Amelioration of Glaisher's Congruence

Scheherazade Zerroukhat, Redha Chellal, Farid Bencherif

Amelioration of Glaisher's Congruence

Scheherazade Zerroukhat, Redha Chellal et Farid Bencherif

USTHB, Fac. Math., P.B. 32, El Alia, 16111, Algiers, Algeria.
szerroukhat@yahoo.fr
chellal.redha@gmail.com
fbencherif@gmail.com

June 18, 2016

Abstract

We prove that if $p \geq 5$ is a prime, then

$$\frac{2^{p-1}-1}{p} \equiv -\frac{1}{2}\left(\sum_{k=1}^{p-1} \frac{2^k}{k}\right) (\mathrm{mod} p^2)$$

Key words: Congruences, harmonic numbers.

1 Introduction

Let p be a prime odd number and for every integer a relatively prime to p , we define Fermat's quotient in basis a as:

$$q_p(a) = \frac{a^{p-1}-1}{p}.$$

The study of Fermat's quotients and the congruances involving these quotients is related to the classical study of the Fermat's last theorem [3]. The following confruence is due to Glaisher [1]:

$$\frac{2^{p-1}-1}{p} \equiv -\frac{1}{2}\left(\sum_{k=1}^{p-1} \frac{2^k}{k}\right) (\pmod p).$$

Using many computations, B. Ronk [4] et [5] has discovered that this congruence was verified modulo p^2 for primes less or equals to 1000. He conjectured that this congruence was verified modulo p^2 for all primes p. In this paper, we will confirm this conjecture. We will also improve (ameliorate) the congruence of Glaisher by proving it

Theorem 1 *For every prime number $p \geq 5$, we have*

$$\frac{2^{p-1}-1}{p} \equiv -\frac{1}{2}\left(\sum_{k=1}^{p-1} \frac{2^k}{k}\right) (\mathrm{mod} p^2).$$

2 Proof of the theorem

We'll give a series of lemmas from which the proof of the result (theorem) will follow.

Lemma 2 *For every prime number $p \geq 5$*

$$\sum_{k=1}^{p-1} \frac{H_k}{k2^k} \equiv 0 \pmod{p}.$$

Proof. This lemma was proved in [6] ■

Lemma 3 *For every prime number p, p odd, we have*

$$H_{p-k} \equiv H_{k-1} \pmod{p}.$$

Proof. This lemma was also proved in [6]. ■

Lemma 4 *For every prime number $p \geq 5$*

$$\sum_{k=1}^{p-2} \frac{2^k H_k}{k+1} \equiv 0 \pmod{p}.$$

Proof.

$$\begin{aligned}
\sum_{k=1}^{p-2} \frac{2^k H_k}{k+1} &= \sum_{k=2}^{p-1} \frac{2^{p-k} H_{p-k}}{p-k+1} \\
&\equiv \sum_{k=2}^{p-1} \frac{2^{p-1-(k-1)} H_{p-k}}{p-(k-1)} \\
&\equiv -\sum_{k=2}^{p-1} \frac{H_{k-1}}{(k-1)2^{k-1}} \\
&\equiv -\sum_{k=1}^{p-2} \frac{H_k}{k2^k} \\
&\equiv -\sum_{k=1}^{p-1} \frac{H_k}{k2^k} + \frac{H_{p-1}}{(p-1)2^{p-1}} \\
&\equiv 0 \pmod{p}.
\end{aligned}$$

■

Lemma 5

$$\frac{x^p-(x-1)^p-1}{p} \equiv -\sum_{k=0}^{p-2} \frac{x^{k+1}}{k+1} + p\sum_{k=1}^{p-2} H_k \, \frac{x^{k+1}}{k+1} (\mathrm{mod} p^2).$$

Proof. We have:

$$\begin{aligned}
\frac{x^p-(x-1)^p-1}{p} &= -\frac{1}{p}\sum_{k=1}^{p-1} \binom{p}{k} x^k (-1)^{p-k} \\
&= \sum_{k=1}^{p-1} (-1)^k \binom{p-1}{k-1} \frac{x^k}{k} \\
&= -\sum_{k=0}^{p-2} (-1)^k \binom{p-1}{k} \frac{x^{k+1}}{k+1}.
\end{aligned}$$

Note that:

$$\begin{aligned}
(-1)^k \binom{p-1}{k} &= \frac{(1-p)(2-p)\dots(k-p)}{1.2.\dots k} \\
&= (1-\frac{p}{1})(1-\frac{p}{2})\dots(1-\frac{p}{k}) \\
&\equiv 1 - p\sum_{j=1}^{k} \frac{1}{j} \ (\mathrm{mod} p^2).
\end{aligned}$$

Hence

$$\begin{aligned}\frac{x^p-(x-1)^p-1}{p} &\equiv -\sum_{k=0}^{p-2}(1-pH_k)\,\frac{x^{k+1}}{k+1}\ (\mathrm{mod} p^2).\\ &\equiv -\sum_{k=0}^{p-2}\frac{x^{k+1}}{k+1}\ +p\sum_{k=1}^{p-2}H_k\ \frac{x^{k+1}}{k+1}\ (\mathrm{mod} p^2).\end{aligned}$$

■

The proof followsimmediately. All we have to do is to replace x by 2. We'll have:

$$\frac{2^p-2}{p}\equiv -\sum_{k=0}^{p-2}\frac{2^{k+1}}{k+1}\ +p\sum_{k=1}^{p-2}H_k\ \frac{2^{k+1}}{k+1}\ (\mathrm{mod} p^2).$$

From which:

$$\frac{2^{p-1}-1}{p}\equiv -\frac{1}{2}\sum_{k=0}^{p-2}\frac{2^{k+1}}{k+1}\ +p\sum_{k=1}^{p-2}H_k\ \frac{2^{k}}{k+1}\ (\mathrm{mod} p^2).$$

References

[1] J.W.L. Glaisher, On the residues of the sums of products of the first $p-1$ numbers and their powers to modulus p^2 or p^3, Quart. J. Math. Oxford 31 (1900), $321-353$.

[2] A. Granville, The square of the Fermat quotient, Integers: Electronic J. of Combinatorial Number Theory 4 (2004), #A22 (electronic)

[3] P. Ribenboim, 13 Lectures on Fermat's Last Theorem. Springer-Verlag, New York, 1979.

[4] http://bernard.ronk.free.fr/Quotient_Fermat_V1.pdf

[5] http://bernard.ronk.free.fr/approche_elementaire_v8_3.pdf

[6] Z. W. Sun, Arithmetic theory of harmonic numbers, Proc. Amer. Math. Soc. 140 (2012), 415–428.