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ABSTRACT. The complex projective space $\mathbb{C}P^2$ of complex dimension 2 has a Spin^c structure carrying Kählerian Killing spinors. The restriction of one of these Kählerian Killing spinors to a Lagrangian or complex surface M^2 characterizes the isometric immersion of M into $\mathbb{C}P^2$.

1. INTRODUCTION

A classical problem in Riemannian geometry is to know when a Riemannian manifold (M^n, g) can be isometrically immersed into a fixed Riemannian manifold $(\bar{M}^{n+p}, \bar{g})$. The case of space forms $\mathbb{R}^{n+1}$, $\mathbb{S}^{n+1}$ and $\mathbb{H}^{n+1}$ is well-known. In fact, the Gauss, Codazzi and Ricci equations are necessary and sufficient conditions. In other ambient spaces, the Gauss, Codazzi and Ricci equations are necessary but not sufficient in general. Some additional conditions may be required like for the case of complex space forms, products, warped products or 3-dimensional homogeneous space for instance (see [8, 9, 19, 22, 30, 33]).

In low dimensions, especially for surfaces, another necessary and sufficient condition is now well-known, namely the existence of a special spinor field called *generalized Killing spinor field* (see [10, 27, 21, 23]). These results are the geometrically invariant versions of previous work on the spinorial Weierstrass representation by R. Kusner and N. Schmidt, B. Konoplechenko, I. Taimanov and many others (see [20, 18, 35]). This representation was expressed by T. Friedrich [10]) for surfaces in $\mathbb{R}^3$ and then extended to other 3-dimensonal (pseudo-)Riemannian manifolds ([27, 33, 32, 24]) as well as for hypersurfaces of 4-dimensional space forms and products [23] or hypersurfaces of 2-dimensional complex space forms by the mean of Spin^c spinors [29].

More precisely, the restriction φ of a parallel spinor field on $\mathbb{R}^{n+1}$ to an oriented Riemannian hypersurface M^n is a solution of a generalized Killing equation

$$\nabla_X \varphi = -\frac{1}{2} A(X)) \varphi, \tag{1}$$

where “ $\cdot$ ” and ∇ are respectively the Clifford multiplication and the spin connection on M^n, and A is the Weingarten tensor of the immersion. Conversely, T. Friedrich proved in [10] that, in the two dimensional case, if there exists a generalized Killing spinor field satisfying Equation (1), where A is an arbitrary field of symmetric endomorphisms of TM, then A satisfies the Codazzi and Gauss equations of hypersurface theory and is consequently the Weingarten tensor of a local isometric immersion of M into $\mathbb{R}^3$. Moreover, in this case, the solution φ of the generalized Killing equation is equivalently a solution of the Dirac equation

$$D\varphi = H\varphi, \tag{2}$$

where $|\varphi|$ is constant and H is a real-valued function (which is the mean curvature of the immersion in $\mathbb{R}^3$).

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More recently, this approach was adapted by the second author, P. Bayard and M.A. Lawn in codimension two, namely, for surfaces in Riemannian 4-dimensional real space forms [4], and then generalized in the pseudo-Riemannian setting [3, 5] as well as for 4-dimensional products [34]. As pointed out in [31], this approach coincides with the Weierstrass type representation for surfaces in $\mathbb{R}^4$ introduced by Konopelchenko and Taimanov [18, 36].

The aim of the present article is to provide an analogue for the complex projective space $\mathbb{C}P^2$. The key point is that contrary to the case of hypersurfaces of $\mathbb{C}P^2$, which is considerered in [29], the use of Spin^c parallel spinors is not sufficient. Indeed, for both canonical or anti-canonical Spin^c structures, the parallel spinors are always in the positive half-part of the spinor bundle. But, as proved in [4] or [34], a spinor with non-vanishing positive and negative parts is required to get the integrability condition to get an immersion in the desired target space. For this reason, Spin^c parallel spinors are not adapted to our problem. Thus, we make use of real Kählerian Killing spinors. Therefore, our argument holds also for the complex projective space and not for the complex hyperbolic space, since $\mathbb{CH}^2$ does not carry a real or imaginary Kählerian Killing spinor.

We will focus on the case of complex and Lagrangian immersions into $\mathbb{C}P^2$. These two cases and especially the Lagrangian case are of particular interest in the study of surfaces in $\mathbb{C}P^2$ (see [14, 15, 37] and references therein for instance). Namely, we prove the two following results.

The first theorem gives a spinorial characterization of complex surfaces in the complex projective space $\mathbb{C}P^2$.

Theorem 1.1. *Let (M^2, g) be an oriented Riemannian surface and E an oriented vector bundle of rank 2 over M with scalar product $\langle\cdot,\cdot\rangle_E$ and compatible connection ∇^E. We denote by $\Sigma = \Sigma M \otimes \Sigma E$ the twisted spinor bundle. Let $B : TM \times TM \longrightarrow E$ a bilinear symmetric map, $j : TM \longrightarrow TM$ a complex structure on M and $t : E \longrightarrow E$ a complex structure on E. Assume moreover that $t(B(X,Y)) = B(X, jY)$, $\forall X \in TM$. Then, the two following statements are equivalent*

(1) *There exists a Spin^c structure on $\Sigma M \otimes \Sigma E$ with $\Omega^{M+E}(e_1, e_2) = 0$ and a spinor field $\varphi \in \Gamma(\Sigma M \otimes \Sigma E)$*
$$\nabla_X \varphi \quad = \quad -\frac{1}{2}\eta(X)\cdot\varphi - \frac{1}{2}X\cdot\varphi + \frac{i}{2}jX\cdot\overline{\varphi}, \tag{3}$$
such that φ^+ and φ^- never vanish and where η is given by
$$\eta(X) = \sum_{j=1}^{2} e_j \cdot B(e_j, X).$$

(2) *There exists a local isometric* **complex** *immersion of (M^2, g) into $\mathbb{C}P^2$ with E as normal bundle and second fundamental form B such the complex structure of $\mathbb{C}P^2$ over M is given by j and t (in the sense of Proposition 3.2).*

The spinors φ^+ and φ^- denote respectiveley the positive and negative half part of φ defined Section 2. They are the projections of φ on the eigensubspaces for the eigenvalues $+1$ and -1 of the complex volume form and $\bar{\varphi}$ is defined by $\bar{\varphi} = \varphi_+ - \varphi_-$.

The second theorem is the analogue of Theorem 1.1 for Lagrangian surfaces into $\mathbb{C}P^2$.

Theorem 1.2. *Let (M^2, g) be an oriented Riemannian surface and E an oriented vector bundle of rank 2 over M with scalar product $\langle\cdot,\cdot\rangle_E$ and compatible connection ∇^E. We denote by $\Sigma = \Sigma M \otimes \Sigma E$ the twisted spinor bundle. Let $B : TM \times TM \longrightarrow E$ a bilinear symmetric map, $h : TM \longrightarrow E$ and $s : E \longrightarrow TM$ the dual map of h. Assume moreover that h and s are parallel, $hs = -\mathrm{id}_E$ and $A_{hY}X + s(B(X,Y)) = 0$, for all $X \in TM$, where $A_\nu : TM \longrightarrow TM$ if defined by $g(A_\nu X, Y) = \langle B(X,Y), \nu\rangle_E$ for all $X, Y \in TM$ and $\nu \in E$. Then, the two following statements are equivalent*

(1) *There exists a* Spin^c *structure on* $\Sigma M \otimes \Sigma E$ *with* $\Omega^{M+N}(e_1, e_2) = -2$ *and a spinor field* $\varphi \in \Gamma(\Sigma M \otimes \Sigma E)$ *satisfying for all* $X \in \mathfrak{X}(M)$

$$\nabla_X \varphi = -\frac{1}{2}\eta(X)\cdot\varphi - \frac{1}{2}X\cdot\varphi + \frac{i}{2}hX\cdot\overline{\varphi}, \tag{4}$$

such that φ^+ *and* φ^- *never vanish and where* η *is given by*

$$\eta(X) = \sum_{j=1}^{2} e_j \cdot B(e_j, X).$$

(2) *There exists a local isometric* **Lagrangian** *immersion of* (M^2, g) *into* $\mathbb{C}P^2$ *with* E *as normal bundle and second fundamental form* B *such that over* M *the complex structure of* $\mathbb{C}P^2$ *is given by* h *and* s *(in the sense of Proposition 3.2).*

Note that in the statements of both theorem φ^+ and φ^- denotes respectiveley the positive and negative half part of φ defined Section . They are the projections of φ on the eigensubspaces for the eigenvalues $+1$ and -1 of the complex volume form.

2. Preliminaries and Notations

In this section, we briefly review some basic facts about Kähler geometry and Spin^c structures on manifolds and their submanifolds. For more details we refer to [28, 2, 6, 25, 26, 16, 17, 7, 1, 11, 12, 13].

2.1. Spin^c structures on Kähler-Einstein manifolds.

Let (M^n, g) be an n-dimensional closed Riemannian Spin^c manifold and denote by ΣM its complex spinor bundle, which has complex rank equal to $2^{[\frac{n}{2}]}$. The bundle ΣM is endowed with a Clifford multiplication denoted by "$\cdot$" and a scalar product denoted by $\langle \cdot, \cdot \rangle$.

Given a Spin^c structure on (M^n, g), one can check that the determinant line bundle $\det(\Sigma M)$ has a root L of index $2^{[\frac{n}{2}]-1}$. This line bundle L over M is called the auxiliary line bundle associated with the Spin^c structure. From a topological point of view, a Riemannian manifold has a Spin^c structure if and only if there exists a complex line bundle L (which will be the auxiliary line bundle) on M such that

$$\omega_2(M) = [c_1(L)]_{mod\ 2},$$

where $\omega_2(M)$ is the second Steifel-Whitney class of M and $c_1(L)$ is the first Chern class of L. In the particular case, when the line bundle has a square root, i.e., $\omega_2(M) = 0$, the manifold is called a Spin manifold. In this case, we denote by $\Sigma^{'} M$ the spinor bundle or the Spin bundle. It can be chosen such that the auxiliary line bundle is trivial.

Locally, a Spin^c structure always exists. In fact, the square root of the auxiliary line bundle L and $\Sigma^{'} M$ always exist locally. But, $\Sigma M = \Sigma^{'} M \otimes L^{\frac{1}{2}}$ is defined globally. This essentially means that, while the spinor bundle and $L^{\frac{1}{2}}$ may not exist globally, their tensor product (the Spin^c bundle) is defined globally. Thus, the connection ∇ on SSM is the twisted connection of the one on the spinor bundle (induced by the Levi-Civita connection) and a fixed connection A on L. The Spin^c Dirac operator D acting on the space of sections of ΣM is defined by the composition of the connection ∇ with the Clifford multiplication. In local coordinates:

$$D = \sum_{j=1}^{n} e_j \cdot \nabla_{e_j},$$

where $\{e_j\}_{j=1,\ldots,n}$ is a local orthonormal basis of TM. D is a first-order elliptic operator and is formally self-adjoint with respect to the L^2-scalar product.

We recall that the complex volume element $\omega_{\mathbb{C}} = 1^{[\frac{n+1}{2}]} e_1 \wedge \ldots \wedge e_n$ acts as the identity on the spinor bundle if n is odd. If n is even, $\omega_{\mathbb{C}}^2 = 1$. Thus, under the action of the complex volume element, the spinor bundle decomposes into the eigenspaces $\Sigma^{\pm}M$ corresponding to the ± 1 eigenspaces, the *positive* (resp. *negative*) spinors.

Every spin manifold has a trivial Spin^c structure, by choosing the trivial line bundle with the trivial connection whose curvature F_A vanishes. Every Kähler manifold (M^{2m}, g, J) has a canonical Spin^c structure induced by the complex structure J. The complexified tangent bundle decomposes into $T^{\mathbb{C}}M = T_{1,0}M \oplus T_{0,1}M$, the i-eigenbundle (resp. $(-i)$-eigenbundle) of the complex linear extension of J. For any vector field X, we denote by $X^{\pm} := \frac{1}{2}(X \mp iJX)$ its component in $T_{1,0}M$, resp. $T_{0,1}M$. The spinor bundle of the canonical Spin^c structure is defined by

$$\Sigma M = \Lambda^{0,*}M = \mathop{\oplus}_{r=0}^{m} \Lambda^r(T^*_{0,1}M),$$

and its auxiliary line bundle is $L = (K_M)^{-1} = \Lambda^m(T^*_{0,1}M)$, where $K_M = \Lambda^{m,0}M$ is the canonical bundle of M. The line bundle L has a canonical holomorphic connection, whose curvature form is given by $-i\rho$, where ρ is the Ricci form defined, for all vector fields X and Y, by $\rho(X, Y) = \mathrm{Ric}(JX, Y)$ and Ric denotes the Ricci tensor. Similarly, one defines the so called anti-canonical Spin^c structure, whose spinor bundle is given by $\Lambda^{*,0}M = \oplus_{r=0}^{m}\Lambda^r(T^*_{1,0}M)$ and the auxiliary line bundle by K_M. The spinor bundle of any other Spin^c structure on M can be written as:

$$\Sigma M = \Lambda^{0,*}M \otimes \mathbb{L},$$

where $\mathbb{L}^2 = K_M \otimes L$ and L is the auxiliary line bundle associated with this Spin^c structure. The Kähler form Ω, defined as $\Omega(X, Y) = g(JX, Y)$, acts on ΣM via Clifford multiplication and this action is locally given by:

$$\Omega \cdot \psi = \frac{1}{2}\sum_{j=1}^{2m} e_j \cdot Je_j \cdot \psi,$$

for all $\psi \in \Gamma(\Sigma M)$, where $\{e_1, \dots, e_{2m}\}$ is a local orthonormal basis of TM. Under this action, the spinor bundle decomposes as follows:

$$\Sigma M = \mathop{\oplus}_{r=0}^{m} \Sigma_r M, \tag{5}$$

where $\Sigma_r M$ denotes the eigenbundle to the eigenvalue $i(2r - m)$ of Ω, of complex rank $\binom{m}{k}$. It is easy to see that $\Sigma_r M \subset \Sigma^+ M$ (resp. $\Sigma_r M \subset \Sigma^- M$) if and only if r is even (resp. r is odd). Moreover, for any $X \in \Gamma(TM)$ and $\varphi \in \Gamma(\Sigma_r M)$, we have $X^+ \cdot \varphi \in \Gamma(\Sigma_{r+1}M)$ and $X^- \cdot \varphi \in \Gamma(\Sigma_{r-1}M)$, with the convention $\Sigma_{-1}\mathrm{M} = \Sigma_{m+1}\mathrm{M} = M \times \{0\}$. Thus, for any Spin^c structure, we have $\Sigma_r M = \Lambda^{0,r}M \otimes \Sigma_0 M$. Hence, $(\Sigma_0 M)^2 = K_M \otimes L$, where L is the auxiliary line bundle associated with the Spin^c structure. For example, when the manifold is spin, we have $(\Sigma_0 M)^2 = K_M$ [38, 39]. For the canonical Spin^c structure, since $L = (K_M)^{-1}$, it follows that $\Sigma_0 M$ is trivial. This yields the existence of parallel spinors (the constant functions) lying in $\Sigma_0 M$, *cf.* [40].

On a Kähler manifold (M, g, J) endowed with any Spin^c structure, a spinor of the form $\varphi_r + \varphi_{r+1} \in \Gamma(\Sigma_r M \oplus \Sigma_{r+1}M)$, for some $0 \leq r \leq m$, is called a *Kählerian Killing* Spin^c *spinor* if there exists a non-zero real constant α, such that the following equations are satisfied, for all vector fields X,

$$\begin{cases} \nabla_X \varphi_r = \alpha\, X^- \cdot \varphi_{r+1}, \\ \nabla_X \varphi_{r+1} = \alpha\, X^+ \cdot \varphi_r. \end{cases} \tag{6}$$

Kählerian Killing spinors lying in $\Gamma(\Sigma_m M \oplus \Sigma_{m+1}M) = \Gamma(\Sigma_m M)$ or in $\Gamma(\Sigma_{-1}M \oplus \Sigma_0 M) = \Gamma(\Sigma_0 M)$ are just parallel spinors. In [16], the authors gave examples of Spin^c structures on compact Kähler-Einstein manifolds of positive scalar curvature, which carry Kählerian Killing Spin^c spinors lying in $\Sigma_r M \oplus \Sigma_{r+1}M$, for $r \neq \frac{m \pm 1}{2}$, in contrast to the spin case, where Kählerian Killing spinors may only exist for m odd in the middle of the decomposition (5). We briefly describe these Spin^c structures here. If the first Chern class $c_1(K_M)$ of the canonical bundle of the Kähler M is a non-zero cohomology class, the greatest number $p \in \mathbb{N}^*$ such that $\frac{1}{p}c_1(K_M) \in H^2(M,\mathbb{Z})$, is called the *Maslov index* of the Kähler manifold. One can thus consider a p-th root of the canonical bundle K_M, *i.e.* a complex line bundle $\mathcal{L}$, such that $\mathcal{L}^p = K_M$. In [16], O. Hijazi, S. Montiel and F. Urbano proved the following:

Theorem 2.1 (Theorem 14, [16]). *Let M be a $2m$-dimensional Kähler-Einstein compact manifold with scalar curvature $4m(m+1)$ and index $p \in \mathbb{N}^*$. For each $0 \leq r \leq m+1$, there exists on M a Spin^c structure with auxiliary line bundle given by $\mathcal{L}^q$, where $q = \frac{p}{m+1}(2r-m-1) \in \mathbb{Z}$, and carrying a Kählerian Killing spinor $\psi_{r-1}+\psi_r \in \Gamma(\Sigma_{r-1}M \oplus \Sigma_r M)$,* i.e. *for all $X \in \Gamma(TM)$, it satisfies the first order system*

$$\begin{cases} \nabla_X \psi_r = -X^+ \cdot \psi_{r-1}, \\ \nabla_X \psi_{r-1} = -X^- \cdot \psi_r, \end{cases}$$

For example, if M is the complex projective space $\mathbb{C}P^m$ of complex dimension m, then $p = m+1$ and $\mathcal{L}$ is just the tautological line bundle. We fix $0 \leq r \leq m+1$ and we endow $\mathbb{C}P^m$ with the Spin^c structure whose auxiliary line bundle is given by $\mathcal{L}^q$ where $q = \frac{p}{m+1}(2r-m-1) = 2r-m-1 \in \mathbb{Z}$. For this Spin^c structure, the space of Kählerian Killing spinors in $\Gamma(\Sigma_{r-1}M \oplus \Sigma_r M)$ has dimension $\binom{m+1}{r}$. In this example, for $r = 0$ (resp. $r = m+1$), we get the canonical (resp. anticanonical) Spin^c structure for which Kählerian Killing spinors are just parallel spinors.

2.2. **Submanifolds of Spin^c manifolds.** Let (M^2, g) be an oriented Riemannian surface, with a given Spin^c structure, and E an oriented Spin^c vector bundle of rank 2 on M with an Hermitian product $\langle\cdot,\cdot\rangle_E$ and a compatible connection ∇^E. We consider the spinor bundle Σ over M twisted by E and defined by $\Sigma = \Sigma M \otimes \Sigma E$, where ΣM and ΣE are the spinor bundles of M and E respectively. We endow Σ with the spinorial connection ∇ defined by

$$\nabla = \nabla^{\Sigma M} \otimes Id_{\Sigma E} + Id_{\Sigma M} \otimes \nabla^{\Sigma E},$$

where $\nabla^{\Sigma M}$ and $\nabla^{\Sigma E}$ are respectively the spinorial connections on ΣM and ΣE. We also define the Clifford product $\cdot$ by

$$\begin{cases} X \cdot \varphi = (X \cdot_M \alpha) \otimes \overline{\sigma} & \text{if } X \in \Gamma(TM) \\ X \cdot \varphi = \alpha \otimes (X \cdot_E \sigma) & \text{if } X \in \Gamma(E) \end{cases}$$

for all $\varphi = \alpha \otimes \sigma \in \Sigma M \otimes \Sigma E$, where $\cdot_M$ and $\cdot_E$ denote the Clifford products on ΣM and on ΣE respectively and where $\overline{\sigma} = \sigma^+ - \sigma^-$ for the natural decomposition of $\Sigma E = \Sigma^+ E \oplus \Sigma^- E$. Here, $\Sigma^+ E$ and $\Sigma^- E$ are the eigensubbundles (for the eigenvalue 1 and -1) of ΣE for the action of the normal volume element $\omega_\perp = i\xi_1 \cdot_E \xi_2$, where $\{\xi_1, \xi_2\}$ is a local orthonormal frame of E. Note that $\Sigma^+ M$ and $\Sigma^- M$ are defined similarly by for the tangent volume element $\omega = ie_1 \cdot_M e_2$. If $\{e_1, e_2\}$ is an orthonormal basis of TM, we define the twisted Dirac operator D on $\Gamma(\Sigma)$ by

$$D\varphi = e_1 \cdot \nabla_{e_1}\varphi + e_2 \cdot \nabla_{e_2}\varphi.$$

We note that Σ is also naturally equipped with a hermitian scalar product $\langle .,. \rangle$ which is compatible with the connection ∇, and thus also with a compatible real scalar product $\mathrm{Re}\langle .,. \rangle$. We also note that the Clifford product $\cdot$ of vectors belonging to $TM \oplus E$ is antihermitian with respect to this hermitian product. Finally, we stress that the four subbundles $\Sigma^{\pm\pm} := \Sigma^\pm M \otimes \Sigma^\pm E$ are orthogonal with respect to the hermitian product. We will also consider $\Sigma^+ = \Sigma^{++} \oplus \Sigma^{--}$ and $\Sigma^- = \Sigma^{+-} \oplus \Sigma^{-+}$.Throughout the paper we will assume that the hermitian product is $\mathbb{C}-$linear w.r.t. the first entry, and $\mathbb{C}-$antilinear w.r.t. the second entry.

Let $(\widetilde{M}^4, \widetilde{g})$ be a Riemannian Spin^c manifold and (M^2, g) an oriented surface isometrically immersed into $\widetilde{M}$. We denote by NM the normal bundle of M into $\widetilde{M}$. As M is an oriented surface, it is also Spin^c. We denote by $i\widetilde{F}$ (resp. iF) the curvature 2-form of the auxiliary line bundle $L^{\widetilde{M}}$ (resp. L) associated with the Spin^c structure on $\widetilde{M}$ (resp. M). Since the manifolds M and $\widetilde{M}$ are Spin^c, there exists a Spin^c structure on the bundle NM whose auxiliary line bundle L_N is given by $L_N := (L)^{-1} \otimes L^{\widetilde{M}}{}_{|M}$. We denote by ΣN the Spin^c bundle of NM and let $\Sigma = \Sigma M \otimes \Sigma N$ the spinor bundle over M twisted by NM constructed as above with the associated connection and Clifford multiplication. It is a classical fact that the spinor bundle of $\widetilde{M}$ over M, $\Sigma\widetilde{M}_{|M}$ identifies with Σ. Moreover the connections on each bundle are linked by the so-called spinorial Gauss formula: for any $\varphi \in \Gamma(\Sigma)$ and any $X \in TM$,

$$\widetilde{\nabla}_X \varphi = \nabla_X \varphi + \frac{1}{2} \sum_{j=1,2} e_j \cdot II(X, e_j) \cdot \varphi \tag{7}$$

where II is the second fundamental form, $\widetilde{\nabla}$ is the spinorial connection of $\Sigma\widetilde{M}$ and ∇ is the spinoral connection of Σ defined as above and $\{e_1, e_2\}$ is a local orthonormal frame of TM. Here $\cdot$ is the Clifford product on $\Sigma\widetilde{M}$ which identifies with the Clifford mulitplication on Σ.

3. Immersed surfaces into the complex projective space

In this section, we will give the basic facts about immersed surfaces in the complex projective plane and in particular derive a sequence of necessary and sufficent conditions for the existence of such immersions.

3.1. **Compatibility equations.** Let (M^2, g) be a Riemannian surface isometrically immersed itn the 2-dimensional complex projective space of constant holomorphic sectional curvature $4c > 0$. We denote by ∇ the Levi-Civita connection of (M^2, g), $\widetilde{g}$ the Fubini-Study metric of $\mathbb{C}P^2(4c)$ and $\widetilde{\nabla}$ its Levi-Civita connection. First of all, we recall that the curvature tensor of $\mathbb{C}P^2(4c)$ is given by

$$\begin{aligned} \widetilde{R}(X,Y,Z,W) \quad = \quad c\Big[& \langle X,W\rangle\langle Y,Z\rangle - \langle X,Z\rangle\langle Y,W\rangle + \langle JX,W\rangle\langle JY,Z\rangle \\ & - \langle JX,Z\rangle\langle JY,W\rangle + 2\langle X,JY\rangle\langle JZ,W\rangle \Big]. \end{aligned} \tag{8}$$

The complex structure J induces the existence of the following four operators

$$j : TM \longrightarrow TM,\ h : TM \longrightarrow NM,\ s : NM \longrightarrow TM \text{ and } t : NM \longrightarrow NM$$

defined for any $X \in TM$ and $\xi \in NM$ by

$$JX = jX + hX \quad \text{and} \quad J\xi = s\xi + t\xi. \tag{9}$$

From the fact that $J^2 = -\mathrm{Id}$ and J is antisymmetric, we get that j and t are antisymmetric and we have the following relations between these four operators

$$j^2 X = -X - shX, \tag{10}$$

$$t^2 \xi = -\xi - hs\xi, \tag{11}$$

$$js\xi + st\xi = 0, \tag{12}$$

$$hjX + thX = 0, \tag{13}$$

$$\widetilde{g}(hX, \xi) = -\widetilde{g}(X, s\xi), \tag{14}$$

for any $X \in \Gamma(TM)$ and $\xi \in \Gamma(NM)$. Moreover, from the fact that J is parallel, we have

$$(\nabla_X j)Y = A_{hY}X + s(B(X,Y)), \tag{15}$$

$$\nabla^{\perp}_X(hY) - h(\nabla_X Y) = t(B(X,Y)) - B(X, jY), \tag{16}$$

$$\nabla^{\perp}(t\xi) - t(\nabla^{\perp}_X \xi) = -B(s\xi, X) - h(A_\xi X), \tag{17}$$

$$\nabla_X(s\xi) - s(\nabla^{\perp}_X \xi) = -j(A_\xi X) + A_{t\xi}X, \tag{18}$$

where $B : TM \times TM \longrightarrow NM$ is the second fundamental form and for any $\xi \in TM$, A_ξ is the Weingarten operator associated with ξ and defined by $\widetilde{g}(A_\xi X, Y) = \widetilde{g}(B(X,Y), \xi)$ for any vectors X, Y tangent to M. Finally, from (8), we deduce that the Gauss, Codazzi and Ricci equations are respectively given by

$$\begin{aligned} R(X,Y)Z &= c\bigg[\langle Y,Z\rangle X - \langle X,Z\rangle Y + \langle jY,Z\rangle jX - \langle jX,Z\rangle jY \\ &\quad +2\langle X,jY\rangle jZ\bigg] + A_{B(Y,Z)}X - A_{B(X,Z)}Y, \\ (\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) &= c\bigg[\langle jY,Z\rangle hX - \langle jX,Z\rangle hY + 2\langle jX,Y\rangle hZ\bigg], \\ R^{\perp}(X,Y)\xi &= c\bigg[\langle hY,\xi\rangle hX - \langle hX,\xi\rangle hY + 2\langle jX,Y\rangle t\xi\bigg] \\ &\quad +B(A_\xi Y, X) - B(A_\xi X, Y). \end{aligned}$$

In the local orthonormal frames $\{e_1, e_2\}$ and $\{\nu_1, \nu_2\}$, these equations become (for $c = 1$):

$$K_M = 1 + < B_{22}, B_{11} > - |B_{12}|^2 + 3 < je_1, e_2 >^2, \tag{19}$$

$$K_N = - < [S_{\nu_1}, S_{\nu_2}](e_1), e_2 > + h_{21}h_{12} - h_{11}h_{22} + 2j_{12}t_{12}, \tag{20}$$

$$(\nabla_{e_1} B)(e_2, e_k) - (\nabla_{e_2} B)(e_1, e_k) = j_{2k}he_1 - j_{1k}he_2 + 2j_{12}he_k \tag{21}$$

It is clear that equations (10) to (21) are necessary condition for surfaces in $\mathbb{C}P^2(4c)$. Conversely, given (M^2, g) a Reimannian surface, E a 2-dimensional vector bundle over M endowed with a scalar product $\overline{g}$ and a compatible connection ∇^E. Let $j : TM \longrightarrow TM$, $h : TM \longrightarrow E$, $s : E \longrightarrow TM$ and $t : E \longrightarrow E$ four tensors.

Definition 3.1. *We say that $(M, g, E, \overline{g}, \nabla^E, B, j, h, s, t)$ satisfies the compatibility equations for $\mathbb{C}P^2(4c)$ if j and t are antisymmetric the Gauss, Codazzi and Ricci equations* (19) (20) (21) *and equations* (10)-(17) *are fulfilled.*

Now, we can state the following classical *Fundamental Theorem* for surfaces of $\mathbb{C}P^2$, which can be found for instance as a special case of the general result of P. Piccione and D. Tausk [30].

Proposition 3.2. *If $(M, g, E, \overline{g}, \nabla^E, B, j, h, s, t)$ satisfies the compatibility equations for $\mathbb{C}P^2(4c)$ then, there exists an isometric immersion $\Phi : M \longrightarrow \mathbb{C}P^2(4c)$ such that the normal bundle of M for this immersion is isomorphic to E and such that the second fundamental form II and the normal connection $\nabla^{\perp}$ are given by B and ∇^E. Precisely, there exists a vector bundle isometry $\widetilde{\Phi} : E \longrightarrow NM$ so that*

$$II = \widetilde{\Phi} \circ B,$$
$$\nabla^{\perp}\widetilde{\Phi} = \widetilde{\Phi}\nabla^E.$$

Moreover, we have

$$J(\Phi_* X) = \Phi_*(jX) + \widetilde{\Phi}(hX),$$
$$J(\widetilde{\Phi}\xi) = \Phi_*(sX) + \widetilde{\Phi}(t\xi),$$

where J is the canonical complex structure of $\mathbb{C}P^2(4c)$ and this isometric immersion is unique up to an isometry of $\mathbb{C}P^2(4c)$.

3.2. **Special cases.** Two special cases are of particular interest and have been widely studied, the complex and Lagrangian surfaces.

A surface (M^2, g) of $\mathbb{C}P^2(4c)$ is said **complex** if the tangent bundle of M is stable by the complex structure of $\mathbb{C}P^2(4c)$, that is, $J(TM) = TM$. Note that we have necessarily $J(NM) = NM$. Hence, in that case, with the above notations, we have $h = 0$, $s = 0$ and so j and t are respectively almost complex structures on TM and NM. The compatibility equations for complex surfaces of $\mathbb{C}P^2(4c)$ becomes

$$\begin{cases} h = 0,\ s = 0,\ j^2 = -\mathrm{id}_{TM},\ t^2 = -\mathrm{id}_E \\ \nabla j = 0,\ \nabla^\perp t = 0 \\ t(B(X,Y)) - B(X, jY) = 0,\ \forall X \in TM \\ K_M = 4 + < B_{22}, B_{11} > - |B_{12}|^2 \\ K_N = - < [S_{\nu_1}, S_{\nu_2}](e_1), e_2 > + 2 \\ (\nabla_{e_1} B)(e_2, e_k) - (\nabla_{e_2} B)(e_1, e_k) = 0 \end{cases} \tag{22}$$

A surface (M^2, g) of $\mathbb{C}P^2(4c)$ is said totally real if $J(TM)$ is transversal to TM. In the particular case when $J(TM) = NM$, we say that (M^2, g) is **Lagrangian**. In that case, we have $j = 0$ and $t = 0$. Hence, the compatibility equations for Lagranigan surfaces of $\mathbb{C}P^2(4c)$ are

$$\begin{cases} j = 0,\ t = 0,\ sh = -\mathrm{id}_{TM},\ hs = -\mathrm{id}_E \\ \nabla s = 0,\ \nabla^\perp h = 0 \\ A_{hY} X + s(B(X,Y)) = 0,\ \forall X \in TM \\ K_M = 1 + < B_{22}, B_{11} > - |B_{12}|^2 \\ K_N = - < [S_{\nu_1}, S_{\nu_2}](e_1), e_2 > + h_{21}h_{12} - h_{11}h_{22} \\ (\nabla_{e_1} B)(e_2, e_k) - (\nabla_{e_2} B)(e_1, e_k) = 0 \end{cases} \tag{23}$$

4. Restriction of a Kählerian Killing Spin^c spinor and curvature computation

We consider a special Spin^c structure on $\mathbb{C}P^2$ carrying a (real) Kählerian Killing spinor φ. For example, on can take $q = -1$ and hence $r = 1$. For this structure, the curvature of the line bundle is given by $F_A(X, Y) = -2ig(JX, Y)$. The spinor $\varphi = \varphi_0 + \varphi_1 \in \Gamma(\Sigma_0 \mathbb{C}P^2 \oplus \Sigma_1 \mathbb{C}P^2)$ satisfies the following:

$$\begin{cases} \widetilde{\nabla}_X \varphi_0 = -X^- \cdot \varphi_1, \\ \widetilde{\nabla}_X \varphi_1 = -X^+ \cdot \varphi_0, \end{cases}$$

Thus, we have

$$\nabla_X \varphi = -\frac{1}{2} X \cdot \varphi + \frac{i}{2} JX \cdot \overline{\varphi},$$

where $\overline{\varphi} = \varphi_0 - \varphi_1$ is the conjugate of φ for the action of the complex volume element $\omega_4^{\mathbb{C}} = -e_1 \cdot e_2 \cdot e_3 \cdot e_4$. Indeed, $\Sigma_0 \mathbb{C}P^2 \subset \Sigma^+ \mathbb{C}P^2$ and $\Sigma_1 \mathbb{C}P^2 = \Sigma^- \mathbb{C}P^2$. Note also that such as spinor is of constant norm and each part φ_0 and φ_1 does not have any zeros. Indeed, for instance, if φ_0 vanishes at one point, then it must vanish everywhere (as it is obtained by parallel transport) and φ_1 is as a parallel spinor which is not the case for this Spin^c structure.

Now, let M be a surface of $\mathbb{C}P^2$ with normal bundle denoted by NM. By the identification of the Clifford multiplications and the Spin^c Gauss formula, we have

$$\nabla_X \varphi = -\frac{1}{2} \eta(X) \cdot \varphi - \frac{1}{2} X \cdot \varphi + \frac{i}{2} JX \cdot \overline{\varphi}.$$

In intrinsic terms, it can be written as

$$\nabla_X \varphi = -\frac{1}{2} \eta(X) \cdot \varphi - \frac{1}{2} X \cdot \varphi + \frac{i}{2} jX \cdot \overline{\varphi} + \frac{i}{2} hX \cdot \overline{\varphi}, \tag{24}$$

where η is given by

$$\eta(X)=\sum_{j=1}^{2}e_j\cdot B(e_j,X). \tag{25}$$

Here B is the second fundamental form of the immersion, and the operators j and h are those introduces in Section 3. We deduce immediately that

$$\nabla_X\overline{\varphi}=-\frac{1}{2}\eta(X)\cdot\overline{\varphi}+\frac{1}{2}X\cdot\overline{\varphi}-\frac{i}{2}jX\cdot\varphi-\frac{i}{2}hX\cdot\varphi.$$

Now, let us go back to an instrinsic setting by considering (M^2,g) an oriented Riemannian surface and E an oriented vector bundle of rank 2 over M with scalar product $\langle\cdot,\cdot\rangle_E$ and compatible connection ∇^E. We denote by $\Sigma=\Sigma M\otimes\Sigma E$ the twisted spinor bundle. Let $B:TM\times TM\longrightarrow E$ a bilinear symmetric map and $j:TM\longrightarrow TM,\ h:TM\longrightarrow E$ two tensors. We assume that the spinor field $\varphi\in\Gamma(\Sigma)$ satisfies Equation (24). We will compute the spinorial curvature for this spinor field φ. For this, let $\{e_1,e_2\}$ be a normal local orthonormal frame of TM and $\{\nu_1,\nu_2\}$ a local orthonormal frame of E. We have

$$\begin{aligned}
\nabla_{e_1}\nabla_{e_2}\varphi &= -\frac{1}{2}\nabla_{e_1}(\eta(e_2))\cdot\varphi+\frac{1}{4}\eta(e_2)\cdot\eta(e_1)\cdot\varphi+\frac{1}{4}\eta(e_2)\cdot e_1\cdot\varphi\\
&\quad -\frac{i}{4}\eta(e_2)\cdot j(e_1)\cdot\overline{\varphi}-\frac{i}{4}\eta(e_2)\cdot h(e_1)\cdot\overline{\varphi}+\frac{1}{4}e_2\cdot\eta(e_1)\cdot\varphi-\frac{1}{2}\nabla_{e_1}e_2\cdot\varphi\\
&\quad +\frac{1}{4}e_2\cdot e_1\cdot\varphi-\frac{i}{4}e_2\cdot j(e_1)\cdot\overline{\varphi}-\frac{i}{4}e_2\cdot h(e_1)\cdot\overline{\varphi}\\
&\quad +\frac{i}{2}\nabla_{e_1}(j(e_2))\cdot\overline{\varphi}-\frac{i}{4}j(e_2)\cdot\eta(e_1)\cdot\overline{\varphi}+\frac{i}{4}j(e_2)\cdot e_1\cdot\overline{\varphi}\\
&\quad +\frac{1}{4}j(e_2)\cdot j(e_1)\cdot\varphi+\frac{1}{4}j(e_2)\cdot h(e_1)\cdot\varphi+\frac{i}{2}\nabla^{\perp}_{e_1}(h(e_2))\cdot\overline{\varphi}\\
&\quad -\frac{i}{4}h(e_2)\cdot\eta(e_1)\cdot\overline{\varphi}+\frac{i}{4}h(e_2)\cdot e_1\cdot\overline{\varphi}+\frac{1}{4}h(e_2)\cdot j(e_1)\cdot\varphi\\
&\quad +\frac{1}{4}h(e_2)\cdot h(e_1)\cdot\varphi\\
&= -\frac{1}{2}\nabla_{e_1}(\eta(e_2))\cdot\varphi+\frac{1}{4}\eta(e_2)\cdot\eta(e_1)\cdot\varphi+\frac{1}{4}e_2\cdot e_1\cdot\varphi\\
&\quad -\frac{1}{2}\nabla_{e_1}e_2\cdot\varphi+\frac{1}{4}\eta(e_2)\cdot e_1\cdot\varphi+\frac{1}{4}e_2\cdot\eta(e_1)\cdot\varphi\\
&\quad +\frac{i}{2}\nabla_{e_1}(j(e_2))\cdot\overline{\varphi}+\frac{i}{2}\nabla^{\perp}_{e_1}(h(e_2))\cdot\overline{\varphi}\\
&\quad -\frac{i}{4}e_2\cdot j(e_1)\cdot\overline{\varphi}+\frac{i}{4}j(e_2)\cdot e_1\cdot\overline{\varphi}\\
&\quad -\frac{i}{4}e_2\cdot h(e_1)\cdot\overline{\varphi}+\frac{i}{4}h(e_2)\cdot e_1\cdot\overline{\varphi}\\
&\quad +\frac{1}{4}j(e_2)\cdot je_1\cdot\varphi\\
&\quad +\frac{1}{4}h(e_2)\cdot he_1\cdot\varphi\\
&\quad +\frac{1}{4}\big(j(e_2)\cdot h(e_1)\cdot\varphi+h(e_2)\cdot je_1\cdot\varphi\big)\\
&\quad -\frac{i}{4}\big(\eta(e_2)\cdot h(e_1)\cdot\overline{\varphi}+h(e_2)\cdot\eta(e_1)\cdot\overline{\varphi}\big)\\
&\quad -\frac{i}{4}\big(\eta(e_2)\cdot j(e_1)\cdot\overline{\varphi}+j(e_2)\cdot\eta(e_1)\cdot\overline{\varphi}\big)\\
&= I+II+III+IV+V+VI+VII+IIX+IX+X,
\end{aligned}$$

where

$$I = -\frac{1}{2}\nabla_{e_1}(\eta(e_2))\cdot\varphi + \frac{1}{4}\eta(e_2)\cdot\eta(e_1)\cdot\varphi + \frac{1}{4}e_2\cdot e_1\cdot\varphi - \frac{1}{2}\nabla_{e_1}e_2\cdot\varphi$$

$$II = +\frac{i}{2}\nabla_{e_1}(j(e_2))\cdot\overline{\varphi} + \frac{i}{2}\nabla^{\perp}_{e_1}(h(e_2))\cdot\overline{\varphi}$$

$$III = -\frac{i}{4}e_2\cdot j(e_1)\cdot\overline{\varphi} + \frac{i}{4}j(e_2)\cdot e_1\cdot\overline{\varphi}$$

$$IV = -\frac{i}{4}e_2\cdot h(e_1)\cdot\overline{\varphi} + \frac{i}{4}h(e_2)\cdot e_1\cdot\overline{\varphi}$$

$$V = +\frac{1}{4}j(e_2)\cdot je_1\cdot\varphi$$

$$VI = +\frac{1}{4}h(e_2)\cdot he_1\cdot\varphi$$

$$VII = +\frac{1}{4}\big(j(e_2)\cdot h(e_1)\cdot\varphi + h(e_2)\cdot je_1\cdot\varphi\big)$$

$$IIX = -\frac{i}{4}\big(\eta(e_2)\cdot h(e_1)\cdot\overline{\varphi} + h(e_2)\cdot\eta(e_1)\cdot\overline{\varphi}\big)$$

$$IX = -\frac{i}{4}\big(\eta(e_2)\cdot j(e_1)\cdot\overline{\varphi} + j(e_2)\cdot\eta(e_1)\cdot\overline{\varphi}\big)$$

$$X = \frac{1}{4}\eta(e_2)\cdot e_1\cdot\varphi + \frac{1}{4}e_2\cdot\eta(e_1)\cdot\varphi$$

We point out that

$$\nabla_{[e_1,e_2]}\varphi = \underbrace{-\frac{1}{2}\eta([e_1,e_2])\cdot\varphi - \frac{1}{2}[e_1,e_2]\cdot\varphi}_{\widetilde{I}([e_1,e_2])} + \underbrace{\frac{i}{2}j([e_1,e_2])\cdot\overline{\varphi} + \frac{i}{2}h([e_1,e_2])\cdot\overline{\varphi}}_{\widetilde{II}([e_1,e_2])}.$$

Some terms are vanishing as shown in the following lemma.

Lemma 4.1. *We have*

$$X(e_1,e_2) - X(e_2,e_1) = 0 \tag{26}$$

$$IV(e_1,e_2) - IV(e_2,e_1) = 0 \tag{27}$$

$$(II + IIX + IX)(e_1,e_2) - (II + IIX + IX)(e_2,e_1) - \widetilde{II}([e_1,e_2]) = 0 \tag{28}$$

Proof:

(1) Using the definition of η, we get $-\frac{1}{2}B(e_j,X) = e_j\cdot\eta(X) - \eta(X)\cdot e_j$. Hence

$$X(e_1,e_2) - X(e_2,e_1) = -\frac{1}{8}B(e_2,e_1) + \frac{1}{8}B(e_1,e_2) = 0.$$

(2)

$$
\begin{aligned}
& IV(e_1,e_2)-IV(e_2,e_1)= \\
& -\frac{i}{4}(e_2\cdot h(e_1)-h(e_2)\cdot e_1)\cdot\overline{\varphi}+\frac{i}{4}(e_1\cdot h(e_2)-h(e_1)\cdot e_2)\cdot\overline{\varphi} \\
= \; & \frac{i}{4}(2g(e_2,h(e_1))-2g(h(e_2),e_1))\cdot\overline{\varphi} \\
= \; & 0, \qquad (29)
\end{aligned}
$$

because X and $h(X)$ are orthogonal for $X\in\Gamma(TM)$.

(3) First we have

$$
\begin{aligned}
& II(e_1,e_2)-II(e_2,e_1)-\widetilde{II}([e_1,e_2]) \\
= \; & \frac{i}{2}\nabla_{e_1}(j(e_2))\cdot\overline{\varphi}+\frac{i}{2}\nabla^{\perp}_{e_1}(h(e_2))\cdot\overline{\varphi}-\frac{i}{2}\nabla_{e_2}(j(e_1))\cdot\overline{\varphi}-\frac{i}{2}\nabla^{\perp}_{e_2}(h(e_1))\cdot\overline{\varphi} \\
& -\frac{i}{2}j([e_1,e_2])\cdot\overline{\varphi}-\frac{i}{2}h([e_1,e_2])\cdot\overline{\varphi} \\
= \; & \frac{i}{2}\Big((\nabla_{e_1}j)e_2\cdot\overline{\varphi}+(\nabla_{e_1}h)e_2\cdot\overline{\varphi}-(\nabla_{e_2}j)e_1\cdot\overline{\varphi}-(\nabla_{e_2}h)e_1\cdot\overline{\varphi}\Big) \\
= \; & \frac{i}{2}\Big(s(B(e_1,e_2))\cdot\overline{\varphi}+S_{h(e_2)}e_1\cdot\overline{\varphi}+t(B(e_1,e_2))\cdot\overline{\varphi}-B(e_1,j(e_2))\cdot\overline{\varphi} \\
& -s(B(e_1,e_2))\cdot\overline{\varphi}-S_{h(e_1)}e_2\cdot\overline{\varphi}-t(B(e_1,e_2))\cdot\overline{\varphi}+B(e_2,je_1)\cdot\overline{\varphi}\Big) \\
= \; & \frac{i}{2}\Big(S_{h(e_2)}e_1\cdot\overline{\varphi}-S_{h(e_1)}e_2\cdot\overline{\varphi}-B(e_1,j(e_2))\cdot\overline{\varphi}+B(e_2,je_1)\cdot\overline{\varphi}\Big) \qquad (30)
\end{aligned}
$$

Moreover, we calculate

$$
\begin{aligned}
& -B(e_1,j(e_2))\cdot\overline{\varphi}+B(e_2,je_1)\cdot\overline{\varphi} \\
= \; & -g(j(e_2),e_1)B(e_1,e_1)\cdot\overline{\varphi}+g(e_2,j(e_1))B(e_2,e_2)\cdot\overline{\varphi} \\
= \; & 2g(j(e_1),e_2)H\cdot\overline{\varphi} \qquad (31)
\end{aligned}
$$

and

$$
\begin{aligned}
& S_{h(e_2)}e_1\cdot\overline{\varphi}-S_{h(e_1)}e_2\cdot\overline{\varphi} \\
= \; & -<S_{h(e_1)}e_2,e_1>e_1\cdot\overline{\varphi}-<S_{h(e_1)}e_2,e_2>e_2\cdot\overline{\varphi} \\
& +<S_{h(e_2)}e_1,e_1>e_1\cdot\overline{\varphi}+<S_{h(e_2)}e_1,e_2>e_2\cdot\overline{\varphi} \\
= \; & -<B(e_2,e_1),h(e_1)>e_1\cdot\overline{\varphi}-<B(e_2,e_2),h(e_1)>e_2\cdot\overline{\varphi} \\
& +<B(e_1,e_1),h(e_2)>e_1\cdot\overline{\varphi}+<B(e_1,e_2),h(e_2)>e_2\cdot\overline{\varphi} \qquad (32)
\end{aligned}
$$

In addition we have

$$
\begin{aligned}
& IIX(e_1,e_2)-IIX(e_2,e_1) \\
= \; & \frac{i}{4}\Big(-e_1\cdot B(e_1,e_2)\cdot h(e_1)-e_2\cdot B(e_2,e_2)\cdot h(e_1)-h(e_2)\cdot e_1\cdot B(e_1,e_1)-h(e_2)\cdot e_2\cdot B(e_1,e_2) \\
& +e_1\cdot B(e_1,e_1)\cdot h(e_2)+e_2\cdot B(e_1,e_2)\cdot h(e_2)+h(e_1)\cdot e_1\cdot B(e_1,e_2)+h(e_1)\cdot e_2\cdot B(e_2,e_2)\Big)\cdot\overline{\varphi} \\
= \; & \frac{i}{4}\Big(2<B(e_1,e_2),h(e_1)>e_1+2<B(e_2,e_2),h(e_1)>e_2 \\
& -2<B(e_1,e_2),h(e_2)>e_2-2<B(e_1,e_1),h(e_2)>e_1\Big)\cdot\overline{\varphi} \qquad (33)
\end{aligned}
$$

and

$$
\begin{aligned}
& IX(e_1,e_2)-IX(e_2,e_1)\\
= \;& -\frac{i}{4}\eta(e_2)\cdot j(e_1)\cdot\overline{\varphi}-\frac{i}{4}j(e_2)\cdot\eta(e_1)\cdot\overline{\varphi}+\frac{i}{4}\eta(e_1)\cdot j(e_2)\cdot\overline{\varphi}+\frac{i}{4}j(e_1)\cdot\eta(e_2)\cdot\overline{\varphi}\\
= \;& \frac{i}{4}\Big(-e_1\cdot B(e_1,e_2)\cdot j(e_1)\cdot -e_2\cdot B(e_2,e_2)\cdot j(e_1)\cdot -j(e_2)\cdot e_1\cdot B(e_1,e_1)\cdot\\
& -j(e_2)\cdot e_2\cdot B(e_1,e_2)\cdot +e_1\cdot B(e_1,e_1)\cdot j(e_2)\cdot +e_2\cdot B(e_1,e_2)\cdot j(e_2)\cdot\\
& +j(e_1)\cdot e_1\cdot B(e_2,e_1)\cdot +j(e_1)\cdot e_2\cdot B(e_2,e_2)\cdot\Big)\overline{\varphi}\\
= \;& \frac{i}{4}\Big(e_1\cdot j(e_1)\cdot B(e_1,e_1)+e_2\cdot j(e_1)\cdot B(e_2,e_2)-j(e_2)\cdot e_1\cdot B(e_1,e_1)-j(e_2)\cdot e_2\cdot B(e_1,e_2)\\
& -e_1\cdot j(e_2)\cdot B(e_1,e_1)-e_2\cdot j(e_2)\cdot B(e_1,e_2)+j(e_1)\cdot e_1\cdot B(e_1,e_2)+j(e_1)\cdot e_2\cdot B(e_2,e_2)\Big)\cdot\overline{\varphi}\\
= \;& \frac{i}{4}\big(-2g(j(e_1),e_2)B(e_2,e_2)+2g(j(e_2,e_1))B(e_1,e_1)\big)\cdot\overline{\varphi}\\
= \;& -ig(j(e_1,e_2))H\cdot\overline{\varphi}. && (34)
\end{aligned}
$$

Now, replacing (31) and (32) in (30) and combining together with (33) and (34), we get the desired result.

□

Now, we have this second lemma.

Lemma 4.2. *We have*

(1)

$$V(e_1,e_2)-V(e_2,e_1)=-\frac{1}{2}\langle j(e_2),e_1)\rangle^2 e_1\cdot e_2,$$

(2)

$$VI(e_1,e_2)-VI(e_2,e_1)=\frac{1}{2}\left[\langle h(e_2),\nu_1\rangle\langle h(e_1),\nu_2\rangle-\langle h(e_1),\nu_1\rangle\langle h(e_2),\nu_2\rangle\right]\nu_1\cdot\nu_2.$$

(3)

$$III(e_1,e_2)-III(e_2,e_1)=ig(e_2,je_1)\overline{\varphi} \tag{35}$$

(4)

$$VII(e1,e_2)-VII(e_2,e_1)=\frac{1}{2}\big(j_{21}h_{11}e_1\cdot\nu_1+j_{21}h_{12}e_1\cdot\nu_2+j_{21}h_{21}e_2\cdot\nu_1+j_{12}h_{22}e_2\cdot\nu_2\big)\cdot\varphi \tag{36}$$

Proof.

(1) We denote by $j_{kl}=g(je_k,e_l)$. Since j is antisymmetric, we have $j_{kl}=-j_{lk}$ and so

$$
\begin{aligned}
& V(e_1,e_2)-V(e_2,e_1)\\
= \;& \frac{1}{4}\big(j(e_2)\cdot j(e_1)-j(e_1)\cdot j(e_2)\big)\cdot\varphi\\
= \;& \frac{1}{4}(j_{21}j_{12}e_1\cdot e_2-j_{12}j_{21}e_2\cdot e_1)\cdot\varphi\\
= \;& \frac{1}{2}j_{21}j_{12}e_1\cdot e_2\cdot\varphi\\
= \;& -\frac{1}{2}g(j(e_1),e_2)^2 e_1\cdot e_2\cdot\varphi && (37)
\end{aligned}
$$

(2)

$$
\begin{aligned}
& VI(e_1,e_2)-VI(e_2,e_1)\\
= & \frac{1}{4}(h(e_2)\cdot h(e_1)-h(e_1)\cdot h(e_2))\\
= & \frac{1}{4}\big(-h_{21}h_{11}+h_{21}h_{12}\nu_1\cdot\nu_2+h_{22}h_{11}\nu_2\cdot\nu_1-h_{22}h_{12}+h_{11}h_{21}\\
& -h_{11}h_{22}\nu_1\cdot\nu_2-h_{12}h_{21}-h_{11}h_{22}\nu_1\cdot\nu_2-h_{12}h_{21}\nu_2\cdot\nu_1+h_{12}h_{22}\big)\cdot\varphi\\
= & \frac{1}{2}[g(h(e_2),\nu_1)g(h(e_1),\nu_2)-g(h(e_1),\nu_1)g(h(e_2),\nu_2)]\nu_1\cdot\nu_2\cdot\varphi
\end{aligned}
$$

(3)

$$
\begin{aligned}
III(e_1,e_2)-III(e_2,e_1) &= -\frac{i}{4}\big(e_2\cdot j(e_1)-j(e_2)\cdot e_1-e_1\cdot j(e_2)+j(e_1)\cdot e_2\big)\cdot\overline{\varphi}\\
&= -\frac{i}{4}\big(-j_{12}+j_{21}+j_{21}-j_{12}\big)\cdot\overline{\varphi}\\
&= ig(e_2,j(e_1))\overline{\varphi} \qquad (38)
\end{aligned}
$$

(4) We have

$$
\begin{aligned}
& VII(e_1,e_2)-VII(e_2,e_1)\\
= & \frac{1}{4}\big(j(e_2)\cdot h(e_1)+h(e_2)\cdot je_1-j(e_1)\cdot h(e_2)-h(e_1)\cdot je_2\big)\cdot\varphi\\
= & \frac{1}{4}\big(j_{21}e_1\cdot(h_{11}\nu_1+h_{12}\nu_2)+j_{12}(h_{21}\nu_1+h_{22}\nu_2)e_2\\
& -j_{12}e_2\cdot(h_{21}\nu_1+h_{22}\nu_2)-j_{21}(h_{11}\nu_1+h_{12}\nu_2)e_1\big)\cdot\varphi\\
= & \frac{1}{2}\big(j_{21}h_{11}e_1\cdot\nu_1+j_{21}h_{12}e_1\cdot\nu_2+j_{21}h_{21}e_2\cdot\nu_1+j_{12}h_{22}e_2\cdot\nu_2\big)\cdot\varphi \qquad (39)
\end{aligned}
$$

□

Finally, we have this last lemma obtained by a straightforward calculation and using Lemma 3.3 of [4].

Lemma 4.3.

$$
\begin{aligned}
& I(e_1,e_2)-I(e_2,e_1)-\widetilde{I}(e_1,e_2)\\
= & -\frac{1}{2}\sum_{j=1}^{2}e_j\cdot\big((\nabla'_{e_1}B)(e_2,e_j))-(\nabla'_{e_2}B)(e_1,e_j)\big)\\
& +\frac{1}{2}g([S_{\nu_1},S_{\nu_2}])(e_1),e_2)\nu_1\cdot\nu_2\cdot\varphi\\
& -\frac{1}{2}e_1\cdot e_2\cdot\varphi, \qquad (40)
\end{aligned}
$$

where ∇' is the natural connection on $T^*M\otimes T^*M\otimes E$ and $B_{kl}=B(e_k,e_l)$

5. Lagrangian case, proof of Theorem 1.2

Now, we have all the ingredients to prove Theorems 1.1 and 1.2. We begin by the Lagragian case. Assume that $j=0$ and $t=0$. So we have

$$
\mathcal{R}_{e_1,e_2}\varphi = \frac{1}{2}K_Me_1\cdot e_2\cdot\varphi-\frac{1}{2}K_E\nu_1\cdot\nu_2\cdot\varphi+\frac{i}{2}\alpha^{M+E}(e_1,e_2)\varphi, \qquad (41)
$$

with $\alpha^{M+E}(e_1,e_2)=0$ because $j=0$. From the other hand, we have

$$\begin{aligned}
& \mathcal{R}_{e_1,e_2}\varphi && (42)\\
= & -\frac{1}{2}\sum_{j=1}^{2} e_j\cdot((\nabla^{'}_{e_1}B)(e_2,e_j)-(\nabla^{'}_{e_2}B)(e_1,e_j))\cdot\varphi \\
& +\frac{1}{2}(|B_{12}|^2-<B_{11},B_{22}>)e_1\cdot e_2\cdot\varphi \\
& +\frac{1}{2}<[S_{\nu_1},S_{\nu_2}](e_1),e_2>\nu_1\cdot\nu_2\cdot\varphi \\
& -\frac{1}{2}e_1\cdot e_2\cdot\varphi+\frac{1}{2}(h_{21}h_{12}-h_{11}h_{22})\nu_1\cdot\nu_2\cdot\varphi && (43)
\end{aligned}$$

We get finally that $T\cdot\varphi=0$, where $T\in(\Lambda^2M\otimes 1\oplus TM\otimes E\oplus 1\otimes\Lambda^2E)$. Thus, since φ^+ and φ^- never vanish, we have $T=0$ by [4][Lemma 3.4] and it gives

$$K_M=<B_{11},B_{22}>-|B_{12}|^2+1,$$

$$K_E=-<[S_{\nu_1},S_{\nu_2}](e_1),e_2>-(h_{21}h_{12}-h_{22}h_{11}),$$

$$(\nabla_{e_1}B)(e_2,e_j)-(\nabla_{e_2}B)(e_1,e_j)=0,$$

which are Gauss, Ricci and Codazzi equations for a Lagrangian surface in $\mathbb{C}P^2$ and so the conditions (23) are fulfuilled. Hence, by Proposition 3.2, we conclude that there exists a Lagrangian isomertic immersion from (M,g) into $\mathbb{C}P^2$ with E as normal bundle and B as second fundamental form. This proves that assertion (2) of Theorem 1.2 implies assertion (1). The converse is immediate by the discussions of Sections 3 and 4. Theorem 1.2 is proven.

6. Complex case, proof of Theorem 1.1

Assume that $s=0$; $h=0$ so $\alpha^{M+E}(e_1,e_2)=-2$. We take $je_1=e_2$ and $t\nu_1=\nu_2$, i.e, $g(je_1,e_2)=g(t\nu_1,\nu_2)=1$.

We calculate and we get

$$\begin{aligned}
\mathcal{R}_{e_1,e_2}\varphi &= -\frac{1}{2}K_Me_1\cdot e_2\cdot\varphi-\frac{1}{2}K_N\nu_1\cdot\nu_2\cdot\varphi+\frac{i}{2}\alpha^{M+N}(e_1,e_2)\varphi \\
&= -\frac{1}{2}K_Me_1\cdot e_2\cdot\varphi-\frac{1}{2}K_N\nu_1\cdot\nu_2\cdot\varphi-i\varphi \\
&= \overline{T}\cdot\varphi-i\varphi, && (44)
\end{aligned}$$

where $\overline{T}$ is a 2-form. From the other hand, we have

$$\begin{aligned}
\mathcal{R}_{e_1,e_2}\varphi = & -e_1\cdot e_2\cdot\varphi+i\overline{\varphi} \\
& -\frac{1}{2}\sum_{j=1}^{2} e_j\cdot((\nabla_{e_1}B)(e_2,e_j)-(\nabla_{e_2}B)(e_1,e_j))\cdot\varphi \\
& +\frac{1}{2}(|B_{12}|^2-<B_{11},B_{22}>)e_1\cdot e_2\cdot\varphi \\
& +\frac{1}{2}<[S_{\nu_1},S_{\nu_2}](e_1),e_2>\nu_1\cdot\nu_2\cdot\varphi \\
& =\tilde{T}\cdot\varphi+i\overline{\varphi}. && (45)
\end{aligned}$$

Together, it gives $\overline{T} \cdot \varphi - \tilde{T} \cdot \overline{\varphi} - i\varphi - i\overline{\varphi} = 0$, which means that $\mathcal{T} \cdot \varphi - i\varphi - i\overline{\varphi} = 0$, where $\mathcal{T}$ is again a 2 form given by

$$\begin{aligned} \mathcal{T} &= -\frac{1}{2}K_M e_1 \cdot e_2 \cdot \varphi - \frac{1}{2}K_N \nu_1 \cdot \nu_2 \cdot \varphi + e_1 \cdot e_2 \cdot \varphi \\ &\quad -\frac{1}{2}(|B_{12}|^2 - < B_{11}, B_{22} >) e_1 \cdot e_2 \cdot \varphi \\ &\quad -\frac{1}{2} < [S_{\nu_1}, S_{\nu_2}](e_1), e_2 > \nu_1 \cdot \nu_2 \cdot \varphi \\ &\quad +\frac{1}{2}\sum_{j=1}^{2} e_j \cdot ((\nabla_{e_1} B)(e_2, e_j) - (\nabla_{e_2} B)(e_1, e_j)) \cdot \varphi \end{aligned} \tag{46}$$

We give now the following Lemma

Lemma 6.1. *Let $\mathcal{T}$ be a 2 form, i.e, $\mathcal{T} \in \Lambda^2 M \otimes 1 \oplus \Lambda^1 M \otimes \Lambda^1 E \oplus 1 \otimes \Lambda^2 E$. Assume that*

$$\mathcal{T} \cdot \varphi - i\varphi - i\overline{\varphi} = 0,$$

and write $\mathcal{T} = T^t e_1 \cdot e_2 + T^n \nu_1 \cdot \nu_2 + T^m$, where $T^m \in \Lambda^1 M \otimes \Lambda^1 E$. Then,

$$T^t = -1, T^n = 0, \quad \textit{and} \quad T^m = 0.$$

Proof. Let $\varphi = \varphi^+ + \varphi^-$, with

$$\varphi^+ = \varphi^{++} + \varphi^{--},$$

$$\varphi^- = \varphi^{-+} + \varphi^{+-},$$

a solution of (24) with $h = 0$. This means that

$$\begin{aligned} \nabla_X \varphi^{++} &= -\frac{1}{2}X \cdot \varphi^{-+} - \frac{i}{2}jX \cdot \varphi^{-+} \\ \nabla_X \varphi^{+-} &= -\frac{1}{2}X \cdot \varphi^{--} + \frac{i}{2}jX \cdot \varphi^{--} \\ \nabla_X \varphi^{-+} &= -\frac{1}{2}X \cdot \varphi^{++} + \frac{i}{2}jX \cdot \varphi^{++} \\ \nabla_X \varphi^{--} &= -\frac{1}{2}X \cdot \varphi^{+-} - \frac{i}{2}jX \cdot \varphi^{+-}. \end{aligned}$$

For a sake of simplicity, and without lost of generality, we can restrict only $\varphi^+ = \varphi^{++}$ and $\varphi^- = \varphi^{-+}$ which which have no zeros by assumption. The equation

$$\mathcal{T} \cdot \varphi - i\varphi - i\overline{\varphi} = 0,$$

becomes

$$T^t e_1 \cdot e_2 \cdot (\varphi^{++} + \varphi^{-+}) + (T^n + 1)\nu_1 \cdot \nu_2 \cdot (\varphi^{++} + \varphi^{-+}) + T^m \cdot (\varphi^{++} + \varphi^{-+}) = i\overline{\varphi} = i(\varphi^{++} - \varphi^{-+})$$

Taking the scalar product with φ^{++} then with φ^{-+}, we get

$$T^t + T^n + 1 = -1,$$

$$-T^t + T^n + 1 = 1,$$

which gives $T^n = -1$, $T^t = -1$ and $T^m = 0$. These are Gauss, Codazzi and Ricci equations and so the conditions (22) are fulfilled. There are exactly the conditions ofr a complex immersion. Hence, by Proposition 3.2, we conclude that there exists a complex isometric immersion from (M, g) into $\mathbb{C}P^2$ with E as normal bundle and B as second fundamental form. As for the Lagrangian case, this proves that assertion (2) of Theorem 1.1 implies assertion (1). Here again, the converse is immediate by the discussions of Sections 3 and 4, which concludes the proof of Theorem 1.1.

7. The Dirac equation

Let φ be a spinor field satisfying Equation (24), then it satisfies the following Dirac equation

$$D\varphi = \vec{H}\cdot\varphi - \varphi + \frac{i}{2}\beta\cdot\overline{\varphi}, \tag{47}$$

where β is the 2-form defined by $\beta = \sum_{i=1,2} e_i\cdot he_i = \sum_{i,j=1}^{2} h_{ij}e_i\cdot\xi_j$, where $h_{i,j} = \langle he_i, \xi_j\rangle$. As in [4] and [32], we will show that this equation with an appropiate condition on the norm of both φ^+ and φ^- is equivalent to Equation (24), where the tensor B is expressed in terms of the spinor field φ and such that $\mathrm{tr}(B) = 2\vec{H}$. Moreover, from Equation (24) we deduce the following conditions on the derivatives of $|\varphi^+|^2$ and $|\varphi^-|^2$. Indeed, after decomposition onto Σ^+ and Σ^-, (24) becomes

$$\nabla_X\varphi^\pm = -\frac{1}{2}\eta(X)\cdot\varphi^\pm - \frac{1}{2}X\cdot\varphi^\mp \mp \frac{i}{2}jX\varphi^\mp \mp \frac{i}{2}hX\varphi^\mp.$$

From this we deduce that

$$X(|\varphi^\pm|^2) = \mathrm{Re}\left\langle -\frac{1}{2}X\cdot\varphi^\mp \mp \frac{i}{2}jX\cdot\varphi^\mp \mp \frac{i}{2}hX\cdot\varphi^\mp, \varphi^\pm\right\rangle \tag{48}$$

Now, let φ a spinor field solution of the Dirac equation (47) with φ^+ and φ^- nowhere vanishing and satisying the norm condition (48), we set for any vector fields X and Y tangent to M and $\xi\in E$

$$\begin{aligned}\langle B(X,Y),\xi\rangle \quad = \quad & \frac{1}{|\varphi^+|^2}\mathrm{Re}\left\langle X\cdot\nabla_Y\varphi^+ - \frac{1}{2}\left(X + ijX + ihX\right)\cdot Y\cdot\varphi^-, \xi\cdot\varphi^+\right\rangle \\ & + \frac{1}{|\varphi^-|^2}\mathrm{Re}\left\langle X\cdot\nabla_Y\varphi^- - \frac{1}{2}\left(X - ijX - ihX\right)\cdot Y\cdot\varphi^-, \xi\cdot\varphi^+\right\rangle\end{aligned} \tag{49}$$

Then, we have the following

Proposition 7.1. *Let $\varphi\in\Gamma(\Sigma)$ satisfying the Dirac equation* (47)

$$D\varphi = \vec{H}\cdot\varphi - \varphi - \beta\cdot\overline{\varphi}$$

such that

$$X(|\varphi^\pm|^2) = \mathrm{Re}\left\langle -\frac{1}{2}X\cdot\varphi^\mp \mp \frac{i}{2}jX\cdot\varphi^\mp \mp \frac{i}{2}hX\cdot\varphi^\mp, \varphi^\pm\right\rangle$$

then φ is solution of Equation (24)

$$\nabla_X\varphi = -\frac{1}{2}\eta(X)\cdot\varphi - \frac{1}{2}X\cdot\varphi + \frac{i}{2}jX\cdot\overline{\varphi} + \frac{i}{2}hX\cdot\overline{\varphi}$$

where η is defined by $\eta(X) = -\frac{1}{2}\sum_{j=1}^{2} e_j\cdot B(e_j, X)$.

Moreover, B is symmetric.

The proof of this proposition will not be given, since it is completely similar to the case of Riemannian products [32]. Now, combining this proposition with Theorems 1.1 and 1.2, we get the following corollaries. We have this first one for complex surfaces.

Corollary 7.2. *Let (M^2, g) be an oriented Riemannian surface and E an oriented vector bundle of rank 2 over M with scalar product $<\cdot,\cdot>_E$ and compatible connection ∇^E. We denote by $\Sigma = \Sigma M\otimes\Sigma E$ the twisted spinor bundle. Let j be a complex structure on M and t a complex structure on E. Let $\vec{H}$ be a section of E. Then, the two following statements are equivalent*

(1) *There exists a* Spin^c *structure on* $\Sigma M \otimes \Sigma E$ *with* $\alpha^{M+E}(e_1, e_2) = 0$ *and a spinor field* φ *in* Σ *solution of the Dirac equation*

$$D\varphi = \vec{H} \cdot \varphi - \varphi$$

such that φ^+ *and* φ^- *never vanish, satisfy the norm condition*

$$X(|\varphi^\pm|^2) = \mathrm{Re}\left\langle -\frac{1}{2}X \cdot \varphi^\mp \mp \frac{i}{2}jX \cdot \varphi^\mp \varphi^\pm \right\rangle$$

and such that the maps j, t *and the tensor* B *defined by* (49) *satisfy* $t(B(X,Y)) = B(X, jY)$ *for all* $X, Y \in \mathfrak{X}(M)$.

(2) *There exists an isometric* **complex** *immersion of* (M^2, g) *into* $\mathbb{C}P^2$ *with* E *as normal bundle and mean curvature* $\vec{H}$ *such that over* M *the complex strcuture of* $\mathbb{C}P^2$ *is given by* j *and* t *(in the sense of Proposition 3.2).*

We have this second corollary for Lagrangian surfaces.

Corollary 7.3. *Let* (M^2, g) *be an oriented Riemannian surface and* E *an oriented vector bundle of rank 2 over* M *with scalar product* $< \cdot, \cdot >_E$ *and compatible connection* ∇^E. *We denote by* $\Sigma = \Sigma M \otimes \Sigma E$ *the twisted spinor bundle. Let* $B : TM \times TM \longrightarrow E$ *a bilinear symmetric map,* $h : TM \longrightarrow E$ *and* $s : E \longrightarrow TM$ *the dual map of* h. *Assume that the maps* h, s *are parallel and satisfy* $hs = -\mathrm{id}_E$. *Let* $\vec{H}$ *be a section of* E. *Then, the two following statements are equivalent*

(1) *There exists a* Spin^c *structure on* $\Sigma M \otimes \Sigma E$ *with* $\alpha^{M+E}(e_1, e_2) = -2$ *and a spinor field* φ *in* Σ *solution of the Dirac equation*

$$D\varphi = \vec{H} \cdot \varphi - \varphi + \frac{i}{2}\beta \cdot \overline{\varphi}$$

such that φ^+ *and* φ^- *never vanish, satisfy the norm condition*

$$X(|\varphi^\pm|^2) = \mathrm{Re}\left\langle -\frac{1}{2}X \cdot \varphi^\mp \mp \frac{i}{2}hX \cdot \varphi^\mp \varphi^\pm \right\rangle$$

and such that the tensor B *defined by* (49) *satisfy* $A_{hY}X + s(B(X,Y)) = 0$, *for all* $X \in TM$, *where* $A_\nu : TM \longrightarrow TM$ *if defined by* $g(A_\nu X, Y) = \langle B(X,Y), \nu \rangle_E$ *for all* $X, Y \in TM$ *and* $\nu \in E$.

(2) *There exists an isometric* **Lagrangian** *immersion of* (M^2, g) *into* $\mathbb{C}P^2$ *with* E *as normal bundle and mean curvature* $\vec{H}$ *such that over* M *the complex strcuture of* $\mathbb{C}P^2$ *is given by* h *and* s *(in the sense of Proposition 3.2).*

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