Jumplets
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functions are designed towards the following observation. 

\[ f(t) \text{ if } (3) \]

\[ \epsilon_{-0} f(t_n \pm |\epsilon|) \text{ for all } f \in F \text{ and } n = 1, \ldots, N. \]

\[ (\psi^{-}_n, f)_J = \lim_{\epsilon \to 0} f(t_n \pm |\epsilon|) \text{ for all } f \in F \text{ and } n = 1, \ldots, N. \]

**Proof.** If \( f \in F \) then \( q : t \mapsto f(t_{n-1} + s/a_n) \) is a polynomial of degree \( p_n \) on the unit interval \([0, 1]\). Since \( \psi^{-}_n \) is supported on \( J_n \), we employ the definition of \( \psi^{-}_n \) and the properties of \( R_d \) to obtain \( (\psi^{-}_n, f)_J = (R_{p_n}, q)_{[0,1]} = q(1) = \lim_{\epsilon \to 0} f(t_n - |\epsilon|). \) The proof for \( \psi^{+}_n \) is analogous. \( \square \)
The announced basis for the discontinuous part of $F$ is now defined by

\[ \psi_n := \frac{\psi_n^- - \psi_n^+}{\sqrt{\beta_n^- + \beta_n^+}}, \quad n = 1, \ldots, N. \]

**Proposition 2.2.** Assume that $p_n \geq \frac{p_{\text{min}}}{2} \geq 1$ in (2). Then, $\Psi := \{\psi_n : n = 1, \ldots, N\}$ is a Riesz basis for the $L_2$-orthogonal complement of $E$ in $F$. More precisely, for all vectors $c \in \mathbb{R}^N$,

\[ C_- |c|^2 \leq \|\Psi^T c\|^2 \leq C_+ |c|^2 \quad \text{with} \quad C_{\pm} = 1 \pm \frac{1}{1 + p_{\text{min}}}. \]

**Proof.** Set $\psi_n^o := \psi_n^- - \psi_n^+$. Then $(\psi_n^o, e)_J = \lim_{e \to 0} e(t_n - |e|) - \lim_{e \to 0} e(t_n + |e|) = 0$ for any $e \in E$ by continuity of $e$. Hence, each $\psi_n$ is $L_2$-orthogonal to $E$. By definition, $\psi_n \in F$. Therefore, once (5) has been established, the fact that the $L_2$-orthogonal complement of $E$ in $F$ is $N$-dimensional shows that $\Psi$ is a basis for it. Now we compute

\[ \gamma_{n,n-1}^o := (\psi_n^o, \psi_{n-1}^o)_J = -(\psi_n^-, \psi_{n-1}^+)_J = -\lim_{e \to 0} \psi_n^-(t_n-1 + |e|) = a_n(-1)^{1+p}(1 + p_n), \]

and similarly

\[ \beta_n^o := (\psi_n^o, \psi_n^o)_J = \|\psi_n^o\|^2 = \beta_n^- + \beta_n^+. \]

Set $\delta_n := \gamma_{n,n-1}^o / \sqrt{\beta_n^- \beta_{n-1}^+}$. Then the Gramian of $\Psi$ is the tridiagonal symmetric $N \times N$ matrix

\[ M_{\Psi} := \begin{pmatrix} 1 & \delta_2 & \delta_3 & \cdots \\ \delta_2 & 1 & \delta_3 & \cdots \\ \delta_3 & \delta_2 & 1 & \cdots \\ \vdots & \vdots & \cdots & \ddots \end{pmatrix}. \]

Note $c^T M_{\Psi} c = |c|^2 + \sum_{n=2}^N 2\delta_n c_n c_{n-1}$ for any $c \in \mathbb{R}^N$, so we wish to compare the magnitude of the second term to $|c|^2$. Observe

\[ |2\delta_n c_n c_{n-1}| = |\gamma_{n,n-1}^o| \times 2 \frac{|c_n|}{\sqrt{\beta_n^-}} \frac{|c_{n-1}|}{\sqrt{\beta_{n-1}^+}} \leq \frac{\beta_n}{1 + p_{\text{min}}} \left( \frac{c_n^2}{\beta_n^-} + \frac{c_{n-1}^2}{\beta_{n-1}^+} \right), \]

so that $\sum_{n=2}^N |2\delta_n c_n c_{n-1}| \leq \frac{1}{1 + p_{\text{min}}} |c|^2$. Therefore, $C_- |c|^2 \leq c^T M_{\Psi} c \leq C_+ |c|^2$, with $C_{\pm} = 1 \pm \frac{1}{1 + p_{\text{min}}}$. \hfill $\Box$

On a uniform mesh $T$ and with uniform polynomial degree $p$ one obtains $\delta_n = -\frac{1}{2}(-1)^p$ for each $n$ in the Gramian (8). It is a tridiagonal symmetric Toeplitz matrix with eigenvalues $\{1 + \frac{1}{1+p} \cos \frac{k\pi}{N+1} : k = 1, \ldots, N\}$. Letting $N \to \infty$ shows that the constants in (5) cannot be improved in general.

The fact that the Gramian (8) is well-conditioned (5), together with Lemma 2.1, allows for fast computation of the discontinuous part of a function $f \in F$. Let $\Delta \in \mathbb{R}^N$ denote the vector of jumps of $f$, whose components are $\Delta_n = \lim_{e \to 0} \{f(t_n - |e|) - f(t_n + |e|)\}$ for $n = 1, \ldots, N$. Then Lemma
The coefficients $c \in \mathbb{R}^N$ of the discontinuous part of $f$ with respect to the basis (4) satisfy the linear system $M_c b = h$, which can be quickly solved approximatively by the conjugate gradient method. This may be cheaper than projecting onto $E$ and then taking the difference because the dimension of $E$ is significantly larger than $N$ when the polynomial degrees $p_n$ in (2) are large.

3. Application: Adaptive Approximation

As an application of the above construction we describe an algorithm for the adaptive approximation of a given function $g \in L^2(J)$. Suppose $\mathcal{T}_i$, $i = 0, 1, 2, \ldots$, is a sequence of meshes as in the introduction, which are nested, $\mathcal{T}_i \subset \mathcal{T}_{i+1}$. Let $E_i \subset Y$ be the space of piecewise affine splines on with respect to $\mathcal{T}_i$, and set $E_i := E_i \cap H^1(J)$, which is then the space of continuous piecewise affine splines. Note that $E_i = E_i + E_i$, where the prime denotes the derivative. Let $E_i^\perp$ denote the $L^2$-orthogonal complement of $E_i$ in $E_i$, and let $Q_i: L^2(J) \to E_i^\perp$ be the surjective $L^2$-orthogonal projection.

Suppose $g_i \in E_i$ is an approximation of $g$. Then we consider $g_i^\perp := Q_i^\perp g_i \in E_i^\perp$, and use the coefficients of $g_i^\perp$ with respect to the Riesz basis (4) for $E_i^\perp$ as error indicators for the marking of subintervals to be adaptively refined. The adaptive algorithm is as follows. Let $\mathcal{T}_0$ be given. Fix a threshold parameter $\theta \in (0, 1]$. For each $i = 0, 1, 2, \ldots$, do: 1) Compute the $L^2$-orthogonal projection $g_i$ of $g_i$ onto $E_i$, or an approximation thereof. 2) Compute the projection $g_i^\perp := Q_i^\perp g_i$. Set $N_i = \#(\mathcal{T} \cap (0, T))$ for the number of interior nodes in $\mathcal{T}_i$, and $[N_i] := \{1, \ldots, N_i\}$. Let $c \in \mathbb{R}^{N_i}$ be the vector of the coefficients of $g_i^\perp \in E_i^\perp$ with respect to the Riesz basis (4) for $E_i^\perp$. Select $M_i \subset [N_i]$ of minimal size such that $\sum_{n \in M_i} c_n^2 \geq \theta^2 |c|^2$. 3) If $\mathcal{T}_i = \{0 = t_0 < t_1 < \ldots < t_{N_i} < t_{N_i+1} = T\}$, let the new mesh $\mathcal{T}_{i+1}$ contain $\mathcal{T}_i$, and the new nodes $\frac{1}{2}(t_{n-1} + t_n)$ and $\frac{1}{2}(t_n + t_{n+1})$ for all $n \in M_i$.

The Riesz basis property guarantees that the part of the indicator $g_i^\perp$ corresponding to the marked subset $M_i$ carries a fraction of its total $L^2(J)$ norm that is comparable to $\theta$.

As an example we take $g : t \mapsto t^{-1/3}$ on $J := [0, 1]$ with $T := 1$. We set $\mathcal{T}_0 := \{0, 1/4, 1/2, 3/4, 1\}$ for the initial mesh. Instead of the $L^2$-orthogonal projection we use the 4-point Gauss–Legendre quadrature rule on each subinterval to obtain $g_i$. The error $\|g - g_i\|_J$ for the above adaptive is shown in Figure 2. Due to the lack of smoothness of the given function $g$, it is approximated by piecewise constant functions on a uniform mesh with a rate of $\approx 1/6$ with respect to the mesh size $\# \mathcal{T}_i$, while adaptivity recovers the asymptotic rate of one.