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Non-failable approximation method for conditioned distributions

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Abstract

We consider a general method for the approximation of the distribution of a process conditioned to not hit a given set. Existing methods are based on particle system that are failable, in the sense that, in many situations, they are not well defined after a given random time. We present a method based on a new particle system which is always well define. Moreover, we provide sufficient conditions ensuring that the particle method converges uniformly in time. We also show that this method provides an approximation method for the quasi-stationary distribution of Markov processes. Our results are illustrated by their application to a neutron transport model.

Keywords: Particle system; process with absorption; Approximation method for degenerate processes

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1 Introduction

This article is concerned with the approximation of the distribution of Markov processes conditioned to not hit a given absorbing state. Let X be a discrete

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time Markov process evolving in a state space $E \cup \{\partial\}$, where $\partial \notin E$ is an absorbing state, which means that

$$X_n = \partial, \ \forall n \geq \tau_\partial,$$

where $\tau_\partial = \min\{n \geq 0, \ X_n = \partial\}$. Our first aim is to provide an approximation method based on an interacting particle system for the conditional distribution

$$\mathbb{P}_\mu(X_n \in \cdot \mid n < \tau_\partial) = \frac{\mathbb{P}_\mu(X_n \in \cdot \text{ and } n < \tau_\partial)}{\mathbb{P}_\mu(n < \tau_\partial)}, \tag{1}$$

where $\mathbb{P}_\mu$ denotes the law of X with initial distribution μ on E. Our only assumption to achieve our aim will be that survival during a given finite time is possible from any state $x \in E$, which means that

$$\mathbb{P}_x(\tau_\partial > 1) = \mathbb{P}_x(X_1 \in E) > 0, \ \forall x \in E. \tag{2}$$

Our second aim is to provide a general condition ensuring that the approximation method is uniform in time. The main assumption will be that there exist positive constants C and γ such that, for any initial distributions μ_1 and μ_2,

$$\left\| \mathbb{P}_{\mu_1}(X_n \in \cdot \mid n < \tau_\partial) - \mathbb{P}_{\mu_2}(X_n \in \cdot \mid n < \tau_\partial) \right\|_{TV} \leq C e^{-\gamma n}, \ \forall n \geq 0.$$

This property has been extensively studied in [9]. In particular, it is known to imply the existence of a unique quasi-stationary distribution for the process X on E. Another main result of our paper is that, under mild assumptions, the approximation method can be used to estimate this quasi-stationary distribution.

The naïve Monte-Carlo approach to approximate such distributions would be to consider $N \gg 1$ independent interacting particles $X^1, \ldots, X^N$ evolving following the law of X under $\mathbb{P}_\mu$ and to use the following asymptotic relation

$$\mathbb{P}_\mu(X_n \in \cdot \text{ and } n < \tau_\partial) \simeq_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \delta_{X_n^i}(\cdot \cap E)$$

and then

$$\mathbb{P}_\mu(X_n \in \cdot \mid n < \tau_\partial) \simeq_{N \to \infty} \frac{\sum_{i=1}^{N} \delta_{X_n^i}(\cdot \cap E)}{\sum_{i=1}^{N} \delta_{X_n^i}(E)}.$$

However, the number of particles remaining in E typically decreases exponentially fast, so that, at any time $n \geq 0$, the actual number of particles that are used to approximate $\mathbb{P}_\mu(X_n \in \cdot \mid n < \tau_\partial)$ is of order $e^{-\lambda n} N$ for some $\lambda > 0$. As a consequence, the variance of the right hand term typically grows exponentially fast and then the precision of the Monte-Carlo method worsens dramatically over time. In fact, for a finite number of particles N, the number of particles X_n^i belonging to E eventually vanishes in finite time with probability one. Thus the right hand term in the above equation eventually becomes undefined. Since we're typically interested in the long time behavior of (1) or in methods that need to evolve without interruption for a long time, the naïve Monte Carlo method is definitely not well suited to fulfill our objective.

In order to overcome this difficulty, modified Monte-Carlo methods have been introduced in the recent past years by Del Moral for discrete time Markov processes (see for instance [14, 15] or the well documented web page [13], with many applications of such modified Monte-Carlo method). The main idea is to consider independent particles $X_1, \ldots, X_N$ evolving in E following the law of X, but such that, at each time $n \in \mathbb{N}$, any absorbed particle is re-introduced to the position of one other particle, chosen uniformly among those remaining in E; then the particles evolve independently from each others and so on. While this method is powerful, one drawback is that, at some random time T, all the particles will eventually be absorbed simultaneously. At this time, the interacting particle system is stopped and there is no natural way to reintroduce all the particles at time $T+1$. When the number of particles is large and the probability of absorption is uniformly bounded away from zero, the time T is typically very large and this explain the great success of this method. However, many situations does not enter the scope of these assumptions, such as diffusion processes picked at discrete times or the neutron transport approximation (see Section 4). Our method is non-failable in these situations. Moreover the uniform convergence theorem provided in Section 3 also holds in these cases, under suitable assumptions.

When the underlying process is a continuous time process, one alternative to the methods of [14] has been introduced recently. The idea is to consider a continuous time N-particles system, where the particles evolve independently until one (and only one) of them is absorbed. At this time, the unique absorbed particle is re-introduced to the position of one other particle, chosen uniformly among those remaining in E. This continuous time system, introduced by Burdzy, Holyst, Ingermann and March (see for instance [3]), can be used to approximate the distribution of diffusion processes conditioned not to hit a boundary.

Unfortunately, it yields two new difficulties. The first one is that it only works if the number of jumps does not explode in finite time almost surely (which is not always the case even in non-trivial situations, see for instance [2]). The second one is that, when it is implemented numerically, one has to compute the exact absorption time of each particles, which can be cumbersome for diffusion processes and complicated boundaries. Note that, when this difficulties are overcome, the empirical distribution of the process is known to converge to the conditional distribution (see for instance the general result [23] and the particular cases handled in [4, 17, 20, 16, 22, 24, 1]).

Finally, it appears that both methods are not applicable in the generality we aim to achieve in the present paper and, in some cases, both method will fail (as in the case of the neutron transport example of Section 4). Let us now describe the original algorithm studied in the present paper.

Fix $N \geq 2$. The particle system that we introduce is a discrete time Markov process $(X_n^1, \dots, X_n^N)_{n\in\mathbb{N}}$ evolving in E^N. We describe its dynamic between two successive times n and $n+1$, knowing $(X_n^1, \dots, X_n^N) \in E^N$, by considering the following random algorithm which act on any N-uplet of the form

$$y = ((x_1, b_1), \dots, (x_N, b_N)) \in (E \times \{0,1\})^N.$$

Algorithm 1. Initiate y by setting $x_i = X_n^i$ and $b_i = 0$ for all $i \in \{1, \dots, N\}$ and repeat the following steps until $b_i = 1$ for all $i \in \{1, \dots, N\}$.

1. Choose randomly an index i_0 uniformly among $\{i \in \{1, \dots, N\},\ b_i = 0\}$
2. Choose randomly a position $Z \in E$ according to $\mathbb{P}_{x_{i_0}}(X_1 \in \cdot)$. Then
   - If $Z = \partial$, chose an index j_0 among $\{1, \dots, N\} \setminus \{i_0\}$ and replace (x_{i_0}, b_{i_0}) by (x_{j_0}, b_{j_0}) in y.
   - If $Z \neq \partial$, replace (x_{i_0}, b_{i_0}) by $(Z, 1)$ in y.

After a (random) finite number of iterations, the N-uplet y will satisfy $b_i = 1$ for all $i \in \{1, \dots, N\}$. When this is achieved, we set $(X_{n+1}^1, \dots, X_{n+1}^N) = (x_1, \dots, x_N)$.

Our first main result, stated in Section 2, is that, for all $n \geq 0$, the empirical distribution of the particle system evolving following the above dynamic actually converges to the conditional distribution of the original process X at time n. We prove this result by building a continuous time Markov process

$(Y_t)_{t\in[0,+\infty[}$ such that Y_n is distributed as X_n for all entire time $n \in \mathbb{N}$, and such that the general convergence result of [23] applies.

Our second main result, stated in Section 3, shows that, if the conditional distribution of the process X is exponentially mixing (in the sense of [9] or [10] for the time-inhomogeneous setting) and under a non-degeneracy condition that is usually satisfied, then the approximation method converges uniformly in time.

In Section 4, we illustrate our method by proving that it applies to neutron transport process absorbed at the boundary of an open set D.

2 Convergence of fixed time marginals

In this section, we consider the particle system defined by Algorithm 1. We state and prove our main result in a general setting.

Theorem 1. *Assume that $\mu_0^N := \frac{1}{N}\sum_{i=0}^N \delta_{X_0^i}$ converges in law to a probability measure μ_0 on E. Then, for any $n \in \mathbb{N}$ and any bounded continuous function $f : E \to \mathbb{R}$,*

$$\frac{1}{N}\sum_{i=1}^N f(X_n^i) \xrightarrow[N\to\infty]{law} \mathbb{E}_{\mu_0}(f(X_n)|n < \tau_\partial).$$

Moreover,

$$\mathbb{E}\left|\frac{1}{N}\sum_{i=1}^N f(X_n^i) - \mathbb{E}_{\mu_0^N}(f(X_n)|n<\tau_\partial)\right| \leq \frac{2(1+\sqrt{2})\|f\|_\infty}{\sqrt{N}}\mathbb{E}\left(\frac{1}{\mathbb{P}_{\mu_0^N}(n<\tau_\partial)}\right) \tag{3}$$

We emphasize that our result applies to any process X satisfying (2), overcoming the limitations of all previously cited particle approximation methods, as illustrated by the application to a neutron transport process in Section 4.

Proof of Theorem 1. The proof is divided in two steps. First, we provide an implementation of Algorithm 1 as the discrete time included chain of a continuous time Fleming-Viot type particle system. In particular, this step provides a mathematically tractable implementation of Algorithm 1. In a second step, we use existing results on Fleming-Viot type particle systems to deduce that the empirical distribution of the particle system converges to the conditional distribution (1).

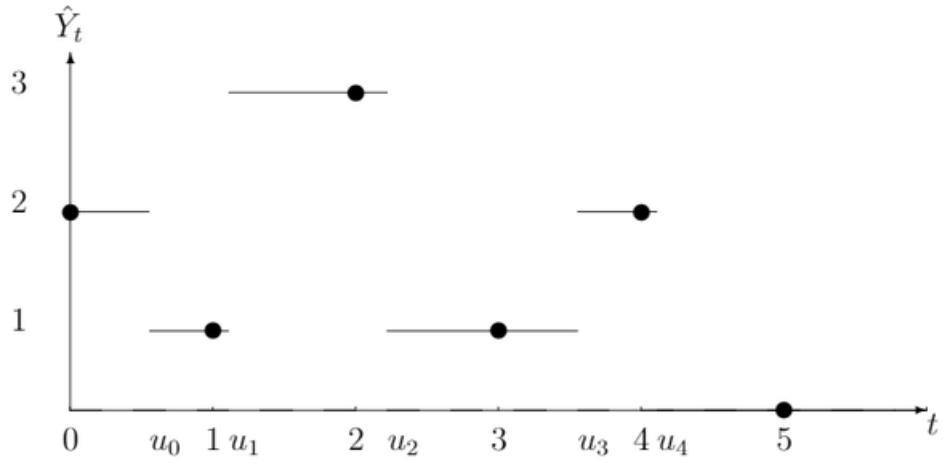


Figure 1: The points represent the values taken by the Markov chain X and the thin lines are the trajectory of Y, which jumps from X_n to X_{n+1} at time u_n. At any time $t \in [n, n+1[$, B_t is equal to 1 if the process has already jumped during any interval $[n, n+1]$ and is equal to 0 otherwise.

Step 1 : Algorithm 1 as a Fleming-Viot type process
Let us introduce the continuous time process $(Y_t)_{t\in[0,+\infty[}$ defined, for any $t \in [0,+\infty[$, by

$$Y_t = X_{[t]} 1_{t<u_{[t]}} + X_{[t]+1} 1_{t\geq u_{[t]}}$$

where $[\cdot]$ denotes the integer part and $(u_n)_{n\in\mathbb{N}}$ is a family of independent random variables such that, for all $n \in \mathbb{N}$, u_n follows a uniform law on $[n, n+1]$. With this definition, $(Y_t)_{t\geq 0}$ is a non-Markovian continuous time process such that Y_n and X_n have the same law for all $n \in \mathbb{N}$.

Now, we define the continuous time process $(B_t)_{t\geq 0}$ by

$$B_t = \sum_{n\in\mathbb{N}} 1_{t\in[u_n,n+1[}, \quad \forall\, t \geq 0.$$

By construction, the continuous-time process $(Z_t)_{t\in[0,+\infty[}$ defined by

$$Z_t = (t, Y_t, B_t),\ \forall\, t \in [0,+\infty[,$$

is a strong Markov process evolving in $F = \mathbb{R}_+ \times (E \cup \{\partial\}) \times \{0,1\}$, with absorbing set $\partial_F := \mathbb{R}_+ \times \{\partial\} \times \{0,1\}$ (see Figure 1 for an illustration when $E = \mathbb{N}$ and $\partial = 0$).

Let us now define a Fleming-Viot type system whose particles evolve as independent copies of Z between their absorption times. More precisely, fix $N \geq 2$ and consider the following continuous time Fleming-Viot type particle system, denoted by $(Z^i)_{i\in\{1,\dots,N\}}$, starting from $(z_1,\dots,z_N) \in F^N$ and evolving as follows.

- The N particles evolve as N independent copies of Z until one of them reaches ∂_F. Note that it is clear from the definition of Z that only one particle jumps at this time.
- Then the unique killed particle is taken from the absorbing point ∂_F and is instantaneously placed at the position of an other particle chosen uniformly between the $N-1$ remaining ones; in this situation we say that the particle undergoes a *rebirth*.
- Then the particles evolve as independent copies of Z until one of them reaches ∂_F and so on.

For all $t \in [0,+\infty[$, we denote by $A_t^{i,N}$ the number of rebirths of the i^{th} particle occurring before time t and by A_t^N the total number of rebirths before the time t. Clearly,

$$A_t^N = \sum_{i=1}^N A_t^{i,N} \text{ almost surely.}$$

Also, for all $t \in [0,+\infty[$, we set $Z_t^i = (t, Y_t^i, B_t^i)$, where Y_t^i and B_t^i are the marginal component of Z^i in E and $\{0,1\}$ respectively.

One can easily check that this Fleming-Viot system (considered at discrete times) is a particular implementation of the informal description of Algorithm 1 in the introduction. Indeed, at any time $n \geq 0$, the Fleming-Viot system is defined so that

$$(Y_n^i, B_n^i) = (Y_n^i, 0).$$

Then, at each time $t \in [n, n+1[$, the index of the next moving particle $i \in \{1,\dots,n\}$ belongs to the set of particles $j \in \{1,\dots,N\}$ such that $B_n^j = 0$. Moreover, conditionally to $B_t^j = 0$, the jumping times of these particles are independent and identically distributed (uniformly on $[t, n+1[$). As a consequence i is chosen uniformly among these indexes (this is the first step of Algorithm 1). Then, at the jumping time τ, the position Y_τ^i of the particle i is chosen according to

$\mathbb{P}_{Y_t^i}(X_1 \in \cdot)$ and B_τ^i is set to 1. If the position at time τ is ∂, then the particle i undergoes a rebirth and hence (Y_τ^i, B_τ^i) is replaced by $(Y_\tau^k, B_\tau^k) = (Y_t^k, B_t^k)$, where k is chosen uniformly among $\{1,\dots,N\}\setminus\{i\}$. Hence the second step of Algorithm 1 is completed. Finally, the procedure is repeated until all the marginals B^i are equal to 1, as in Algorithm 1.

In particular, for any $n \geq 0$, the random variable $(X_n^1,\dots,X_n^N)$ obtained from Algorithm 1 and the variable $(Y_n^1,\dots,Y_n^N)$ obtained from the Fleming-Viot type algorithm have the same law.

Step 2 : Convergence of the empirical system.

In this step, we consider a sequence of initial positions $(Y_0^1,\dots,Y_N^N)_{N\geq 2}$ such that $\frac{1}{N}\sum_{i=0}^N \delta_{Y_0^i}$ converges in law to a probability measure μ_0 on E. Our aim is to prove that, for any $n \in \mathbb{N}$ and any bounded continuous function $f : E \to \mathbb{R}$,

$$\frac{1}{N}\sum_{i=1}^N f(Y_n^i) \xrightarrow[N\to\infty]{law} \mathbb{E}_{\mu_0}(f(Y_n)|n < \tau_\partial).$$

Note that, since X_n and Y_n share the same law, this immediately implies the first part of Theorem 1.

Since $(Z^1,\dots,Z^N)$ is a Fleming-Viot type process without simultaneous killings, [23, Theorem 2.2] implies that it is sufficient to prove that, for all $N \geq 2$ and almost surely,

$$A_t^N < \infty,\ \forall\, t \geq 0, \tag{4}$$

where we recall that A_t^N is number of rebirths undergone by the Fleming-Viot type system with N particles before time t.

First, let us remark that

$$\mathbb{P}(A_t^N = \infty) \leqslant \mathbb{P}(A_{[t]+1}^N = \infty).$$

Moreover,

$$\mathbb{P}(A_{[t]+1}^N = \infty) = \mathbb{P}\left(\bigcup_{i=0}^{[t]}\{A_{i+1}^N - A_i^N = \infty\}\right) \leqslant \sum_{i=0}^{[t]} \mathbb{P}\left(A_{i+1}^N - A_i^N = \infty\right).$$

Using the weak Markov property at time i, it is sufficient to prove that

$$\mathbb{P}(A_1^N = \infty) = 0 \tag{5}$$

for any initial distribution of the Fleming-Viot type process $(Y^1,\dots,Y^N)$ in order to conclude that

$$\mathbb{P}\left(A^N_{i+1}-A^N_i=\infty\right)=0,\ \forall i\geq 0.$$

and hence that (4) holds true.

But $A^N_1=\infty$ if and only if there exists at least one particle for which there is an infinity of rebirths, hence

$$\mathbb{P}(A^N_t=\infty)\leqslant\sum_{i=1}^{N}\mathbb{P}\left(A^{i,N}_1=\infty\right) \tag{6}$$

Now, when a particle undergoes a rebirth, it jumps on the position of one of the $N-1$ remaining particles. As a consequence, at any time $t\in[0,1[$, the position Y^i_t of the particles $i\in\{1,\dots,N\}$ such that $B^i_t=0$ are included in the set $\{Y^1_0,\dots,Y^N_0\}$. In particular, the probability that such a particle undergoes a rebirth during its next move is bounded above by

$$c:=\max_{i\in\{1,\dots,N\}}\mathbb{P}_{Y^i_0}(X_1=\partial)<1.$$

Hence, a classical renewal argument shows that the probability that a particle undergoes n rebirths is bounded above by c^n. This implies that the probability that a particle undergoes an infinity of rebirths is zero. This, together with (6) implies (5), which concludes the proof of the first part of Theorem 1.

In order to conclude the proof, let us simply remark that, for a deterministic value of μ^N_0, the inequality of Theorem 1 is directly provided by [23, Theorem 2.2]. Now, if μ^N_0 is a random measure, the inequality is obtained by integrating the deterministic case inequality with respect to the law of μ^N_0. This concludes the proof of Theorem 1.

□

3 Uniform convergence for uniformly mixing conditioned semi-groups

In a recent paper [9], necessary and sufficient conditions on an absorbed Markov process X were obtained to ensure that a process satisfies

$$\left\|\mathbb{P}_{\mu_1}(X_n\in\cdot\mid n<\tau_\partial)-\mathbb{P}_{\mu_2}(X_n\in\cdot\mid n<\tau_\partial)\right\|_{TV}\leq Ce^{-\gamma n}, \tag{7}$$

where γ and C are positive constants. In particular this implies the existence of a unique quasi-stationary distribution ν_{QSD} for X, that is a unique probability measure on E such that $\mathbb{P}_{\nu_{QSD}}(X_n \in \cdot \mid n < \tau_\partial) = \nu_{QSD}$, for all $n \geq 0$. General and classical results on quasi-stationary distributions (see for instance [18, 21, 11]) implies that there exists $\lambda_0 > 0$ such that

$$\mathbb{P}_{\nu_{QSD}}(n < \tau_\partial) = e^{-\lambda_0 n}, \ \forall n \geq 0. \tag{8}$$

The exponential convergence property 7 holds for a large class of processes, including birth and death processes with catastrophe, branching Brownian particles, neutron transport approximations processes (see [9]), one dimensional diffusion with or without killing (see [8, 6]), multi-dimensional birth and death processes (see [7]) and multi-dimensional diffusion processes (see [5]). Also, similar properties can be proved for time-inhomogeneous processes, as stressed in the recent paper [10], with applications to time-inhomogeneous diffusion processes and birth and death processes in a quenched random environment.

In this section, we state and prove our second main result, which states that, if (7) holds and if, for any $n_0 \geq 1$, there exists $\varepsilon_0 > 0$ such that

$$\inf_{x \in E} \mathbb{P}_x(\mathbb{P}_{X_1}(n_0 < \tau_\partial) \geq \varepsilon_0 \mid 1 < \tau_\partial) = c_0 > 0, \tag{9}$$

then the convergence of the empirical distribution of the particle system described in Algorithm 1 converges uniformly in time to the conditional distribution of the process X.

We emphasize that the additional assumption (9) is true for many processes satisfying (7), for instance in the case of one-dimensional diffusion processes, multidimensional diffusion processes or piecewise deterministic Markov processes (this the detailed examples of Section 4). As far as we know, none of this processes were covered in this generality by previous methods. In particular, this is the first method that allows the approximation of the conditional distribution of the neutron transport approximation process (see Section 4), since in this case it easy to check that, with probability one, all the particles will eventually hit the boundary at the same time when using previous algorithms. The methods also allows to handle the case of the diffusion process on $E = (0,2]$ killed at 0, reflected at 2 and solution to the following stochastic differential equation

$$dX_t = dW_t + \frac{1}{\beta X_t^{\beta-1}}, \ X_0 \in (0,2], \ \beta > 2.$$

In this case, the continuous time Fleming-Viot approximation method introduced in [3] explodes in finite time almost surely, as proved in [2]. As a matter of fact, it is not known if the Fleming-Viot type particle system is well defined as soon as the diffusion coefficient is degenerated or not regular toward the boundary 0. On the contrary, our assumption holds true for fairly general one dimensional diffusion processes, thanks to the study provided in [8]. Hence our approximation method is valid and, using the next results, converges uniformly in time for both neutron transport processes and degenerate diffusion processes.

For any $n \in \mathbb{N}$, we define the empirical distribution of the process at time n as $\mu_n^N = \frac{1}{N}\sum_{i=1}^{N} \delta_{X_n^i}$, and for any bounded measurable function f on E, we set

$$\mu_n^N(f) = \frac{1}{N}\sum_{i=1}^{N} f(X_n^i).$$

Theorem 2. *Assume that* (7) *and* (9) *holds true. Then there exist two constants* $C > 0$ *and* $\alpha < 0$*, such that, for all* $\delta > 0$ *and all measurable function* $f : E \to \mathbb{R}$ *bounded by* 1*,*

$$\mathbb{E}\left(\left|\mu_n^N(f) - \mathbb{E}_{\mu_0^N}(f(X_n) \mid n < \tau_\partial)\right|\right) \leq \frac{CN^\alpha}{\delta} + C\,\mathbb{P}\left(\mathbb{P}_{\mu_0^N}(n_0 < \tau_\partial) \leq \varepsilon_0\delta\right),\ \forall n \geq 0,$$

with

$$\alpha = \frac{-\gamma}{2\left(\lambda_0 + \gamma\right)} < 0.$$

In the case where the initial position of the particle system are drawn as independent random variables distributed following the same law μ, then, choosing $\delta > 0$ small enough so that $\mathbb{P}_\mu(n_0 < \tau_\partial) \geq 2\varepsilon_0\delta$, basic concentration inequalities and the equality $\mathbb{E}(\mathbb{P}_{X_0^i}(n_0 < \tau_\partial)) = \mathbb{P}_\mu(n_0 < \tau_\partial)$ imply that

$$\begin{aligned}\mathbb{P}\left(\mathbb{P}_{\mu_0^N}(n_0 < \tau_\partial) \leq \varepsilon_0\delta\right) &\leq \mathbb{P}\left(\mathbb{P}_\mu(n_0 < \tau_\partial) - \frac{1}{N}\sum_{i=1}^{N}\mathbb{P}_{X_0^i}(n_0 < \tau_\partial) \geq \varepsilon_0\delta\right)\\ &\leq e^{-N\beta},\end{aligned}$$

for some $\beta > 0$. This implies the following corollary.

Corollary 1. *Assume that* $(X_0^1,\dots,X_0^N)$ *are independent and identically distributed following a given law* μ *on* E. *If* (7) *and* (9) *hold true, then there exist two constants* $C_\mu > 0$ *and* $\alpha < 0$ *such that, and all measurable function* $f : E \to \mathbb{R}$ *bounded by* 1,

$$\mathbb{E}\left(\left|\mu_n^N(f) - \mathbb{E}_{\mu_0^N}(f(X_n) \mid n < \tau_\partial)\right|\right) \leq C_\mu N^\alpha,\ \forall n \geq 0,$$

where α *is the same constant as in Theorem 2.*

We emphasize that the above results and their proofs can be adapted to the time-inhomogeneous setting of [10], with appropriate modifications of Assumption (9).

The following result is specific to the time-homogeneous setting and is proved at the end of this section.

Theorem 3. *Under the assumptions of Theorem 2, the particle system* $(X^1,\dots,X^N)$ *is exponentially ergodic, which means that it admits a stationary distribution* M^N *(which is a probability measure on* E^N*) and that there exists positive constants* C_N *and* γ_N *such that*

$$\left\| M^N - Law(X_n^1,\dots,X_n^N) \right\|_{TV} \leq C_N e^{-\gamma_N t}.$$

Moreover, there exists a positive constant $C > 0$ *such that, for all measurable function* $f : E \to \mathbb{R}$ *bounded by* 1,

$$\mathbb{E}_{M^N}\left(\left|\mu_0^N(f) - \nu_{QSD}(f)\right|\right) \leq CN^\alpha, \tag{10}$$

where α *is the same as in Theorem 2 and* $(X_0^1,\dots,X_0^N)$ *is distributed following* M^N.

Proof of Theorem 2. Using the exponential convergence assumption (7), we deduce that, for any function $f : E \to \mathbb{R}$ such that $\|f\|_\infty = 1$ and all $n \geq n_1 \geq 0$,

$$\begin{aligned}
\mathbb{E}\left(\left|\mu_n^N(f) - \mathbb{E}_{\mu_0^N}(f(X_n) \mid n < \tau_\partial)\right|\right) &\leq \mathbb{E}\left(\left|\mu_n^N(f) - \mathbb{E}_{\mu_{n-n_1}^N}(f(X_n) \mid n < \tau_\partial)\right|\right) \\
&\quad + \mathbb{E}\left(\left|\mathbb{E}_{\mu_{n-n_1}^N}(f(X_n) \mid n < \tau_\partial) - \mathbb{E}_{\mu_0^N}(f(X_n) \mid n < \tau_\partial)\right|\right) \\
&\leq \mathbb{E}\left(\left|\mu_n^N(f) - \mathbb{E}_{\mu_{n-n_1}^N}(f(X_n) \mid n < \tau_\partial)\right|\right) + Ce^{-\gamma n_1}.
\end{aligned} \tag{11}$$

Denoting by $(\mathscr{F}_n)_{n\in\mathbb{N}}$ the natural filtration of the particle system $(X_n^1,\dots,X_n^N)_{n\in\mathbb{N}}$, we deduce from Theorem 1 that, almost surely,

$$\mathbb{E}\left(\left|\mu_n^N(f)-\mathbb{E}_{\mu_{n-n_1}^N}(f(X_n)\mid n<\tau_\partial)\right| \mid \mathscr{F}_{n-n_1}\right)\le 2\wedge\frac{2(1+\sqrt{2})}{\mathbb{P}_{\mu_{n-n_1}^N}(n_1<\tau_\partial)\sqrt{N}}.$$

Hence, for all $\varepsilon>0$,

$$\mathbb{E}\left(\left|\mu_n^N(f)-\mathbb{E}_{\mu_{n-n_1}^N}(f(X_n)\mid n<\tau_\partial)\right|\right)\le 2\mathbb{P}\left(\mathbb{P}_{\mu_{n-n_1}^N}(n_1<\tau_\partial)\le\varepsilon\right)+\frac{2(1+\sqrt{2})}{\varepsilon\sqrt{N}}\mathbb{P}\left(\mathbb{P}_{\mu_{n-n_1}^N}(n_1<\tau_\partial)>\varepsilon\right).$$

But [9, Theorem 2.1] entails the existence of a measure ν on E and positive constants $c_1,n_0>0$ and $c_2>0$ such that for any $n\in\mathbb{N}$ and $x\in E$,

$$\begin{cases}\mathbb{P}_x(X_{n_0}\in\cdot\mid n_0<\tau_\partial)\ge c_1\nu(\cdot)\\ \mathbb{P}_\nu(n<\tau_\partial)\ge c_2\mathbb{P}_x(n<\tau_\partial)\end{cases}$$

Note that, from now on, n_0 is a fixed constant. This entails

$$\begin{aligned}\mathbb{P}_\mu(n_1<\tau_\partial)&\ge\mathbb{P}_\mu(n_0<\tau_\partial)c_1\mathbb{P}_\nu(n_1-n_0<\tau_\partial)\\&\ge\mathbb{P}_\mu(n_0<\tau_\partial)c_1c_2\max_{x\in E}\mathbb{P}_x(n_1-n_0<\tau_\partial)\\&\ge c_1c_2\mathbb{P}_\mu(n_0<\tau_\partial)e^{-\lambda_0(n_1-n_0)},\end{aligned}$$

where $\lambda_0>0$ is the constant of (8). We deduce that

$$\mathbb{E}\left(\left|\mu_n^N(f)-\mathbb{E}_{\mu_{n-n_1}^N}(f(X_n)\mid n<\tau_\partial)\right|\right)\le 2\mathbb{P}\left(\mathbb{P}_{\mu_{n-n_1}^N}(n_0<\tau_\partial)\le c_3\varepsilon e^{(n_1-n_0)\lambda_0}\right)+\frac{2(1+\sqrt{2})}{\varepsilon\sqrt{N}}. \quad (12)$$

with $c_3=\frac{1}{c_1c_2}$. Our aim is now to control $\mathbb{P}\left(\mathbb{P}_{\mu_n^N}(n_0<\tau_\partial)\le\varepsilon\right)$, uniformly in $n\ge 0$ and for all $\varepsilon\in(0,\varepsilon_0/2)$. In order to do so, we make use of the following lemma, proved at the end of this subsection.

Lemma 1. *There exists $p_0>0$ and $\delta>0$ such that, for any value of μ_0^N,*

$$\mathbb{P}\left(\mu_1^N(\mathbb{P}_\cdot(n_0<\tau_\partial))\le\delta\varepsilon_0\right)\le 1-p_0.$$

Moreover, if $\mathbb{P}_{\mu_0^N}(1<\tau_\partial)>\varepsilon$ for some $\varepsilon\in(0,c_0\varepsilon_0)$, then

$$\mathbb{P}\left(\mu_1^N(\mathbb{P}_\cdot(n_0<\tau_\partial))\le\varepsilon\right)\le\frac{2(1+\sqrt{2})}{\varepsilon(c_0\varepsilon_0-\varepsilon)\sqrt{N}}.$$

From this lemma (where we assume without loss of generality that $\delta \leq 1/2$), from the Markov property applied to the particle system and since $\mathbb{P}_{\mu_0^N}(n_0 < \tau_\partial) > \varepsilon$ implies $\mathbb{P}_{\mu_0^N}(1 < \tau_\partial) > \varepsilon$, we deduce that, for any $\varepsilon \in (0, \delta\varepsilon_0)$,

$$\begin{aligned}
\mathbb{P}\left(\mathbb{P}_{\mu_{n+1}^N}(n_0 < \tau_\partial) \leq \varepsilon\right) &\leq \frac{2(1+\sqrt{2})}{\varepsilon(\varepsilon_0/2-\varepsilon)\sqrt{N}}\,\mathbb{P}\left(\mathbb{P}_{\mu_n^N}(n_0 < \tau_\partial) > \varepsilon\right) + (1-p_0)\mathbb{P}\left(\mathbb{P}_{\mu_n^N}(n_0 < \tau_\partial) \leq \varepsilon\right) \\
&\leq \frac{2(1+\sqrt{2})}{\varepsilon(\varepsilon_0/2-\varepsilon)\sqrt{N}} + (1-p_0)\mathbb{P}\left(\mathbb{P}_{\mu_n^N}(n_0 < \tau_\partial) \leq \varepsilon\right) \\
&\leq \frac{2(1+\sqrt{2})}{p_0\varepsilon(\varepsilon_0/2-\varepsilon)\sqrt{N}} + (1-p_0)^{n+1}\mathbb{P}\left(\mathbb{P}_{\mu_0^N}(n_0 < \tau_\partial) \leq \varepsilon\right),
\end{aligned} \tag{13}$$

where the last line is obtained by iteration over n.

This and equation (12) imply that, for any $\varepsilon \in \left(0, \varepsilon_0\delta/(c_3 e^{(n_1-n_0)\lambda_0})\right)$,

$$\begin{aligned}
\mathbb{E}\left(\left|\mu_n^N(f) - \mathbb{E}_{\mu_{n-n_1}^N}(f(X_n) \mid n < \tau_\partial)\right|\right) \leq &\frac{4(1+\sqrt{2})}{p_0 c_3 \varepsilon e^{(n_1-n_0)\lambda_0}(\varepsilon_0/2 - c_3\varepsilon e^{(n_1-n_0)\lambda_0})\sqrt{N}} \\
&+ 2(1+\sqrt{2})(1-p_0)^{n-n_1}\mathbb{P}\left(\mathbb{P}_{\mu_0^N}(n_0 < \tau_\partial) \leq c_3\varepsilon e^{(n_1-n_0)\lambda_0}\right) \\
&+ \frac{2(1+\sqrt{2})}{\varepsilon\sqrt{N}}.
\end{aligned}$$

Taking $\varepsilon = \varepsilon_0\delta/(c_3 e^{(n_1-n_0)\lambda_0})$ and assuming, without loss of generality, that $\delta \leq 1/4$, we obtain

$$\begin{aligned}
\mathbb{E}\left(\left|\mu_n^N(f) - \mathbb{E}_{\mu_{n-n_1}^N}(f(X_n) \mid n < \tau_\partial)\right|\right) \leq &\frac{16(1+\sqrt{2})}{p_0\varepsilon_0^2\delta\sqrt{N}} \\
&+ 2(1+\sqrt{2})(1-p_0)^{n-n_1}\mathbb{P}\left(\mathbb{P}_{\mu_0^N}(n_0 < \tau_\partial) \leq \varepsilon_0\delta\right) \\
&+ \frac{2(1+\sqrt{2})c_2 e^{n_1\lambda_0}}{\varepsilon_0\delta\sqrt{N}}.
\end{aligned}$$

Finally, using inequality (11) and taking

$$n_1 = \left\lfloor \frac{\ln N}{2\left(\lambda_0+\gamma\right)} \right\rfloor,$$

straightforward computations implies the existence of a constant $C > 0$ such that, for all $n \geq n_1$,

$$\mathbb{E}\left(\left|\mu_n^N(f) - \mathbb{E}_{\mu_0^N}(f(X_n) \mid n < \tau_\partial)\right|\right) \leq \frac{CN^\alpha}{\delta} + C\mathbb{P}\left(\mathbb{P}_{\mu_0^N}(n_0 < \tau_\partial) \leq \varepsilon_0\delta\right)$$

with

$$\alpha = \frac{-\gamma}{2\left(\lambda_0+\gamma\right)} < 0.$$

Now, for $n \le n_1$, we have

$$\begin{aligned}
\mathbb{E}\left(\left|\mu_n^N(f) - \mathbb{E}_{\mu_0^N}(f(X_n) \mid n < \tau_\partial)\right|\right) &\le \mathbb{E}\left(2 \wedge \frac{2(1+\sqrt{2})}{\mathbb{P}_{\mu_0^N}(n_1 < \tau_\partial)\sqrt{N}}\right) \\
&\le \mathbb{E}\left(2 \wedge \frac{2(1+\sqrt{2})e^{n_1\lambda_0}}{c_3\mathbb{P}_{\mu_0^N}(n_0 < \tau_\partial)\sqrt{N}}\right) \\
&\le 2\mathbb{P}\left(\mathbb{P}_{\mu_0^N}(n_0 < \tau_\partial) \le \varepsilon_0\delta\right) + \frac{2(1+\sqrt{2})e^{n_1\lambda_0}}{\varepsilon_0\delta\sqrt{N}}.
\end{aligned}$$

Using the same computations as above, this concludes the proof of Theorem 2. □

Proof of Lemma 1. Assume that $\mathbb{P}_{\mu_0^N}(1 < \tau_\partial) > \varepsilon$. We obtain from Theorem 1 that

$$\mathbb{E}\left(\left|\mu_1^N(\mathbb{P}_\cdot(n_0 < \tau_\partial)) - \mathbb{E}_{\mu_0^N}\left(\mathbb{P}_{X_1}(n_0 < \tau_\partial) \mid 1 < \tau_\partial\right)\right|\right) \le \frac{2(1+\sqrt{2})\|\mathbb{P}_\cdot(n_0 < \tau_\partial)\|_\infty}{\varepsilon\sqrt{N}}.$$

Markov's inequality thus implies that, for all $\delta > 0$,

$$\mathbb{P}\left(\left|\mu_1^N(\mathbb{P}_\cdot(n_0 < \tau_\partial)) - \mathbb{E}_{\mu_0^N}\left(\mathbb{P}_{X_1}(n_0 < \tau_\partial) \mid 1 < \tau_\partial\right)\right| \ge \delta\right) \le \frac{2(1+\sqrt{2})}{\varepsilon\delta\sqrt{N}}$$

and hence that

$$\mathbb{P}\left(\mu_1^N(\mathbb{P}_\cdot(n_0 < \tau_\partial)) \le \mathbb{E}_{\mu_0^N}\left(\mathbb{P}_{X_1}(n_0 < \tau_\partial) \mid 1 < \tau_\partial\right) - \delta\right) \le \frac{2(1+\sqrt{2})}{\varepsilon\delta\sqrt{N}}.$$

But, by Assumption (9), we have $\mathbb{P}_{\mu_0^N}\left(\mathbb{P}_{X_1}(n_0 < \tau_\partial) \ge \varepsilon_0 \mid 1 < \tau_\partial\right) \ge c_0$, so that $\mathbb{E}_{\mu_0^N}\left(\mathbb{P}_{X_1}(n_0 < \tau_\partial) \mid 1 < \tau_\partial\right) \ge c_0\varepsilon_0$. We deduce that

$$\mathbb{P}\left(\mu_1^N(\mathbb{P}_\cdot(n_0 < \tau_\partial)) \le \varepsilon_0 c_0 - \delta\right) \le \frac{2(1+\sqrt{2})}{\varepsilon\delta\sqrt{N}}.$$

Choosing $\delta = c_0\varepsilon_0 - \varepsilon$, we finally obtained

$$\mathbb{P}\left(\mu_1^N(\mathbb{P}_\cdot(n_0 < \tau_\partial)) \leq \varepsilon\right) \leq \frac{2(1+\sqrt{2})}{\varepsilon(c_0\varepsilon_0 - \varepsilon)\sqrt{N}}.$$

In the general case (when one does not have a good control on $\mathbb{P}_{\mu_0^N}(1 < \tau_\partial)$), the above strategy is bound to fail since we do not have a good control on the distance between the conditioned semi-group and the empirical distribution of the particle system. As a consequence, we need to take a closer look at Algorithm 1. As explained in the description of this algorithm, the position of the system at time 1 is computed from the position of the system at time 0 through several steps, each step being composed of two stages. We denote by $\bar{k}$ the number of steps needed to compute the position of the system at time 1.

For any step $k \geq 1$, we denote by $X_0^{i,k}$ the position of the i^{th} particle at the beginning of step k, by b_i^k the state of the i^{th} particle at the beginning of step k, and by $i_k \in \{i \in \{1,\dots,N\},\, b_i^k = 0\}$ the index of the particle chosen during the first stage of step k. With this notation, the process $\left((X_0^{i,k\wedge\bar{k}}, b_i^{k\wedge\bar{k}})_{i\in\{1,\dots,N\}}, i_{k\wedge\bar{k}}\right)_{k\in\mathbb{N}}$ is a Markov chain. In what follows, we denote by $(\mathscr{G}_k)_{k\geq 1}$ the natural filtration of this Markov chain.

We also introduce the quantities

$$\begin{aligned}
N_k &= \sharp\{i \in \{1,\dots,N\},\, b_i^k = 1 \text{ (at the beginning of the } k^{th} \text{ step)}\},\\
N'_k &= \sharp\{i \in \{1,\dots,N\},\, b_i^k = 1 \text{ and } \mathbb{P}_{X_0^{i,k}}(n_0 < \tau_\partial) \geq \varepsilon_0\},\\
N''_k &= \sharp\{i \in \{1,\dots,N\},\, b_i^k = 1 \text{ and } \mathbb{P}_{X_0^{i,k}}(n_0 < \tau_\partial) < \varepsilon_0\}.
\end{aligned}$$

Of course, we have $N_k = N'_k + N''_k$ and, at the beginning of the first step, one has $N_1 = N'_1 = N''_1 = 0$. For any $k \geq 1$, conditionally to $\mathscr{G}_k$ and on the event $k \leq \bar{k}$, the position of $X_0^{i_k,k+1}$ is chosen with respect to

$$\mathbb{P}_{X_0^{i_k,k}}(X_1 \in \cdot) + \mathbb{P}_{X_0^{i_k,k}}(\tau_\partial \leq 1)\frac{1}{N-1}\sum_{i=1, i\neq i_k}^{N} \delta_{X^{i,k}}.$$

Hence, conditionally to $\mathscr{G}_k$ and on the event $k \le \bar{k}$, (N'_{k+1}, N''_{k+1}) is equal to

$$\begin{cases} (N'_k, N''_k) & \text{with prob. } \mathbb{P}_{X_0^{i_k,k}}(\tau_\partial \le 1)\frac{N-1-N_k}{N-1}, \\ (N'_k+1, N''_k) & \text{with prob. } \mathbb{P}_{X_0^{i_k,k}}\left(\mathbb{P}_{X_1}(n_0 < \tau_\partial) \ge \varepsilon_0\right) + \mathbb{P}_{X_0^{i_k,k}}(\tau_\partial \le 1)\frac{N'_k}{N-1}, \\ (N'_k, N''_k+1) & \text{with prob. } \mathbb{P}_{X_0^{i_k,k}}\left(0 < \mathbb{P}_{X_1}(n_0 < \tau_\partial) < \varepsilon_0\right) + \mathbb{P}_{X_0^{i_k,k}}(\tau_\partial \le 1)\frac{N''_k}{N-1}. \end{cases} \tag{14}$$

From Assumption (9), we deduce that, conditionally to $\mathscr{G}_k$ and on the event $k \le \bar{k}$, (N'_{k+1}, N''_{k+1}) is equal to

$$\begin{cases} (N'_k, N''_k) & \text{with prob. } \mathbb{P}_{X_0^{i_k,k}}(\tau_\partial \le 1)\frac{N-1-N_k}{N-1}, \\ (N'_k+1, N''_k) & \text{with prob. } \ge c_0\mathbb{P}_{X_0^{i_k,k}}(1 < \tau_\partial) + \mathbb{P}_{X_0^{i_k,k}}(\tau_\partial \le 1)\frac{N'_k}{N-1}, \\ (N'_k, N''_k+1) & \text{with prob. } \le (1-c_0)\mathbb{P}_{X_0^{i_k,k}}(1 < \tau_\partial) + \mathbb{P}_{X_0^{i_k,k}}(\tau_\partial \le 1)\frac{N''_k}{N-1}. \end{cases} \tag{15}$$

Let us denote by $k_1, \dots, k_n, \dots$ the successive step numbers during which the sequence $(N_k)_{k\ge1}$ jumps, that is

$$\begin{aligned} k_1 &= \inf\{k \ge 1,\, N_{k+1} \ne N_k\}, \\ k_{n+1} &= \inf\{k \ge k_n + 1,\, N_{k+1} \ne N_k\},\ \forall n \ge 1. \end{aligned}$$

It is clear that, for all $n \ge 1$, $k_n + 1$ is a stopping time with respect to the filtration $\mathscr{G}$. We are interested in the sequence of random variables $(U_n)_{n\ge1}$, defined by

$$U_n = N'_{k_n+1},\ \forall n \in \{1, \dots, N\}.$$

Conditionally to $\mathscr{G}_{k_n+1}$ (the filtration $(\mathscr{G}_k)_{k\ge1}$ before the stopping time $k_n + 1$), we deduce from (14) that, for all $x \in E$,

$$\begin{aligned} \mathbb{P}(U_{n+1} = U_n + 1 \mid \mathscr{G}_{k_n+1}, X_0^{i_{k_{n+1}},k_{n+1}} = x) &\ge \frac{\mathbb{P}_x(1 < \tau_\partial)c_0 + \mathbb{P}_x(\tau_\partial \le 1)\frac{N'_{k_n+1}}{N-1}}{\mathbb{P}_x(1 < \tau_\partial) + \mathbb{P}_x(\tau_\partial \le 1)\frac{N_{k_n+1}}{N-1}} \\ &\ge c_0 \wedge \frac{N'_{k_n+1}}{N_{k_n+1}} = c_0 \wedge \frac{U_n}{n}, \end{aligned}$$

since $N_{k_n+1} = n$ almost surely. We deduce that

$$\mathbb{P}(U_{n+1} = U_n + 1 \mid U_1, \dots, U_n) \geq c_0 \wedge \frac{U_n}{n}$$
$$\mathbb{P}(U_{n+1} = U_n \mid U_1, \dots, U_n) = 1 - \mathbb{P}(U_{n+1} = U_n + 1 \mid U_1, \dots, U_n) \leq 1 - c_0 \wedge \frac{U_n}{n}.$$

Moreover, Assumption (9) entails that $\mathbb{P}(U_1 = 1) \geq c_0 > 0$.

As a consequence, there exists a coupling between $(U_n)_{n\in\mathbb{N}}$ and the Markov chain $(V_n)_{n\geq 1}$ with initial law $\mathbb{P}(V_1 = 1) = c_0$ and transition probabilities

$$\mathbb{P}(V_{n+1} = V_n + 1 \mid V_1, \dots, V_n) = c_0 \wedge \frac{V_n}{n},$$
$$\mathbb{P}(V_{n+1} = V_n \mid V_1, \dots, V_n) = 1 - \mathbb{P}(V_{n+1} = V_n + 1 \mid V_1, \dots, V_n)$$

such that $U_n \geq V_n$, for all $n \in \{1, \dots, N\}$. The process $(V_n/n)_{n\in\mathbb{N}}$ is a positive super-martingale (and a martingale if $c_0 = 1$) and hence it converges to a random variable R_∞ almost surely as $n \to \infty$. Let us now prove that R_∞ is not equal to zero almost surely.

Consider a Pólya urn starting with n_0 balls with one white one, that is a Markov chain $(W_n)_{n\in\mathbb{N}}$ in $\mathbb{N}$ such that $W_0 = 1$ and

$$\mathbb{P}(W_{n+1} = W_n + 1 \mid W_n) = \frac{W_n}{n + n_0}, \ \forall n \geq 0,$$
$$\mathbb{P}(W_{n+1} = W_n \mid W_n) = 1 - \frac{W_n}{n + n_0}, \ \forall n \geq 0.$$

It is well known that $(W_n/n)_{n\in\mathbb{N}}$ is a positive and bounded martingale which converges almost surely to a random variable S_∞ distributed following a Beta distribution with parameters $(1, n_0 - 1)$. In particular, this implies that the event $\{W_k/k \leq c_0, \forall k \geq 0\}$ has a positive probability and that, conditionally to this event, W_n/n converges to a positive random variable :

$$\mathbf{1}_{W_k/k\leq c_0, \forall k\geq 0} \frac{W_n}{n} \xrightarrow[n\to\infty]{a.s.} \mathbf{1}_{W_k/k\leq c_0, \forall k\geq 0} S_\infty.$$

Since $(V_{n+n_0})_{n\in\mathbb{N}}$ and $(W_n)_{n\in\mathbb{N}}$ have the same transition probabilities at time n from states $u \in \mathbb{N}$ such that $u/n \leq c_0$, there exists a coupling such that

$$\mathbf{1}_{V_0\geq 1} V_{n+n_0} \geq \mathbf{1}_{V_0\geq 1} \mathbf{1}_{W_k/k\leq c_0, \forall k\geq 0} W_n, \ \forall n \in \mathbb{N} \text{ almost surely,}$$

and hence such that

$$\mathbf{1}_{V_0\geq 1} R_\infty \geq \mathbf{1}_{V_0\geq 1}\, \mathbf{1}_{W_k/k\leq c_0,\, \forall k\geq 0}\, S_\infty \text{ almost surely.}$$

Since the right hand side is positive with positive probability, we deduce that R_∞ is positive with positive probability. But $R_\infty > 0$ implies that $\inf_{n\geq 0} V_n/n > 0$, thus there exists $\delta > 0$ such that

$$\mathbb{P}\left(\inf_{n\geq 0} \frac{V_n}{n} > \delta\right) > 0.$$

Because of the relation between V_n and U_n, we deduce that

$$p_0 := \mathbb{P}\left(\frac{N'_{k_N}}{N} > \delta\right) = \mathbb{P}\left(\frac{U_N}{N} > \delta\right) > 0.$$

By definition of N'_{k_N}, $\frac{N'_{k_N}}{N} > \delta$ implies $\mu_1^N(\mathbb{P}_\cdot(n_0 < \tau_\partial)) > \delta\varepsilon_0$ and hence

$$\mathbb{P}\left(\mu_1^N(\mathbb{P}_\cdot(n_0 < \tau_\partial)) \leq \delta\varepsilon_0\right) \leq 1 - p_0.$$

This concludes the proof of Lemma 1.

□

Proof of Theorem 3. We first prove the exponential ergodicity of the particle system and then deduce (10).

Using Lemma 1, we know that, for any initial distribution of $(X_0^1, \ldots, X_0^N)$,

$$\mathbb{P}(\mathbb{P}_{\mu_1^N}(n_0 < \tau_\partial) > \delta\varepsilon_0) \geq p_0.$$

On the event $\mathbb{P}_{\mu_1^N}(n_0 < \tau_\partial) > \delta\varepsilon_0$, there exists at least one particle satisfying $\mathbb{P}_{X_1^i}(n_0 < \tau_\partial) > \delta\varepsilon_0$. Let us denote by $I \subset \{1, \ldots, N\}$ the set of indexes of such particles and by $J = \{1, \ldots, N\} \setminus I$ the set of indexes j such that $\mathbb{P}_{X_1^j}(n_0 < \tau_\partial) \leq \delta\varepsilon_0$.

The probability that the $\sharp J$ first steps of Algorithm 1 concern the indexes of $\sharp J$ in strictly increasing order is strictly lowered by $1/N^{\sharp J}$ and hence by $1/N^N$. For each of this step, the probability that the chosen particle with index in J is killed and then is sent to the position of a particle with index in I is lowered by $(1-\delta\varepsilon_0)\sharp I/N$ and hence by $(1-\delta\varepsilon_0)/N$. Overall, the probability that, after the $\sharp J$ first steps of Algorithm 1, all the particles with index in J have jumped on a

particle with index in I is bounded below by $1/N^N \times (1-\delta\varepsilon_0)^{\sharp J}/N^{\sharp J}$ and hence by $(1-\delta\varepsilon_0)^N/N^{2N}$.

On this event, the probability that, for each next step in the algorithm up to time n_0+1, the chosen particle jumps without being absorbed is bounded below by $(\delta\varepsilon_0)^N$. But, using [9, Theorem 2.1], we know that, under Assumption 7, there exist a probability measure ν on E and a constant $c_1>0$ such that $\mathbb{P}_x(X_{n_0}\in\cdot \mid n_0<\tau_\partial)\geq c_1\nu(\cdot)$. Since the particles are independent on the event where none of them is killed, we finally deduce that the distribution of the particle system at time n_0+1 satisfies

$$\mathbb{P}\left((X^1_{n_0+1},\dots,X^1_{n_0+1})\in\cdot\right)\geq p_0(\delta\varepsilon_0^N(1-\delta\varepsilon_0)^N/N^{2N}c_1^N\nu^{\otimes N}(\cdot).$$

Classical coupling criteria (see for instance [19]) entails the exponential ergodicity of the particle system.

Let us now prove that (10) holds. Consider $x\in E$ and $\delta>0$ such that $\mathbb{P}_x(n_0<\tau_\partial)>\varepsilon_0\delta$. Then, applying Theorem 2 to the particle system with initial position $(X^1_0,\dots,X^N_0)=(x,\dots,x)$, we deduce that, for all $n\geq 0$,

$$\mathbb{E}\left(\left|\mu^N_n(f)-\mathbb{E}_x(f(X_n)\mid n<\tau_\partial)\right|\right)\leq\frac{CN^\alpha}{\delta}.$$

Using the exponential ergodicity of the particle system and the exponential convergence (7), we deduce that

$$\mathbb{E}_{M^N}\left(\left|\mu^N_0(f)-\nu_{QSD}(f)\right|\right)\leq\frac{CN^\alpha}{\delta}+2C_Ne^{-\gamma_N n}+Ce^{-\gamma n}.$$

Letting n tend toward infinity implies (10) and concludes the proof of Theorem 3. □

4 Example: Absorbed neutron transport process

The propagation of neutrons in fissible media is typically modeled by neutron transport systems, where the trajectory of the particle is composed of straight exponential paths between random changes of directions [12, 25]. The behavior of a neutron before its absorption by a medium is related to the behavior of neutron tranport before extinction, where extinction corresponds to the exit of a neutron from a bounded set D.

We recall the setting of the neutron transport process studied in [9]. Let D be an open connected bounded domain of $\mathbb{R}^2$, let S^2 be the unit sphere of $\mathbb{R}^2$ and $\sigma(du)$ be the uniform probability measure on S^2. We consider the Markov process $(X_t, V_t)_{t\geq 0}$ in $D \times S^2$ constructed as follows: $X_t = \int_0^t V_s\, ds$ and the velocity $V_t \in S^2$ is a pure jump Markov process, with constant jump rate $\lambda > 0$ and uniform jump probability distribution σ. In other words, V_t jumps to i.i.d. uniform values in S^2 at the jump times of a Poisson process. At the first time where $X_t \notin D$, the process immediately jumps to the cemetery point ∂, meaning that the process is absorbed at the boundary of D. An example of path of the process (X, V) is shown in Fig. 2. For all $x \in D$ and $u \in S^2$, we denote by $\mathbb{P}_{x,u}$ (resp. $\mathbb{E}_{x,u}$) the distribution of (X, V) conditionned on $(X_0, V_0) = (x, u)$ (resp. the expectation with respect to $\mathbb{P}_{x,u}$).

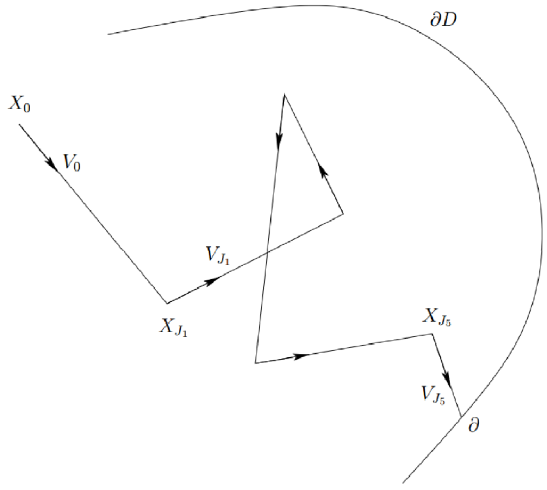


Figure 2: A sample path of the neutron transport process (X, V). The times $J_1 < J_2 < \ldots$ are the successive jump times of V.

We also assume the following condition on the boundary of the bounded open set D. This is an interior cone type condition satisfied for example by convex open sets of $\mathbb{R}^2$ and by open sets with C^2 boundaries.

Assumption (H) on D. We assume that there exists $\varepsilon > 0$ such that

- $D_\varepsilon := \{x \in D : d(x, \partial D) > \varepsilon\}$ is non-empty and connected;

- there exists $0 < s_\varepsilon < t_\varepsilon$ and $\underline{\sigma} > 0$ such that, for all $x \in D \setminus D_\varepsilon$, there exists $K_x \subset S^2$ measurable such that $\sigma(K_x) \geq \underline{\sigma}$ and for all $u \in K_x$, $x + su \in D_\varepsilon$ for all $s \in [s_\varepsilon, t_\varepsilon]$ and $x + su \notin \partial D$ for all $s \in [0, s_\varepsilon]$.

The next proposition implies that the approximation method introduced in this paper converges uniformly in time toward the conditional distribution of (X, V).

Proposition 1. *The absorbed Markov process (X, V) satisfies the assumptions of Theorem 2.*

Proof of Proposition 1. Theorem 4.3 from [9] states that the Markov process (X, V) satisfies the exponential convergence condition (7). It only remains to check that that Assumption (9) is also satisfied.

By [9, (4.3)], for any $n_0 \geq 1$, there exists a constant $c_{n_0} > 0$ such that

$$c_{n_0} := \inf_{(x,u) \in D_\varepsilon \times S^2} \mathbb{P}_{(x,u)}(1 + n_0 < \tau_\partial) > 0. \tag{16}$$

In particular, we deduce that, for all $n_0 \geq 0$,

$$\mathbb{P}_{(x,u)}(\mathbb{P}_{(X_1,V_1)}(n_0 < \tau_\partial) \mid 1 < \tau_\partial) \geq c_{n_0}, \quad \forall (x,u) \in D_\varepsilon \times S^2. \tag{17}$$

Now, fix $(x,u) \notin D_\varepsilon \times S^2$ and consider the first (deterministic) time $s(x,u)$ when the ray starting from x with direction u hits D_ε or ∂D, defined by

$$s(x,u) = \inf\{t \geq 0\,, \text{ such that } x + tu \in D_\varepsilon \cup \partial D\}.$$

Let us first assume that $x + s(x,u)u \in \partial D_\varepsilon$. Then $x + (s(x,u) + \eta)u \in D_\varepsilon$ for some $\eta > 0$ and hence

$$\mathbb{P}_{(x,u)}(X_{s(x,u)+\eta} \in D_\varepsilon) \geq \mathbb{P}\left(J_1 > s(x,u) + \eta\right) = e^{-(s(x,u)+\eta)} \geq e^{-\mathrm{diam}(D)},$$

where $J_1 < J_2 < \cdots$ denotes the successive jump times of the process V. Using the Markov property, we deduce from (16) that

$$\mathbb{P}_{(x,u)}(1 + n_0 < \tau_\partial) \geq \mathbb{P}(s + \eta + 1 + n_0 < \tau_\partial) \geq e^{-\mathrm{diam}(D)}\, c_{n_0}. \tag{18}$$

This implies that, for all $n_0 \geq 1$ and all $(x,u) \notin D_\varepsilon \times S^2$ such that $x + s(x,u)u \in \partial D_\varepsilon$,

$$\mathbb{P}_{(x,u)}(\mathbb{P}_{(X_1,V_1)}(n_0 < \tau_\partial) \mid 1 < \tau_\partial) \geq e^{-\mathrm{diam}(D)} c_{n_0}. \tag{19}$$

Let us now assume that $x + s(x,u)u \in \partial D$. Using Assumption (H), we obtain

$$\begin{aligned}\mathbb{P}_{(x,u)}(\exists t \in [0,s) \text{ s.t. } X_t + s(X_t,V_t)V_t \in \partial D_\varepsilon) &\geq \mathbb{P}_{(x,u)}(J_1 < s,\, V_{J_1} \in K_{x+uJ_1},\, J_2 > s) \\ &\geq (1-e^{-s})\underline{\sigma}e^{-s}.\end{aligned}$$

Using the strong Markov property and (18), we deduce that

$$\mathbb{P}_{(x,u)}(1+n_0 < \tau_\partial) \geq (1-e^{-s})\underline{\sigma}e^{-s}e^{-\mathrm{diam}(D)}c_{n_0} \geq (1-e^{-s\wedge 1})\underline{\sigma}e^{-2\mathrm{diam}(D)}c_{n_0}.$$

If $s \leq 1$, then $1 < \tau_\partial$ implies that the process jumps at least one time before reaching ∂D, that is before time s, so that $\mathbb{P}_{(x,u)}(1 < \tau_\partial) \leq \mathbb{P}(J_1 \leq s) = 1-e^{-s}$. Using the Markov property, we deduce that, for $s \leq 1$ or $s > 1$,

$$\begin{aligned}\mathbb{P}_{(x,u)}(\mathbb{P}_{(X_1,V_1)}(n_0 < \tau_\partial) \mid 1 < \tau_\partial) &= \mathbb{P}_{(x,u)}(1+n_0 < \tau_\partial \mid 1 < \tau_\partial) \\ &\geq (1-e^{-1})\underline{\sigma}e^{-2\mathrm{diam}(D)}c_{n_0}.\end{aligned}$$

This, Equations (17) and (19) together imply that Assumption (9) is fulfilled for all $(x,u) \in D, S^2$. This concludes the prood of Proposition 1. □

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