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Lotka-Volterra with randomly fluctuating environments: a full description

Florent Malrieu, Tran Hoa Phu
July 20, 2016

Abstract

In this note, we study the long time behavior of Lotka-Volterra systems whose coefficients vary randomly. Benaïm and Lobry established that randomly switching between two environments that are both favorable to the same species may lead to four different regimes: almost sure extinction of one of the two species, random extinction of one species or the other and persistence of both species. Our purpose here is to provide a complete description of the model. In particular, we show that any couple of environments may lead to the four different behaviours of the stochastic process depending on the jump rates.

1 Introduction

For a given set of positive parameters \( \varepsilon = (a, b, c, d, \alpha, \beta) \), consider the Lotka-Volterra differential system in \( \mathbb{R}^2_+ \), is given by

\[
\begin{align*}
x' &= \alpha x (1 - ax - by) \\
y' &= \beta y (1 - cx - dy) \\
(x_0, y_0) &\in \mathbb{R}^2_+
\end{align*}
\]

We denote by \( F_\varepsilon \) the associated vector field: \((x', y') = F_\varepsilon(x, y)\). Let us note already that when \( a < c, b < d \), the point \((1/a, 0)\) attracts any path starting in \((0, +\infty)^2\). We say that the environment is favorable to species \( x \). Similarly, when \( a > c, b > d \), the point \((0, 1/d)\) attracts any path starting in \((0, +\infty)^2\). We say that the environment is favorable to species \( y \). See [?] for a detailed presentation of the four generic configurations. The environment is said to be of

- Type 1: if \( a < c, b < d \) (favorable to species \( x \))
- Type 2: if \( a > c, b > d \) (favorable to species \( y \))
- Type 3: if \( a > c, b < d \) (persistence)
- Type 4: if \( a < c, b > d \) (extinction of species \( x \) or \( y \) depending on the starting point)

Consider two such systems \( \varepsilon_0 = (a_0, b_0, c_0, d_0, \alpha_0, \beta_0) \) and \( \varepsilon_1 = (a_1, b_1, c_1, d_1, \alpha_1, \beta_1) \) and introduce the random process \( \{(X_t, Y_t, I_t)\} \) on \( \mathbb{R} \times \{0, 1\} \) obtained by switching between these two deterministic dynamics, at rates \( \lambda_0, \lambda_1 \). More precisely, we consider the Markov process driven by the following generator

\[ Lf(z, i) = F_i(z) \cdot \nabla_z f(z, i) + \lambda_i (f(z, 1 - i) - f(z, i)), \quad (z, i) \in \mathbb{R}^2 \times \{0, 1\}. \]
Equivalently, \((I_t)_{t \geq 0}\) is a Markov process on \(\{0, 1\}\) with jump rate \(\lambda_0\) and \(\lambda_1\), that is
\[
P(I_{t+\delta} = i | I_t = i, F_t) = \lambda_i s + o(s),
\]
where \(F_t\) is the sigma field generated by \(\{I_u, u \leq t\}\). Finally, \((X_t, Y_t)\) is solution of
\[
(X'_t, Y'_t) = F_{I_t}(X_t, Y_t).
\]

This process on \(\mathbb{R}^2 \times \{0, 1\}\) has already been studied in [?, ?]. It belongs to the class of the piecewise deterministic Markov processes introduced by Davis [?]. See also [?] for a recent review of the application areas of such processes. Let us introduce the invasion rates of species \(x\) and \(y\) defined in [?]

\[
\Lambda_y = \int \beta_0(1 - c_0x)\mu(dx, 0) + \int \beta_1(1 - c_1x)\mu(dx, 1),
\]

\[
\Lambda_x = \int \alpha_0(1 - b_0y)\hat{\mu}(dy, 0) + \int \alpha_1(1 - b_1y)\hat{\mu}(dy, 1),
\]

where \(\mu\) is the invariant probability measure of \((X_t, I_t)\) associated to equation:

\[
X'_t = \alpha_{I_t}X_t(1 - a_{I_t}X_t),
\]

and \(\hat{\mu}\) is the invariant probability measure of \((Y_t, I_t)\) associated to equation:

\[
Y'_t = \beta_{I_t}Y_t(1 - d_{I_t}Y_t).
\]

The meaning of \(\Lambda_y\) is the following: when species \(y\) is close to extinction, species \(x\) behaves approximately as \((X'_t, 0) = F_{I_t}(X_t, 0)\) and \(\Lambda_y\) is the growth rate of species \(y\) with respect to invariant measure \(\mu\) of \((X, I)\). Note that the invasion rates depend on the jump rates \((\lambda_0, \lambda_1) \in (0, +\infty)^2\). For every \((\lambda_0, \lambda_1) \in (0, +\infty)^2\), we have two parametrizations of these jump rates:

\[
(s, t) \in [0, 1] \times (0, +\infty) : \quad st = \lambda_0, \quad (1 - s)t = \lambda_1.
\]

\[
(u, v) \in [0, 1] \times (0, +\infty) : \quad uv = \lambda_0/\alpha_0, \quad (1 - u)v = \lambda_1/\alpha_1.
\]

The change of parameters \((u, v) = \xi(s, t)\) is triangular in the sense that \(u\) only depends on \(s\)

\[
(u, v) = \xi(s, t) = \left(\frac{s\alpha_1}{(1 - s)\alpha_0 + s\alpha_1}, \frac{t}{\alpha_0\alpha_1}(1 - s)\alpha_0 + s\alpha_1)\right).
\]

Let us denote the invasion rates in the \((u, v)\) coordinates by

\[
\tilde{\Lambda}_x(u, v) = \Lambda_x(\xi^{-1}(u, v)) \quad \text{and} \quad \tilde{\Lambda}_y(u, v) = \Lambda_y(\xi^{-1}(u, v)).
\]

It is established in [?] that signs of \(\tilde{\Lambda}_x\) and \(\tilde{\Lambda}_y\) determine the long time behavior of \((X_t, Y_t)\).

<table>
<thead>
<tr>
<th>(\Lambda_y)</th>
<th>(\Lambda_x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(&gt; 0)</td>
<td>persistence of the two species</td>
</tr>
<tr>
<td>(&lt; 0)</td>
<td>extinction of species (x) or (y)</td>
</tr>
</tbody>
</table>

Moreover, in [?] it is shown that two environments of Type 1 may lead to four regimes for the stochastic process. This surprising result is reminiscent of switched stable linear ODE studied in [?, ?].
A fundamental property of the model is that, for all $0 \leq s \leq 1$, the vector field $(1-s)F_{\varepsilon_0} + sF_{\varepsilon_1}$ is the Lotka-Volterra system associated to the environment $\varepsilon_s = (a_s, b_s, c_s, s_0, a_s, b_s)$ with

$$\alpha_s = s\alpha_1 + (1-s)\alpha_0, \quad a_s = \frac{s\alpha_1 a_1 + (1-s)a_0 a_0}{\alpha_s}, \quad b_s = \frac{s\alpha_1 b_1 + (1-s)a_0 b_0}{\alpha_s}, \quad (1.1)$$

$$\beta_s = s\beta_1 + (1-s)\beta_0, \quad c_s = s\beta_1 c_1 + (1-s)\beta_0 c_0, \quad d_s = s\beta_1 d_1 + (1-s)\beta_0 d_0. \quad (1.2)$$

Set

$$I = \{0 \leq s \leq 1 : a_s > c_s\} \quad \text{and} \quad J = \{0 \leq s \leq 1 : b_s > d_s\}.$$

We denote by $\tilde{I}$ the image of $I$ for the other parametrization.

**Remark 1.1.** As noticed in [?], if $\varepsilon_0$ and $\varepsilon_1$ are of Type 1 then $I$ or $J$ may generically be empty or an open interval which closure is contained in $(0, 1)$.

Let us recall below the key result in [?] about the expression of the invasion rates.

**Lemma 1.2.** [?, Lemma 1.2] Assume that $\varepsilon_0$ and $\varepsilon_1$ are of Type 1 and, w.l.g., $a_0 < a_1$. The quantity $\Lambda_y$ can be rewritten as:

$$\Lambda_y(u,v) = \frac{1}{(a_0 - a_0)(\frac{1}{a_0} (1-u) + \frac{1}{a_1} u) E[\phi(U_{u,v})]}$$

where $\phi : [0, 1] \to \mathbb{R}$ is defined by

$$\phi(y) = (a_0 + (a_1 - a_0)y) P\left(\frac{1}{a_0 + (a_1 - a_0)y}\right),$$

where

$$P(x) = \left(\frac{\beta_1}{\alpha_1} (1 - c_1 x)(1 - a_0 x) - \frac{\beta_0}{\alpha_0} (1 - c_0 x)(1 - a_1 x)\right) \frac{a_1 - a_0}{[a_1 - a_0]}, \quad (1.3)$$

and $U_{u,v}$ is a Beta distributed Beta$(uv, (1-u)v)$ random variable. Moreover, $\phi$ has the following properties:

- If $I$ is empty then $\phi$ is nonpositive.
- If $I$ is nonempty ($I = (u_1, u_2)$) then $\phi$ is concave, negative on $(0, u_1) \cup (u_2, 1)$ and positive on $\tilde{I} = (u_1, u_2)$.

Our first result is the precise study of the properties of $\Lambda_x$ and $\Lambda_y$ with two environments $\varepsilon_0, \varepsilon_1$ that are respectively of Type 1 and Type 2. In particular, we describe the regions where $\Lambda_x$ and $\Lambda_y$ are positive.

**Theorem 1.3.** (Shape of the regions). Assume that $\varepsilon_0$ and $\varepsilon_1$ are respectively of Type 1 and Type 2. Then, there exists a function $u \mapsto v_y(u)$ from $(0, 1) \to [0, \infty]$, such that $\Lambda_y(u, v) < 0$ when $v < v_y(u)$ and $\Lambda_y(u, v) > 0$ when $v > v_y(u)$. Let $a$ be the coefficient of second degree of polynomial $P$ given by (1.3).

- If $a < 0$, there exists $0 < \alpha < \bar{\alpha} < 1$ such that $v_y$ is infinite on $[0, \alpha]$, is decreasing and continuous on $(\alpha, \bar{\alpha})$, tends to $+\infty$ at $\alpha$, tends to $0$ at $\bar{\alpha}$ and is equal to $0$ on $[\bar{\alpha}, 1]$.

- If $a > 0$, there exists $0 < \bar{\alpha} < \alpha < 1$ such that $v_y$ is equal to $0$ on $[0, \bar{\alpha}]$, is increasing and continuous on $(\bar{\alpha}, \alpha)$, tends to $0$ at $\bar{\alpha}$, tends to $+\infty$ at $\alpha$, and is infinite on $[\alpha, 1]$.

Moreover, $\alpha$ and $\bar{\alpha}$ are explicit.
The second result is the following theorem.

**Theorem 1.4.** For any \((i, j)\) in \([1, 2, 3, 4]^2\), there exist two environments \(\varepsilon_0\) of Type \(i\) and \(\varepsilon_1\) of Type \(j\) such that the associated stochastic process has four possible regimes depending on the jump rates.

The paper is organized as follows. In Section 2 we prove the properties of \(\tilde{\Lambda}_x\) and \(\tilde{\Lambda}_y\). In Section 3 we prove Theorem ???. In Section 4 we present illustrations obtained by numerical simulation. In Section 5 we study the case when the two environments are of Type 3. Finally, in Section 6, we prove Theorem ?? providing, in each case, a good couple of environments.

## 2 Expression of invasion rates

**Lemma 2.1.** If \(\varepsilon_0\) and \(\varepsilon_1\) are respectively of Type 1 and Type 2, then \(\tilde{I}\) is always nonempty and there exists \(0 < \alpha < 1\) (depends on \(\alpha_i, \beta_i, a_i, c_i\)) such that \(\tilde{I} = (\alpha, 1]\).

**Proof.** Set

\[
R = \frac{\beta_0 \alpha_1}{\alpha_0 \beta_1}, \quad u = \frac{\alpha_1}{\alpha_0}, \quad A = (a_1 - a_0)(R - 1), \quad B = (2a_0 - c_0 - a_1)R + (c_1 - a_0), \quad C = (c_0 - a_0)R.
\]

For any \(s \in (0, 1)\), we get that

\[
c_s - a_s = \frac{Au^2 + Bu + C}{R(1 - u) + u}
\]

where \(a_s\) and \(c_s\) are given by (??) and (??). Set

\[
T(u) = Au^2 + Bu + C \quad \forall u \in [0, 1].
\]

We easily get

\[
T(0) = C = (c_0 - a_0)R > 0, \quad T(1) = A + B + C = c_1 - a_1 < 0.
\]

Because \(T\) is a second degree polynomial with \(T(0) > 0\) and \(T(1) < 0\), we conclude that

\[
T(u) < 0 \iff u > \alpha = \frac{-B - \sqrt{B^2 - 4AC}}{2A}.
\]

Therefore \(u \in \tilde{I} \iff T(u) < 0 \iff u > \alpha \iff u \in (\alpha, 1]\). As a consequence, \(\tilde{I} = (\alpha, 1]\). \qed

**Proposition 2.2.** The map \(\tilde{\Lambda}_y(u, v)\) satisfies the following properties:

For all \(u \in [0, 1]\)

\[
\lim_{v \to \infty} \tilde{\Lambda}_y(u, v) = \beta_u(1 - \frac{c_u}{a_u}) \begin{cases} > 0 & \text{if } u \in \tilde{I} = (\alpha, 1], \\ = 0 & \text{if } u \in \partial\tilde{I} = \{\alpha\}, \\ < 0 & \text{if } u \in (0, 1) \setminus \tilde{I} = [0, \alpha), \end{cases}
\]

and

\[
\lim_{v \to 0} \tilde{\Lambda}_y(u, v) = \frac{1}{\frac{1}{\alpha_0(1 - u)} + \frac{1}{u}} \left( \frac{\beta_1}{\alpha_1(1 - c_1)} - \frac{\beta_0}{\alpha_0(1 - c_0)} \right) u + \frac{\beta_0}{\alpha_0(1 - c_0)}. \quad (2.1)
\]
**Proof.** The proposition is obtained by changing variables \((s, t) \leftrightarrow (u, v)\) from [?, Prop. 2.3]. □

**Proposition 2.3.** There exists \(0 < \bar{\alpha} < 1\) such that \(\lim_{v \to 0} \tilde{\Lambda}_g(u, v) > 0\) if \(u > \bar{\alpha}\) and \(\lim_{v \to 0} \tilde{\Lambda}_g(u, v) < 0\) if \(u < \bar{\alpha}\).

**Proof.** The limit in (2.3) has the same sign than
\[
g(u) = \left( \frac{\beta_1}{\alpha_1} (1 - \frac{c_1}{a_1}) - \frac{\beta_0}{\alpha_0} (1 - \frac{c_0}{a_0}) \right) u + \frac{\beta_0}{\alpha_0} (1 - \frac{c_0}{a_0}), \quad \forall u \in [0, 1].
\]
We get
\[
g(0) = \frac{\beta_0}{\alpha_0} (1 - \frac{c_0}{a_0}) < 0 \quad \text{and} \quad g(1) = \frac{\beta_1}{\alpha_1} (1 - \frac{c_1}{a_1}) > 0.
\]
Since \(g\) is a linear function, \(\bar{\alpha}\) is the unique zero of \(g\) and the result is clear. □

**Proposition 2.4.** Let \(a\) be the coefficient of degree 2 of polynomial \(P\) given by (2.2)
\[
a = \left( \frac{\beta_1}{\alpha_1} c_1 a_0 - \frac{\beta_0}{\alpha_0} c_0 a_1 \right) \frac{a_1 - a_0}{|a_1 - a_0|}.
\]
If \(a < 0\) (resp. \(a > 0\) or \(a = 0\)) then \(\alpha < \bar{\alpha}\) (resp. \(\alpha > \bar{\alpha}\) or \(\alpha = \bar{\alpha}\)).

**Proof.** By symmetry we only consider the case \(a < 0\). Without loss of generality, we assume that \(a_1 > a_0\) and \(a\) becomes:
\[
a = \frac{\beta_1}{\alpha_1} c_1 a_0 - \frac{\beta_0}{\alpha_0} c_0 a_1.
\]
To prove that \(\alpha < \bar{\alpha}\), it is sufficient to prove \(A\bar{\alpha}^2 + B\bar{\alpha} + C < 0\). Since, by definition of \(\bar{\alpha}\),
\[
\left( \frac{\beta_1}{\alpha_1} (1 - \frac{c_1}{a_1}) - \frac{\beta_0}{\alpha_0} (1 - \frac{c_0}{a_0}) \right) \bar{\alpha} + \frac{\beta_0}{\alpha_0} (1 - \frac{c_0}{a_0}) = 0,
\]
we get, multiplying by \(a_0 a_1 \alpha_1 / \beta_1\), that
\[
(a_0 a_1 - c_1 a_0 - Ra_1 a_0 + Ra_1 c_0)\bar{\alpha} + Ra_1 (a_0 - c_0) = 0. \tag{2.2}
\]
Replacing \(\bar{\alpha}\) by its expression in (2.2), we get:
\[
A\bar{\alpha}^2 + B\bar{\alpha} + C = \frac{R(a_1 - c_1)(a_0 - a_1)(a_0 - c_0)(a_0 c_1 - Ra_1 c_0)}{(a_0 a_1 - a_0 c_1 - Ra_0 a_1 + Ra_1 c_0)^2}.
\]
Since \(c_0 > a_0, a_1 > c_1, a_1 > a_0\) and \(a_0 c_1 - Ra_1 c_0 = \frac{\alpha_1}{\beta_1} a < 0\), we conclude \(A\bar{\alpha}^2 + B\bar{\alpha} + C < 0\). □

### 3 Shape of the positivity region

Recall \(a = \left( \frac{\beta_1}{\alpha_1} a_0 c_1 - \frac{\beta_0}{\alpha_0} a_1 c_0 \right) \frac{a_1 - a_0}{|a_1 - a_0|}\) is the coefficient of degree 2 of polynomial \(P\) given by (2.2).

**Lemma 3.1.** [?], Lemma 4.1] Assume \(\varepsilon_0\) and \(\varepsilon_1\) are of Type 1. If \(\tilde{I}\) is nonempty, then the map \((u, v) \rightarrow E[\phi(U_{u,v})]\) is increasing in \(v\) and concave in \(u\).
Remark 3.2. In Benaïm and Lobry’s case, if $I$ is nonempty, $\phi$ is concave and the parameter $a$ is always negative. In the present case, $a$ may be negative, positive or zero. Therefore, we have the following lemma.

Lemma 3.3. Assume $\varepsilon_0$ and $\varepsilon_1$ are respectively of Type 1 and Type 2, then the shape of $\phi$ depends on the sign of $a$:

- If $a$ is negative, then $\phi$ is strongly concave and $(u, v) \to \mathbb{E}[\phi(U_{u,v})]$ is increasing in $v$ and concave in $u$.
- If $a$ is positive, then $\phi$ is strongly convex and $(u, v) \to \mathbb{E}[\phi(U_{u,v})]$ is decreasing in $v$ and convex in $u$.
- If $a$ is zero, then $\phi$ is linear and $(u, v) \to \mathbb{E}[\phi(U_{u,v})]$ is constant in $v$ and linear in $u$.

Proof. This is a straightforward adaptation of [?, Lem 4.1].

Let us conclude this section with the proof of Theorem ??.

Proof of Theorem ?? . We consider only the case $a < 0$. Set $K = (\alpha, \bar{\alpha})$. We know clearly that $v \to \tilde{\Lambda}_y(u, v)$ admits:

- negative limits at 0 and $\infty$ if $u \in [0, \alpha)$,
- positive limits at 0 and $\infty$ if $u \in (\bar{\alpha}, 1]$,
- a negative limit at 0 and a positive limit at $\infty$ if $u \in (\alpha, \bar{\alpha})$.

The fact that $v \mapsto \tilde{\Lambda}_y(u, v)$ is increasing justifies the existence of $v_y$, and we have

$$\tilde{\Lambda}_y(u, v) = 0 \iff u \in K, v = v_y(u).$$

Let us prove that $v_y$ is decreasing in $K$. Let $\delta_1 < \delta_2$ be two points in $K$. Choose any $\delta_3 \in (\bar{\alpha}, 1)$, we get $\tilde{\Lambda}_y(\delta_1, v_y(\delta_1)) = 0$ and $\tilde{\Lambda}_y(\delta_3, v_y(\delta_1)) > 0$. Since $\tilde{\Lambda}_x(\cdot, v_y(\delta_1))$ is concave and $\delta_1 < \delta_2 < \delta_3$, we get $\tilde{\Lambda}_y(\delta_2, v_y(\delta_1)) > 0$. Since $\tilde{\Lambda}_x(\cdot, \cdot)$ is increasing, we obtain $v_y(\delta_2) < v_y(\delta_1)$.

The continuity of $v_y$ on $K$ is a straightforward consequence of the continuity of the function $\tilde{\Lambda}_y$, which is obvious from the expression (??).

Let us show $v_y$ tends to $\infty$ on $\alpha$. Let $\{u_n\} \subset K : u_n \downarrow \alpha$. Since $v_y$ is decreasing in $K$, we get $v_y(u_n) \uparrow v \in [0, \alpha]$. If $v$ is finite, since the zero set of $\tilde{\Lambda}_y$ is closed, by continuity, $\alpha \in K$ (impossible). So $v_y(u_n) \uparrow \infty$.

Let us prove $v_y$ tends to 0 on $\bar{\alpha}$. Let $\{u_n\} \subset K : u_n \uparrow \bar{\alpha}$. Since $v_y$ is decreasing in $K$, we get $v_y(u_n) \downarrow v \in [0, \alpha]$. If $v > 0$, since $u_n < \bar{\alpha}$, we obtain $\tilde{\Lambda}_y(u_n, v/e) < 0$ $\forall n$. Therefore $0 < \tilde{\Lambda}_y(\bar{\alpha}, v/e) = \lim_{n \to \infty} \tilde{\Lambda}_y(u_n, v/e) \leq 0$ (impossible). As a consequence, $v = 0$ and $v_y(u_n) \downarrow 0$. 

4 Numerical illustrations

Recall that for all $u \in [0, 1]$, $v_y(u)$ and $v_x(u)$ are the unique respective solutions of

$$\tilde{\Lambda}_y(u, v) = 0 \quad \text{and} \quad \tilde{\Lambda}_x(u, v) = 0.$$

We now consider, for a varying parameter $\rho$, the environments

$$\varepsilon_0 = (1, 5, 2, 8, 3, 3) \quad \text{and} \quad \varepsilon_1 = (2, 11, 1, \rho, 2, 1.8).$$

(4.1)
Figure 1: The blue curve is the graph of $v_y$ (it does not depend on $\rho$); the green and red curves are $v_x$ for the environments given in (??) with $\rho = 10$ and $\rho = 9$ respectively.

Figure ?? represents the "critical" functions $v_y$ and $v_x$ for different choices of the environments. Thanks to [?], these plots give us information about how many regimes we can observe when the jump rates are modified. For example, the plot for $\rho = 10$ has three regimes: extinction of $x$ (on the right of the green curve), persistence (between the green and blue curves) and extinction of $y$ (on the left of the blue curve). For $\rho = 9$, there is an additional zone (above the red curve and below the blue curve) that corresponds to jump rates leading to random extinction of a species.

5 Switching between two persistent Lotka-Volterra systems

Let us assume that $\varepsilon_0$ and $\varepsilon_1$ are of Type 3. In this case, one can easily get that extinction of species $y$ is not possible if $u$ is close to 0 or 1; in other words, $[0, 1] \setminus \tilde{I}$ is either empty or is an open interval which closure is contained in $[0, 1]$. Recall

$$R = \frac{\beta_0 \alpha_1}{a_0 \beta_1}, \quad A = (a_1 - a_0)(R - 1), \quad B = (2a_0 - c_0 - a_1)R + (c_1 - a_0), \quad C = (c_0 - a_0)R.$$

Then, we get that

$$[0, 1] \setminus \tilde{I} \neq \emptyset \iff \begin{cases} A < 0 \\ \Delta = B^2 - 4AC > 0 \\ 0 < -B - \sqrt{\Delta} < 1. \end{cases}$$

Moreover, if $[0, 1] \setminus \tilde{I}$ is nonempty, then the map $(u, v) \to \mathbb{E}[\phi(U_u,v)]$ is (strictly) decreasing in $v$ and convex in $u$. This is a straightforward adaptation of Lemma 4.1 in [?].

Figure ?? provides the shape of $v_x$ and $v_y$ for the environments $\varepsilon_0 = (6, 1, 4, 2, 1, 5)$ and $\varepsilon_1 = (3, 3, 2, 5, 5, 1)$. Once again, the switched process has four regimes depending on the jump rates.

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Remark 5.1. We see a surprising result: although both vector fields are persistent, the stochastic process may lead to the extinction of one of the two species.

Figure 2: Graph of $v_y$ (blue curve) and $v_x$ (red curve) for the environments $\varepsilon_0 = (6, 1, 4, 2, 1, 5)$ and $\varepsilon_1 = (3, 3, 2, 5.5, 5, 1)$.

6 General case: proof of Theorem ??

The following array presents, for any couple of types, an example of two environments that are associated to a stochastic process with four regimes depending on the jump rates. The first line has been obtained in [?]. The second line is studied in Section 2. The fifth line is studied in Section 5. The reader can easily check that the other cases correspond to Figure ??.

<table>
<thead>
<tr>
<th>$(F_0, F_1)$</th>
<th>$a_0$</th>
<th>$b_0$</th>
<th>$c_0$</th>
<th>$d_0$</th>
<th>$a_0$</th>
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<tbody>
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<td>1</td>
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<td>2</td>
<td>1</td>
<td>5</td>
<td>3</td>
<td>3</td>
<td>4</td>
<td>3.5</td>
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<tr>
<td>Type 1-2</td>
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<td>8</td>
<td>3</td>
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<td>2</td>
<td>11</td>
<td>1</td>
<td>9</td>
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Figure 3: Graph of $v_y$ (blue curve) and $v_x$ (red curve) for the four last cases.
References


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