JUMPLETS

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ABSTRACT. In this note we describe a locally supported Riesz basis consisting of "jumplets" for the orthogonal complement of continuous splines on an interval in the space of discontinuous ones.

1. INTRODUCTION

Let $T > 0$ and write $J := [0, T]$. For $N \geq 1$ let a finite partition of J be given by

$$\mathscr{T} = \{0 =: t_0 < t_1 < \ldots < t_N < t_{N+1} := T\}. \tag{1}$$

Let $\mathbb{P}_d$ denote the space of real-valued polynomials on $\mathbb{R}$ of degree at most d. We write

$$F := \{f \in L_2(J) : f|_{(t_{n-1},t_n)} \in \mathbb{P}_{p_n}\} \quad \text{and} \quad E := F \cap H^1(J) \tag{2}$$

for the space F of piecewise polynomials on the interval $[0, T]$ of given polynomial degrees $p_n \geq 0$ and for its subset $E \subset F$ of the continuous ones. Here and throughout, L_2 and H^1 refer to the usual Lebesgue and Sobolev spaces of real-valued functions on J.

In the next Section 2 we construct a locally supported Riesz basis for the L_2 orthogonal complement of E in F. Its dimension is obviously N, which is the number of interior nodes of $\mathscr{T}$. We refer to E as the continuous part of F, and to its L_2-orthogonal complement in F as the discontinuous part of F. An application to adaptive approximation in L_2 is discussed in Section 3.

2. CONSTRUCTION OF THE BASIS

If $I \subset \mathbb{R}$ is an interval, we write $(\cdot, \cdot)_I$ for the $L_2(I)$ scalar product, and $\|\cdot\|_I$ for the norm. The euclidean norm is denoted by $|\cdot|$. For integer $d \geq 0$ let $\ell_d \in \mathbb{P}_d$ be the Legendre polynomial on the unit interval $[0, 1]$ characterized by $\ell_d(1) = \sqrt{2d + 1}$ and $\int_0^1 \ell_d(t)q(t)\mathrm{d}t = 0$ for any $q \in \mathbb{P}_{d-1}$. Then $\{\ell_k : 0 \leq k \leq d\}$ is an orthonormal basis for $\mathbb{P}_d \cap L_2([0, 1])$.

For integer $d \geq 0$ define the polynomial $R_d := \sum_{k=0}^d \ell_k(1)\ell_k$. Its key property is that $(R_d, q)_{[0,1]} = q(1)$ for any polynomial $q \in \mathbb{P}_d$, as can be verified by expanding q into the Legendre basis. Further, $\|R_d\|_{[0,1]} = 1 + d$. Together, these observations imply $R_d(1) = (R_d, R_d)_{[0,1]} = (1 + d)^2$. Using $\ell_k(0) = (-1)^k \ell_k(1)$ in the definition of R_d we also obtain $R_d(0) = (-1)^d(1 + d)$.

For each $n = 1, \ldots, N + 1$, let $J_n := (t_{n-1}, t_n)$ be the n-th subinterval in the partition $\mathscr{T}$ from (1). Write $\alpha_n := |t_n - t_{n-1}|^{-1}$ for the inverse of its length, and define the constants $\beta_n := \alpha_n(1 + p_n)^2$, where p_n is the polynomial degree from (2). Define the functions ψ_n^- and ψ_{n-1}^+ in F having the values

$$\psi_n^-(t) := \alpha_n R_{p_n}(\alpha_n(t - t_{n-1})) \quad \text{and} \quad \psi_{n-1}^+(t) := \alpha_n R_{p_n}(\alpha_n(t_n - t)) \tag{3}$$

if $t \in J_n$, and zero else. Note that ψ_{n-1}^+ is the reflection of ψ_n^- about the vertical at the middle of the subinterval J_n. Since β_n is associated with both, ψ_{n-1}^+ and ψ_n^-, we also write $\beta_{n-1}^+ := \beta_n^- := \beta_n$. These functions are designed towards the following observation.

Lemma 2.1. $(\psi_n^\pm, f)_J = \lim_{\epsilon\to 0} f(t_n \pm |\epsilon|)$ *for all* $f \in F$ *and* $n = 1, \ldots, N$.

Proof. If $f \in F$ then $q : t \mapsto f(t_{n-1} + s/\alpha_n)$ is a polynomial of degree p_n on the unit interval $[0, 1]$. Since ψ_n^- is supported on J_n, we employ the definition of ψ_n^- and the properties of R_d to obtain $(\psi_n^-, f)_J = (R_{p_n}, q)_{[0,1]} = q(1) = \lim_{\epsilon\to 0} f(t_n - |\epsilon|)$. The proof for ψ_n^+ is analogous. □

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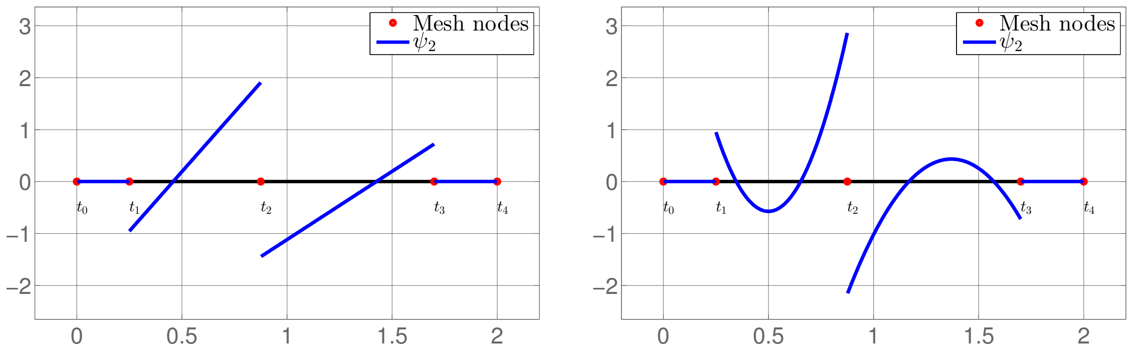

FIGURE 1. The basis function ψ_2 from (4) for polynomial degrees $p=1$ (left) / $p=2$ (right).

The announced basis for the discontinuous part of F is now defined by

$$\psi_n := \frac{\psi_n^- - \psi_n^+}{\sqrt{\beta_n^- + \beta_n^+}}, \quad n = 1, \ldots, N. \tag{4}$$

Proposition 2.2. *Assume that $p_n \geq p_{\min} \geq 1$ in (2). Then, $\Psi := \{\psi_n : n = 1, \ldots, N\}$ is a Riesz basis for the L_2-orthogonal complement of E in F. More precisely, for all vectors $c \in \mathbb{R}^N$,*

$$C_-|c|^2 \leq \|\Psi^\top c\|_J^2 \leq C_+|c|^2 \quad \textit{with} \quad C_\pm = 1 \pm \frac{1}{1+p_{\min}}. \tag{5}$$

Proof. Set $\psi_n^\circ := \psi_n^- - \psi_n^+$. Then $(\psi_n^\circ, e)_J = \lim_{\epsilon\to 0} e(t_n - |\epsilon|) - \lim_{\epsilon \to 0} e(t_n + |\epsilon|) = 0$ for any $e \in E$ by continuity of e. Hence, each ψ_n is L_2-orthogonal to E. By definition, $\psi_n \in F$. Therefore, once (5) has been established, the fact that the L_2-orthogonal complement of E in F is N-dimensional shows that Ψ is a basis for it. Now we compute

$$\gamma_{n,n-1}^\circ := (\psi_n^\circ, \psi_{n-1}^\circ)_J = -(\psi_n^-, \psi_{n-1}^+)_J = -\lim_{\epsilon\to 0} \psi_n^-(t_{n-1} + |\epsilon|) = \alpha_n(-1)^{1+p_n}(1+p_n), \tag{6}$$

and similarly

$$\beta_n^\circ := (\psi_n^\circ, \psi_n^\circ)_J = \|\psi_n^-\|_J^2 + \|\psi_n^+\|_J^2 = \beta_n^- + \beta_n^+. \tag{7}$$

Set $\delta_n := \gamma_{n,n-1}^\circ / \sqrt{\beta_n^\circ \beta_{n-1}^\circ}$. Then the Gramian of Ψ is the tridiagonal symmetric $N \times N$ matrix

$$M_\Psi := \begin{pmatrix} 1 & \delta_2 & & \\ \delta_2 & 1 & \delta_3 & \\ & \delta_3 & 1 & \ddots \\ & & \ddots & \ddots \end{pmatrix}. \tag{8}$$

Note $c^\top M_\Psi c = |c|^2 + \sum_{n=2}^N 2\delta_n c_n c_{n-1}$ for any $c \in \mathbb{R}^N$, so we wish to compare the magnitude of the second term to $|c|^2$. Observe

$$|2\delta_n c_n c_{n-1}| = |\gamma_{n,n-1}^\circ| \times 2 \frac{|c_n|}{\sqrt{\beta_n^\circ}} \frac{|c_{n-1}|}{\sqrt{\beta_{n-1}^\circ}} \leq \frac{\beta_n}{1+p_{\min}} \times \left(\frac{c_n^2}{\beta_n^\circ} + \frac{c_{n-1}^2}{\beta_{n-1}^\circ} \right), \tag{9}$$

so that $\sum_{n=2}^N |2\delta_n c_n c_{n-1}| \leq \frac{1}{1+p_{\min}} |c|^2$. Therefore, $C_-|c|^2 \leq c^\top M_\Psi c \leq C_+|c|^2$, with $C_\pm = 1 \pm \frac{1}{1+p_{\min}}$. □

On a uniform mesh $\mathcal{T}$ and with uniform polynomial degree p one obtains $\delta_n = -\frac{1}{2}\frac{(-1)^p}{1+p}$ for each n in the Gramian (8). It is a tridiagonal symmetric Toeplitz matrix with eigenvalues $\{1 + \frac{1}{1+p} \cos \frac{k\pi}{N+1} : k = 1, \ldots, N\}$. Letting $N \to \infty$ shows that the constants in (5) cannot be improved in general.

The fact that the Gramian (8) is well-conditioned (5), together with Lemma 2.1, allows for fast computation of the discontinuous part of a function $f \in F$. Let $\Delta \in \mathbb{R}^N$ denote the vector of jumps of f, whose components are $\Delta_n = \lim_{\epsilon \to 0}\{f(t_n - |\epsilon|) - f(t_n + |\epsilon|)\}$ for $n = 1, \ldots, N$. Then Lemma

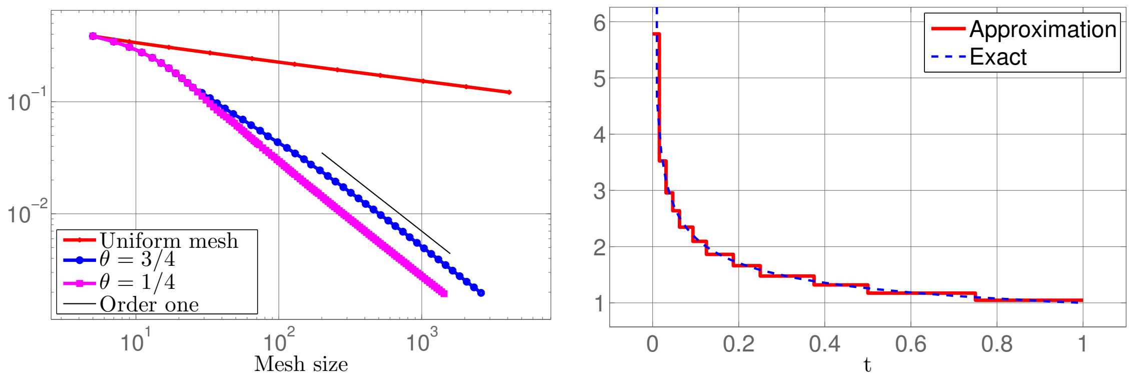

FIGURE 2. Left: L_2 error achieved by the adaptive approximation algorithm for $g : t \mapsto t^{-1/3}$ on $J = (0,1)$ by piecewise constant functions, and the error of the L_2 best approximation of g on a uniform mesh, as a function of the mesh size. Right: g and its adaptive approximation g_4 for $\theta = 3/4$.

2.1 and definition (4) of ψ_n imply $b_n := (\psi_n, f)_J = \Delta_n / \sqrt{\beta_n^- + \beta_n^+}$. The coefficients $c \in \mathbb{R}^N$ of the discontinuous part of f with respect to the basis (4) satisfy the linear system $M_\Psi c = b$, which can be quickly solved approximatively by the conjugate gradient method. This may be cheaper than projecting onto E and then taking the difference because the dimension of E is significantly larger than N when the polynomial degrees p_n in (2) are large.

3. APPLICATION: ADAPTIVE APPROXIMATION

As an application of the above construction we describe an algorithm for the adaptive approximation of a given function $g \in L_2(J)$. Suppose $\mathscr{T}_i$, $i = 0, 1, 2, \ldots$, is a sequence of meshes as in the introduction, which are nested, $\mathscr{T}_i \subset \mathscr{T}_{i+1}$. Let $F_i \subset Y$ be the space of piecewise affine splines on with respect to $\mathscr{T}_i$, and set $E_i := F_i \cap H^1(J)$, which is then the space of continuous piecewise affine splines. Note that $F_i = E_i' + E_i$, where the prime denotes the derivative. Let $E_i^\perp$ denote the L_2-orthogonal complement of E_i in F_i, and let $Q_i^\perp : L_2(J) \to E_i^\perp$ be the surjective L_2-orthogonal projection.

Suppose $g_i \in E_i'$ is an approximation of g. Then we consider $g_i^\perp := Q_i^\perp g_i \in E_i^\perp$, and use the coefficients of $g_i^\perp$ with respect to the Riesz basis (4) for $E_i^\perp$ as error indicators for the marking of subintervals to be adaptively refined. The adaptive algorithm is as follows. Let $\mathscr{T}_0$ be given. Fix a threshold parameter $\theta \in (0, 1]$. For each $i = 0, 1, 2, \ldots$, do: **1)** Compute the L_2-orthogonal projection g_i of g onto E_i', or an approximation thereof. **2)** Compute the projection $g_i^\perp := Q_i^\perp g_i$. Set $N_i = \#(\mathscr{T}_i \cap (0, T))$ for the number of interior nodes in $\mathscr{T}_i$, and $[N_i] := \{1, \ldots, N_i\}$. Let $c \in \mathbb{R}^{N_i}$ be the vector of the coefficients of $g_i^\perp \in E_i^\perp$ with respect to the Riesz basis (4) for $E_i^\perp$. Select $M_i \subset [N_i]$ of minimal size such that $\sum_{n \in M_i} c_n^2 \geq \theta^2 |c|^2$. **3)** If $\mathscr{T}_i = \{0 = t_0 < t_1 < \ldots < t_{N_i} < t_{N_i+1} = T\}$, let the new mesh $\mathscr{T}_{i+1}$ contain $\mathscr{T}_i$, and the new nodes $\frac{1}{2}(t_{n-1} + t_n)$ and $\frac{1}{2}(t_n + t_{n+1})$ for all $n \in M_i$.

The Riesz basis property guarantees that the part of the indicator $g_i^\perp$ corresponding to the marked subset M_i carries a fraction of its total $L_2(J)$ norm that is comparable to θ.

As an example we take $g : t \mapsto t^{-1/3}$ on $J := [0, T]$ with $T := 1$. We set $\mathscr{T}_0 := \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$ for the initial mesh. Instead of the L_2-orthogonal projection we use the 4-point Gauss–Legendre quadrature rule on each subinterval to obtain g_i. The error $\|g - g_i\|_J$ for the above adaptive is shown in Figure 2. Due to the lack of smoothness of the given function g, it is approximated by piecewise constant functions on a uniform mesh with a rate of $\approx 1/6$ with respect to the mesh size $\#\mathscr{T}_i$, while adaptivity recovers the asymptotic rate of one.

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