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Amelioration of Glaisher's Congruence

Scheherazade Zerroukhat, Redha Chellal et Farid Bencherif

USTHB, Fac. Math., P.B. 32, El Alia, 16111, Algiers, Algeria.

szerroukhat@yahoo.fr

chellal.redha@gmail.com

fbencherif@gmail.com

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Abstract

We prove that if $p \geq 5$ is a prime, then

$$\frac{2^{p-1} - 1}{p} \equiv -\frac{1}{2} \left(\sum_{k=1}^{p-1} \frac{2^k}{k} \right) \pmod{p^2}$$

Key words: Congruences, harmonic numbers.

1 Introduction

Let p be a prime odd number and for every integer a relatively prime to p , we define Fermat's quotient in basis a as:

$$q_p(a) = \frac{a^{p-1} - 1}{p}.$$

The study of Fermat's quotients and the congruences involving these quotients is related to the classical study of the Fermat's last theorem [3]. The following congruence is due to Glaisher [1]:

$$\frac{2^{p-1} - 1}{p} \equiv -\frac{1}{2} \left(\sum_{k=1}^{p-1} \frac{2^k}{k} \right) \pmod{p}.$$

Using many computations, B. Ronk [4] et [5] has discovered that this congruence was verified modulo p^2 for primes less or equals to 1000. He conjectured that this congruence was verified modulo p^2 for all primes p . In this paper, we will confirm this conjecture. We will also improve (ameliorate) the congruence of Glaisher by proving it

Theorem 1 *For every prime number $p \geq 5$, we have*

$$\frac{2^{p-1} - 1}{p} \equiv -\frac{1}{2} \left(\sum_{k=1}^{p-1} \frac{2^k}{k} \right) \pmod{p^2}.$$

2 Proof of the theorem

We'll give a series of lemmas from which the proof of the result (theorem) will follow.

Lemma 2 For every prime number $p \geq 5$

$$\sum_{k=1}^{p-1} \frac{H_k}{k2^k} \equiv 0 \pmod{p}.$$

Proof. This lemma was proved in [6] ■

Lemma 3 For every prime number p , p odd, we have

$$H_{p-k} \equiv H_{k-1} \pmod{p}.$$

Proof. This lemma was also proved in [6]. ■

Lemma 4 For every prime number $p \geq 5$

$$\sum_{k=1}^{p-2} \frac{2^k H_k}{k+1} \equiv 0 \pmod{p}.$$

Proof.

$$\begin{aligned} \sum_{k=1}^{p-2} \frac{2^k H_k}{k+1} &= \sum_{k=2}^{p-1} \frac{2^{p-k} H_{p-k}}{p-k+1} \\ &\equiv \sum_{k=2}^{p-1} \frac{2^{p-1-(k-1)} H_{p-k}}{p-(k-1)} \\ &\equiv - \sum_{k=2}^{p-1} \frac{H_{k-1}}{(k-1)2^{k-1}} \\ &\equiv - \sum_{k=1}^{p-2} \frac{H_k}{k2^k} \\ &\equiv - \sum_{k=1}^{p-1} \frac{H_k}{k2^k} + \frac{H_{p-1}}{(p-1)2^{p-1}} \\ &\equiv 0 \pmod{p}. \end{aligned}$$

■

Lemma 5

$$\frac{x^p - (x-1)^p - 1}{p} \equiv - \sum_{k=0}^{p-2} \frac{x^{k+1}}{k+1} + p \sum_{k=1}^{p-2} H_k \frac{x^{k+1}}{k+1} \pmod{p^2}.$$

Proof. We have:

$$\begin{aligned} \frac{x^p - (x-1)^p - 1}{p} &= -\frac{1}{p} \sum_{k=1}^{p-1} \binom{p}{k} x^k (-1)^{p-k} \\ &= \sum_{k=1}^{p-1} (-1)^k \binom{p-1}{k-1} \frac{x^k}{k} \\ &= - \sum_{k=0}^{p-2} (-1)^k \binom{p-1}{k} \frac{x^{k+1}}{k+1}. \end{aligned}$$

Note that:

$$\begin{aligned} (-1)^k \binom{p-1}{k} &= \frac{(1-p)(2-p)\dots(k-p)}{1 \cdot 2 \cdot \dots \cdot k} \\ &= \left(1 - \frac{p}{1}\right) \left(1 - \frac{p}{2}\right) \dots \left(1 - \frac{p}{k}\right) \\ &\equiv 1 - p \sum_{j=1}^k \frac{1}{j} \pmod{p^2}. \end{aligned}$$

Hence

$$\begin{aligned} \frac{x^p - (x-1)^p - 1}{p} &\equiv -\sum_{k=0}^{p-2} (1 - pH_k) \frac{x^{k+1}}{k+1} \pmod{p^2}. \\ &\equiv -\sum_{k=0}^{p-2} \frac{x^{k+1}}{k+1} + p \sum_{k=1}^{p-2} H_k \frac{x^{k+1}}{k+1} \pmod{p^2}. \end{aligned}$$

■

The proof follows immediately. All we have to do is to replace x by 2. We'll have:

$$\frac{2^p - 2}{p} \equiv -\sum_{k=0}^{p-2} \frac{2^{k+1}}{k+1} + p \sum_{k=1}^{p-2} H_k \frac{2^{k+1}}{k+1} \pmod{p^2}.$$

From which:

$$\frac{2^{p-1} - 1}{p} \equiv -\frac{1}{2} \sum_{k=0}^{p-2} \frac{2^{k+1}}{k+1} + p \sum_{k=1}^{p-2} H_k \frac{2^k}{k+1} \pmod{p^2}.$$

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