation for the matrix generalized function system
\[ \partial \bar{z} \Psi + A \Psi + B \Psi = 0, \] (1)
where \( \partial \bar{z} = \frac{\partial}{\partial \bar{z}} \), the coefficients \( A \) and \( B \) and solutions \( \Psi \) are \((N \times N)\)-matrix functions on \( D \), with \( D \) an open simply connected domain in \( \mathbb{C} \). In particular, this generalizes the transform for \( N = 1 \) found in [4] with \( A = 0 \). In addition, we show that the Moutard type transform for system (1) with \( B = 0 \) is equivalent to a gauge transform for the connection \( \nabla \bar{z} = \partial \bar{z} + A \). In turn, our studies show that the Moutard type transform for system (1) with \( A = 0 \) can be treated as a proper analog of the aforementioned gauge transform.

As for \( N = 1 \), system (1) is reduced to the system
\[ \partial \bar{z} \Psi + B \Psi = 0, \] (2)
i.e. to system (1) with \( A = 0 \), by the gauge transform
\[ \Psi \rightarrow \tilde{\Psi} = g^{-1} \Psi, \quad B \rightarrow \tilde{B} = g^{-1} B \bar{g}, \quad \partial \bar{z} g + A g = 0, \quad \det g \neq 0. \]

We say that the system
\[ \partial \bar{z} \Psi^+ - \bar{\Psi}^+ B = 0 \] (3)
is conjugate to system (2) (see [5] for a similar definition for \( N = 1 \)).

We have the following result.

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Theorem 1 Systems (2) and (3) are covariant, i.e. mapped into the systems of the same type, with respect to the Moutard type transform
\[ \Psi \rightarrow \tilde{\Psi} = \Psi - F \omega_{F,F+}^{-1} \omega_{F,F+}, \]
\[ \Psi^+ \rightarrow \tilde{\Psi}^+ = \Psi^+ - \omega_{F,F+} \omega_{F,F+}^{-1} F^+, \]
\[ B \rightarrow \tilde{B} = B + F \omega_{F,F+}^{-1} F^+, \]

where \( F \) and \( F^+ \) are arbitrary fixed solutions of (2) and (3), respectively,
\[ \partial \bar{z} \omega_{F,F+} + \Phi = 0, \]
\[ \text{Re} \omega_{F,F+} = 0, \]

for \( \Phi \) and \( \Phi^+ \) meeting equations (2) and (3), and \( \det \omega_{F,F+} \neq 0. \)

For finding \( \omega_{F,F+} \) satisfying (5) we use also that \( \partial \bar{z} \omega_{F,F+} = -\Phi^+ \Phi. \) In addition, our definition of \( \omega_{F,F+} \) is self-consistent up to a pure imaginary matrix integration constant in view of the identity \( \partial \bar{z} \Phi^+ \Phi = -\partial \bar{z} \Phi \Phi. \) The latter equality follows from systems (2) and (3) for \( \Phi \) and \( \Phi^+ \), respectively. We recall that the domain \( D \) is simply connected.

Given \( \omega_{F,F+}, \omega_{F,F+}, \) and \( \omega_{F,F+}, \) Theorem 1 is proved by straightforward computations.

In addition, for the system
\[ \partial \bar{z} \Psi + A \Psi = 0, \]

i.e., for system (1) with \( B = 0, \) the following result also holds.

Proposition 1 System (6) is covariant under the following Moutard type transform
\[ \Psi \rightarrow \tilde{\Psi} = \Psi - F \hat{\omega}_{F,F+}^{-1} \hat{\omega}_{F,F+}, \]
\[ A \rightarrow \hat{A} = A + F \hat{\omega}_{F,F+}^{-1} F^+, \]

where \( F \) is an arbitrary fixed solution of (6), \( F^+ \) is an arbitrary fixed matrix function,
\[ \partial \bar{z} \hat{\omega}_{F,F+} = F^+ \Phi \]

for any matrix function \( \Phi, \) and \( \det \hat{\omega}_{F,F+} \neq 0. \)

Equations (7) and (8) are analogs of equations (4) and (5). However, in difference with (5), we do not require that the matrix functions \( \hat{\omega}_{F,F+} \) would be pure imaginary. Equation (8) is solvable for \( \hat{\omega}_{F,F+} \) and Proposition 1 is proved by straightforward computations.

Remark. Let \( A, \hat{A}, \Psi, F, F^+, \) and \( \hat{\omega}_{F,F+} \) be the same as in Proposition 1. Let
\[ g = 1 - F \hat{\omega}_{F,F+}^{-1} A, \]
\[ A \hat{\omega}_{F,F+} = \Lambda A + F^+. \]

Then
\[ \partial \bar{z} (g \Psi) + \hat{A} (g \Psi) = 0. \]

It is proved by straightforward computations and it shows that for invertible \( g \) the transform \( A \rightarrow \hat{A} \) reduces to a gauge transform.
References


