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BIFURCATION ANALYSIS OF A VIBROPERCUSSION SYSTEM BY THE METHOD OF AMPLITUDE SURFACES

K. V. Avramov

An amplitude surface is introduced to analyze the bifurcation behavior of a system upon change in two characteristic parameters. A bifurcation analysis of a vibropercussion system with changing force amplitude and frequency is performed as an example of using amplitude surfaces. Bifurcations of codimension two are revealed.

1. Introduction and Description of a Mechanical System. Amplitude surfaces are used here to analyze the bifurcations of nonlinear mechanical systems upon change in two characteristic parameters. In mechanics, the bifurcation analysis is frequently based on amplitude–frequency characteristics [1–3, 5]. An amplitude surface is a three-dimensional generalization of an amplitude–frequency characteristic. The amplitude surfaces of a vibropercussion system can be constructed by a numerical method. Such a surface makes it possible to reveal various codimension two bifurcations formed by joining tangent-bifurcation lines.

Let us consider a vibropercussion system with two constraints whose reactions are determined by the nonlinear dependence $R(\xi) = \xi + c\xi^n$. In terms of ordinary dimensionless variables [4, 12], the equations of motion of the vibropercussion system has the form

$$\begin{aligned} \omega^2 x'' + 2\alpha\omega x' + H(x) &= f \sin(t), \\ H(x) &= \begin{cases} x + c(x+1)^n, & x > -1, \\ -1, & -2\lambda - 1 < x < -1, \\ x + 2\lambda + (-1)^{n+1} c(x + 2\lambda + 1)^n, & x < -2\lambda - 1, \end{cases} \end{aligned} \quad (1)$$

where ω is the frequency of perturbation, α is the coefficient of viscous resistance, and 2λ is the distance between the constraints.

Note that the function $H(x) = -1$ for $-2\lambda - 1 < x < -1$ describes motion between the two constraints. Equation (1) accounts for the action of a constant force on a material point. System (1) can be rearranged for the state variables $(y_1, y_2) = (x', x)$, which are used below.

2. The Method of Amplitude Surfaces. Amplitude surfaces are convenient for the geometrical representation of the bifurcation behavior of limit cycles upon change in two characteristic parameters of the system. Let such parameters be ω and f and the amplitude surface be the dependence of the semi-range S of periodic motions on ω and f . Therefore, a function $S = G(\omega, f)$ represents the amplitude surface in the parameter space $(S, \omega, f) \in R^3$. Each point on the amplitude surface corresponds to a certain limit cycle.

Following [9], let us group the limit cycles of system (1) according to the values of the multipliers (ρ_1, ρ_2) as follows: directly unstable cycles ($\rho_1 > 1, 0 < \rho_2 < 1$), inversely unstable cycles ($\rho_1 < -1, -1 < \rho_2 < 0$), and stable cycles ($|\rho_i| < 1, i = 1, 2$). Cycles of one type can transform into cycles of other type through local bifurcations. As is known, system (1) can give rise to codimension one tangent bifurcations and period-doubling bifurcations. Note that $\rho_1 = 1$ for tangent bifurcations

and $\rho_1 = -1$ for period-doubling bifurcations. Let us find what the geometrical representation of bifurcations is from the standpoint of the amplitude surface. Note that amplitude surfaces contain period-doubling and tangent-bifurcation lines. It is clear that the tangent-bifurcation line is one along which stable cycles of order N join with directly unstable cycles of the same order and the period-doubling line is where three types of cycles join together: stable cycles of order $2N$, inversely unstable cycles of the same order, and stable cycles of order $2N$. Note that bifurcation lines can join at codimension two bifurcation points [7, 8, 10].

Let us now describe the amplitude-surface algorithm. First, the amplitude–frequency characteristic is designed to find bifurcation points. Second, these bifurcation points are used to draw bifurcation lines on the parameter plane $(\omega, f) \in R^2$. Finally, the amplitude surface is constructed based on the amplitude–frequency characteristic and the bifurcation lines. Thus, the amplitude-surface algorithm includes two algorithms: one is intended for designing amplitude–frequency characteristics and the other for drawing bifurcation lines on the plane (ω, f) . To plot the amplitude–frequency characteristic, let us take advantage of the algorithm from [5, 11]. Omitting the description of this algorithm, let us dwell on the bifurcation-line algorithm for system (1). The solutions of Eqs. (1) can be presented as $y_i(\omega, f, y_{01}, y_{02}, t)$, $i = 1, 2$, where y_{0i} are the initial states of the system. Then, cycles of order N satisfy the nonlinear equations

$$y_i(\omega, f, y_{01}, y_{02}, 2\pi N) - y_{0i} = 0. \quad (2)$$

Equations (2) together with the additional equation

$$W(\omega, f, y_{01}, y_{02}) = \det \left[\frac{\partial(y_1, y_2)}{\partial(y_{01}, y_{02})} - E \right] = 0, \quad (3)$$

where $\left[\frac{\partial(y_1, y_2)}{\partial(y_{01}, y_{02})} \right]$ is the Jacobian matrix and E is a unit matrix, describe cycles undergoing bifurcations.

A bifurcation line can be represented by a mesh function $(\omega_1, f_1), (\omega_2, f_2), \dots, (\omega_M, f_M)$. Each point of the mesh function is the solution of the system of equations (2), (3). Therefore, to obtain the bifurcation line M , it is necessary to solve system (2), (3). It can be solved numerically as follows. First, system (2), (3) is resolved for one of the vectors $v_1 = (\omega, y_{01}, y_{02})$ and $v_2 = (\omega, y_{01}, y_{02})$. If the vector of unknowns is either v_1 or v_2 , then either f or ω respectively, is specified prior to solving the equations. The variable specified is called a parametrizing variable [11]. The parametrizing variable is selected before the computation of each point of the bifurcation line by the algorithm described in [11]. Equations (2) and (3) are solved by Newton's method. Let us determine the Jacobian matrix appearing in Newton's method. Let $\pi_1 = \omega$ and $\pi_2 = f$ for brevity; then the Jacobian matrix can be written in the form

$$\left[\frac{\partial(y_1 - y_{01}, y_2 - y_{02}, W)}{\partial(y_{01}, y_{02}, \pi_i)} \right] = \begin{bmatrix} \frac{\partial(y_1, y_2)}{\partial(y_{01}, y_{02})} - E & P \\ Q & R \end{bmatrix} = \begin{bmatrix} \frac{\partial y_1}{\partial y_{01}} - 1 & \frac{\partial y_1}{\partial y_{02}} & \frac{\partial y_1}{\partial \pi_i} \\ \frac{\partial y_2}{\partial y_{01}} & \frac{\partial y_2}{\partial y_{02}} - 1 & \frac{\partial y_2}{\partial \pi_i} \\ \frac{\partial W}{\partial y_{01}} & \frac{\partial W}{\partial y_{02}} & \frac{\partial W}{\partial \pi_i} \end{bmatrix}. \quad (4)$$

Since the finite-difference formulas used to find the derivatives in (4) degrade the convergence of Newton's method, we will take advantage of the following method to determine the elements of the Jacobian matrix.

It is well known that the submatrix $\left[\frac{\partial(y_1, y_2)}{\partial(y_{01}, y_{02})} \right]$ is the solution of the system of differential variational equations

$$\frac{d}{dt} \left[\frac{\partial(y_1, y_2)}{\partial(y_{01}, y_{02})} \right] = [V(t)] \left[\frac{\partial(y_1, y_2)}{\partial(y_{01}, y_{02})} \right],$$

$$\left[\frac{\partial(y_1, y_2)}{\partial(y_{01}, y_{02})} \right]_{t=0} = [E], \quad [V(t)] = \begin{bmatrix} -\frac{2\alpha}{\omega} & -\frac{1}{\omega^2} \frac{dH}{dy_2} \\ 1 & 0 \end{bmatrix}. \quad (5)$$

To derive the submatrix $[P]$, let us set up two systems of differential equations for the vectors $\left[\frac{\partial y_1}{\partial \pi_i} \frac{\partial y_1}{\partial \pi_i} \right] i = 1 \text{ or } 2$ by differentiating (1) with respect to π_i . Written in a matrix form, these systems look like this:

$$\frac{d}{dt} \begin{bmatrix} \frac{\partial y_1}{\partial \omega} & \frac{\partial y_1}{\partial f} \\ \frac{\partial y_2}{\partial \omega} & \frac{\partial y_2}{\partial f} \end{bmatrix} = [V(t)] \begin{bmatrix} \frac{\partial y_1}{\partial \omega} & \frac{\partial y_1}{\partial f} \\ \frac{\partial y_2}{\partial \omega} & \frac{\partial y_2}{\partial f} \end{bmatrix} + \begin{bmatrix} \frac{2\alpha}{\omega^2} y_1 + \frac{2}{\omega^3} H(y_2) - \frac{2f}{\omega^3} \sin(t) & \frac{1}{\omega^2} \sin(t) \\ 0 & 0 \end{bmatrix}. \quad (6)$$

The solutions of Eqs. (6) with zero initial conditions for $t = 2\pi N$ are the elements of the submatrix $[P]$ from (4). Let us now derive the submatrices $[Q]$ and $[R]$ appearing in (5). Using (4), we find

$$\frac{\partial W}{\partial y_{0v}} = \det \begin{bmatrix} \frac{\partial^2 y_1}{\partial y_{01} \partial y_{0v}} & \frac{\partial y_1}{\partial y_{02}} \\ \frac{\partial^2 y_2}{\partial y_{01} \partial y_{0v}} & \frac{\partial y_2}{\partial y_{02}} - \rho \end{bmatrix} + \det \begin{bmatrix} \frac{\partial y_1}{\partial y_{01}} - \rho & \frac{\partial^2 y_1}{\partial y_{0v} \partial y_{02}} \\ \frac{\partial y_2}{\partial y_{01}} & \frac{\partial^2 y_2}{\partial y_{0v} \partial y_{02}} \end{bmatrix}, \quad v = 1, 2. \quad (7)$$

The formula for $\frac{\partial W}{\partial \pi_i}$ ($i = 1, 2$) is similar. To determine the second derivatives appearing in (7), let us set up systems of differential equations for the vectors consisting of these second derivatives by differentiating (5) with respect to $y_{0\mu}$ and π_i . Written in a matrix form, these systems look like this:

$$\begin{aligned} \frac{d}{dt} \begin{bmatrix} \frac{\partial^2 y_1}{\partial y_{0\mu} \partial y_{0v}} & \frac{\partial^2 y_1}{\partial f \partial y_{0v}} & \frac{\partial^2 y_1}{\partial \omega \partial y_{0v}} \\ \frac{\partial^2 y_2}{\partial y_{0\mu} \partial y_{0v}} & \frac{\partial^2 y_2}{\partial f \partial y_{0v}} & \frac{\partial^2 y_2}{\partial \omega \partial y_{0v}} \end{bmatrix} &= [V(t)] \begin{bmatrix} \frac{\partial^2 y_1}{\partial y_{0\mu} \partial y_{0v}} & \frac{\partial^2 y_1}{\partial f \partial y_{0v}} & \frac{\partial^2 y_1}{\partial \omega \partial y_{0v}} \\ \frac{\partial^2 y_2}{\partial y_{0\mu} \partial y_{0v}} & \frac{\partial^2 y_2}{\partial f \partial y_{0v}} & \frac{\partial^2 y_2}{\partial \omega \partial y_{0v}} \end{bmatrix} \\ &+ \frac{1}{\omega^2} \begin{bmatrix} -\frac{d^2 H}{dy_2^2} \frac{\partial y_2}{\partial y_{0v}} \frac{\partial y_2}{\partial y_{0\mu}} & -\frac{d^2 H}{dy_2^2} \frac{\partial y_2}{\partial f} \frac{\partial y_2}{\partial y_{0v}} & 2\alpha \frac{\partial y_1}{\partial y_{0v}} + \frac{2}{\omega} \frac{dH}{dy_2} \frac{\partial y_2}{\partial y_{0v}} - \frac{d^2 H}{dy_2^2} \frac{\partial y_2}{\partial \omega} \frac{\partial y_2}{\partial y_{0v}} \\ 0 & 0 & 0 \end{bmatrix}. \end{aligned} \quad (8)$$

Equations (8) have zero initial conditions. Let us solve Eqs. (8). The function $\frac{d^2 H}{dy_2^2}$ can be represented in the form

$$\frac{d^2 H}{dy_2^2} = \sigma(t) + q(y_2), \quad \sigma(t) = \sum_{j=1}^{p_1} \frac{\delta(t - \tau_j)}{|y_1(\tau_j)|} - \sum_{v=1}^{p_2} \frac{\delta(t - T_v)}{|y_1(T_v)|}, \quad (9)$$

where $q(y_2)$ is a nonlinear function, $\delta(\cdot)$ is the delta function, τ_j ($j = \overline{1, p_1}$), and T_v ($v = \overline{1, p_2}$) are values of t found from the nonlinear equations

$$y_2(\tau_j) = -1, \quad y_2(T_v) = -1 - 2\lambda. \quad (10)$$

The solutions of Eqs. (8) are represented as

$$\begin{aligned} \frac{\partial^2 y_i}{\partial y_{0\mu} \partial y_{0v}} &= u_{\mu v}^{(i)} + v_{\mu v}^{(i)}, \quad \frac{\partial^2 y_i}{\partial f \partial y_{0v}} = w_v^{(i)} + p_v^{(i)}, \quad \frac{\partial^2 y_i}{\partial \omega \partial y_{0v}} = r_v^{(i)} + s_v^{(i)}, \\ i &= 1, 2, \quad \mu = 1, 2, \quad v = 1, 2. \end{aligned} \quad (11)$$

The right-hand sides of relations (1) satisfy the matrix equations

Fig. 1

Fig. 2

$$\frac{d}{dt} \begin{bmatrix} u_{\mu\nu}^{(1)} & w_{\nu}^{(1)} & r_{\nu}^{(1)} \\ u_{\mu\nu}^{(2)} & w_{\nu}^{(2)} & r_{\nu}^{(2)} \end{bmatrix} = [V(t)] \begin{bmatrix} u_{\mu\nu}^{(1)} & w_{\nu}^{(1)} & r_{\nu}^{(1)} \\ u_{\mu\nu}^{(2)} & w_{\nu}^{(2)} & r_{\nu}^{(2)} \end{bmatrix} - \frac{\sigma(t)}{\omega^2} \begin{bmatrix} \frac{\partial y_2}{\partial y_{0\nu}} & \frac{\partial y_2}{\partial y_{0\mu}} & \frac{\partial y_2}{\partial f} & \frac{\partial y_2}{\partial y_{0\nu}} & \frac{\partial y_2}{\partial \omega} & \frac{\partial y_2}{\partial y_{0\nu}} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad (12)$$

$$\begin{aligned} \frac{d}{dt} \begin{bmatrix} v_{\mu\nu}^{(1)} & p_{\nu}^{(1)} & s_{\nu}^{(1)} \\ v_{\mu\nu}^{(2)} & p_{\nu}^{(2)} & s_{\nu}^{(2)} \end{bmatrix} &= [V(t)] \begin{bmatrix} v_{\mu\nu}^{(1)} & p_{\nu}^{(1)} & s_{\nu}^{(1)} \\ v_{\mu\nu}^{(2)} & p_{\nu}^{(2)} & s_{\nu}^{(2)} \end{bmatrix} \\ &+ \left[\frac{q(y_2)}{\omega^2} \frac{\partial y_2}{\partial y_{0\nu}} \frac{\partial y_2}{\partial y_{0\mu}} - \frac{q(y_2)}{\omega^2} \frac{\partial y_2}{\partial f} \frac{\partial y_2}{\partial y_{0\nu}} \frac{2\alpha}{\omega^2} \frac{\partial y_1}{\partial y_{0\nu}} + \frac{2}{\omega^3} \frac{dH}{dy_2} \frac{\partial y_2}{\partial y_{0\nu}} - \frac{q(y_2)}{\omega^2} \frac{\partial y_2}{\partial \omega} \frac{\partial y_2}{\partial y_{0\nu}} \right]. \end{aligned} \quad (13)$$

The linear differential equations (12) have semianalytic solutions owing to the delta functions appearing in them. To write the solutions of Eqs. (12), we use the method of variations of constants and the Ostrogradskii–Liouville formula,

$$u_{\mu\nu}^i(2\pi N) = \sum_{k=1}^{p_2} \frac{\partial y_2}{\partial y_{0\nu}|_{t=T_k}} \frac{\partial y_2}{\partial y_{0\mu}|_{t=T_k}} \Delta_i(T_k) - \sum_{j=1}^{p_1} \frac{\partial y_2}{\partial y_{0\nu}|_{t=\tau_j}} \frac{\partial y_2}{\partial y_{0\mu}|_{t=\tau_j}} \Delta_i(\tau_j), \quad (14)$$

$$w_{\nu}^i(2\pi N) = \sum_{k=1}^{p_2} \frac{\partial y_2}{\partial f|_{t=T_k}} \frac{\partial y_2}{\partial y_{0\nu}|_{t=T_k}} \Delta_i(T_k) - \sum_{j=1}^{p_1} \frac{\partial y_2}{\partial f|_{t=\tau_j}} \frac{\partial y_2}{\partial y_{0\nu}|_{t=\tau_j}} \Delta_i(\tau_j), \quad (15)$$

$$r_{\nu}^i(2\pi N) = \sum_{k=1}^{p_2} \frac{\partial y_2}{\partial \omega|_{t=T_k}} \frac{\partial y_2}{\partial y_{0\nu}|_{t=T_k}} \Delta_i(T_k) - \sum_{j=1}^{p_1} \frac{\partial y_2}{\partial \omega|_{t=\tau_j}} \frac{\partial y_2}{\partial y_{0\nu}|_{t=\tau_j}} \Delta_i(\tau_j), \quad (16)$$

Fig. 3

where

$$\Delta_i(z) = \frac{\exp\left(\frac{2\alpha z}{\omega}\right)}{\omega^2 |y_1(z)|} \left(\frac{\partial y_i}{\partial y_{01}|_{t=2\pi N}} \frac{\partial y_2}{\partial y_{02}|_{t=z}} - \frac{\partial y_i}{\partial y_{02}|_{t=2\pi N}} \frac{\partial y_2}{\partial y_{01}|_{t=z}} \right). \quad (17)$$

Thus, formulas (4)–(17) allow determining numerically the Jacobian matrix. The solutions of system (2), (3) and the bifurcation line in the form of a mesh function are found by Newton's method. Let us further analyze the bifurcation behavior of a vibropercussion system with the following parameters: $c = 0$, $\alpha = 0.005$, and $\lambda = 5$. In this case, the system remains essentially nonlinear and approximately describes the oscillations of the power train in an internal-combustion engine [3]. The function $H(x)$ becomes simpler and formulas (5)–(17) remain the same. The software package used to analyze the vibropercussion system allows any value of c .

3. Codimension Two Bifurcation Points. Codimension two bifurcation points emerge on the parametric plane (ω, f) where the bifurcation lines join [8]. To help visualizing the situation, Fig. 1 and 2 show fragments of amplitude surfaces. The bifurcation points shown have been obtained in a numerical analysis of limit cycles of the principal resonance and subharmonic second-order oscillations. These oscillations occur because of period-doubling bifurcations of limit cycles of the first order. As is seen from Figs. 1 and 2, the amplitude surface includes joined sheets. A sheet with directly unstable cycles is denoted by D and a sheet with inversely unstable cycles by I .

The points L_1 , L_2 , and L_3 were earlier discovered in mappings [7, 10] and are called codimension two tangent bifurcation points. The tangent-bifurcation lines at these points are denoted by $tg1$ – $tg6$ and the period-doubling lines by $pd1$ – $pd6$. Note that the points L_1 , L_2 , and L_3 differ by the arrangement of the period-doubling lines. According to [7], the region on the amplitude surface near L_1 is called a spring area. This area contains a sheet of directly unstable cycles bounded by tangent-bifurcation lines $tg1$ and $tg2$. According to [7], the regions near L_2 and L_3 are called crossroad and saddle areas, respectively. The amplitude surface near L_4 was earlier observed in a one-dimensional polynomial mapping [10]. The point L_5 was earlier discovered in a mapping of dimension two. This point is where the two tangent-bifurcation lines $tg9$ and $tg10$ join. And the point L_6 is where the four tangent-bifurcation lines E_1 , E_2 , E_3 , and E_4 meet. These lines and the corresponding sheets satisfy the following relations: $E_1 = P_1 \cap P_4$, $E_2 = P_2 \cap P_3$, $E_3 = P_1 \cap P_2$, and $E_4 = P_3 \cap P_4$.

4. Principal Resonance. Let us consider bifurcations of limit cycles with the period of a perturbation action. A fragment of the amplitude surface representing this bifurcation behavior is shown in Fig. 3. Let us select the parameters so that the cycle appears on the right of the period-doubling line D_6 . As ω decreases, the cycle intersects this line, and stable cycles become inversely unstable. The bifurcations D_5 and D_6 give rise to period-doubling cycles, which are not analyzed here. Because of the tangent bifurcation K_4 , the stable cycles near the period-doubling line D_5 turn into directly unstable cycles. The sheet of these cycles is bounded by the tangent-bifurcation lines K_3 and K_4 , which join at the codimension two bifurcation point P_1 . The stable cycles near the bifurcation line K_3 have the maximum amplitude of oscillations compared with the other cycles shown on the amplitude surface. Crossing the period-doubling line D_4 , these cycles go over into inversely unstable cycles. Analyzing Fig. 3, let us trace the bifurcation behavior of the system. Note that the tangent-bifurcation lines K_1 and K_2 meet at the codimension two bifurcation point P_3 and the lines K_5 and K_6 at the codimension two bifurcation point P_5 . The domain of the amplitude surface containing the bifurcation lines K_5 , K_6 , D_2 , and D_7 and the codimension two bifurcation point P_5 is called a crossroad area. The lines K_5 and K_7 join at the codimension two bifurcation point P_6 .

5. Conclusions. A new method has been proposed for the bifurcation analysis of mechanical systems. The method is based on amplitude surfaces, which allow visualizing the bifurcations of mechanical systems. The amplitude-surface algorithm has been detailed and used to analyze the bifurcation behavior of a vibropercussion system, as an example. Owing to the results presented here, various bifurcations of vibropercussion systems have been revealed. They are codimension two bifurcations formed by joined tangent-bifurcation lines on the amplitude surface and corresponding to the principal resonance and subharmonic oscillations of the second order.

REFERENCES

1. K. V. Avramov, A. S. Belomyttsev, and V. N. Karaban, "Regions of random oscillations of discrete mechanical systems with piecewise-linear elastic characteristics," *Prikl. Mekh.*, **30**, No. 5, 81–88 (1994).
2. K. V. Avramov and V. N. Karaban, "Resonance in random oscillations of discrete mechanical systems with piecewise-linear elastic characteristics," *Prikl. Mekh.*, **33**, No. 7, 84–88 (1997).
3. K. V. Avramov and V. N. Karaban, "Features of forced oscillations in nonlinear power trains," *Izv. Vuzov, Mashinost.*, No. 4–6, 3–8 (1994).
4. V. I. Babitskii, *The Theory of Vibropercussion Systems* [in Russian], Nauka, Moscow (1978).
5. A. S. Belomyttsev and V. N. Karaban, "An algorithm for solution of a nonlinear boundary-value problem for ordinary differential equations in the nonuniqueness region," *Zh. Vych. Mat. Mat. Fiz.*, **26**, No. 7, 1099–1102 (1986).
6. V. I. Gulyaev, V. A. Bazhenov et al., *Stability of Periodic Processes in Nonlinear Systems* [in Russian], Vyshcha Shkola, Lvov (1983).
7. R. Allam and C. Mira, "Crossroad area–dissymmetrical spring area–symmetrical spring area and double crossroad area–double spring area transitions," *Int. J. Bifurc. Chaos*, **3**, 429–435 (1993).
8. J. Guckenheimer and P. J. Holmes, *Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields*, Springer-Verlag, New York (1983).
9. H. Kawakami, "Bifurcation of periodic responses in forced dynamic nonlinear circuits: computation of bifurcation values of the system parameters," *IEEE Trans. Circuits and Systems*, **3**, 212–227 (1984).
10. C. Mira, H. Kawakami, and R. Allam, "The dovetail bifurcation structure and its qualitative changes," *Int. J. Bifurc. Chaos*, **3**, 903–919 (1993).
11. R. Seydel, Nonlinear Computation, *Int. J. Bifurc. Chaos*, **7**, 2105–2126 (1997).
12. S. W. Shaw and P. J. Holmes, "A periodically forced piecewise linear oscillator," *J. Sound Vibr.*, **1**, 129–155 (1983).