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To cite this version:
Julien Roth. A DDVV INEQUALITY FOR SUBMANIFOLDS OF WARPED PRODUCTS. Bulletin of the Australian Mathematical Society, 2017, 95 (3), pp.495-499. hal-01338102

HAL Id: hal-01338102
https://hal.archives-ouvertes.fr/hal-01338102
Submitted on 27 Jun 2016
A DDVV INEQUALITY FOR SUBMANIFOLDS OF WARPED PRODUCTS

JULIEN ROTH

ABSTRACT. We prove a DDVV inequality for submanifolds of warped products of the form \( I \times_a \mathbb{M}^n(c) \) where \( I \) is an interval and \( \mathbb{M}^n(c) \) a real space form of curvature \( c \). As an application, we give a rigidity result for submanifolds of \( \mathbb{R} \times_a \mathbb{H}^n(c) \).

RÉSUMÉ. Une inégalité de type DDVV pour les sous-variétés des produits tordus.
Nous donnons une inégalité de type DDVV pour les sous-variétés des produits tordus de la forme \( I \times_a \mathbb{M}^n(c) \) où \( I \) est un interval et \( \mathbb{M}^n(c) \) un espace modélisé réel de courbure constante \( c \). Nous en déduisons un résultat de rigidité pour les sous-variétés de \( \mathbb{R} \times_a \mathbb{H}^n(c) \).

Version française abrégée.

Soit \( (M^m, g) \) une variété riemannienne immergée isométriquement dans une variété riemannienne ambiante \( (\mathbb{N}^{m+p}, \bar{g}) \), de dimension \( m + p \). Lorsque \( N \) est un espace modélisé simplement connexe à courbure sectionnelle constante \( c \), l’inégalité suivante est vérifiée :

\[
||H||^2 \geq \rho + \rho^\perp - c,
\]

où \( \rho = \frac{2}{n(n-1)} \sum_{i<j} \langle R(e_i, e_j) e_j, e_i \rangle \) est la courbure scalaire normalisée de \( (M, g) \) et \( \rho^\perp = \frac{2}{n(n-1)} \sum_{i<j} \sum_{\alpha<\beta} \langle R^\perp(e_i, e_j) \xi_\alpha, \xi_\beta \rangle \) la courbure scalaire normale (également normalisée), \( \{e_1, \ldots, e_n\} \) et \( \{\xi_1, \ldots, \xi_p\} \) étant respectivement des bases orthonormées locales de \( T M \) et \( T^\perp M \). Cette inégalité est connue sous le nom de conjecture de DDVV car conjecturée par De Smets-Dillen-Verstraelen-Vrancken \[2\]. La conjecture a été démontrée récemment par Lu \[6\] et par Ge-Tang \[4\] indépendamment. Plus récemment, Chen et Cui \[1\] ont obtenu une inégalité comparable dans le cas des espaces produits \( \mathbb{S}^n \times \mathbb{R} \) et \( \mathbb{H}^n \times \mathbb{R} \).

Le but de cette note est d’étendre le résultat de Chen-Cui pour les sous-variétés des produits tordus \( I \times_a \mathbb{M}^n(c) \), c’est-à-dire \( I \times \mathbb{M}^n(c) \) muni de la métrique \( \bar{g} = dt^2 + a(t)^2 g_{\mathbb{M}^n(c)} \), \( I \) étant un intervalle de \( \mathbb{R} \) et \( a : I \to \mathbb{R} \) une fonction lisse ne s’annulant pas. Nous démontrons le résultat suivant.

**Théorème 1.** Soient \( n > m \geq 2 \) deux entiers. Soit \( M^m \) une sous-variété du produit tordu \( I \times_a \mathbb{M}^n(c) \) avec courbure scalaire et courbure scalaire normale normalisées \( \rho \) et \( \rho^\perp \) et courbure moyenne \( H \). L’inégalité suivante est vérifiée :

\[
||H||^2 \geq \rho + \rho^\perp + \frac{(a')^2}{a^2} - \frac{c}{a^2} \left(1 - \frac{2}{n} ||T||^2 \right) - \frac{2a''}{na} ||T||^2.
\]

Nous en déduisons un résultat pour les surfaces des produits tordus \( \mathbb{R} \times_a \mathbb{H}^n(c) \).

**Corollaire 1.** Soit \( M^m \) une sous-variété complète sans bord du produit tordu \( \mathbb{R} \times_a \mathbb{H}^n(c) \) avec courbure scalaire et courbure scalaire normale normalisées \( \rho \) et \( \rho^\perp \) et courbure

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*Date: 30 mars 2016.*
warped products and \( \rho \) of product spaces Smets-Dillen-Verstrealen-Vrancken in [2] and proved recently by Lu [6] and by Ge-Tang for space forms. Indeed, \( T \) where scalar and normal scalar curvatures

\[
\text{Remark 1. Note that, of course, we recover the DDVV inequality of [1] for product spaces } S^n \times \mathbb{R} \text{ and } H^n \times \mathbb{R} \text{ as well as for } \mathbb{R}^{n+1} \text{ by taking } a = 1, \text{ but we also recover the inequality for space forms. Indeed, } S^n \text{ and } H^n \text{ can be expressed in term of warped products. Namely, we have}
\]

1. INTRODUCTION

Let \( (M^n, g) \) be a \( n \)-dimensional Riemannian manifold isometrically immersed into a \((n + p)\)-dimensional Riemannian manifold \((N^{n+p}, \bar{g})\). When the ambient space is a real space form of constant sectional curvature \( c \), we have the following pointwise inequality

\[
||H||^2 \geq \rho + \rho^\perp - c, \quad \text{and } \quad (1)
\]

where \( \rho = \frac{2}{n(n-1)} \sum_{i<j} (R(e_i, e_j)e_i, e_j) \) is the normalized scalar curvature of \((M, g)\) and \( \rho^\perp = \frac{2}{n(n-1)} \sum \sum_{i<j, \alpha<\beta} (R^\perp(e_i, e_j)\xi_\alpha, \xi_\beta) \) is the normalized normal curvature of the immersion, where \( \{e_1, \cdots, e_n\} \) and \( \{\xi_1, \cdots, \xi_p\} \) are respectively orthonormal frames of \( TM \) and \( T^\perp M \). This inequality, known as DDVV conjecture, was conjectured by De Smets-Dillen-Verstrealen-Vrancken in [2] and proved recently by Lu [6] and by Ge-Tang [4] independently. More recently, Chen and Cui generalized this inequality in the setting of product spaces \( S^n \times \mathbb{R} \) and \( H^n \times \mathbb{R} \).

In this note, we extend Chen-Cui result by proving a DDVV inequality for submanifolds of warped products \( I \times_a M^n(c) \) where \( I \subset \mathbb{R} \) is an interval and \( a : I \longrightarrow \mathbb{R} \) is a nowhere vanishing smooth function. We denote by \( \partial_t = \frac{\partial}{\partial t} \) the unit vector field tangent to the factor \( I \). We prove the following result.

**Theorem 1.** Let \( M^n \) be a submanifold of the warped product \( I \times_a M^n(c) \) with normalized scalar and normal scalar curvatures \( \rho \) and \( \rho^\perp \) and mean curvature \( H \). Then, we have

\[
||H||^2 \geq \rho + \rho^\perp + \left( \frac{(a')^2}{a^2} - \frac{c}{a^2} \right) \left( 1 - \frac{2}{n} ||T||^2 \right) - \frac{2a''}{na} ||T||^2, \quad \text{where } T \text{ is the part of } \partial t \text{ tangent to } M.
\]

**Remark 1.** Note that, of course, we recover the DDVV inequality of [1] for product spaces \( S^n \times \mathbb{R} \) and \( H^n \times \mathbb{R} \) as well as for \( \mathbb{R}^{n+1} \) by taking \( a = 1 \), but we also recover the inequality for space forms. Indeed, \( S^n \) and \( H^n \) can be expressed in term of warped products. Namely, we have

1. \( S^n = [0, 2\pi] \times_a S^{n-1} \text{ with } a(t) = \sin(t). \text{ Hence the inequality of Theorem 1 becomes } ||H||^2 \geq \rho + \rho^\perp - 1. \)
2. \( H^n = [0, +\infty[ \times_a S^{n-1} \text{ with } a(t) = \sinh(t) \text{ or } H^n = \mathbb{R} \times_a \mathbb{R}^{n-1} \text{ with } a(t) = \cosh(t). \text{ For both cases, the inequality of Theorem 1 becomes } ||H||^2 \geq \rho + \rho^\perp + 1. \)

2. PRELIMINARIES

Let \( M^n(c) \) be the simply connected real space form of dimension \( n \) and constant curvature \( c \). Let \( I \subset \mathbb{R} \) an interval and \( a : I \longrightarrow \mathbb{R} \) be a nowhere vanishing smooth function. We consider the warped \( \mathbb{R}^{n+1} = I \times_a M^n(c) \), that the product \( I \times M^n(c) \) endowed the metric \( \bar{g} = dt^2 + a(t)^2 g_{a(t)(c)} \). We denote by \( \partial_t = \frac{\partial}{\partial t} \) the unit vector field tangent to the factor \( I \). We recall (see [5] for instance) that the curvature tensor of \( (\mathbb{R}^{n+1}, \bar{g}) \) is given by

\[
\bar{R}(X, Y)Z = \left( \frac{(a')^2}{a^2} - \frac{c}{a^2} \right) \left( \langle X, Z \rangle Y - \langle Y, Z \rangle X \right) + \frac{a''}{a} - \left( \frac{(a')^2}{a^2} + \frac{c}{a^2} \right) \left( \langle X, Z \rangle \langle Y, \partial_t \rangle \partial_t - \langle Y, Z \rangle \langle X, \partial_t \rangle \partial_t \right) - \langle Y, \partial_t \rangle \langle Z, \partial_t \rangle X + \langle X, \partial_t \rangle \langle Z, \partial_t \rangle Y.
\]
Let \((M^m, g)\) be a Riemannian manifold isometrically immersed into \(\tilde{P}\). We denote by \(B\) its second fundamental form and \(A\) the shape operator defined for any \(X, Y \in \Gamma(TM)\) and \(\xi \in \Gamma(T^\perp M)\) by \(\langle B(X, Y), \xi \rangle = \langle A_\xi X, Y \rangle\). Moreover, \(\partial_t\) can be written

\[
\partial_t = T + \sum_{\alpha=1}^{p} f_\alpha \xi_\alpha,
\]

where \(T\) is a vector field tangent to \(M\), \(\{\xi_1, \cdots, \xi_p\}\) is a local orthonormal frame of \(T^\perp M\) and \(f_1, \cdots, f_p\) are smooth functions over \(M\). We will simply denote \(A_\xi\) by \(A_\alpha\).

From the expression of the curvature tensor of \(\tilde{P}\), we get immediately the Gauss, Codazzi and Ricci equations for a submanifold of \(\tilde{P}\). Namely, if we denote by \(R\) and \(R^\perp\) the curvature tensor of \((M, g)\) and the normal curvature respectively, we have the following

**Proposition 2.1.** The Gauss, Codazzi and Ricci equations of the immersion of \(M\) into \(\tilde{P}\) are respectively

\[
\langle R(X, Y)Z, W \rangle = \langle B(Y, Z), B(X, W) \rangle - \langle B(Y, W), B(X, Z) \rangle + \left(\frac{(a')^2}{a^2} - \frac{c}{a^2}\right) \left(\langle X, Z \rangle \langle Y, W \rangle - \langle Y, Z \rangle \langle X, W \rangle\right) \\
+ \left(\frac{a''}{a} - \frac{(a')^2}{a^2} + \frac{c}{a^2}\right) \left(\langle X, Z \rangle \langle Y, T \rangle \langle W, T \rangle - \langle Y, Z \rangle \langle X, T \rangle \langle W, T \rangle - \langle Y, T \rangle \langle Z, T \rangle \langle X, W \rangle + \langle X, T \rangle \langle Z, T \rangle \langle Y, W \rangle\right),
\]

\[
\langle (\overline{\nabla}_X B)(Y, Z), \xi_\alpha \rangle = \left(\frac{a''}{a} - \frac{(a')^2}{a^2} + \frac{c}{a^2}\right) f_\alpha \left(\langle Y, T \rangle \langle X, Z \rangle - \langle X, T \rangle \langle Y, Z \rangle\right),
\]

\[
\langle R^\perp(X, Y)\nu, \xi \rangle = \langle [A_\nu, A_\xi]X, Y \rangle.
\]

The proof is straightforward form the expression of \(\tilde{R}\).

Finally, we recall that the DDVV conjecture was reduced to the following algebraic result (see [3]) proved by Lu.

**Theorem ([6]).** Let \(n, p \geq 2\) be two integers and \(M_1, M_2, \cdots, M_p\) be some \(n \times n\) real symmetric and trace-free matrices. Then, we have

\[
\sum_{\alpha, \beta=1}^{p} \| [M_\alpha, M_\beta] \|^2 \leq \left(\sum_{\alpha=1}^{p} \| M_\alpha \|^2\right)^2.
\]

Now, we are able to prove Theorem 1.
3. Proof of Theorem 1

First, from the definition of \( \rho \) and using the Gauss equation, we have

\[
\rho = \frac{2}{n(n-1)} \sum_{i<j} (R(e_i, e_j)e_j, e_i)
\]

\[
= \frac{1}{n(n-1)} \sum_{i<j} (R(e_i, e_j)e_j, e_i)
\]

\[
= \frac{1}{n(n-1)} \sum_{i\neq j} \left( \langle B(e_j, e_j), B(e_i, e_i) \rangle - ||B(e_i, e_j)||^2 + \left( \frac{(a')^2}{a^2} - \frac{c}{a^2} \right) \langle T, e_i \rangle^2 + \langle T, e_j \rangle^2 \right)
\]

\[
= \left( \frac{(a')^2}{a^2} - \frac{c}{a^2} \right) + \frac{1}{n(n-1)} \left( n^2||H||^2 - ||B||^2 + 2(n-1) \left( \frac{a''}{a} - \frac{(a')^2}{a^2} + \frac{c}{a^2} \right) ||T||^2 \right)
\]

Now, we set \( \tau = B - Hq \) the traceless part of the second fundamental form. Clearly, we have \( ||\tau||^2 = ||B||^2 - n||H||^2 \). Hence we get

\[
(4) \quad \rho = \left( \frac{(a')^2}{a^2} - \frac{c}{a^2} \right) \left( 1 - \frac{2}{n} ||T||^2 \right) - \frac{2a''}{na} ||T||^2 + ||H||^2 - \frac{1}{n(n-1)} ||\tau||^2.
\]

Moreover, for any \( \alpha \in \{1, \ldots, p\} \), we set \( S_\alpha : TM \to TM \) the operator defined by \( \langle S_\alpha X, Y \rangle = \langle \tau(X, Y), \xi_\alpha \rangle \). Obviously, we have \( S_\alpha = A_\alpha - \langle H, \xi_\alpha \rangle I_d \) and \( [A_\alpha, A_\beta] = [S_\alpha, S_\beta] \). From the Ricci Equation, given in Proposition 2.1, we have

\[
\rho^\perp = \frac{1}{n(n-1)} \sum_{\alpha, \beta=1}^p \frac{1}{||A_\alpha, A_\beta||^2} \sum_{\alpha, \beta=1}^p \frac{1}{||S_\alpha, S_\beta||^2}.
\]

Since, the operators \( S_\alpha \) are symmetric and trace-free, we can apply the theorem of Lu at any point of \( M \) to get

\[
\sum_{\alpha, \beta=1}^p ||S_\alpha, S_\beta||^2 \leq \left( \sum_{\alpha=1}^p ||S_\alpha||^2 \right)^2.
\]

Thus,

\[
\rho^\perp \leq \frac{1}{n(n-1)} \sum_{\alpha=1}^n ||S_\alpha||^2 = \frac{1}{n(n-1)} ||\tau||^2.
\]

Reporting this in (4), we obtain

\[
||H||^2 \geq \rho + \rho^\perp + \left( \frac{(a')^2}{a^2} - \frac{c}{a^2} \right) \left( 1 - \frac{2}{n} ||T||^2 \right) - \frac{2a''}{na} ||T||^2
\]

which concludes the proof. \( \square \)

4. An Application to Submanifolds of \( \mathbb{R} \times_{e^t} \mathbb{H}^n(c) \)

We finish this note by the following application of Theorem 1 to submanifolds of warped product of the type \( \mathbb{R} \times_{e^t} \mathbb{H}^n(c) \) where \( a \) is the real function defined by \( a(t) = e^{\lambda t} \) with \( \lambda \) a real constant.

**Corollary 1.** Let \( M^n \) be a submanifold of the warped product \( \mathbb{R} \times_{e^t} \mathbb{H}^n(c) \) with normalized scalar and normal scalar curvatures \( \rho \) and \( \rho^\perp \) and mean curvature \( H \). Then, we have

\[
||H||^2 \geq \rho + \rho^\perp + \lambda^2 - ce^{-2\lambda t} \left( 1 - \frac{2}{n} ||T||^2 \right).
\]
Proof: This comes directly from Theorem 1 with the fact that \( \frac{\langle a' \rangle^2}{\sigma^2} - \frac{c}{\sigma} = \lambda^2 - ce^{-2M} \) and \( \frac{\langle a'' \rangle^2}{\sigma^2} = \lambda^2 \). Hence the term \( \left( \frac{\langle a' \rangle^2}{\sigma^2} - \frac{c}{\sigma} \right) \left( 1 - \frac{2}{n} ||T||^2 \right) - \frac{2\langle a'' \rangle^2}{\sigma^2} ||T||^2 \) becomes \( \lambda^2 - ce^{-2M} \left( 1 - \frac{2}{n} ||T||^2 \right) \).

Comparing \( ||H||^2 \) with \( \rho \) is a natural question which leads to rigidity results. Indeed, by the Gauss formula, we know that, for hypersurfaces of space forms, \( \rho \) is up to a constant (which is the sectional curvature \( k \) of the ambiant space form) the second mean curvature \( H_2 \), that is the second elementary symmetric polynomial in the principal curvatures. Moreover, it is a classical fact that \( H^2 \geq H_2 \) with equality at umbilical points. Hence, assuming \( H^2 \leq \rho - k \) implies that \( M \) is a hypersphere. In this spirit, and using the above DDVV inequality, we give the following rigidity result.

Corollary 2. Let \( M^n \) be a complete submanifold without boundary of the warped product \( \mathbb{R} \times_{\gamma, M} \mathbb{H}^n(c) \) with normalized scalar and normal scalar curvatures \( \rho \) and \( \rho^\perp \) and mean curvature \( H \). If \( ||H||^2 \leq \rho + \lambda^2 \), then
\[
||H||^2 = \rho + \lambda^2, \quad \rho^\perp = 0, \quad n = 2 \quad \text{and} \quad ||T|| = 1.
\]
Hence, \( M \) is a surface of the type \( \mathbb{R} \times_{\gamma, \mu} \gamma \), where \( \gamma \) is a curve in \( \mathbb{H}^n(c) \).

Proof: First note that since \( n \geq 2 \) and \( ||T||^2 \leq 1 \) and \( c < 0 \), we have \( ce^{-2M} \left( 1 - \frac{2}{n} ||T||^2 \right) \leq 0 \). Note also that, by definition, \( \rho^\perp \geq 0 \). Hence, from Corollary 1 \( ||H||^2 \leq \rho + \lambda^2 \) is possible if and only if \( ||H||^2 = \rho + \lambda^2 \), \( \rho^\perp = 0 \), \( n = 2 \) and \( ||T|| = 1 \). Since \( n = 2 \), then \( M \) is a surface and the fact that \( ||T|| = 1 \) implies that \( T = \partial_t \) and so \( M \) is of the type \( I \times_{\gamma, M} \gamma \), where \( \gamma \) is a curve in \( \mathbb{H}^n(c) \). Since we assume that \( M \) is complete and without boundary, \( I = \mathbb{R} \). This concludes the proof.

References

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