Finding good 2-partitions of digraphs II. Enumerable properties

Jørgen Bang-Jensen, Nathann Cohen, Frédéric Havet

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J. Bang-Jensen*   Nathann Cohen†   Frédéric Havet‡

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Abstract

We continue the study, initiated in [3], of the complexity of deciding whether a given digraph D has a vertex-partition into two disjoint subdigraphs with given structural properties and given minimum cardinality. Let $\mathcal{E}$ be the following set of properties of digraphs: $\mathcal{E}$ ={strongly connected, connected, minimum out-degree at least 1, minimum in-degree at least 1, minimum semi-degree at least 1, minimum degree at least 1, having an out-branching, having an in-branching}. In this paper we determine, for all choices of $\mathbb{P}_1, \mathbb{P}_2$ from $\mathcal{E}$ and all pairs of fixed possitive integers k_1, k_2, the complexity of deciding whether a digraph has a vertex partition into two digraphs D_1, D_2 such that D_i has property $\mathbb{P}_i$ and $|V(D_i)| \geq k_i$, $i = 1, 2$. We also classify the complexity of the same problems when restricted to strongly connected digraphs. The complexity of the analogous problems when $\mathbb{P}_1 \in \mathcal{H}$ and $\mathbb{P}_2 \in \mathcal{H} \cup \mathcal{E}$, where $\mathcal{H}$ ={acyclic, complete, arcless, oriented (no 2-cycle), semicomplete, symmetric, tournament} were completely characterized in [3]

Keywords: oriented, NP-complete, polynomial, partition , splitting digraphs, acyclic, semicomplete digraph, tournament, out-branching, feedback vertex set, 2-partition, minimum degree.

1 Introduction

A k**-partition** of a (di)graph D is a partition of $V(D)$ into k disjoint sets. Let $\mathbb{P}_1, \mathbb{P}_2$ be two (di)graph properties, then a $(\mathbb{P}_1, \mathbb{P}_2)$**-partition** of a (di)graph D is a 2-partition (V_1, V_2) where V_1 induces a (di)graph with property $\mathbb{P}_1$ and V_2 a (di)graph with property $\mathbb{P}_2$. For example a $(\delta^+ \geq 1, \delta^+ \geq 1)$-partition is a 2-partition of a digraph where each partition induces a subdigraph with minimum out-degree at least 1.

There are many papers dealing with vertex-partition problems on (di)graphs. Examples (from a long list) are [1, 3, 8, 9, 12, 13, 14, 15, 17, 18, 19, 20, 21, 23, 24, 26, 27, 28, 30]. In [3] a systematic study of the complexity 2-partition problems for digraphs was initiated and a full characterization was given for the case where one part in the 2-partition has a property from the set $\mathcal{H}$ defined in the abstract and the other from $\mathcal{H} \cup \mathcal{E}$ (also defined in the abstract). See Tables 1 and 2. In this paper we provide the last entries in those tables by determining the complexity of those partition problems where both parts are required to have a given property from $\mathcal{E}$. Each of these properties $\mathbb{P}$ are **enumerable** : any given digraph D has only a polynomial number of inclusionwise maximal induced subdigraphs with property $\mathbb{P}$ and all of those can be found in polynomial time (see [3, Lemma 2.2]).

For each of the 36 distinct 2-partition problems that we study, it can be checked in linear time whether the given digraph has this property. Hence all of them are in NP. Several of these 36 $(\mathbb{P}_1, \mathbb{P}_2)$-partition problems are NP-complete and some results are somewhat surprising. For example, we show

*Department of Mathematics and Computer Science, University of Southern Denmark, Odense DK-5230, Denmark (email:jbj@imada.sdu.dk). This work was done while the first author was visiting INRIA, Sophia Antipolis, France, project COATI. Hospitality and financial support from Labex UCN@Sophia, Sophia Antipolis is gratefully acknowledged. The research of Bang-Jensen was also supported by the Danish research council under grant number 1323-00178B

†CNRS – Université Paris-Sud (email: nathann.cohen@gmail.com)

‡Project Coati, I3S (CNRS, UNSA) and INRIA, Sophia Antipolis, France (email: frederic.havet@inria.fr). Partially supported by ANR under contract STINT ANR-13-BS02-0007.

that the $(\delta^+ \geq 1, \delta \geq 1)$-partition problem is NP-complete. Some other problems are polynomial-time solvable because (under certain conditions) there are trivial $(\mathbb{P}_1, \mathbb{P}_2)$-partitions (V_1, V_2) with $|V_1| = 1$ (or $|V_2| = 1$). Therefore, in order to avoid such trivial partitions we consider $[k_1, k_2]$-**partitions**, that is, partitions (V_1, V_2) of V such that $|V_1| \geq k_1$ and $|V_2| \geq k_2$. Consequently, for each pair of above-mentioned properties and all pairs (k_1, k_2) of positive integers, we consider the $(\mathbb{P}_1, \mathbb{P}_2)$-$[k_1, k_2]$-partition problem, which consists in deciding whether a given digraph D has a $(\mathbb{P}_1, \mathbb{P}_2)$-$[k_1, k_2]$-partition. The results are summarized in the upper-left 8×8 subtable of Table 1 (all other results, in grey, are proved in [3]).

$\mathbb{P}_1 \setminus \mathbb{P}_2$	strong	conn.	$\mathbb{B}^+$	$\mathbb{B}^-$	$\delta \geq 1$	$\delta^+ \geq 1$	$\delta^- \geq 1$	$\delta^0 \geq 1$	$\mathbb{A}$	$\mathbb{C}$	$\mathbb{X}$
strong	NPc	NPc^L	NPc^L	NPc^L	NPc^L	NPc^L	NPc^L	NPc	P	P	P
conn.	NPc^R	P	P	P	P	NPc	NPc	NPc	P	P	P
$\mathbb{B}^+$	NPc^R	P	P	NPc	P	NPc	P	NPc	P	P	P
$\mathbb{B}^-$	NPc^R	P	NPc	P	P	P	NPc	NPc	P	P	P
$\delta \geq 1$	NPc^R	P	P	P	P	NPc	NPc	NPc	P	P	P
$\delta^+ \geq 1$	NPc^R	NPc	NPc	P	NPc	P	NPc	NPc	P	P	P
$\delta^- \geq 1$	NPc^R	NPc	P	NPc	NPc	NPc	P	NPc	P	P	P
$\delta^0 \geq 1$	NPc	NPc	NPc	NPc	NPc	NPc	NPc	NPc	P	P	P
$\mathbb{A}$	P	P	P	P	P	P	P	P	NPc	P	NPc
$\mathbb{C}$	P	P	P	P	P	P	P	P	P	P	P
$\mathbb{X}$	P	P	P	P	P	P	P	P	NPc	P	P

Properties: conn. : connected; $\mathbb{B}^+$: out-branchable; $\mathbb{B}^-$: in-branchable; $\mathbb{A}$: acyclic; $\mathbb{C}$: complete; $\mathbb{X}$: any property in 'being independent', 'being oriented', 'being semi-complete', 'being a tournament' and 'being symmetric'.
Complexities: P: polynomial-time solvable; NPc : NP-complete for all values of k_1, k_2; NPc^L : NP-complete for $k_1 \geq 2$, and polynomial-time solvable for $k_1 = 1$. NPc^R : NP-complete for $k_2 \geq 2$, and polynomial-time solvable for $k_2 = 1$.

Table 1: Complexity of the $(\mathbb{P}_1, \mathbb{P}_2)$-$[k_1, k_2]$-partition problem for some properties $\mathbb{P}_1, \mathbb{P}_2$.

The paper is organized as follows. We first introduce the necessary terminology. In Section 3 we handle the polynomial-time solvable cases. Then in Section 4 we handle the NP-complete cases and in Section 5 we investigate the impact of strong connectivity on the complexity of the partition problems. We prove that several of the NP-complete problems become polynomial-time solvable when the input is a strong digraph. We also point out that others are still NP-complete on strong digraphs. Our results are summarized in the upper 8×8 submatrix of Table 2. Finally we discuss a few other natural 2-partition problems and pose some open problems.

2 Notation and definitions

Notation follows [2]. In this paper graphs and digraphs have no parallel edges/arcs and no loops. We use the shorthand notation $[k]$ for the set $\{1, 2, \ldots, k\}$. Let $D = (V, A)$ be a digraph with vertex set V and arc set A. We use $|D|$ to denote $|V(D)|$. Given an arc $uv \in A$ we say that u **dominates** v and v is **dominated** by u. If uv or vu (or both) are arcs of D, then u and v are **adjacent**. If none of the arcs exist in D, then u and v are **non-adjacent**. The **underlying graph** of a digraph D, denoted $UG(D)$, is obtained from D by suppressing the orientation of each arc and deleting multiple copies of the same edge (coming from directed 2-cycles). A digraph D is **connected** if $UG(D)$ is a connected graph, and the **connected components** of D are those of $UG(D)$.

A (u, v)-path is a directed path from u to v, and for two disjoint non-empty subsets X, Y of V an (X, Y)-path is a directed path which starts in a vertex $x \in X$ and ends in a vertex $y \in Y$ and whose internal vertices are not in $X \cup Y$. A digraph is **strongly connected** (or **strong**) if it contains a (u, v)-path for every ordered pair of distinct vertices u, v. A **strong component** of a digraph D is a maximal subdigraph of D which is strong. An **initial** (resp. **terminal**) strong component of D is a strong component X with no arcs entering (resp. leaving) X in D.

$\mathbb{P}_1 \setminus \mathbb{P}_2$	strong	conn.	$\mathbb{B}^+$	$\mathbb{B}^-$	$\delta \geq 1$	$\delta^+ \geq 1$	$\delta^- \geq 1$	$\delta^0 \geq 1$	$\mathbb{A}$	$\mathbb{C}$	$\mathbb{X}$
strong	NPc	**P**	NPc*	NPc*	**P**	NPc^L	NPc^L	NPc	P	P	P
conn.	**P**	P	P	P	P	**P**	**P**	**P**	P	P	P
$\mathbb{B}^+$	NPc*	P	P	NPc*	P	NPc^L	P	NPc^L	P	P	P
$\mathbb{B}^-$	NPc*	P	NPc*	P	P	P	NPc^L	NPc^L	P	P	P
$\delta \geq 1$	**P**	P	P	P	P	**P**	**P**	**P**	P	P	P
$\delta^+ \geq 1$	NPc^R	**P**	NPc^R	P	**P**	P	NPc	NPc	P	P	P
$\delta^- \geq 1$	NPc^R	**P**	P	NPc^R	**P**	NPc	P	NPc	P	P	P
$\delta^0 \geq 1$	NPc	**P**	NPc^R	NPc^R	**P**	NPc	NPc	NPc	P	P	P
$\mathbb{A}$	P	P	P	P	P	P	P	P	NPc	P	NPc
$\mathbb{C}$	P	P	P	P	P	P	P	P	P	P	P
$\mathbb{X}$	P	P	P	P	P	P	P	P	NPc	P	P

The legend is the same as in Table 1, but we have one more complexity type: NPc* : NP-complete for $k_1, k_2 \geq 2$, and polynomial-time solvable for $k_1 = 1$ or $k_2 = 1$. We also emphasize with a bold **P**, the problems that are polynomial-time solvable on strong digraphs and NP-complete in the general case.

Table 2: Complexity of the $(\mathbb{P}_1, \mathbb{P}_2)$-$[k_1, k_2]$-partition problem on strong digraphs.

The **subdigraph induced** by a set of vertices X in a digraph D, denoted $D\langle X\rangle$, is the digraph with vertex set X and which contains those arcs from D that have both end-vertices in X. When X is a subset of the vertices of D, we denote by $D - X$ the subdigraph $D\langle V - X\rangle$. If D' is a subdigraph of D, for convenience we abbreviate $D - V(D')$ to $D - D'$.

A digraph is **acyclic** if it does not contain any directed cycles. An **oriented** graph is a digraph without directed 2-cycles. A **semicomplete** digraph is a digraph with no non-adjacent vertices and a **tournament** is a semicomplete digraph which is also an oriented graph. Finally, a **complete** digraph is a digraph in which every pair of distinct vertices induce a directed 2-cycle.

The **in-degree** (resp. **out-degree**) of v, denoted by $d^-_D(v)$ ($d^+_D(v)$), is the number of arcs from $V \setminus \{v\}$ to v (v to $V \setminus \{v\}$). The **degree** of v, denoted by $d_D(v)$ is given by $d_D(v) = d^+_D(v) + d^-_D(v)$. Finally the **minimum out-degree**, respectively **minimum in-degree**, **minimum degree** is denoted by $\delta^+(D)$, respectively $\delta^-(D)$, $\delta(D)$ and the **minimum semi-degree** of D, denoted by $\delta^0(D)$, is defined as $\delta^0(D) = \min\{\delta^+(D), \delta^-(D)\}$. A vertex is **isolated** if it has degree 0.

An **out-tree** rooted at the vertex s, also called an **s-out-tree**, is a connected digraph T such that $d^-_T(s) = 0$ and $d^-_T(v) = 1$ for every vertex v different from s. Equivalently, for every $v \in V(T) \setminus \{s\}$ there is a unique (s, v)-path in T. The directional dual notion is the one of **in-tree**. An **in-tree** rooted at the vertex s, or **s-in-tree**, is a digraph T such that $d^+_T(s) = 0$ and $d^+_T(v) = 1$ for every vertex v different from s.

An **s-out-branching** (resp. **s-in-branching**) is a spanning s-out-tree (resp. s-in-tree). We say that a subset $X \subseteq V(D)$ is **out-branchable** (resp. **in-branchable**) if $D\langle X\rangle$ has an s-out-branching (resp. s-in-branching) for some $s \in X$.

Let D be a digraph. For a set S of vertices of D, we denote by $\mathrm{Reach}^+_D(S)$, or simply $\mathrm{Reach}^+(S)$ if D is clear form the context, the set of vertices that can be reached from S in D, that is, the set of vertices v for which there exists an (S, v)-path in D. Similarly, we denote by $\mathrm{Reach}^-_D(S)$, or simply $\mathrm{Reach}^-(S)$, the set of vertices that can reach S in D, that is, the set of vertices v for which there exists a (v, S)-path in D. For sake of clarity, we write $\mathrm{Reach}^+_D(x)$ (resp. $\mathrm{Reach}^-_D(x)$ in place of $\mathrm{Reach}^+_D(\{x\})$ (resp. $\mathrm{Reach}^-_D(\{x\})$. The following lemma is well-known and easy to prove.

Lemma 2.1 *Let D be a digraph. If S is a set of vertices such that $D\langle S\rangle$ is out-branchable and $\mathrm{Reach}^+_D(S) = V(D)$, then D has an out-branching with root in S.*

3 Polynomial cases

In all of this subsection (k_1, k_2) denotes a fixed (not part of the input) pair of positive integers.

Several results are stated for two pairs of properties and in that case the corresponding 2-partition problems transform into each other by a reversal of all arcs in the digraph. Hence it suffices to show that the problem corresponding to the first pair is polynomial-time solvable. Also, in order to save some space, we make no effort to obtain the best possible complexity but establish only that the problem at hand is polynomially solvable.

3.1 Polynomial-time solvable $[k_1, k_2]$-partition problems

We first deal with the cases where the partition problem in question is polynomial for all fixed lower bounds k_1, k_2 on the sizes of the two sets in the partition.

We first prove that the (connected, connected)-$[k_1, k_2]$-partition problem is polynomial-time solvable. The algorithm relies on the following lemma.

Lemma 3.1 *Let k_1, k_2 be fixed positive integers and let D be a connected digraph. D admits a (connected, connected)-$[k_1, k_2]$-partition if and only if it contains two disjoint connected subdigraphs D_1 and D_2 of order k_1 and k_2 respectively.*

Proof: Clearly, every connected digraph or order at least k contains a connected subdigraph of order k. Therefore, if D admits a (connected, connected)-$[k_1, k_2]$-partition, then D contains two disjoint connected subdigraphs D_1 and D_2 or order k_1 and k_2 respectively.

Assume now that D contains two disjoint connected subdigraphs D_1 and D_2 or order k_1 and k_2 respectively.

Set $U_1 := V(D_1)$ and $U_2 := V(D_2)$. As long as there $U_1 \cup U_2 \neq V(D)$, there must be a vertex $x \in V(D) \setminus (U_1 \cup U_2)$ adjacent to a vertex y in $U_1 \cup U_2$. If x is adjacent to a vertex in U_1, then we set $U_1 := U_1 \cup \{x\}$. Otherwise, x is adjacent to a vertex in U_2, and we set $U_1 := U_1 \cup \{x\}$. Observe that doing so, at each step $D\langle U_1\rangle$ and $D\langle U_2\rangle$ are connected. Hence at the end of the procedure, we obtain a (connected, connected)-$[k_1, k_2]$-partition of D. $\diamond$

Theorem 3.2 *Deciding whether a digraph has a (connected, connected)-$[k_1, k_2]$-partition is polynomial-time solvable.*

Proof: The algorithm relies on Lemma 3.1. Given a digraph D, it proceeds as follows. We first compute the connected components of D.

- If there are more than two of them, then D is clearly a 'no'-instance, so we return 'No'.
- If there are two connected components C_1, C_2, then the only (connected, connected)-partitions of D are $(V(C_1), V(C_2))$ and $(V(C_2), V(C_1))$. Therefore we check whether one of this two is a $[k_1, k_2]$-partition and return 'Yes' in the affirmative and 'No' otherwise.
- If D is connected, then for all pair of disjoint subsets U_1, U_2 with $|U_1| = k_1$ and $|U_2| = k_2$, we check whether both $D\langle U_1\rangle$ and $D\langle U_2\rangle$ are connected. Then, by Lemma 3.1, we can return 'Yes' in the affirmative and 'No' otherwise. $\diamond$

We now prove that the $(\delta \geq 1,\ \delta \geq 1)$-$[k_1, k_2]$-partition problem is polynomial-time solvable. The algorithm relies on Corollary 3.5, which derives from the following two lemmas.

Lemma 3.3 *Let k be positive integer. Every digraph D with $\delta(D) \geq 1$ and $|D| \geq k$ has a subdigraph D' with $\delta(D') \geq 1$ and $|D'| \in \{k, k+1\}$.*

Proof: By induction on $|D|$, the result holding trivially when $|D| \in \{k, k+1\}$. Assume now that $|D| \geq k+2$. Let C be a connected component of D, let T be a spanning tree of C and let v be a leaf of T. If $|C| = 2$, then $\delta(D - C) \geq 1$ and $|D - C| \geq k$. Hence by the induction hypothesis, there is a subdigraph D' of $D - C$, and thus also of D, with $\delta(D') \geq 1$ and $|D'| \in \{k, k+1\}$.

If $|C| \geq 3$, then $\delta(C - v) \geq 1$, because every vertex in $C - v$ has a neighbour distinct from v in T. Thus $\delta(D - v) \geq 1$ and $|D - v| \geq k$. Hence by the induction hypothesis, there is a subdigraph D' of $D - v$, and thus also of D, with $\delta(D') \geq 1$ and $|D'| \in \{k, k+1\}$. $\diamond$

Lemma 3.4 *Let D be a digraph with with $\delta(D) \geq 1$. If there are two disjoint subsets (U_1, U_2) such that $\delta(D\langle U_1\rangle) \geq 1$ and $\delta(D\langle U_2\rangle) \geq 1$, then there exists a $(\delta \geq 1,\ \delta \geq 1)$-partition (V_1, V_2) of D such that $U_1 \subseteq V_1$ and $U_2 \subseteq V_2$.*

Proof: Assume that there are two disjoint subsets (U_1, U_2) such that $\delta(D\langle U_1\rangle) \geq 1$ and $\delta(D\langle U_2\rangle) \geq 1$. Initialize $V_1 := U_1$ and $V_2 := U_2$. As long as $V_1 \cup V_2 \neq V(D)$, pick a vertex $v \in V(D) \setminus (V_1 \cup V_2)$. Since $\delta(D) \geq 1$, v has a neighbour w in D. If $w \in V_1$, then set $V_1 := V_1 \cup \{v\}$. Otherwise set $V_2 := V_2 \cup \{v, w\}$. Observe that doing so at each step we have $\delta(D\langle V_1\rangle) \geq 1$ and $\delta(D\langle V_2\rangle) \geq 1$. Hence, at the end, (V_1, V_2) is a $(\delta \geq 1,\ \delta \geq 1)$-partition of D. $\diamond$

Lemmas 3.3 and 3.4 directly imply the following.

Corollary 3.5 *Let k_1, k_2 be fixed positive integers and let D be a digraph with $\delta(D) \geq 1$. D admits a $(\delta \geq 1,\ \delta \geq 1)$-$[k_1, k_2]$-partition if and only if it contains two disjoint subdigraphs D_1 and D_2 such that $|D_1| \in \{k_1, k_1 + 1\}$ and $|D_2| \in \{k_2, k_2 + 1\}$ respectively.*

Theorem 3.6 *Deciding whether a digraph has a $(\delta \geq 1,\ \delta \geq 1)$-$[k_1, k_2]$-partition is polynomial-time solvable.*

Proof: The algorithm on a given digraph D is the following. For every pair U_1, U_2 of disjoints subsets of $V(D)$ such that $|U_1| \in \{k_1, k_1 + 1\}$ and $|U_2| \in \{k_2, k_2 + 1\}$, we check whether $\delta(D\langle U_1\rangle) \geq 1$ and $\delta(D\langle U_2\rangle) \geq 1$. In the affirmative for one such pair we return 'Yes', otherwise we return 'No'.

As there are $O(n^{k_1+k_2+2})$ possible such pairs, the algorithm runs in polynomial time. Its validity follows from Corollary 3.5. $\diamond$

Theorem 3.7 *Deciding whether a digraph has an (out-branchable, $\delta \geq 1$)-$[k_1, k_2]$-partition (resp. (in-branchable, $\delta \geq 1$)-$[k_1, k_2]$-partition) is polynomial-time solvable.*

Proof: By directional duality, it suffices to prove that the (out-branchable, $\delta \geq 1$)-$[k_1, k_2]$-partition problem is polynomial-time solvable.

Let D be a digraph. We shall describe a polynomial-time procedure that, given a set U_1 of k_1 vertices such that $D\langle U_1\rangle$ is out-branchable, decides whether there is an (out-branchable, $\delta \geq 1$)-$[k_1, k_2]$-partition (V_1, V_2) with $U_1 \subseteq V_1 \subseteq \text{Reach}^+(U_1)$. Then taking the $O(n^{k_1})$ k_1-subsets of $V(D)$, one after another checking whether $D\langle U_1\rangle$ is out-branchable, and if yes running this procedure, we obtain the desired algorithm. Note that the condition $V_1 \subseteq \text{Reach}^+(U_1)$ and the fact that $D\langle U_1\rangle$ is out-branchable implies that $D\langle V_1\rangle$ has an out-branching with root in U_1.

The procedure proceeds as follows. Let U_1 be a set of k_1 vertices such that $D\langle U_1\rangle$ has an out-branching B_1^+. Let S be the set of isolated vertices in $D - U_1$. If each vertex of S has in-degree at least 1, and $|D - (U_1 \cup S)| \geq k_2$, we return 'Yes'. Otherwise we return 'No'.

The procedure clearly runs in polynomial time. Let us now prove that it is valid.

Observe first that in any (out-branchable, $\delta^- \geq 1$)-$[k_1, k_2]$-partition (V_1, V_2) with $U_1 \subseteq V_1 \subseteq \text{Reach}^+(U_1)$, the vertices of S cannot be in V_2 because their degree in $D - U_1$ is 0. Hence $V_2 \subseteq V(D) \setminus (U_1 \cup S)$, so $|D - (U_1 \cup S)| \geq k_2$. Moreover, all vertices of S must be in V_1. So they must have an in-neighbour (in the path from U_1 to them) and so they must have in-degree at least 1. Hence when the procedure returns 'No', there is no (out-branchable, $\delta^- \geq 1$)-$[k_1, k_2]$-partition (V_1, V_2) with $U_1 \subseteq V_1 \subseteq \text{Reach}^+(U_1)$.

Assume now that each vertex of S has in-degree at least 1, and $|D - (U_1 \cup S)| \geq k_2$. The procedure returns 'Yes'. Let us show that this is valid. Each vertex s of S has in-degree at least 1. Since it is isolated in $D - U_1$ its in-neighbours in D are in U_1. Hence one can extend B_1^+ into an out-branching of $W_1 = V_1 \cup S$ by adding S and one arc entering s for each vertex $s \in S$. Now all vertices of $W_2 = V(D) \setminus (U_1 \cup S)$ have degree at least 1 in $D - U_1$, and thus also in $D\langle W_2\rangle$, because the vertices of S are isolated. Hence $\delta(D\langle W_2\rangle) \geq 1$. Therefore, since $|W_2| \geq k_2$, (W_1, W_2) is an (out-branchable, $\delta^- \geq 1$)-$[k_1, k_2]$-partition with $U_1 \subseteq W_1 \subseteq \text{Reach}^+(U_1)$. $\diamond$

Theorem 3.8 *Deciding whether a digraph has an (out-branchable, connected)-$[k_1, k_2]$-partition (resp. (in-branchable, connected)-$[k_1, k_2]$-partition) is polynomial-time solvable.*

Proof: Again it suffices to describe a polynomial-time algorithm that given a digraph D decides whether it has an (out-branchable, connected)-$[k_1, k_2]$-partition is polynomial-time solvable.

Let D be a digraph. We shall describe a polynomial-time procedure that, given a set U_1 of k_1 vertices such that $D\langle U_1\rangle$ is out-branchable, decides whether there is an (out-branchable, connected)-$[k_1, k_2]$-partition (V_1, V_2) with $U_1 \subseteq V_1 \subseteq \text{Reach}^+(U_1)$. Then taking the $O(n^{k_1})$ k_1-subsets of $V(D)$, one after another checking if $D\langle U_1\rangle$ is out-branchable, and if yes running this procedure, we obtain the desired algorithm.

The procedure proceeds as follows. Let U_1 be a set of k_1 vertices such that $D\langle U_1\rangle$ is out-branchable. We compute the connected components of $D - U_1$. For each such component C we check whether $|C| \geq k_2$ and $\text{Reach}^+_{D-C}(U_1) = V(D - C)$. In the affirmative for one of the components, we return 'Yes', otherwise we return 'No'.

The procedure clearly runs in polynomial time. Let us now prove that it is valid. We need to prove that D has an (out-branchable, connected)-$[k_1, k_2]$-partition (V_1, V_2) with $U_1 \subseteq V_1 \subseteq \text{Reach}^+(U_1)$ if and only if there exists a connected component C of $D - U_1$ such that $|C| \geq k_2$ and $\text{Reach}^+_{D-C}(U_1) = V(D - C)$.

Assume first that there exists a connected component C of $D - U_1$ such that $|C| \geq k_2$ and $\text{Reach}^+_{D-C}(U_1) = V(D - C)$. Since $U_1 \subseteq V(D - C)$, we have $|V(D - C)| \geq k_1$. Moreover, as $\text{Reach}^+_{D-C}(U_1) = V(D - C)$ and U_1 is out-branchable, by Lemma 2.1, $D - C$ is out-branchable. Hence $(V(D - C), V(C))$ is an (out-branchable, connected)-$[k_1, k_2]$-partition with $U_1 \subseteq V(D - C) \subseteq \text{Reach}^+(U_1)$.

Assume now that D has an (out-branchable, connected)-$[k_1, k_2]$-partition (V_1, V_2) with $U_1 \subseteq V_1 \subseteq \text{Reach}^+(U_1)$. Since $D\langle V_2\rangle$ is a connected subgraph of $D - U_1$, it is contained in a connected component C of $D - U_1$. Thus $|C| \geq |V_2| \geq k_2$. Now $V(D - C) \subseteq V_1$. Let v be a vertex of $D - C$. As $V_1 \subseteq \text{Reach}^+(U_1)$, there is a (U_1, v)-path in D. Observe that since C is a connected component of $D - U_1$, a shortest (U_1, v)-path contains no arc in C. Therefore $v \in \text{Reach}^+_{D-C}(U_1)$. Hence $\text{Reach}^+_{D-C}(U_1) = V(D - C)$ and the algorithm above correctly answers 'Yes'. ◇

Theorem 3.9 *Deciding whether a digraph has an (out-branchable, $\delta^- \geq 1$)-$[k_1, k_2]$-partition ((in-branchable, $\delta^+ \geq 1$)-$[k_1, k_2]$-partition) is polynomial-time solvable.*

Proof: It suffices to describe a polynomial-time algorithm that given a digraph D decides whether it has an (out-branchable, $\delta^- \geq 1$)-$[k_1, k_2]$-partition is polynomial-time solvable.

Observe that if the digraph has such a partition, then every vertex except possibly one (the root of the out-branching) has in-degree at least 1. Hence if D has more than one vertex with in-degree 0, we return 'No'. Henceforth we may assume that all vertices except possibly one, denoted by s, have in-degree at least 1.

We shall describe a procedure that given a set U_1 of k_1 vertices, containing s if it exists, and such that $D\langle U_1\rangle$ is out-branchable, decides whether there is an (out-branchable, $\delta^- \geq 1$)-$[k_1, k_2]$-partition (V_1, V_2) with $U_1 \subseteq V_1 \subseteq \text{Reach}^+(U_1)$. Then taking the $O(n^{k_1})$ possible such U_1 one after another and running this procedure, we obtain the desired algorithm.

The procedure proceeds as follows. We initialize $W_1 := U_1$. As long as there is a vertex v with in-degree 0 in $D - U_1$, we add it to W_1. This is valid as such vertices v cannot be in V_2 in a desired partition (V_1, V_2). Observe moreover that each time we add such a v, it has in-degree at least 1 in D, because it is not s. Hence, there is an arc from W_1 to v. So by Lemma 2.1, at each step $D\langle W_1\rangle$ is out-branchable.

This process will terminate with either $\delta^-(D - W_1) \geq 1$ or $D - W_1$ is empty. If $|D - W_1| \geq k_2$, then we return 'Yes', because $(W_1, V(D)\setminus W_1)$ is an (out-branchable, $\delta^- \geq 1$)-$[k_1, k_2]$-partition (V_1, V_2) with $U_1 \subseteq V_1 \subseteq \text{Reach}^+(U_1)$. If $|D - W_1| < k_2$, then we return 'No'. Indeed for every (out-branchable, $\delta^- \geq 1$)-$[k_1, k_2]$-partition (V_1, V_2) with $U_1 \subseteq V_1 \subseteq \text{Reach}^+(U_1)$, the set V_1 must contain W_1, so $V_2 \subseteq V(D - W_1)$ so $k_2 \leq |V_2| \leq |D - W_1| < k_2$. Hence no such partition exists. ◇

Theorem 3.10 *Deciding whether a digraph has an (out-branchable,out-branchable)-$[k_1, k_2]$-partition ((in-branchable, in-branchable)-$[k_1, k_2]$-partition) is polynomial-time solvable.*

Proof: An s-out-tree with at least k vertices clearly contains as an induced subdigraph an s-out-tree with exactly k vertices. Hence, by Lemma 2.1, D has an (out-branchable, out-branchable)-partition (V_1, V_2) such that $|V_i| \geq k_i$ and $D\langle V_i\rangle$ is out-branchable for $i = 1, 2$ if and only if D contains two vertex-disjoint out-trees T_1, T_2 such that $|V(T_1)| = k_1$, $|V(T_2)| = k_2$ and $\text{Reach}^+_D(V(T_1) \cup V(T_2)) = V(D)$. Thus, by trying, for (at most) all possible choices of disjoint vertex sets X_1, X_2 with $|X_i| = k_i$ whether $D\langle X_1\rangle$ and $D\langle X_2\rangle$ are out-branchable and $\text{Reach}^+_D(X_1 \cup X_2) = V(D)$, we can decide in polynomial time (since k_1, k_2 are constants) whether D has an (out-branchable, out-branchable)-$[k_1, k_2]$-partition. $\diamond$

Theorem 3.11 *Deciding whether a digraph has a (connected, $\delta \geq 1$)-$[k_1, k_2]$-partition is polynomial-time solvable.*

Proof: Let us describe a polynomial-time algorithm that given a digraph D decides whether it has a (connected, $\delta \geq 1$)-$[k_1, k_2]$-partition.

Observe that if D has an isolated vertex v, then necessarily v must be in the connected part of a (connected, $\delta \geq 1$)-partition, and so $\{v\}$ must be the connected part. Henceforth, if $k_1 \geq 2$, we return 'No', and if $k_1 = 1$ we check whether $D - v$ is connected or not. In the affirmative, we return 'Yes', otherwise we return 'No'. Henceforth, we may assume that D has no isolated vertices.

We shall describe a procedure that given a set U_1 of k_1 vertices such that $D\langle U_1\rangle$ is connected, decides whether there is a (connected, $\delta \geq 1$)-$[k_1, k_2]$-partition (V_1, V_2) with $U_1 \subseteq V_1$. Since every connected graph of order at least k_1 contains a connected subdigraph of order exactly k_1, taking the $O(n^{k_1})$ U_1 sets of order k_1, checking if they are connected and in the affirmative running this procedure, we obtain the desired algorithm.

The procedure proceeds as follows. We initialize $W_1 := U_1$. As long as there is a vertex v with degree 0 in $D - W_1$, we add it to W_1. This is valid as such vertices v cannot be in V_1 in a desired partition (V_1, V_2). Observe moreover that each time we add such a v, it has a neighbour in W_1, because it is not isolated in D. So, at each step $D\langle W_1\rangle$ is connected.

Once at this stage, $\delta(D - W_1) \geq 1$ or $D - W_1$ is empty. If $|D - W_1| \geq k_2$, then we return 'Yes', because $(W_1, V(D) \setminus W_1)$ is a (connected, $\delta \geq 1$)-$[k_1, k_2]$-partition with $U_1 \subseteq W_1$. If $|D - W_1| < k_2$, then we return 'No'. Indeed for every (connected, $\delta \geq 1$)-$[k_1, k_2]$-partition (V_1, V_2) with $U_1 \subseteq V_1$, the set V_1 must contain W_1. Hence $V_2 \subseteq V(D - W_1)$ so $k_2 \leq |V_2| \leq |D - W_1| < k_2$. Hence no such partition exists. $\diamond$

Lemma 3.12 *Let D be a digraph with $\delta^+(D) \geq 1$ and let $k \leq |D|$ be an integer. One of the following holds :*

(i) D has a directed cycle of order at least k;

(ii) D has a subdigraph D' with $\delta^+(D') \geq 1$ and $k \leq |D'| \leq 2k - 2$.

Proof: We prove the result by induction on $|V(D)| + |A(D)|$, the result holding trivially when $|D| \leq 2k - 2$. Henceforth we assume that $|D| \geq 2k - 1$.

If there is a vertex v with out-degree at least 2, then applying applying the induction to $D \setminus a$ for any arc a leaving v, we obtain the result. So we may assume that $d^+(v) = 1$ for all $v \in V(D)$.

If there is a vertex v with in-degree 0, then $\delta^+(D - v) \geq 1$. By the induction hypothesis the result holds for $D - v$ and so for D. Henceforth, we may assume that $d^-(v) \geq 1$ for all $v \in V(D)$. Since $d^+(v) = 1$ for all $v \in V(D)$, and $\sum_{v \in V(D)} d^+(v) = \sum_{v \in V(D)} d^-(v)$, we have $d^+(v) = d^-(v) = 1$ for all $v \in V(D)$.

Consequently, D is the disjoint union of directed cycles. Let $C_1, \ldots, C_p$ be those cycles, with $|C_1| \geq \cdots \geq |C_p|$. Let q be the smallest integer such that $\sum_{i=1}^q |C_i| \geq k$. If $q = 1$, then C_1 is a cycle of length at least k and (i) holds. If $q \geq 2$, then $|C_q| \leq |C_1| \leq k - 1$ and $\sum_{i=1}^{q-1} |C_i| \leq k - 1$. Thus $\sum_{i=1}^q |C_i| \leq 2k - 2$. Then $D' = \bigcup_{i=1}^q C_i$ satisfies $\delta^+(D') \geq 1$ and $|D'| \leq 2k - 2$, so (ii) holds. $\diamond$

We shall now prove that the ($\delta^+ \geq 1$, $\delta^+ \geq 1$)-$[k_1, k_2]$-partition problem is polynomial-time solvable. By directional duality, this implies that the ($\delta^- \geq 1$, $\delta^- \geq 1$)-$[k_1, k_2]$-partition problem is polynomial-time solvable. In order to prove this result, we need some preliminary results.

Lemma 3.13 *Let k_1 and k_2 be two positive integers, and let D be a digraph with $\delta^+(D) \geq 1$. D has a $(\delta^+ \geq 1, \delta^+ \geq 1)$-$[k_1, k_2]$-partition if and only if one of its subdigraphs has a $(\delta^+ \geq 1, \delta^+ \geq 1)$-$[k_1, k_2]$-partition.*

Proof: The necessity is trivial since D is a subdigraph of itself. Let us now prove the sufficiency.

Assume that a subdigraph of D has a $(\delta^+ \geq 1, \delta^+ \geq 1)$-$[k_1, k_2]$-partition. Consider the largest subdigraph D' of D that has a $(\delta^+ \geq 1, \delta^+ \geq 1)$-$[k_1, k_2]$-partition (U_1, U_2).

Suppose for a contradiction that $D' \neq D$. Set $S = V(D) \setminus V(D')$. Then every vertex $s \in S$ has no out-neighbour in U_1 otherwise $D\langle V(D') \cup \{s\}\rangle$ would have the $(\delta^+ \geq 1, \delta^+ \geq 1)$-$[k_1, k_2]$-partition $(U_1 \cup \{s\}, U_2)$, a contradiction. Similarly, every $s \in S$ has no out-neighbour in U_2. It follows that $D\langle S\rangle$ has minimum out-degree 1. Hence $(U_1 \cup S, U_2)$ is a $(\delta^+ \geq 1, \delta^+ \geq 1)$-$[k_1, k_2]$-partition of D, a contradiction. $\diamond$

Lemmas 3.12 and 3.13 immediately yield the following.

Corollary 3.14 *A digraph D has a $(\delta^+ \geq 1, \delta^+ \geq 1)$-$[k_1, k_2]$-partition if and only if one of the following holds:*

(i) D has two disjoint directed cycles C_1 and C_2 of length at least k_1 and k_2 respectively;

(ii) D has a subdigraph D'_1 such that $\delta^+(D'_1) \geq 1$, $k_1 \leq |D'_1| \leq 2k_1 - 2$, and $D - D'_1$ has a directed cycle of length at least k_2;

(iii) D has a subdigraph D'_2 such that $\delta^+(D'_2) \geq 1$, $k_2 \leq |D'_2| \leq 2k_2 - 2$, and $D - D'_2$ has a directed cycle of length at least k_1;

(iv) D has two disjoint subdigraphs D'_1 and D'_2 such that $\delta^+(D'_i) \geq 1$ and $k_i \leq |D'_i| \leq 2k_i - 2$ for $i = 1, 2$.

A vertex v is said to be *big* if $d^+(v) \geq 3$, or $d^-(v) \geq 3$, or $d^+(v) = d^-(v) = 2$. The Directed Grid Theorem proved by Kawayarabashi and Kreutzer [16] implies the following (See [4].).

Theorem 3.15 *Let F be a planar digraph with no big vertices. Deciding whether a given digraph contains a subdivision of F is polynomial-time solvable.*

Theorem 3.16 *Let k_1 and k_2 be two positive integers. Deciding whether a digraph has a $(\delta^+ \geq 1, \delta^+ \geq 1)$-$[k_1, k_2]$-partition is polynomial-time solvable.*

Proof: We rely on Corollary 3.14. In fact, we show that each of the statement (i)–(iv) can be checked in polynomial time.

The fact that (i) can be checked in polynomial time follows from Theorem 3.15. It suffices to apply it to the disjoint union of a directed cycle of length k_1 and a directed cycle of length k_2, which is clearly a planar graph with no big vertices.

(ii) To check (ii), we proceed as follows. For every set S such that $k_1 \leq |S| \leq 2k_1 - 2$, we check whether $\delta^+(D\langle S\rangle) \geq 1$ and $D - S$ contains a directed cycle of length at least k_2, which can be done by Theorem 3.15. If the answer is 'Yes' for one such S, then we return 'Yes', otherwise we return 'No'. Checking whether $\delta^+(D\langle S\rangle) \geq 1$ can be one in constant time since S has bounded size, and as there are $O(n^{2k_1-2})$ possible sets S_1, the procedure runs in polynomial time.

(iii) can be checked in the same way as (ii)

(iv) can be checked in time $O(n^{2k_1+2k_2-4})$ by brute force. For each pair of disjoints sets S_1 and S_2 such that $k_1 \leq |S_1| \leq 2k_1 - 2$ and $k_2 \leq |S_2| \leq 2k_2 - 2$, we check whether $\delta^+(D\langle S_1\rangle) \geq 1$ and $\delta^+(D\langle S_2\rangle) \geq 1$. Each check can be done in constant time and there are $O(n^{2k_1+2k_2-4})$ possible pairs of sets (S_1, S_2). $\diamond$

3.2 Polynomial [1,k]-partition problems

We now deal with those cases where the one vertex graph satisfies one of the two properties and where this implies that the [1,k]-versions of the partition problem are polynomial-time solvable while the $[k_1, k_2]$-partition version are NP-complete for $k_1 \geq 2$. The NP-completeness proofs will be given in the next subsection.

Proposition 3.17 *Let k be a positive integer. Deciding whether a digraph has a (strong, out-branchable)-$[1, k]$-partition (resp. (strong, in-branchable)-$[1, k]$-partition) is polynomial-time solvable.*

Proof: By the remark above, it suffices to describe a polynomial-time algorithm that decides whether a digraph D has a (strong, out-branchable)-$[1,k]$-partition.

Let D be a digraph. If $|D| < k+1$, then the answer is clearly 'No'. Henceforth, we may assume that $|D| \geq k+1$.

Observe that the strong part of a (strong, out-branchable)-partition is contained in a strong component of D and that its out-branchable part contains vertices of at most one initial strong component. In particular, a digraph with a (strong, out-branchable)-partition has at most two initial strong components.

We compute the strong components of D.

- If D has more than two initial strong components, then we return 'No'.
- If D has two initial components C_1 and C_2, then by the above observation, the strong part of a (strong, out-branchable)-partition must be C_1 or C_2. So for $i = 1, 2$, we check whether $(V(C_i), V(D - C_i))$ is a (strong, out-branchable)-$[1,k]$-partition.
- If D has a unique initial strong component, then D has an out-branching. Take a leaf v of this out-branching. $(\{v\}, V(D-v))$ is is a (strong, out-branchable)-$[1,k]$-partition. Thus we return 'Yes'. $\diamond$

Proposition 3.18 *Let k be a positive integer. The (strong, $\delta^+ \geq 1$)-$[1,k]$-partition problem (resp. (strong, $\delta^- \geq 1$)-$[1,k]$-partition problem) is polynomial-time solvable.*

Proof: Let D be a digraph. If $|D| < k+1$, then the answer is clearly 'No'. Henceforth, we may assume that $|D| \geq k+1$.

If a digraph D has a (strong, $\delta^+ \geq 1$)-partition, then it has at most one vertex with out-degree 0. Moreover, if it has such a vertex x, then either $(\{x\}, V(D) \setminus \{x\})$ is a (strong, $\delta^+ \geq 1$)-$[1,k]$-partition or x is the unique out-neighbour of some vertex $y \neq x$ in which case D is a 'No'-instance.

So we may assume that $\delta^+(D) \geq 1$. For every vertex x, we check whether $\delta^+(D - x) \geq 1$. In the affirmative for some x, then we return 'Yes' because $(\{x\}, V(D) \setminus \{x\})$ is a (strong, $\delta^+ \geq 1$)-$[1,k]$-partition.

Otherwise, $\delta^+(D-x) = 0$ for every $x \in V(D)$. In that case, we claim that D is a disjoint union of directed cycles $Z_1, \dots, Z_p$, $p \geq 1$. Indeed, consider a spanning subdigraph D' such that $d^+_{D'}(v) = 1$ for every vertex v. D' contains no vertex x with in-degree 0 because $\delta^+(D-x) = 0$. Hence D' is the disjoint union of cycles $Z_1, \dots, Z_p$. Now there is no arc uv in $A(D) \setminus A(D')$ for otherwise $\delta^+(D-x) \neq 0$ for x the out-neighbour of u in D'. This proves our claim. Consequently D is a 'Yes'-instance if and only if one of the Z_j has length at most $|D| - k$, which can be checked in linear time. $\diamond$

Proposition 3.19 *Let k be a positive integer. The (strong, connected)-$[1,k]$-partition problem and the (strong, $\delta \geq 1$)-$[1,k]$-partition problems are polynomial-time solvable.*

Proof: We first consider the (strong, connected)-$[1,k]$-partition problem. Let D be given. We compute the connected components of D. If $|D| < k+1$, then the answer is clearly 'No'. Henceforth, we may assume that $|D| \geq k+1$.

- If D has more than two connected components, then it is clearly a 'No'-instance, so we return 'No'.
- If D has exactly two connected components C_1 and C_2, then the only possible (strong,connected)-partition of D are $(V(C_1), V(C_2))$ and $(V(C_2), V(C_1))$. Hence D is a 'Yes'-instance if and only if one of these two partition is a (strong,connected)-$[1,k]$-partition, which can be checked in linear time.
- If D is connected, then for any leaf s of a spanning tree of $V(D)$, we have $(\{x\}, V(D) \setminus \{x\})$ is a (strong,connected)-$[1,k]$-partition. So we return 'Yes'.

The algorithm is similar for the (strong, $\delta \geq 1$)-partition problem. If $|D| < k+1$, then the answer is clearly 'No'. Henceforth, we may assume that $|D| \geq k+1$. We determine the isolated vertices of D.

- If D has two or more isolated vertices, then D is clearly a 'No'-instance, so we return 'No'.
- If D has precisely one isolated vertex x, then $(\{x\}, V(D)\backslash\{x\})$ is a (strong, $\delta \geq 1$)-$[1,k]$-partition, so we return 'Yes'.
- Assume now D has no isolated vertex, that is $\delta(D) \geq 1$. If D has a connected component C with at least 3 vertices, then for any leaf x of a spanning tree of C, $(\{x\}, V(D) \setminus \{x\})$ is a (strong, $\delta \geq 1$)-$[1,k]$-partition, so we return 'Yes'. Otherwise, all connected components of D have order 2. In such a case D is a 'Yes'-instance if and only if $|D| \geq k+2$ and one its components is strong (i.e. a directed 2-cycle). This can be easily checked in linear time. $\diamond$

4 NP-complete cases

All the NP-completeness proofs below are variations of the same idea. They are based on ring digraphs which we define below.

4.1 Variants of 3-SAT used in the proofs

Let us recall the definition of the 3-SAT problem(s): An instance is a boolean formula $\mathcal{F} = C_1 \wedge C_2 \wedge \ldots \wedge C_m$ over the set of n boolean variables $x_1, \ldots, x_n$. Each clause C_i is of the form $C_i = (\ell_{i,1} \vee \ell_{i,2} \vee \ell_{i,3})$ where each $\ell_{i,j}$ belongs to $\{x_1, x_2, \ldots, x_n, \bar{x}_1, \bar{x}_2, \ldots \bar{x}_n\}$ and $\bar{x}_i$ is the negation of variable x_i. Our NP-completeness proofs will use reductions from the 3-SAT problem and the two following variants of the 3-SAT problem: a) 2-IN-3-SAT, where exactly two of the three literals in each clause should be safisfied; and b) NOT-ALL-EQUAL-3-SAT (NAE-3-SAT), where every clause must have at least one true and at least one false literal. These variants are both NP-complete [25].

In all of the NP-completeness proofs below we will use the following easy fact: for any pair of fixed integers k, k' and any given instance $\mathcal{F}$ of 3-SAT, 2-IN-3-SAT or NAE-3-SAT, we can always add new variables and clauses whose number only depends on k, k' such that the resulting formula $\mathcal{F}'$ has at least $\max\{k, k'\}$ clauses and at least $\max\{k, k'\}$ variables and $\mathcal{F}'$ is satisfiable if and only if $\mathcal{F}$ is satisfiable. In all the proofs below we may hence assume that the 3-SAT instances that we use in the reductions satisfy that $\min\{n, m\} \geq \max\{k_1, k_2\}$, where k_1, k_2 are the lower bounds on the two sides of the partition. It will be clear from the proofs that this ensures that the partitions (V_1, V_2) that we obtain from a satisfying truth assignment will always satisfy that $|V_i| \geq k_i$ for $i = 1, 2$.

4.2 Ring digraphs and 3-SAT

A **ring digraph** (see Fig. 1) is the digraph that one obtains by taking two or more copies of the complete bipartite digraph on four vertices $\{a_1, a_2, b_1, b_2\}$ and arcs $\{a_1b_1, a_1b_2, a_2b_1, a_2b_2\}$ and joining these in a circular manner by adding a path $P_{i,1}$ from the vertex $b_{i,1}$ to $a_{i+1,1}$ and a path $P_{i,2}$ from $b_{i,2}$ to $a_{i+1,2}$ where $b_{i,1}$ is the ith copy of b_1 etc and indices are 'modulo' n ($b_{n+1,j} = b_{1,j}$ for $j \in [2]$ etc). Ring digraphs form an important structure in many of our NP-completeness proofs below.

We start by showing how we can associate a ring digraph to a given 3-SAT formula. Let $\mathcal{F} = C_1 \wedge C_2 \wedge \ldots \wedge C_m$ be an instance of 3-SAT consisting of m clauses $C_1, \ldots, C_m$ over the same set of n boolean variables $x_1, \ldots, x_n$.

For each variable x_i the ordering of the clauses above induces an ordering of the occurrences of x_i, resp $\bar{x}_i$ in the clauses. Let q_i (resp. p_i), $i \in [n]$, denote the number of times x_i (resp. $\bar{x}_i$) occurs in the clauses Let $R(\mathcal{F}) = (V, A)$ be the ring digraph defined as follows. Its vertex set is

$$\begin{aligned} V = &\{a_{1,1}, \ldots a_{n,1}, a_{1,2} \ldots, a_{n,2}\} \cup \\ &\{b_{1,1}, \ldots, b_{n,1}, b_{1,2}, \ldots, b_{n,2}\} \cup \\ &\bigcup_{i=1}^{n} \{v_{i,0}, v_{i,1}, \ldots, v_{i,q_i+1}, v'_{i,0}, v'_{i,1}, \ldots, v'_{i,p_i+1}\} \end{aligned}$$

Its arc set A consists of the following arcs:

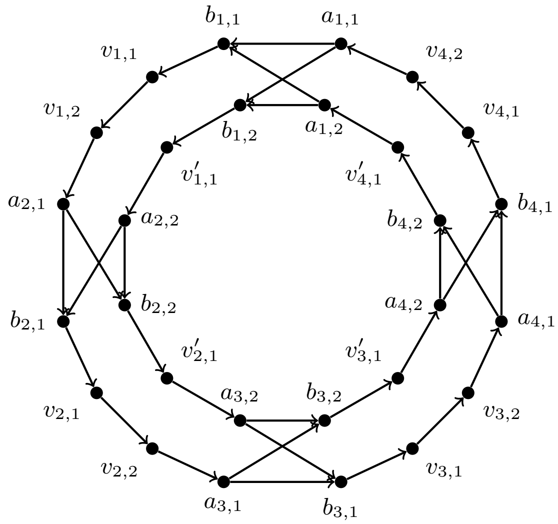

Figure 1: A ring digraph

- $\bigcup_{i=1}^{n}\{a_{i,1}b_{i,1}, a_{i,1}b_{i,2}, a_{i,2}b_{i,1}, a_{i,2}b_{i,2}\}$ (the directed complete bipartite graphs)
- the arcs of the paths $P_{i,1}, P_{i,2}$, $i \in [n]$ where $P_{i,1} = b_{i,1}v_{i,0}v_{i,1}\dots v_{i,q_i+1}a_{i+1,1}$ and $P_{i,2} = b_{i,2}v'_{i,0}v'_{i,1}\dots v'_{i,p_i+1}a_{i+1,2}$, where $a_{n+1,j} = a_{1,j}$ for $j = 1, 2$.

For $1 \leq j \leq m$, we associate to the jth clause $C_j = (\ell_{j,1} \vee \ell_{j,2} \vee \ell_{j,3})$ the set W_j consisting of three vertices of $R(\mathcal{F})$ representing the occurrences of the literals of C_j in $\mathcal{F}$: if $\ell_{j,1} = x_i$ for some $i \in [n]$ and this is the rth occurrence of x_i in the clauses, then W_j contains the vertex $v_{i,r}$. If $\ell_{j,1} = \bar{x}_i$ for some $i \in [n]$ and this is the rth occurrence of $\bar{x}_i$ in the clauses, then W_j contains the vertex $v'_{i,r}$. The other two vertices of W_j are defined similarly.

The following observation which forms the base of many of our proofs is easy to prove (for a proof of result a very similar to this see [7, Claim 1]).

Theorem 4.1 *Let $\mathcal{F}$ be a 3-SAT formula and let $R(\mathcal{F})$ be the corresponding ring digraph. Then the following holds:*

- *$R(\mathcal{F})$ contains a directed cycle which intersects all the sets $W_1, \dots, W_m$ if and only if $\mathcal{F}$ is a 'Yes'-instance of 3-SAT.*
- *$R(\mathcal{F})$ contains two disjoint directed cycles R_1, R_2, each of which intersects all the sets $W_1, \dots, W_m$ if and only if $\mathcal{F}$ is a 'Yes'-instance of NAE-3-SAT.* $\diamond$

4.3 The reductions

In our proofs we shall use different superdigraphs of the ring digraph $R(\mathcal{F})$ for a given 3-SAT formula. It will be clear from their definitions that each of these digraphs can be constructed in polynomial time in the size of the given 3-SAT instance $\mathcal{F}$. Thus in the proofs below we will not mention that the reductions are polynomial but only prove the correctness of the reductions.

We group together those problems for which the same polynomial reduction works. Note that each time we state two problems, they are equivalent in terms of complexity by reversing all arcs in the digraphs. Below, we only consider the first part of those claims.

Theorem 4.2 *The following 2-partition problems are NP-complete.*

(i) (strong, connected)-$[k_1, k_2]$-partition for $k_1 \geq 2$ and $k_2 \geq 1$,

(ii) (strong, $\delta \geq 1$)-$[k_1, k_2]$-partition for $k_1 \geq 2$ and $k_2 \geq 1$,

(iii) (strong, $\delta^- \geq 1$)-$[k_1, k_2]$-partition and (strong, $\delta^+ \geq 1$)-$[k_1, k_2]$-partition for $k_1 \geq 2$ and $k_2 \geq 1$,

(iv) ($\delta^0 \geq 1, \delta^- \geq 1$)-$[k_1, k_2]$-partition and ($\delta^0 \geq 1, \delta^+ \geq 1$)-partition for $k_1, k_2 \geq 1$,

(v) ($\delta^0 \geq 1, \delta \geq 1$)-$[k_1, k_2]$-partition for $k_1, k_2 \geq 1$,

(vi) ($\delta^+ \geq 1, \delta \geq 1$)-$[k_1, k_2]$-partition and ($\delta^- \geq 1, \delta \geq 1$)-partition for $k_1, k_2 \geq 1$,

(vii) ($\delta^+ \geq 1$, connected)-$[k_1, k_2]$-partition and ($\delta^- \geq 1$,connected)-partition for $k_1, k_2 \geq 1$, and

(viii) ($\delta^0 \geq 1$, connected)-$[k_1, k_2]$-partition for $k_1, k_2 \geq 1$.

Proof: We show how to reduce 3-SAT to each of these problems. Let $\mathcal{F} = C_1 \wedge C_2 \wedge \ldots \wedge C_m$ be an instance of 3-SAT consisting of clauses $C_1, \ldots, C_m$ over n boolean variables $x_1, \ldots, x_n$. By the remark in Section 2, we may assume that $n, m \geq \max\{k_1, k_2\}$.

The digraph $R_1(\mathcal{F})$ is obtained from $R(\mathcal{F})$ with the following modifications:

- For all $i \in [n]$, add a new vertex d_i and two arcs $a_{i,1}d_i, a_{i,2}d_i$.
- For all $j \in [m]$, add a new vertex c_j and all arcs from W_j to c_j.

We shall use the digraph $R_1(\mathcal{F})$ in all the proofs. We only give the full details in the proof of (i) as all the other proofs are very similar.

We show that $R_1(\mathcal{F})$ has a (strong,connected)-$[k_1, k_2]$-partition if and only if $\mathcal{F}$ is satisfiable. First observe that, for all $i \in [n]$, every strong subdigraph S of D of order at least 2 contains at least one vertex in $\{a_{i,1}, a_{i,2}\}$ and at least one in $\{b_{i,1}, b_{i,2}\}$ since each of these sets is a cycle transversal (kills all directed cycles) of D. Also no strong subdigraph can contain any of the vertices $d_1, \ldots, d_n, c_1, \ldots, c_m$ as these all have out-degree zero. Every strong subdigraph S contains either all or none of the internal vertices of $P_{i,1}$ and similarly for $P_{i,2}$ for $i \in [n]$. Furthermore, if some strong subdigraph S contained both $P_{i,1}$ and $P_{i,2}$ for some $i \in [n]$, then $D - S$ would not be connected since the vertex d_i would be isolated. So, if (V_1, V_2) is a (strong, connected)-$[k_1, k_2]$-partition of $R_1(\mathcal{F})$, then, for all $i \in [n]$, either $V(P_{i,1}) \subseteq V_1$ and $V(P_{i,2}) \subseteq V_2$, or $V(P_{i,1}) \subseteq V_2$ and $V(P_{i,2}) \subseteq V_1$. Suppose now that D has a (strong,connected)-$[k_1, k_2]$-partition (V_1, V_2). Let $\phi : \{x_1, \ldots, x_n\} \rightarrow \{true, false\}$ be the truth assignment defined by $\phi(x_i) = true$ whenever $D\langle V_1\rangle$ does not contain the path $P_{i,1}$ (in which case, it contains $P_{i,2}$). Since the vertex c_j, $j \in [m]$ belongs to the connected digraph $D\langle V_2\rangle$, at least one of the three paths that c_j is connected to (via the arcs from W_j) is not in $D\langle V_1\rangle$, implying that C_j is satisfied by ϕ and we have a satisfying truth assignment. For the other direction, starting from a satisfying truth assignment ϕ for $\mathcal{F}$, we simply form V_1 by putting $V(P_{i,2})$ in V_1 if $t(x_i) = false$ and otherwise we put $V(P_{i,1})$ in V_1. In both cases, exactly one of the four arcs from $\{a_{i,1}, a_{i,2}\}$ to $\{b_{i,1}, b_{i,2}\}$ belongs to $D\langle V_1\rangle$ and these link up the paths we put in $D\langle V_1\rangle$ so that $D\langle V_1\rangle$ is, in fact, an induced directed cycle. Set $V_2 = V(D) \setminus V_1$. It is easy to see that $D\langle V_2\rangle$ is connected, and that both V_1 and V_2 have size at least $n \geq \max\{k_1, k_2\}$. This completes the proof of (i).

The proofs of (ii) and (iii) are very similar to the proof above: in every desired 2-partition (V_1, V_2) all the vertices $d_1, \ldots, d_n, c_1, \ldots, c_m$ must belong to V_2. Now, if both of $a_{i,1}, a_{i,2}$ were in V_1, then d_i would be an isolated vertex in $D\langle V_2\rangle$ and if they were both in V_2, then $D\langle V_1\rangle$ would not be strong. So exactly one of $a_{i,1}, a_{i,2}$ is in V_1 for every $i \in [n]$. This again implies that precisely one of $b_{i,1}, b_{i,2}$ is in V_1 for each $i \in [n]$. Hence again, if we let vertices of the path $P_{i,2}$ be in V_1 if and only if x_i is true and conversely, we see that $\mathcal{F}$ is satisfiable if and only if we have the desired 2-partition.

The proofs of (iv)-(viii) are also almost identical to the proof above. It suffices to note that the part V_1 which should satisfy $\delta^0 \geq 1$, respectively $\delta^+ \geq 1$ plays the same role as the strong part V_1 did above, because if $D\langle V_1\rangle$ has minimum out-degree or minimum semi-degree at least 1, then $|V_1| \geq 2$. Note that once we have put the vertex $a_{i,g}$, $g \in [2]$ in V_2, all the vertices of $P_{i-1,g}$ must also be in V_2 since they have no out-neighbours left in $V(D) \setminus V_2$. ◇

Theorem 4.3 *The following 2-partition problems are NP-complete.*

(a) *(strong, out-branchable)-$[k_1, k_2]$-partition and (strong, in-branchable)-$[k_1, k_2]$-partition for $k_1 \geq 2$ and $k_2 \geq 1$,*

(b) *($\delta^0 \geq 1$, out-branchable)-$[k_1, k_2]$-partition and ($\delta^0 \geq 1$, in-branchable)-$[k_1, k_2]$-partition for $k_1, k_2 \geq 1$,*

Proof: Let $\mathcal{F} = C_1 \wedge C_2 \wedge \ldots \wedge C_m$ be an instance of 3-SAT consisting of clauses $C_1, \ldots, C_m$ over n boolean variables $x_1, \ldots, x_n$. Let $R_2(\mathcal{F})$ be the digraph obtained from $R(\mathcal{F})$ with the following modifications:

- Add four vertices s, t, x, y and arcs $\{st, sa_{1,1}, sa_{1,2}, b_{n,1}x, b_{n,1}y, b_{n.2}x, b_{n,2}y\}$.
- For all $j \in [m]$, add a new vertex c_j and all arcs from W_j to c_j.
- For all $j \in [m]$, add a vertex f_j and an arc c_jf_j.

We first prove (a) by showing that $\mathcal{F}$ is satisfiable if and only if $R_2(\mathcal{F})$ has a (strong, out-branchable)-$[k_1, k_2]$-partition for some fixed $k_1 \geq 2$ and $k_2 \geq 1$.

By construction of $R_2(\mathcal{F})$, for any (strong, out-branchable)-$[k_1, k_2]$-partition (V_1, V_2) we have the following properties: all vertices of $\{s, t, x, y\} \cup \{c_1, \ldots, c_m, f_1, \ldots, f_m\}$ must belong to V_2 and (as in the proof of Theorem 4.2) V_1 (being of size larger than 1) must contain a directed cycle which uses either $P_{i,1}$ or $P_{i,2}$ for $i \in [n]$. The vertex s must be the root of the out-branching in $D\langle V_2\rangle$ since it is the unique in-neighbour of t. As $a_{1,1}, a_{1,2}, t$ are the only out-neighbours of s, exactly one of the vertices $a_{1,1}, a_{1,2}$ belongs to V_2 and the other to V_1. As $x, y \in V_2$, there is a directed path from s to $\{x, y\}$ in V_2, implying that precisely one of $b_{n,1}, b_{n,2}$ belongs to V_2 and if $b_{n,j} \in V_2$ then all vertices of $P_{n,j}$ are also in V_2 for $j \in [2]$. Now the claim follows as in the proof of Theorem 4.2(i): we assign to the variable x_i the value *true* if and only if $P_{i,1}$ is in V_2.

Conversely, if there is a truth assignment satisfying $\mathcal{F}$, we obtain a (strong, out-branchable)-$[k_1, k_2]$-partition by the above correspondence. For each $j \in [m]$, at least one vertex of W_j is in V_2 and now it is easy to see that s can reach all vertices in $D\langle V_2\rangle$.

The proof of (b) is very similar as the same arguments as above imply that in any ($\delta^0 \geq 1$, out-branchable)-partition (V_1, V_2) the digraph $D\langle V_1\rangle$ is a directed cycle. ◇

Theorem 4.4 *The ($\delta^+ \geq 1$, out-branchable)-$[k_1, k_2]$-partition (($\delta^- \geq 1$, in-branchable)-$[k_1, k_2]$-partition) problem is NP-complete for any two positive integers k_1, k_2.*

Proof: Let k_1, k_2 be two positive integers. Let $\mathcal{F} = C_1 \wedge C_2 \wedge \ldots \wedge C_m$ be an instance of 3-SAT consisting of clauses $C_1, \ldots, C_m$ over n boolean variables $x_1, \ldots, x_n$. Let $R_3(\mathcal{F})$ be the digraph obtained from $R(\mathcal{F})$ with the following modifications:

- For all $i \in [n]$, add a new vertex d_i and two arcs $a_{i,1}d_i, a_{i,2}d_i$.
- For all $j \in [m]$, add a new vertex c_j and all arcs from c_j to W_j.
- Add two vertices s^*, t^* and the arcs $s^*t^*, s^*b_{1,1}, s^*b_{1,2}$.

We show that the digraph $R_3(\mathcal{F})$ has a ($\delta^+ \geq 1$, out-branchable)-$[k_1, k_2]$-partition if and only if $\mathcal{F}$ is satisfiable. By construction of $R_3(\mathcal{F})$, for any ($\delta^+ \geq 1$, out-branchable)-partition (V_1, V_2) of $R_3(\mathcal{F})$, the vertices s^*, t^* and all the vertices $d_1, \ldots, d_n$ belong to V_2 and for each $j \in [m]$ at least one vertex in W_j is in V_1 (because the c_j vertices all have in-degree 0 and they cannot be root of the out-branching as s^* has no in-neighbour). As in the proof above we now see that $D\langle V_1\rangle$ must contain precisely one directed cycle W and this must meet all of the sets $W_1, \ldots, W_m$ (in order to c_j have out-degree at least 1). The rest of the proof should be clear now (it follows from Theorem 4.1). ◇

Theorem 4.5 *The (out-branchable, in-branchable)-$[k_1, k_2]$-partition problem is NP-complete for any two positive integers k_1, k_2.*

Proof: Let k_1, k_2 be two positive integers. Let $\mathcal{F} = C_1 \wedge C_2 \wedge \ldots \wedge C_m$ be an instance of NAE-3-SAT consisting of clauses $C_1, \ldots, C_m$ over n boolean variables $x_1, \ldots, x_n$.

Let $R_4(\mathcal{F})$ be the digraph obtained from $R(\mathcal{F})$ as follows :

- delete the arcs $\{a_{1,1}b_{1,1}, a_{1,1}b_{1,2}, a_{1,2}b_{1,1}, a_{1,2}b_{1,2}\}$;

- add new vertices $\{s,s',t,t',x',y'\}\cup\{c_1,\dots,c_m,\bar{c}_1,\dots,\bar{c}_m\}$;
- add new arcs $\{ss',sb_{1,1},sb_{1,2},x'b_{1,1}x'b_{1,2},t't,a_{1,1}t,a_{1,2}t,a_{1,1}y',a_{1,2}y'\}\cup(\bigcup_{j\in[m]}\{c_jv,v\bar{c}_j \mid v\in W_j\})$.

We shall show that the digraph $R_4(\mathcal{F})$ has an (out-branchable,in-branchable)-$[k_1,k_2]$-partition if and only if $\mathcal{F}$ is a 'Yes'-instance of NAE-3-SAT.

By construction of $R_4(\mathcal{F})$, for any (out-branchable,in-branchable)-$[k_1,k_2]$-partition (V_1,V_2) of $R_4(\mathcal{F})$, we have that $t'\in V_2$ and $s'\in V_1$. Thus the vertex s (resp. t) must be the root of the out-branching (resp. in-branching) in $D\langle V_1\rangle$ (resp. $D\langle V_2\rangle$). Furthermore, we must have $x'\in V_2, y'\in V_1$ and for eevry $j\in[m]$, $c_j\in V_2$ and $\bar{c}_j$ in V_1. Consequently, for each $i\in[n]$ exactly one of the vertices $a_{i,1},a_{i,2}$ and exactly one of the vertices $b_{i,1},b_{i,2}$ is in V_1 (resp. V_2) ($D\langle V_1\rangle$ contains a path from s to y' and $D\langle V_2\rangle$ contains a path from x' to t). Combined with the distribution of the clause vertices this implies that each of $D\langle V_1\rangle, D\langle V_2\rangle$ contains a directed path from $\{b_{1,1},b_{1,2}\}$ to $\{a_{1,1},a_{1,2}\}$ which intersect all the sets $W_1,\dots,W_m$. Now the result follows from Theorem 4.1. $\diamond$

Theorem 4.6 *Let k_1,k_2 be two positive integers. The following partition problems are all NP-complete:*

(A) $(\delta^0\geq 1,\delta^0\geq 1)$-$[k_1,k_2]$-partition,

(B) (strong, $\delta^0\geq 1$)-$[k_1,k_2]$-partition,

(C) (strong, strong)-$[k_1,k_2]$-partition.

Proof: Let $\mathcal{F}=C_1\wedge C_2\wedge\ldots\wedge C_m$ be an instance of 3-SAT consisting of clauses $C_1,\dots,C_m$ over n boolean variables $x_1,\dots,x_n$. We may assume that every variable x_i appears in some clause as the literal x_i and in some other clause as the literal $\bar{x}_i$ and no variable appears twice in the same clause (negated or not). Clauses of the last kind can be removed and if x_i does not appear, say, as literal x_i we can add two clauses $C(x_i)=(x_i\vee y_i\vee y'_i), C'(x_i)=(x_i\vee\bar{y}_i\vee\bar{y}'_i)$ to $\mathcal{F}$ where y_i,y'_i are new variables only used in these two clauses. Clearly the new formula is satisfiable if and only if the old one is.

Let $R_5(\mathcal{F})$ be the digraph obtained from $R(\mathcal{F})$ by

- adding new vertices $\{s,t,t',t'',\alpha,\alpha',\alpha'',\beta,\beta',\beta''\}\cup\{c_1,\dots,c_m\}\cup\{c'_1,\dots,c'_m\}\cup\{c''_1,\dots,c''_m\}$, and
- adding new arcs $\{sb_{1,1},sb_{1,2},tb_{1,1},tb_{1,2},a_{1,1}\alpha'',\alpha''\alpha',\alpha'\alpha,a_{1,2}\beta'',\beta''\beta',\beta'\beta,\alpha s,\alpha t',\beta s,\beta t',t't'',t''t\}\cup(\bigcup_{j\in[m]}\{sc''_j,c''_jc'_j,c'_jc_j\})\cup(\bigcup_{j\in[m]}\{c_jv\mid v\in W_j\})$.

We first show that the digraph $R_5(\mathcal{F})$ represents a reduction from 3-SAT to the $(\delta^0\geq 1,\delta^0\geq 1)$-$[k_1,k_2]$-partition problem. Note that for every $(\delta^0\geq 1,\delta^0\geq 1)$-$[k_1,k_2]$-partition (V_1,V_2) of $R_5(\mathcal{F})$ all the vertices $c_1,c'_1,c''_1\dots,c_m,c'_m,c''_m$ and s belong to the same set V_i because every path into a clause vertex c_j goes through the vertex s and uses the path $sc''_jc'_jc_j$. W.l.o.g. $i=1$. Starting from $b_{n,1},b_{n,2}$ we see that these vertices must belong to different sets of the partition: if they belong to the same set V_j, then first observe that we would have $V(P_{n,1}),V(P_{n,2})\subseteq V_2$ since $b_{n,1},b_{n,2}$ have out-degree at least 1 in $D\langle V_2\rangle$. Now, arguing backwards, we first see that all vertices of $P_{n-1,1},P_{n-1,2}$ must also be in V_j and then that all vertices of $P_{n-2,1},P_{n-2,2}$ must belong to V_j and so on, implying that all vertices of the ring subdigraph $R(\mathcal{F})$ are in V_j. But then, one after another in the following order, $\alpha'',\alpha',\alpha,\beta'',\beta',\beta,s,t',t'',t$ are also in V_j because all their in-neighbours are in V_j. Hence $j=1$ and $V_2=\emptyset$, a contradiction. Hence $b_{n,1},b_{n,2}$ are in different sets and together with our assumption that $s\in V_1$ this easily implies that $t,t',t''\in V_2$. Hence (V_1,V_2) is a $(\delta^0\geq 1,\delta^0\geq 1)$-$[k_1,k_2]$-partition if and only if V_1 contains precisely one of the paths $P_{i,1},P_{i,2}$ for each i and $V_1\cap W_j\neq\emptyset$ for each $j\in[m]$. By Theorem 4.1, this implies that $R_5(\mathcal{F})$ has a $(\delta^0\geq 1,\delta^0\geq 1)$-$[k_1,k_2]$-partition if and only if $\mathcal{F}$ is satisfiable.

Note that a strong digraph D has minimum semi-degree 1, unless it has order 1. Observe moreover that all $(\delta^0\geq 1,\delta^0\geq 1)$-partitions described above are also $(strong, strong)$-partitions. Hence to prove (B) and (C) we just need to prove that $R_5(\mathcal{F})$ has no $(strong,\delta^0\geq 1)$-partition in which the strong part has cardinality 1. In other words, we need to prove that $\delta^0(R_5(\mathcal{F}) - x) = 0$ for every vertex $x \in V(R_5(\mathcal{F}))$.

Suppose for a contradiction that there is a vertex x such that $\delta^0(R_5(\mathcal{F}) - x) \geq 1$. Then x must be a vertex of the ring subdigraph $R(\mathcal{F})$: indeed deleting s leaves each c''_j with in-degree 0, deleting t leaves t'' with out-degree 0, deleting α (resp. β, t'') leaves α' (resp. β', t') with out-degree 0, deleting t', (resp. α', β', α'', β'') leaves t'', (resp. α'', β'', α', β') with degree 1, and finally deleting one of the vertices c'_j, c''_j, c_j clearly leaves a vertex of degree 1. So x must be a vertex on one of the paths $P_{i,j}$, $i \in [n], j \in [2]$ of $R(\mathcal{F})$. But then either the predecessor of x on $P_{i,j}$ has out-degree 0 in $R_5(\mathcal{F}) - x$, or its successor on $P_{i,j}$ has in-degree 0 in $R_5(\mathcal{F}) - x$. Here we used that the vertices $v_{i,0}, v_{i,q_i+1}, v'_{i,0}, v'_{i,p_i+1}$ have degree 2 in $R(\mathcal{F})$, because $p_i, q_i \geq 1$ since both literals x_i and $\bar{x}_i$ appear at least once in the clauses. $\diamond$

Theorem 4.7 *The following 2-partition problems are NP-complete.*

(i) $(\delta^+ \geq 1, \delta^- \geq 1)$-$[k_1, k_2]$-partition for $k_1, k_2 \geq 1$;

(ii) (strong, $\delta^+ \geq 1$)-$[k_1, k_2]$-partition and (strong, $\delta^- \geq 1$)-$[k_1, k_2]$-partition for $k_1 \geq 2$ and $k_2 \geq 1$;

(iii) ($\delta^0 \geq 1$, $\delta^+ \geq 1$)-$[k_1, k_2]$-partition and ($\delta^0 \geq 1$, $\delta^- \geq 1$)-$[k_1, k_2]$-partition for $k_1, k_2 \geq 1$.

Proof: We show how to reduce 3-SAT to each of these problems. Let $\mathcal{F} = C_1 \wedge C_2 \wedge \ldots \wedge C_m$ be an instance of 3-SAT consisting of clauses $C_1, \ldots, C_m$ over n boolean variables $x_1, \ldots, x_n$.

The digraph $R_6(\mathcal{F})$ is obtained from $R(\mathcal{F})$ by adding new vertices $\{c^-_1, \ldots, c^-_m\} \cup \{c^+_1, \ldots, c^+_m\} \cup \{a^+, a^-, b_1, b_2\}$ and arcs $(\bigcup_{j\in[m]}\{c^+_j v \mid v \in W_j\}) \cup (\bigcup_{j\in[m]}\{v c^-_j \mid v \in W_j\}) \cup \{c^-_j a^- \mid j \in [m]\} \cup \{a^+ c^+_j \mid j \in [m]\} \cup \{a^+ b_1, b_1 a^-, a^- b_2, b_2 a^+\}$.

We shall use the digraph $R_6(\mathcal{F})$ in all the proofs. We only give the full details in the proof of (i) as the two other proofs are very similar.

Let us prove that $R_6(\mathcal{F})$ has a $(\delta^+ \geq 1, \delta^- \geq 1)$-$[k_1, k_2]$-partition if and only if $\mathcal{F}$ is satisfiable. By Theorem 4.1, it suffices to prove that $R_6(\mathcal{F})$ has a $(\delta^+ \geq 1, \delta^- \geq 1)$-$[k_1, k_2]$-partition if and only if $R(\mathcal{F})$ has a directed cycle intersecting all the sets $W_1, \ldots, W_m$.

Assume first that $R_6(\mathcal{F})$ has a $(\delta^+ \geq 1, \delta^- \geq 1)$-$[k_1, k_2]$-partition.
Observe that a^+ and a^- are either both in V_1 or both in V_2. Indeed if $a^+ \in V_1$ and $a^- \in V_2$ (resp. $a^+ \in V_2$ and $a^- \in V_1$) , then b_1 (resp. b_2) cannot be in any part. By reversing all arcs and interchanging k_1, k_2 if necessary, we may assume that a^- and a^+ are both in V_1. Then every $c^+_j \in V_1$ because its unique in-neighbour is a^+. Now $D\langle V_2\rangle$ contains a directed cycle C_2. But this cycle must be in the ring digraph $R(\mathcal{F})$ because all directed cycles in $R_6(\mathcal{F}) - \{a^+, a^-\}$ are in $R(\mathcal{F})$. Consider $C_1 = R(\mathcal{F}) - C_2$. It is a directed cycle and $V_1 \cap R(\mathcal{F}) \subseteq V(C_1)$. In particular, the out-neighbour of each c^+_j in $D\langle V_1\rangle$ is in $V(C_1)$. Hence C_1 intersects all the sets $W_1, \ldots, W_m$.

Reciprocally, assume that there exists a directed cycle C_1 which intersects all the sets $W_1, \ldots, W_m$. Set $V_2 = V(R(\mathcal{F})) \setminus V(C_1)$ and $V_1 = V(R_6(\mathcal{F})) \setminus V_2$. Clearly, $D\langle V_2\rangle$ is a directed cycle and $D\langle V_1\rangle$ is strong (and thus has minimum out-degree at least 1) because it is covered by the directed cycles C_1, $(a^+, b_1, a^-, b_2, a^+)$ and the directed paths $(a^+, c^+_j, v, c^-_j, a^-)$ for all $j \in [m]$ and $v \in W_j$.

This proves (i)

The proofs of (ii) and (iii) are very similar. We just need to observe that a strong digraph with order at least 2 and every digraph with minimum semi-degree at least 1 contains a directed cycle. $\diamond$

5 The impact of strong connectivity

In this section, we revisit some of the NP-complete problems from the previous sections to see what happens in the strong case. Several of our NP-completeness proofs in the previous sections involved digraphs (e.g. $R_1(\mathcal{F})$) which are very far from being strongly connected. For some of the proofs the digraphs in the reduction where in fact strong (e.g. $R_5(\mathcal{F})$ or $R_6(\mathcal{F})$), so it is natural to ask which of the problems remain NP-complete when we restrict to strongly connected digraphs.

There are many problems where demanding the input digraph to be strong changes the complexity from NP-complete to polynomial. One example is the problem of deciding whether the underlying graph of a digraph D contains two cycles C, C' so that C is a directed cycle in D whereas C' may not be a directed cycle. This problem was shown to be polynomial for strong digraphs in [5] and NP-complete for non-strong digraphs in [6]. Another example is the problem of deciding whether a digraph contains a spanning galaxy [10].

5.1 Handle decompositions

Let D be a digraph. A *handle* h of D is a directed path $(s, v_1, \dots, v_\ell, t)$ from s to t or a directed cycle when $s = t$, such that $d^-(v_i) = d^+(v_i) = 1$ for all $1 \le i \le \ell$.

The vertices s and t are the **end-vertices** of h while the vertices v_i are its **internal vertices**. The set of internal vertices of h is denoted $\mathrm{Int}(h)$. The vertex s is the **initial vertex** of h and t its **terminal vertex**. The **length** of a handle is the number of its arcs, here $\ell + 1$. A handle of length 1 is said to be **trivial**.

Given a strongly connected digraph D and a subdigraph D' of D, a **handle decomposition** of D starting at D' is a pair $((h_i)_{1 \le i \le p}, (D_i)_{0 \le i \le p})$, where $(D_i)_{0 \le i \le p}$ is a sequence of digraphs and $(h_i)_{1 \le i \le p}$ is a sequence of paths or cycles such that:

- $D_0 = D'$,
- for $1 \le i \le p$, h_i is a handle of D_i and D_i is the (arc-disjoint) union of D_{i-1} and h_i, and
- $D = D_p$.

A handle decomposition is uniquely determined by either D' and $(h_i)_{1 \le i \le p}$, or $(D_i)_{0 \le i \le p}$.

The number of handles p in any handle decomposition of D starting at D' is exactly $|A(D)| - |A(D')| - |V(D)| + |V(D')|$. Observe that $p = 0$ when $D = D'$ is a singleton and $p = 1$ when D is a directed cycle.

The following lemma is easy and well known (see e.g. [2, Section 5.3]).

Lemma 5.1 *Let D be a strong digraph. For any subdigraph D', there exists a handle decomposition of D starting at D', say $(h_i)_{1 \le i \le p}, (D_i)_{0 \le i \le p})$. Moreover, if D' is strong, then D_i is strong for all $0 \le i \le p$.*

5.2 Polynomial-time solvable problems for all k_1, k_2

Lemma 5.2 *Let D be a strong digraph, and let $\mathbb{P}_1 \in \{strong, \delta^+ \ge 1, \delta^- \ge 1, \delta^0 \ge 1\}$ and $\mathbb{P}_2 \in \{connected, \delta \ge 1\}$. If a subdigraph D' of D admits a $(\mathbb{P}_1, \mathbb{P}_2)$-partition (V'_1, V'_2), then D contains a $(\mathbb{P}_1, \mathbb{P}_2)$-partition (V_1, V_2) such that $V'_1 \subseteq V_1$ and $V'_2 \subseteq V_2$.*

Proof: Let (V'_1, V'_2) be a $(\mathbb{P}_1, \mathbb{P}_2)$–partition of a subdigraph D' of D. By Lemma 5.1, there is a handle decomposition $((h_i)_{1 \le i \le p}, (D_i)_{0 \le i \le p})$ of D starting at D'. Set $(X_0, Y_0) = (V'_1, V'_2)$. Now for every $i > 0$, we can extend the $(\mathbb{P}_1, \mathbb{P}_2)$-partition (X_{i-1}, Y_{i-1}) of D_{i-1} into a $(\mathbb{P}_1, \mathbb{P}_2)$-partition (X_i, Y_i) of D_i as follows: if an end-vertex of h_i is in Y_{i-1}, then set $(X_i, Y_i) = (X_{i-1}, Y_{i-1} \cup \mathrm{Int}(h_i))$, otherwise h_i avoids Y_{i-1} and we set $(X_i, Y_i) = (X_{i-1} \cup \mathrm{Int}(h_i), Y_{i-1})$. This is trivially correct. Moreover, for all $1 \le i \le p$, we have $X_{i-1} \subseteq X_i$ and $Y_{i-1} \subseteq Y_i$. So $V'_1 \subseteq X_p$ and $V'_2 \subseteq Y_p$. Hence (X_p, Y_p) is the desired partition (V_1, V_2) of D. ◇

Corollary 5.3 *Let k_1 and k_2 be two positive integers and let $\mathbb{P} \in \{strong, \delta^+ \ge 1, \delta^- \ge 1, \delta^0 \ge 1\}$. A strong digraph D admits a ($\mathbb{P}$, connected)-$[k_1, k_2]$-partition if and only if there is a set V'_2 with $|V'_2| = k_2$ such that $D\langle V'_2 \rangle$ is connected and $D - V'_2$ has a maximal subdigraph with property $\mathbb{P}$ of size at least k_1.*

Proof: Assume that (V_1, V_2) is a ($\mathbb{P}$, connected)-$[k_1, k_2]$-partition of D. Trivially (by trimming leaves from a spanning tree), V_2 contains a subset V'_2 of cardinality k_2 such that $D\langle V'_2 \rangle$ is connected. Moreover, $D - V'_2$ contains $D\langle V_1 \rangle$, which has property $\mathbb{P}$. Hence, in $D - V'_2$, the maximal subdigraph with property $\mathbb{P}$ containing all vertices of V_1 has size at least k_1.

Conversely, assume that there is a set V'_2 with $|V'_2| \ge k_2$ such that $D\langle V'_2 \rangle$ is connected and $D - V'_2$ has a maximal subdigraph D'_1 with property $\mathbb{P}$ of size at least k_1. Set $V'_1 = V(D'_1)$. Then (V'_1, V'_2) is a ($\mathbb{P}$, connected)-$[k_1, k_2]$-partition of $D\langle V'_1 \cup V'_2 \rangle$. Thus, by Lemma 5.2, (V'_1, V'_2) can be extended into a ($\mathbb{P}$, connected)-$[k_1,k_2]$-partition (V_1,V_2) of D with $V_1' \subseteq V_1$ and $V_2' \subseteq V_2$. In particular, $|V_1| \geq |V_1'| \geq k_1$ and $|V_2| \geq |V_2'| \geq k_2$.

$\diamond$

Corollary 5.4 *Let k_1 and k_2 be two positive integers and let $\mathbb{P} \in \{strong, \delta^+ \geq 1, \delta^- \geq 1, \delta^0 \geq 1\}$. There is a $O(n^{k_2+2})$)-time algorithm that solves the ($\mathbb{P}$, connected)-$[k_1,k_2]$-partition problem for strong digraphs.*

Proof: The algorithm proceeds as follows. For every set V_2' of size k_2, we first check whether $D\langle V_2'\rangle$ is connected (This can be done in constant time). If not, we proceed to the next k_2-subset. Assume now that $D\langle V_2'\rangle$ is connected. We use the $O(n^2)$-time algorithm of [3, Lemma 2.2] to find the maximal subdigraph D^* of $D - V_2'$ with property $\mathbb{P}$. If D^* has size at least k_1, then we return 'Yes'. If not, then we proceed to the next k_2-subset Y'.

If after examining all sets Y', no partition is returned, we return 'No'.

The validity of this algorithm follows Corollary 5.3. Its running time is clearly $O(n^{k_2+2})$. $\diamond$

Similarly to the proof of Corollary 5.3, Lemma 5.2 and Proposition 3.3 yield the following corollary.

Corollary 5.5 *Let k_1 and k_2 be two fixed positive integers, and let $\mathbb{P} \in \{strong, \delta^+ \geq 1, \delta^- \geq 1, \delta^0 \geq 1\}$. A strong digraph D admits a $(\mathbb{P}, \delta \geq 1)$-$[k_1,k_2]$-partition if and only if there is a set V_2' with $|V_2'| \in \{k_2, k_2+1\}$ such that $\delta(D\langle V_2'\rangle) \geq 1$ and $D - V_2'$ has a maximal subdigraph with property $\mathbb{P}$ of size at least k_1.*

In the very same way as Corollary 5.3 implied Corollary 5.4, Corollary 5.5 imply the following.

Corollary 5.6 *Let k_1 and k_2 be two fixed positive integers, and let $\mathbb{P} \in \{strong, \delta^+ \geq 1, \delta^- \geq 1, \delta^0 \geq 1\}$. There is a $O(n^{k_2+3})$-time algorithm that solves the $(\mathbb{P}, \delta \geq 1)$-$[k_1,k_2]$-partition problem for strong digraphs.*

5.3 Polynomial-time solvable $[1,k]$-partition problems

Lemma 5.7 *Let k be a positive integer and let D be a strong digraph. If D has a strong subdigraph of order at least k and less than $|D|$, then D admits an (out-branchable, strong)-$[1,k]$-partition of D.*

Proof: Let D' be a strong subdigraph of D of order at least k and less than $|D|$. Consider a handle decomposition $((h_i)_{1\leq i\leq p}, (D_i)_{0\leq i\leq p})$ of D starting at D'. Let q be the last index such that h_q has length at least 2. Observe that q exists since $V(D') \neq V(D)$. Then D_{q-1} is strong, and $|D_{q-1}| \geq k$ since $D' \subseteq D_{q-1}$. Moreover $V_1 = V(D) \setminus V(D_{q-1})$ is the set of internal vertices of h_q. Since h_q has length at least 2, it is not empty and it is clearly out-branchable. Therefore $(V_1, V(D_{q-1}))$ is an (out-branchable, strong)-$[1,k]$-partition of D. $\diamond$

If D is a directed cycle, and u,v two distinct vertices in C, we denote by $C[u,v]$, the directed (u,v)-path in C.

Lemma 5.8 *Let k be a positive integer, let $\mathbb{P}$ be a property in $\{strong, \delta^0 \geq 1, \delta^+ \geq 1, in\text{-}branchable\}$, and let D be a strong digraph having a Hamiltonian directed cycle C. There is an (out-branchable, $\mathbb{P}$)-$[1,k]$-partition of D if and only if there exist two vertices u and v such that u is not the out-neighbour of v in C, $D\langle V(C[u,v])\rangle$ has property $\mathbb{P}$ and has order at least k.*

Proof: By definition, if u is not the out-neighbour of v in C, then $P = C - C[u,v]$ is a non-empty directed path. If, in addition, $D\langle V(C[u,v])\rangle$ has property $\mathbb{P}$ and has order at least k, then $(V(P), V(C[u,v])$ is a ($\mathbb{P}$, out-branchable)-$[1,k]$-partition.

Reciprocally, assume that D has an (out-branchable, $\mathbb{P}$)-$[1,k]$-partition (V_1,V_2). Let vw be an arc of C such that $v \in V_2$ and $w \in V_1$. Let u be the first vertex along $C[w,v]$ that is in V_2. Clearly, $D\langle V(C[u,v])\rangle$ has property $\mathbb{P}$. Moreover, by definition of u, all vertices of V_2 are in $V(C[u,v])$. So $|C[u,v]| \geq |V_2| \geq k$. $\diamond$

Proposition 5.9 *Let k be a positive integer and $\mathbb{P}$ be a property in $\{strong, \delta^0 \geq 1, \delta^+ \geq 1, in\text{-}branchable\}$ (resp. $\{strong, \delta^0 \geq 1, \delta^- \geq 1, out\text{-}branchable\}$). Deciding whether a strong digraph has an (out-branchable, $\mathbb{P}$)-$[1,k]$-partition (resp. (in-branchable,$\mathbb{P}$)-$[1,k]$-partition) is polynomial-time solvable.*

Proof: Let us describe a polynomial-time algorithm that decides whether a digraph D has an (out-branchable, $\mathbb{P}$)-$[1,k]$-partition for $\mathbb{P} \in \{\text{strong}, \delta^0 \geq 1, \delta^+ \geq 1, \text{in-branchable}\}$.

If $|D| \leq 2k-2$, we use brute force and try all possible partitions and check whether one of them is an (out-branchable, $\mathbb{P}$)-$[1,k]$-partition. Henceforth we assume that $|D| \geq 2k-1$.

We compute a handle decomposition $((h_i)_{1\leq i\leq p}, (D_i)_{0\leq i\leq p})$. Let q be the last index such that h_q has length at least 2. If $|D_{q-1}| \geq k$, then by Lemma 5.7, D has an (out-branchable, strong)-$[1,k]$-partition, which is also an (out-branchable, $\mathbb{P}$)-$[1,k]$-partition, so we return 'Yes'. Henceforth, we assume $|D_{q-1}| \leq k-1$, and so h_q has length at least k.

Let u and v be respectively the initial and terminal vertices of h_q. Since D_{q-1} is strong, there is a directed (v,u)-path in D_{q-1}, whose union with h_q is a cycle C. Now C has length at least k since it contains h_q. If C is not Hamiltonian in D, then by Lemma 5.7, D has an (out-branchable, strong)-$[1,k]$-partition, so we return 'Yes'. Henceforth, we assume that C is a Hamiltonian directed cycle of D. For each pair u,v of vertices such that $vu \notin A(C)$, we check whether $C[u,v]$ has property $\mathbb{P}$ and order at least k. If it is the case for at least one such pair, then we return 'Yes', otherwise we return 'No'. This is valid by Lemma 5.8. $\diamond$

Remark 5.10 For sake of simplicity, we made no attempt to optimize the complexity of the above algorithm. But faster algorithms may be designed. For example, in the above algorithm for (out-branchable, strong)-partition, we might have to check for $\Omega(|V(D)|^2)$ pairs whether $C[u,v]$ is strong, which can be done in linear time. We can design a faster algorithm: similarly to Lemma 5.8, one can prove the following: *There is an (out-branchable, strong)-$[1,k]$-partition of D if and only if there is an arc uv of C such that $|\mathrm{Reach}^+_{D-v}(u)| \geq k$.* Using this, we only need to check for the $|V(D)|$ arcs uv whether $|\mathrm{Reach}^+_{D-v}(u)| \geq k$, each such test being possible in linear time.

5.4 NP-completeness

The aim of this subsection is to prove for any $\mathbb{P} \in \{$ strong, $\delta^0 \geq 1, \delta^+ \geq 1$ in-branchable$\}$, and any $k_1, k_2 \geq 2$, the (out-branchable, $\mathbb{P}$)-$[k_1,k_2]$-partition problem is NP-complete. We prove it for $\mathbb{P} \in \{$ strong, $\delta^0 \geq 1\}$ in Theorem 5.11, for $\mathbb{P} = (\delta^+ \geq 1)$ in Theorem 5.12 and for $\mathbb{P}$ =in-branchable in Theorem 5.13.

Theorem 5.11 *(i) Deciding whether a strong digraph has an (out-branchable, strong)-$[k_1,k_2]$-partition (resp. (in-branchable, strong)-$[k_1,k_2]$-partition) is NP-complete for all $k_1, k_2 \geq 2$.*

(ii) Deciding whether a strong digraph has an (out-branchable, $\delta^0 \geq 1$)-$[k_1,k_2]$-partition (resp. (in-branchable, $\delta^0 \geq 1$)-$[k_1,k_2]$-partition) is NP-complete for all $k_1 \geq 1$ and $k_2 \geq 1$.

Proof: The two proofs are identical, so we only give the proof of (i). (Observe that a digraph with $\delta^0 \geq 1$ has necessarily at least two vertices so the condition $k_2 \geq 1$ is equivalent to $k_2 \geq 2$ in (ii).)

We shall show a reduction from 3-SAT using ring digraphs and the following concept. An (a,b)-*bond* in a digraph D is an induced subdigraph with vertex set $\{a, b, z_1, z_2\}$ and arc set $\{az_1, az_2, z_1b, z_2b\}$ such that $d^+_D(z_i) = d^-_D(z_i) = 1$ for $i = 1,2$. The vertices z_1 and z_2 are the *internal vertices* of the bond.

Let $\mathcal{F} = C_1 \wedge C_2 \wedge \ldots \wedge C_m$ be an instance of 3-SAT consisting of clauses $C_1, \ldots, C_m$ over n boolean variables $x_1, \ldots, x_n$. As mentioned in Section 2, we may assume that $m \geq \max\{k_1, k_2\}$.

Let $\mathcal{R}_7(\mathcal{F})$ be the digraph from $R(\mathcal{F})$ by adding a vertex x and the set of vertices $\bigcup_{j=1}^m \{c_j, c'_j\}$, the arc sets $\bigcup_{j=1}^m \{c_jx, c'_jx\}$ and $\bigcup_{j=1}^m \{vc_j, vc'_j \mid v \in W_j\}$, and finally adding an (x,y)-bond for every $y \in V(R(\mathcal{F}))$.

Let us prove that $R_7(\mathcal{F})$ has an (out-branchable, strong)-$[k_1,k_2]$-partition if and only if $\mathcal{F}$ is satisfiable. By Theorem 4.1, it suffices to prove that $R_7(\mathcal{F})$ has an (out-branchable, strong)-$[k_1,k_2]$-partition if and only if $R(\mathcal{F})$ has a directed cycle intersecting all the sets $W_1, \ldots, W_m$.

Assume first that $R_7(\mathcal{F})$ has an (out-branchable, strong)-$[k_1, k_2]$-partition (V_1, V_2). Let B_1 be an out-branching of $D\langle V_1\rangle$.
We claim that $x \in V_1$. Indeed, all arcs of $D - V(R(\mathcal{F}))$ are incident to x; hence if $V_1 \cap V(R(\mathcal{F})) = \emptyset$, then V_1 necessarily contains x because $|V_1| \geq 2$ and $D\langle V_1\rangle$ is connected. Now if there is a vertex y in $V_1 \cap V(R(\mathcal{F}))$, then consider the (x, y)-bond with vertex set $\{x, y, z_1, z_2\}$. The two vertices z_1 and z_2 must be in V_1 because their out-degree in $D - y$ is 0. One of the vertices z_1, z_2 is not the root of B_1 and thus has an in-neighbour in B_1. As it is the unique in-neighbour of z_1 and z_2, vertex x is in V_1. This proves our claim.
Now the vertices of $\bigcup_{j=1}^{m}\{c_j, c'_j\}$ are in V_1 because their out-degree in $D - x$ is 0, and for each $y \in V(R(\mathcal{F}))$ all internal vertices of the (x, y)-bond are also in V_1, because their in-degree in $D - x$ is 0. Consequently, $D\langle V_2\rangle$ is a subdigraph of $R(\mathcal{F})$. Moreover, $D\langle V_2\rangle$ is strong and of order at least 2, so it contains a directed cycle D_2. Let $D_1 = R(\mathcal{F}) - D_2$. Since $\mathcal{R}(\mathcal{F})$ is a ring digraph, D_1 is a directed cycle. In addition, $V_1 \cap V(R(\mathcal{F})) \subseteq V(D_1)$. Now, for all $j \in [m]$, one of the two vertices c_j, c'_j is not the root of B_1, and so has an in-neighbour in D_1. Thus at least one vertex of W_j is in V_1. Consequently, D_1 intersects all the sets $W_1, \dots, W_m$.

Reciprocally, assume that there exists a directed cycle D_1 in $R(\mathcal{F})$ which intersects all the sets $W_1, \dots, W_m$. Set $V_2 = V(R(\mathcal{F})) \setminus V(D_1)$ and $V_1 = V(R_7(\mathcal{F})) \setminus V_2$. Clearly, $D\langle V_2\rangle$ is a directed cycle and $D\langle V_1\rangle$ is strong (and thus is out-branchable). $\diamond$

Theorem 5.12 *Deciding whether a strong digraph has an (out-branchable, $\delta^+ \geq 1$)-$[k_1, k_2]$-partition (resp. (in-branchable, $\delta^- \geq 1$)-$[k_1, k_2]$-partition) is NP-complete for all $k_1, k_2 \geq 2$.*

Proof: The reduction is from 3-SAT, and we here assume that we are given an instance $\mathcal{F}$ consisting of m clauses $C_1, \dots, C_m$ over n variables $x_1, .., x_n$ and m clauses $C_1, ..., C_m$. As mentioned in Section 2, we may assume that $\min\{n, m\} \geq \max\{k_1, k_2\}$. Furthermore, free to add copies of clauses, we may assume that every clause appears twice. We build a strong digraph $D(\mathcal{F})$ starting from a circular strip on the vertices vertices $\{b_i, \overline{b}_i\}$, $1 \leq i \leq n$, where all arcs exist from $\{b_i, \overline{b}_i\}$ to $\{b_{i+1}, \overline{b}_{i+1}\}$ (and from $\{b_n, \overline{b}_n\}$ to $\{b_1, \overline{b}_1\}$). The vertex b_i represents the literal x_i and the vertex $\overline{b}_i$ represents the literal $\overline{x}_i$. For each clause C_j of $\mathcal{F}$, we add a vertex c_j and three arcs from c_j to the vertices corresponding to its three literals. Then add a new vertex s and the arcs $\{sc_j : j \in [m]\}$. Now we add the following two gadgets:

- First gadget: from each vertex from $\{b_i, \overline{b}_i\}$, we add two directed paths of length 2 toward s.
- Second gadget: (depicted Figure 2) from every pair $(b_i, \overline{b}_i)$ to both b_{i+1} and $\overline{b}_{i+1}$ (indices are modulo n).

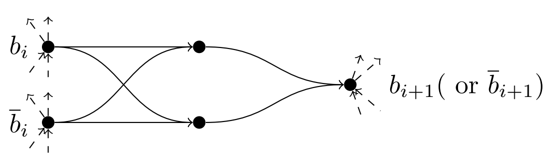

Figure 2: The second gadget

See Figure 3 for a partial view of $D(\mathcal{F})$.

Let us now prove that $D(\mathcal{F})$ has an (out-branchable, $\delta^+ \geq 1$)-$[k_1, k_2]$-partition if and only if $\mathcal{F}$ is satisfiable.

- A satisfying truth assignment to $\mathcal{F}$ yields a good bipartition, by putting in V_2 all the vertices of $\{s\} \cup \{c_1, \dots, c_m\}$, as well as all b_i (resp. $\overline{b}_i$) such that x_i (resp. $\overline{x}_i$) is *true*. We then put in V_1 all b_i (resp. $\overline{b}_i$) such that x_i (resp. $\overline{x}_i$) is false. This easily extends in an (out-branchable, $\delta^+ \geq 1$)-$[k_1, k_2]$-partition of $D(\mathcal{F})$.
- Assume now that $D(\mathcal{F})$ has an (out-branchable, $\delta^+ \geq 1$)-$[k_1, k_2]$-partition (V_1, V_2). Note that if b_{i+1} (resp. $\overline{b}_{i+1}$) is in V_1, then its two in-neighbours in this second gadget are also in V_1, and

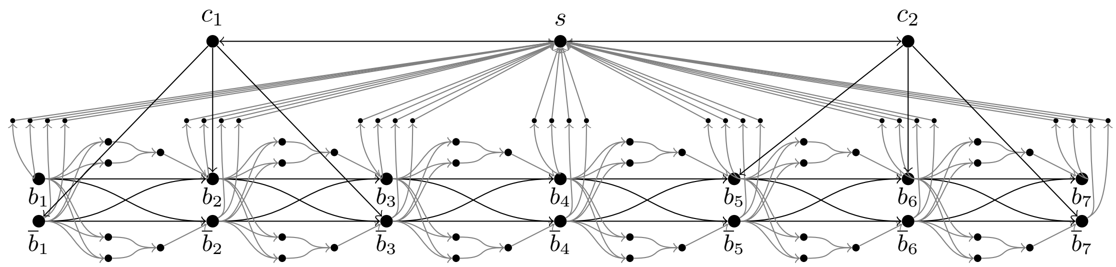

Figure 3: The digraph $D(\mathcal{F})$ when $\mathcal{F}$ has the clauses $C_1 = \overline{x}_1 \vee x_2 \vee \overline{x}_3$ and $C_2 = \overline{x}_5 \vee x_6 \vee \overline{x}_7$

thus one vertex in $\{b_i, \overline{b}_i\}$ is also in V_1. This ensures that V_1 contains at least one of $b_i, \overline{b}_i$ for each $i \in [n]$.

The first gadget ensures that s belongs to V_2: suppose $s \in V_1$, then all in-neighbours of s and then in turn all of $\bigcup_{i\in[n]}\{b_i, \overline{b}_i\}$ are in V_1. By the remark above, this implies that every vertex of the second gadget is in V_1. This implies that $V_2 \subseteq \{c_1, \ldots, c_m\}$, a contradiction since this is an independent set in $D(\mathcal{F})$. So $s \in V_2$ and this implies that all the c_i vertices (except at most one) also belong to V_2. Because each clause appears twice, at least one of the vertices corresponding to each such pair is in V_2 and has an out-neighbour in V_2. Hence, assigning *true* to each variable x_i such that $b_i \in V_2$ and $false$ to the other, we obtain a truth assignment satisfying $\mathcal{F}$. $\diamond$

Theorem 5.13 *Deciding whether a strong digraph has an (out-branchable, in-branchable)-$[k_1, k_2]$-partition is NP-complete for all $k_1, k_2 \geq 2$.*

Proof: Reduction from 3-SAT. Let $\mathcal{F} = C_1 \wedge C_2 \wedge \ldots \wedge C_m$ be an instance of 3-SAT consisting of clauses $C_1, \ldots, C_m$ over n boolean variables $x_1, \ldots, x_n$. Let $R_8(\mathcal{F})$ be the digraph obtained from $R_6(\mathcal{F})$ as follows. For each vertex v of $V(R_6(\mathcal{F}))$, we create two disjoint sets $A^+(v)$ and $A^-(v)$, each of three vertices. Hence $V(R_8(\mathcal{F})) = V(R_6(\mathcal{F})) \cup \bigcup_{v\in V(R_6(\mathcal{F}))}(A^+(v) \cup A^-(v))$. $R_8(\mathcal{F}))$ has the following arcs: for each vertex $v \in V(R_6(\mathcal{F}))$, it contains all arcs from v to $A^+(v)$ and all arcs from $A^+(v)$ to $N^+_{R_6(\mathcal{F})}(v)$. See Figure 4. Similarly, it contains all arcs from $A^-(v)$ to v and all arcs from $N^-_{R_6(\mathcal{F})}(v)$ to $A^-(v)$.

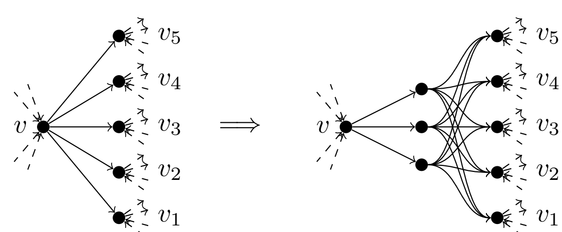

Figure 4: Constructing $R_8(\mathcal{F})$ from $R_6(\mathcal{F})$

Let us first assume that $R_8(\mathcal{F})$ admits an (out-branchable, in-branchable)-$[k_1, k_2]$-partition (V_1, V_2).
Claim: If $v \in V_2$, then at least one vertex of $N^+_{R_6(\mathcal{F})}(v)$ also belongs to V_2.
Proof of the claim: Assume to the contrary that all vertices of $N^+_{R_6(\mathcal{F})}(v)$ are in V_1. Then at most one vertex of $A^+(v)$ of the three central vertices can belong to V_1, because an out-branchable digraph has at most one source. Thus, at least two of them belong to V_2, which is impossible as the two of them would be sinks the in-branchable digraph induced by V_2. $\diamond$

Similarly to the claim, if $v \in V_1$, then at least one vertex of $N^-_{R_6(\mathcal{F})}(v)$ also belongs to V_1. Consequently, the partition $(V_2 \cap V(R_6(\mathcal{F})), V_1 \cap V(R_6(\mathcal{F}))$ is a $(\delta^+ \geq 1, \delta^- \geq 1)$-partition of $R_6(\mathcal{F})$, which yields a truth assignment satisfying $\mathcal{F}$ as shown in the proof of Theorem 4.7.

Reciprocally, assume that $\mathcal{F}$ is satisfied by a truth assignment ϕ. Let (V_1', V_2') be the $(\delta^+ \geq 1, \delta^- \geq 1)$-partition of $R_6(\mathcal{F})$ obtained as follows : let C_1 be a directed cycle C_1 which intersects all the sets $W_1, \dots, W_m$. (Such a cycle can be obtained from ϕ as shown in the proof of Theorem 4.7.) Set $V_2' = V(R(\mathcal{F})) \setminus V(C_1)$ and $V_1' = V(R_6(\mathcal{F})) \setminus V_2$. We now show that such a partition may be extended into an (out-branchable, in-branchable)-$[k_1, k_2]$-partition (V_1, V_2) of $R_8(\mathcal{F})$, by adding the vertex of the sets $A^+(v)$ and $A^-(v)$ appropriately: By construction $R_6(\mathcal{F})\langle V_2' \rangle$ is a directed cycle and thus has an in-branching B^- rooted at some vertex that we call r. Let r' be the out-neighbour of r in $R_6(\mathcal{F})\langle V_2' \rangle$. Furthermore $R_6(\mathcal{F})\langle V_1' \rangle$ is strong (see the proof of Theorem 4.7) and hence has an out-branching rooted at some vertex s. Let s' be any in-neighbour of s in $R_6(\mathcal{F})\langle V_1' \rangle$. Now we can extend (V_1', V_2') to a partition (V_1, V_2) of $R_8(\mathcal{F})$ by adding all vertices of $A^+(v), A^-(v)$ to V_i whenever $v \in V_i'$. We claim that this partition is (out-branchable, in-branchable): to show this, it is sufficient to show that every vertex in V_1 can be reached from s in $R_8(\mathcal{F})\langle V_1 \rangle$ and every vertex in V_2 can reach r in $R_8(\mathcal{F})\langle V_2 \rangle$. Let us observe the following.

- Every vertex $v \in V_1' - s$ has an arc uv entering it in B^+ so u can reach $A^-(v)$ in $R_8(\mathcal{F})\langle V_1 \rangle$.
- $v \in V_1'$ can reach $A^+(v)$
- s' can reach $A^-(s)$.

Together, these observations imply that $R_8(\mathcal{F})\langle V_1 \rangle$ has an out-branching rooted at s. Analogously one can prove that every vertex in V_2 can reach r in $R_8(\mathcal{F})\langle V_2 \rangle$, implying that $R_8(\mathcal{F})\langle V_2 \rangle$ has an in-branching. $\diamond$

6 Concluding remarks and open problems

6.1 Faster and FPT algorithms

In this paper, we gave polynomial algorithms for many $[k_1, k_2]$-partition problems for k_1 and k_2 fixed. However, the proposed algorithms are only polynomial when k_1 and k_2 are fixed and generally have a typical running time of $O(n^{\alpha \cdot k_1 + \beta \cdot k_2 + \gamma})$ for some constants α, β, γ. This means that the $[k_1, k_2]$-partition problem is in XP with respect to the parameter (k_1, k_2). A natural question is then to ask whether some of those problems can be solved in polynomial time or when this is not the case, then in FPT time when k_1 and k_2 are not fixed.

Problem 6.1 *For which pairs $(\mathbb{P}_1, \mathbb{P}_2)$ of properties among the ones studied in this paper, does there exist an algorithm that, given a digraph D and two positive integers k_1, k_2, decides whether D admits a $(\mathbb{P}_1, \mathbb{P}_2)$-$[k_1, k_2]$-partition in polynomial time? Which ones can be solved in FPT time (i.e. $f(k_1, k_2)n^c$)-time with f a computable function and c a constant.*

6.2 Partitions with other properties

Of course, one can consider $(\mathbb{P}_1, \mathbb{P}_2)$-partition problems for different properties $\mathbb{P}_1, \mathbb{P}_2$ than the ones we considered in this paper and its companion [3]. A first possibility is to consider properties that are the union of several properties we considered as examplified in the next subsection. Again imposing the strong connectivity may change the complexity as shown in Subsection 6.2.2. Another possibility is to consider completely different properties like having bounded maximum degree. This is discussed briefly in Subsection 6.2.3.

6.2.1 (connected and $\delta^+ \geq 1$, connected and $\delta^+ \geq 1$)-partitions

Proposition 6.2 *There exists a polynomial-time algorithm that decides whether a given digraph D has a (connected and $\delta^+ \geq 1$, connected and $\delta^+ \geq 1$)-partition.*

Proof: If D has more than two connected components or there is a vertex with out-degree 0, then we return 'No' because D is clearly a 'No'-instance. Henceforth we assume that D has at most two connected components and minimum out-degree at least 1. First, using McCuaig's algorithm

[22], we check in polynomial time whether D has two disjoint directed cycles C_1, C_2. If there are no such cycles, then D is 'No'-instance. Assume now that two such cycles exist. Let W_1 be the set of vertices that can reach $V(C_1)$ by a directed path in $D - V(C_2)$ and let $D_{2,1}, \dots, D_{2,k}$ be the connected components of $D - W_1$ where $V(C_2) \subseteq V(D_{2,1})$. Note that we have $\delta^+(D_{2,i}) \geq 1$ for $i \in [k]$ since every vertex of $V(D) \setminus W_1$ has all its out-neighbours in $V(D) \setminus W_1$. Hence if $k = 1$ we are done, so assume $k > 1$. If there is at least one arc between W_1 and each of $V(D_{2,2}), \dots, V(D_{2,k})$ (by definition of W_1 such arcs will have tail in W_1), then $(V(D_{2,1}), V \setminus V(D_{2,1}))$ is the desired partition so assume w.l.o.g that there is no arc from W_1 to $V(D_{2,k})$. Then all of $D_{2,1}, \dots, D_{2,k-1}$ are adjacent to W_1 (as D has at most two connected components) and $(V(D_{2,k}), V(D) \setminus V(D_{2,k}))$ is the desired partition. $\diamond$

Problem 6.3 *Is the $|V_i| \geq k_i$, $i = 1, 2$, version of the problem above polynomially solvable for all fixed values of k_1, k_2?*

6.2.2 (strong,connected with an underlying cycle)-partitions

We say that a digraph D has an underlying cycle if $UG(D)$ has a cycle.

Theorem 6.4 *The (strong, connected with an underlying cycle)-partition problem is NP-complete for general digraphs and polynomial-time solvable for strong digraphs*

Proof: Suppose first that D is strong. Then using handle-decompositions as above, we observe that D has the desired partition if and only if its underlying graph $UG(D)$ contains two cycles C, C' such that C is also a (directed) cycle in D. One direction is clear and the other follows by starting from such a pair C, C' and then adding (internal parts of) handles to either the part containing C or the one containing C'. In [5], a polynomial algorithm for deciding the existence of cycles C, C' as above (and constructing such when they exist). Combining this with the handle adding process above, we obtain the desired algorithm. $\diamond$

In [6], it was shown that deciding the existence of cycles C, C' as above is NP-complete for general (non-strong) digraphs so we may expect that the (strong, connected with a cycle)-partition problem is NP-complete for non-strong digraphs. This follows from the proof of Theorem 4.2(i), because the proof actually shows that the non-strong part of the (strong, connected)-partition that exists if and only if $\mathcal{F}$ is satisfiable will also contain a cycle (which is even a directed cycle). $\diamond$

Problem 6.5 *What is the complexity for strong digraphs and fixed positive integers k_1, k_2 of the (strong, connected with an underlying cycle)-$[k_1, k_2]$-partition problem?*

This seems related to the following problem which is also open.

Problem 6.6 *What is the complexity of deciding, for fixed positive integers k_1, k_2 whether the underlying graph of a strong digraph contains cycles C_i such that $|V(C_i)| \geq k_i$ and C_1 is a directed cycle in D?*

It follows from Theorem 3.15 that we can decide the existence of disjoint directed cycles of lengths at least p_1 respectively p_2 in polynomial time.

6.2.3 Partitions where we upper bound (maximum) degrees

One may also consider the properties of having maximum degree at most p. This property is clearly hereditary. Note moreover that inducing a digraph with maximum degree (at most) 0 is equivalent to be an independent set.

Observe that if D is a symmetric digraph, then there is a one-to-one correspondence between the $(\Delta^+ \leq p, \Delta^+ \leq p)$-partition of D and the p-improper 2-colourings of $UG(D)$. Cowen et al. [11] proved that for any $p \geq 1$, deciding if a graph has a p-improper 2-colouring is NP-complete. This implies that the $(\Delta^+ \leq p, \Delta^+ \leq p)$-partition problem is NP-complete for any positive p. In fact, their proof can be easily modified to show the following.

Theorem 6.7 *Let p_1, p_2 be two non-negative integers such that $p_1 + p_2 \geq 1$, and let k_1, k_2 be two positive integers. The $(\Delta^+ \leq p_1, \Delta^+ \leq p_2)$-$[k_1, k_2]$-partition problem is NP-complete.*

Thomassen [29] constructed a class $\mathcal{D}$ of out-regular digraphs (same out-degree at all vertices) that have no even directed cycle. Alon [1] pointed out that the digraphs in $\mathcal{D}$ have no 2-partition (V_1, V_2) such that $\Delta^+(D\langle V_i\rangle) < \Delta^+(D)$ for $i \in [2]$. He also proved that it is always possible to split $V(D)$ into $k \geq 3$ sets such that each of the induced subdigraphs has smaller maximum out-degree than D.

It is an easy fact that every graph G contains a spanning bipartite graph $B(G)$ in which the degree of each vertex is at least half of its degree in G. Alon's observation implies that there are digraphs (in particular all digraphs in $\mathcal{D}$) for which we cannot even guarantee out-degree at least one in any spanning bipartite subdigraph.

Problem 6.8 *What is the complexity of deciding whether a given digraph D has a 2-partition (V_1, V_2) with $\Delta^+(D\langle V_i\rangle) < \Delta^+(D)$?*

References

[1] N. Alon. Splitting digraphs. *Combin. Probab. Comput.*, 15:933–937, 2006.

[2] J. Bang-Jensen and G. Gutin. *Digraphs: Theory, Algorithms and Applications, 2nd Edition.* Springer-Verlag, London, 2009.

[3] J. Bang-Jensen and F. Havet. Finding good 2-partitions of digraphs I. Hereditary properties. submitted, 2016.

[4] J. Bang-Jensen, F. Havet, and A. K. Maia de Oliveira. Finding a subdivision of a digraph. *Theoretical Computer Science*, 562:20, 2015.

[5] J. Bang-Jensen and M. Kriesell. On the problem of finding disjoint cycles and dicycles in a digraph . *Combinatorica*, 31:639–638, 2011.

[6] J. Bang-Jensen, M. Kriesell, Maddaloni. A., and S. Simonsen. Vertex-disjoint directed and undirected cycles in general digraphs. *J. Combin. Theory, Ser. B*, 106:1–14, 2014.

[7] J. Bang-Jensen and A. Yeo. Arc-disjoint spanning sub(di)graphs in digraphs. *Theoretical Computer Science*, 438:48–54, 2012.

[8] J. Bensmail. On the complexity of partitioning a graph into a few connected subgraphs. *J. Combin. Optimization*, 30:174–187, 2015.

[9] D. Bokal, G. Fijavz, M. Juvan, Kayl P.M., , and Mohar B. The circular chromatic number of a digraph. *J. Graph Theory*, 46:227–240, 2004.

[10] D. Concalves, F. Havet, Pinlou. A., and S. Thomassé. On spanning galaxies in digraphs. *Discrete Applied Math.*, 160:744–754, 2012.

[11] L. Cowen, W. Goddard, and C. E. Jesurum. Defective coloring revisited. *J. Graph Theory*, 24(3):205–219, 1997.

[12] M.E. Dyer and A.M. Frieze. On the complexity of partitioning graphs into connected subgraphs. *Discrete. Appl. Math.*, 10:139–153, 1985.

[13] S. Földes and P. Hammer. Split graphs. *Congressus Numerantium*, 19:311–315, 1977.

[14] A Grigoriev and R. Sitters. Connected feedback vertex set in planar graphs. In *Graph Theoretical concepts in computer science, 35th international workshop, WG2009*, volume 5911 of *Lecture Notes in Comp. Science*, pages 143–153. Springer Verlag, Berlin, 2009.

[15] P. Hajnal. Partition of graphs with condition on the connectivity and minimum degree. *Combinatorica*, 3:95–99, 1983.

[16] K. Kawarabayashi and S. Kreutzer. The directed grid theorem. *CoRR*, abs/1411.5681, 2014.

[17] D. Kühn and D. Osthus. Partitions of graphs with high minimum degree or connectivity. *J. Combinatorial Theory Ser. B*, 88:29–43, 2003.

[18] D. Kühn, D. Osthus, and T. Townsend. Proof of a tournament partition conjecture and an application to 1-factors with prescribed cycle length. *Combinatorica*, to appear.

[19] H.-O. Le, V.B. Le, and H. Müller. Splitting a graph into disjoint induced paths or cycles. *Discrete. Appl. Math.*, 131:199–212, 2003.

[20] N. Lichiardopol. Vertex-Disjoint Subtournaments of Prescribed Minimum Outdegree or Minimum Semidegree: Proof for Tournaments of a Conjecture of Stiebitz. *International J. Combinatorics*, pages Article ID 273416, 9 pages, 2012.

[21] MH Liu and BG Xu. Bipartition of graph under degree constraints. *Science China Mathematics*, 58:869–874, 2015.

[22] W. McCuaig. Intercyclic digraphs. In *Graph structure theory (Seattle, WA, 1991)*, volume 147 of *Contemp. Math.*, pages 203–245. American Mathematical Society, 1993.

[23] N. Misra, G. Philip, V. Raman, S. Saurabh, and S. Sikdar. FPT algorithms for connected feedback vertex set. *J. Combin. Optimization*, 24:131–146, 2012.

[24] K.B. Reid. Two complementary circuits in two-connected tournaments. In *Cycles in graphs (Burnaby, B.C., 1982)*, volume 115 of *North-Holland Math. Stud.*, pages 321–334. North-Holland, 1985.

[25] T.J. Schaefer. The complexity of satisfiability problems. In *Proceedings of the 10th Annual ACM Symposium on Theory of Computing (STOC 10)*, pages 216–226, New York, 1978. ACM.

[26] M. Stiebitz. Decomposition of graphs and digraphs. In *KAM Series in Discrete Mathematics-Combinatorics-Operations Research-Optimization 95-309.* Charles University Prague, 1995.

[27] H. Suzuki, N. Takahashi, and T. Nishizeki. A linear algorithm for bipartion of biconnected graphs. *Information Processing Letters*, 33:227–231, 1990.

[28] C. Thomassen. Graph decomposition with constraints on the connectivity and minimum degree. *J. Graph Theory*, 7:165–167, 1983.

[29] C. Thomassen. Even cycles in directed graphs. *Eur. J. Combin.*, 6(1):85–89, 1985.

[30] P. van't Hof, D. Paulusma, and G.J. Woeginger. Partitioning graphs into connected parts. *Theoretical Computer Science*, 410:4834–4843, 2009.