Jumplets
Roman Andreev

To cite this version:
Roman Andreev. Jumplets. 2016. hal-01338101
JUMPLETS

ROMAN ANDREEVI

Abstract. In this note we describe a locally supported Riesz basis consisting of “jumplets” for the orthogonal complement of continuous splines on an interval in the space of discontinuous ones.

1. Introduction

Let $T > 0$ and write $J := [0, T]$. For $N \geq 1$ let a finite partition of $J$ be given by

\begin{equation}
\mathcal{T} = \{0 =: t_0 < t_1 < \ldots < t_N < t_{N+1} := T\}.
\end{equation}

Let $\mathbb{P}_d$ denote the space of real-valued polynomials on $\mathbb{R}$ of degree at most $d$. We write

\begin{equation}
F := \{f \in L^2(J) : f|_{(t_{n-1}, t_n)} \in \mathbb{P}_{p_n}\} \quad \text{and} \quad E := F \cap H^1(J)
\end{equation}

for the space $F$ of piecewise polynomials on the interval $[0, T]$ of given polynomial degrees $p_n \geq 0$ and for its subset $E \subset F$ of the continuous ones. Here and throughout, $L^2$ and $H^1$ refer to the usual Lebesgue and Sobolev spaces of real-valued functions on $J$.

In the next Section 2 we construct a locally supported Riesz basis for the $L^2$ orthogonal complement of $E$ in $F$. Its dimension is obviously $N$, which is the number of interior nodes of $\mathcal{T}$. We refer to $E$ as the continuous part of $F$, and to its $L^2$-orthogonal complement in $F$ as the discontinuous part of $F$. An application to adaptive approximation in $L^2$ is discussed in Section 3.

2. Construction of the basis

If $I \subset \mathbb{R}$ is an interval, we write $(\cdot, \cdot)$ for the $L^2(I)$ scalar product, and $\| \cdot \|_I$ for the norm. The euclidean norm is denoted by $| \cdot |$. For integer $d \geq 0$ let $\ell_d \in \mathbb{P}_d$ be the Legendre polynomial on the unit interval $[0, 1]$ characterized by $\ell_d(1) = \sqrt{2d + 1}$ and $\int_0^1 \ell_d(t)q(t)dt = 0$ for any $q \in \mathbb{P}_d$. Then $\{\ell_k : 0 \leq k \leq d\}$ is an orthonormal basis for $\mathbb{P}_d$.

For integer $d \geq 0$ define the polynomial $R_d := \sum_{k=0}^d \ell_k(1)\ell_k$. Its key property is that $(R_d, q)_{[0,1]} = q(1)$ for any polynomial $q \in \mathbb{P}_d$, as can be verified by expanding $q$ into the Legendre basis. Further, $\|R_d\|_{[0,1]} = 1 + d$. Together, these observations imply $R_d(1) = (R_d, R_d)_{[0,1]} = (1 + d)^2$. Using $\ell_k(0) = (-1)^k \ell_k(1)$ in the definition of $R_d$ we also obtain $R_d(0) = (-1)^d(1 + d)$.

For each $n = 1, \ldots, N + 1$, let $J_n := (t_{n-1}, t_n)$ be the $n$-th subinterval in the partition $\mathcal{T}$ from (1). Write $\alpha_n := |t_n - t_{n-1}|^{-1}$ for the inverse of its length, and define the constants $\beta_n := \alpha_n(1 + p_n^2)$, where $p_n$ is the polynomial degree from (2). Define the functions $\psi_n^-$ and $\psi_n^+$ in $F$ having the values

\begin{equation}
\psi_n^-(t) := \alpha_n R_{p_n}(\alpha_n(t - t_{n-1})) \quad \text{and} \quad \psi_{n-1}^+(t) := \alpha_n R_{p_n}(\alpha_n(t_n - t))
\end{equation}

if $t \in J_n$, and zero else. Note that $\psi_{n-1}^+$ is the reflection of $\psi_n^-$ about the vertical at the middle of the subinterval $J_n$. Since $\beta_n$ is associated with both, $\psi_n^-$ and $\psi_n^+$, we also write $\beta_n := \beta_n^+ := \beta_n^-$. These functions are designed towards the following observation.

Lemma 2.1. $(\psi_n^+, f)_J = \lim_{\varepsilon \to 0} f(t_n \pm |\varepsilon|)$ for all $f \in F$ and $n = 1, \ldots, N$.

Proof. If $f \in F$ then $q : t \mapsto f(t_{n-1} + s/\alpha_n)$ is a polynomial of degree $p_n$ on the unit interval $[0, 1]$. Since $\psi_n^-$ is supported on $J_n$, we employ the definition of $\psi_n^-$ and the properties of $R_d$ to obtain

$(\psi_n^-, f)_J = (R_{p_n}, q)_{[0,1]} = q(1) = \lim_{\varepsilon \to 0} f(t_n - |\varepsilon|)$. The proof for $\psi_n^+$ is analogous. \(\square\)
The announced basis for the discontinuous part of $F$ is now defined by
\[
\psi_n := \frac{\psi_n^- - \psi_n^+}{\sqrt{\beta_n^- + \beta_n^+}}, \quad n = 1, \ldots, N.
\]

**Proposition 2.2.** Assume that $p_n \geq p_{\min} \geq 1$ in (2). Then, $\Psi := \{\psi_n : n = 1, \ldots, N\}$ is a Riesz basis for the $L_2$-orthogonal complement of $E$ in $F$. More precisely, for all vectors $c \in \mathbb{R}^N$,
\[
C_- |c|^2 \leq ||\Psi^T c ||_2^2 \leq C_+ |c|^2 \quad \text{with} \quad C_\pm = 1 \pm \frac{1}{1 + p_{\min}}.
\]

**Proof.** Set $\psi_n^o := \psi_n^- - \psi_n^+$. Then $(\psi_n^o, e)_J = \lim_{\varepsilon \to 0} e(t_n - |\varepsilon|) - \lim_{\varepsilon \to 0} e(t_n + |\varepsilon|) = 0$ for any $e \in E$ by continuity of $e$. Hence, each $\psi_n$ is $L_2$-orthogonal to $E$. By definition, $\psi_n \in F$. Therefore, once (5) has been established, the fact that the $L_2$-orthogonal complement of $E$ in $F$ is $N$-dimensional shows that $\Psi$ is a basis for it. Now we compute
\[
\gamma_{n,n-1}^o := (\psi_n^o, \psi_{n-1}^o)_J = -\lim_{\varepsilon \to 0} \psi_n^- (t_{n-1} + |\varepsilon|) = \alpha_n (-1)^{1+p_n} (1 + p_n),
\]
and similarly
\[
\beta_n^o := (\psi_n^o, \psi_n^o)_J = ||\psi_n^-||_2^2 + ||\psi_n^+||_2^2 = \beta_n^- + \beta_n^+.
\]
Set $\delta_n := \gamma_{n,n-1}^o / \sqrt{\beta_n^o \beta_{n-1}^o}$. Then the Gramian of $\Psi$ is the tridiagonal symmetric $N \times N$ matrix
\[
M_\psi := \begin{pmatrix}
1 & \delta_2 & 0 & \cdots \\
\delta_2 & 1 & \delta_3 & \cdots \\
0 & \delta_3 & 1 & \cdots \\
\vdots & \vdots & \ddots & \ddots \\
\end{pmatrix}.
\]

Note $c^T M_\psi c = |c|^2 + \sum_{n=2}^N 2 \delta_n c_n c_{n-1}$ for any $c \in \mathbb{R}^N$, so we wish to compare the magnitude of the second term to $|c|^2$. Observe
\[
|2 \delta_n c_n c_{n-1}| = |\gamma_{n,n-1}^o| \times 2 \left| \frac{c_n}{\sqrt{\beta_n^o}} \frac{c_{n-1}}{\sqrt{\beta_{n-1}^o}} \right| \leq \frac{\beta_n}{1 + p_{\min}} \left( \frac{c_n^2}{\beta_n^o} + \frac{c_{n-1}^2}{\beta_{n-1}^o} \right),
\]
so that $\sum_{n=2}^N |2 \delta_n c_n c_{n-1}| \leq \frac{1}{1 + p_{\min}} |c|^2$. Therefore, $C_- |c|^2 \leq c^T M_\psi c \leq C_+ |c|^2$, with $C_\pm = 1 \pm \frac{1}{1 + p_{\min}}$. \qed

On a uniform mesh $\mathcal{T}$ and with uniform polynomial degree $p$ one obtains $\delta_n = -\frac{1}{2} (-1)^{1+p}$ for each $n$ in the Gramian (8). It is a tridiagonal symmetric Toeplitz matrix with eigenvalues $\{1 + \frac{1}{1+p} \cos \frac{k\pi}{N+1} : k = 1, \ldots, N\}$. Letting $N \to \infty$ shows that the constants in (5) cannot be improved in general.

The fact that the Gramian (8) is well-conditioned (5), together with Lemma 2.1, allows for fast computation of the discontinuous part of a function $f \in F$. Let $\Delta \in \mathbb{R}^N$ denote the vector of jumps of $f$, whose components are $\Delta_n = \lim_{\varepsilon \to 0} \{ f(t_n - |\varepsilon|) - f(t_n + |\varepsilon|) \}$ for $n = 1, \ldots, N$. Then Lemma
2.1 and definition (4) of $\psi_n$ imply $b_n := (\psi_n, f) = \Delta_n / \sqrt{\beta_n - \beta_n^*}$. The coefficients $c \in \mathbb{R}^N$ of the discontinuous part of $f$ with respect to the basis (4) satisfy the linear system $M g = b$, which can be quickly solved approximatively by the conjugate gradient method. This may be cheaper than projecting onto $E$ and then taking the difference because the dimension of $E$ is significantly larger than $N$ when the polynomial degrees $p_n$ in (2) are large.

3. Application: adaptive approximation

As an application of the above construction we describe an algorithm for the adaptive approximation of a given function $g \in L_2(J)$. Suppose $\mathcal{T}_i$, $i = 0, 1, 2, \ldots$, is a sequence of meshes as in the introduction, which are nested, $\mathcal{T}_i \subset \mathcal{T}_{i+1}$. Let $F_i \subset Y$ be the space of piecewise affine splines of order $k$ with respect to $\mathcal{T}_i$, and set $E_i := F_i \cap H^1(J)$, which is then the space of continuous piecewise affine splines. Note that $F_i = E_i' + E_i$, where the prime denotes the derivative. Let $E_i^\perp$ denote the $L_2$-orthogonal complement of $E_i$ in $F_i$, and let $Q_i^\perp : L_2(J) \rightarrow E_i^\perp$ be the surjective $L_2$-orthogonal projection.

Suppose $g_i \in E_i'$ is an approximation of $g$. Then we consider $g_i^\perp = Q_i^\perp g_i \in E_i^\perp$, and use the coefficients of $g_i^\perp$ with respect to the Riesz basis (4) for $E_i^\perp$ as error indicators for the marking of subintervals to be adaptively refined. The adaptive algorithm is as follows. Let $\mathcal{T}_0$ be given. Fix a threshold parameter $\theta \in (0, 1]$. For each $i = 0, 1, 2, \ldots$, do: 1) Compute the $L_2$-orthogonal projection $g_i$ of $g$ onto $E_i'$, or an approximation thereof. 2) Compute the projection $g_i^\perp = Q_i^\perp g_i$. Set $N_i = \#(\mathcal{T}_i \cap (0, T))$ for the number of interior nodes in $\mathcal{T}_i$, and $[N_i] := \{1, \ldots, N_i\}$. Let $c \in \mathbb{R}^{N_i}$ be the vector of the coefficients of $g_i^\perp \in E_i^\perp$ with respect to the Riesz basis (4) for $E_i^\perp$. Select $M_i \subset [N_i]$ of minimal size such that $\sum_{n \in M_i} c_n^2 \geq \theta^2 |c|^2$. 3) If $\mathcal{T}_i = \{0 = t_0 < t_1 < \ldots < t_{N_i} < t_{N_i+1} = T\}$, let the new mesh $\mathcal{T}_{i+1}$ contain $\mathcal{T}_i$, and the new nodes $\frac{1}{2}(t_{n-1} + t_n)$ and $\frac{1}{2}(t_n + t_{n+1})$ for all $n \in M_i$.

The Riesz basis property guarantees that the part of the indicator $g_i^\perp$ corresponding to the marked subset $M_i$ carries a fraction of its total $L_2(J)$ norm that is comparable to $\theta$.

As an example we take $g : t \mapsto t^{-1/3}$ on $J := [0, T]$ with $T := 1$. We set $\mathcal{T}_0 := \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$ for the initial mesh. Instead of the $L_2$-orthogonal projection we use the 4-point Gauss–Legendre quadrature rule on each subinterval to obtain $g_i$. The error $\|g - g_i\|_j$ for the above adaptive is shown in Figure 2. Due to the lack of smoothness of the given function $g$, it is approximated by piecewise constant functions on a uniform mesh with a rate of $\approx 1/6$ with respect to the mesh size $\#\mathcal{T}_i$, while adaptivity recovers the asymptotic rate of one.

This note was mainly written while at RICAM, Linz (AT), 2014.

1Université Paris Diderot, Sorbonne Paris Cité, LJLL (UMR 7598 CNRS), F-75205, Paris, France
E-mail address: roman.andreev@upmc.fr