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Lemma 2.1. \( J \) is analogous.

Proof. If \( f \in F \) then \( q : t \mapsto f(t_{n-1} + s/a_n) \) is a polynomial of degree \( p_n \) on the unit interval \([0, 1] \). Since \( \psi_n^- \) is supported on \( J_n \), we employ the definition of \( \psi_n^- \) and the properties of \( R_d \) to obtain \( (\psi_n^-, f)_J = (R_{p_n}, q)_{[0, 1]} = q(1) = \lim_{\varepsilon \to 0} f(t_n - \varepsilon) \). The proof for \( \psi_n^+ \) is analogous.

JUMPLETS

ROMAN ANDREEV

Abstract. In this note we describe a locally supported Riesz basis consisting of “jumplets” for the orthogonal complement of continuous splines on an interval in the space of discontinuous ones.

1. Introduction

Let \( T > 0 \) and write \( J := [0, T] \). For \( N \geq 1 \) let a finite partition of \( J \) be given by

\[
J = \{0 =: t_0 < t_1 < \ldots < t_N < t_{N+1} := T\}.
\]

Let \( \mathbb{P}_d \) denote the space of real-valued polynomials on \( \mathbb{R} \) of degree at most \( d \). We write

\[
F := \{f \in L_2(J) : f|_{(t_{s-1}, t_s)} \in \mathbb{P}_{p_n}\} \quad \text{and} \quad E := F \cap H^1(J)
\]

for the space \( F \) of piecewise polynomials on the interval \([0, T]\) of given polynomial degrees \( p_n \geq 0 \) and for its subset \( E \subset F \) of the continuous ones. Here and throughout, \( L_2 \) and \( H^1 \) refer to the usual Lebesgue and Sobolev spaces of real-valued functions on \( J \).

In the next Section 2 we construct a locally supported Riesz basis for the \( L_2 \) orthogonal complement of \( E \) in \( F \). Its dimension is obviously \( N \), which is the number of interior nodes of \( J \). We refer to \( E \) as the continuous part of \( F \), and to its \( L_2 \)-orthogonal complement in \( F \) as the discontinuous part of \( F \). An application to adaptive approximation in \( L_2 \) is discussed in Section 3.

2. Construction of the basis

If \( J \subset \mathbb{R} \) is an interval, we write \((\cdot, \cdot)\) for the \( L_2(J) \) scalar product, and \( \|\cdot\|_1 \) for the norm. The euclidean norm is denoted by \( |\cdot| \). For integer \( d \geq 0 \) let \( \ell_d \in \mathbb{P}_d \) be the Legendre polynomial on the unit interval \([0, 1]\) characterized by \( \ell_d(1) = \sqrt{2d + 1} \) and \( \int_{\ell_d(t)q(t)dt = 0} \) for any \( q \in \mathbb{P}_{d-1} \). Then \( \{\ell_k : 0 \leq k \leq d\} \) is an orthonormal basis for \( \mathbb{P}_d \).

For integer \( d \geq 0 \) define the polynomial \( R_d := \sum_{k=0}^{2^d} \ell_k(\ell_k) \). Its key property is that \( (R_d, q)_{[0, 1]} = q(1) \) for any polynomial \( q \in \mathbb{P}_d \), as can be verified by expanding \( q \) into the Legendre basis. Further, \( \|R_d\|_{[0, 1]} = 1 + d \). Together, these observations imply \( R_d(1) = (R_d, R_d)_{[0, 1]} = (1 + d)^2 \). Using \( \ell_k(0) = (-1)^k\ell_k(1) \) in the definition of \( R_d \) we also obtain \( R_d(0) = (-1)^k(1 + d) \).

For each \( n \equiv 1, \ldots, N+1 \), let \( J_n := (t_{n-1}, t_n) \) be the \( n \)-th subinterval in the partition \( \mathcal{J} \) from (1). Write \( a_n := |t_n - t_{n-1}|^{-1} \) for the inverse of its length, and define the constants \( \beta_n := \alpha_n(1 + p_n)^2 \), where \( p_n \) is the polynomial degree from (2). Define the functions \( \psi_n^- \) and \( \psi_n^+ \) in \( F \) having the values

\[
\psi_n^-(t) := \alpha_n R_{p_n}(\alpha_n(t - t_{n-1})) \quad \text{and} \quad \psi_n^+(t) := \alpha_n R_{p_n}(\alpha_n(t_n - t))
\]

if \( t \in J_n \), and zero else. Note that \( \psi_n^{-1} \) is the reflection of \( \psi_n^- \) about the vertical at the middle of the subinterval \( J_n \). Since \( \beta_n \) is associated with both, \( \psi_n^{-1} \) and \( \psi_n^- \), we also write \( \beta_n^{-1} := \beta_n^- := \beta_n \). These functions are designed towards the following observation.

Lemma 2.1. \( \lim_{\varepsilon \to 0} f(t_n \pm |\varepsilon|) = \lim_{\varepsilon \to 0} f(t_n \pm |\varepsilon|) \) for all \( f \in F \) and \( n = 1, \ldots, N \).

Proof. If \( f \in F \) then \( q : t \mapsto f(t_{n-1} + s/a_n) \) is a polynomial of degree \( p_n \) on the unit interval \([0, 1] \). Since \( \psi_n^- \) is supported on \( J_n \), we employ the definition of \( \psi_n^- \) and the properties of \( R_d \) to obtain \( (\psi_n^-, f)_J = (R_{p_n}, q)_{[0, 1]} = q(1) = \lim_{\varepsilon \to 0} f(t_n - |\varepsilon|) \). The proof for \( \psi_n^+ \) is analogous. 

The announced basis for the discontinuous part of \( F \) is now defined by

\[
\psi_n := \frac{\psi_n^- - \psi_n^+}{\sqrt{\beta_n^- + \beta_n^+}}, \quad n = 1, \ldots, N.
\]

**Proposition 2.2.** Assume that \( p_n \geq p_{\text{min}} \geq 1 \) in (2). Then, \( \Psi := \{\psi_n : n = 1, \ldots, N\} \) is a Riesz basis for the \( L_2 \)-orthogonal complement of \( E \) in \( F \). More precisely, for all vectors \( c \in \mathbb{R}^N \),

\[
C_-|c|^2 \leq \|\Psi^T c\|^2 \leq C_+|c|^2 \quad \text{with} \quad C_\pm = 1 \pm \frac{1}{1 + p_{\text{min}}}.
\]

**Proof.** Let \( \psi_n := \psi_n^- - \psi_n^+ \). Then \( (\psi_n^-, \epsilon) = \lim_{\epsilon \to 0} e(t_n - |\epsilon|) - \lim_{\epsilon \to 0} e(t_n + |\epsilon|) = 0 \) for any \( \epsilon \in E \) by continuity of \( e \). Hence, each \( \psi_n \) is \( L_2 \)-orthogonal to \( E \). By definition, \( \psi_n \in F \). Therefore, once (5) has been established, the fact that the \( L_2 \)-orthogonal complement of \( E \) in \( F \) is \( N \)-dimensional shows that \( \Psi \) is a basis for it. Now we compute

\[
\gamma_{n,n-1}^o := (\psi_n^o, \psi_{n-1}^o)_J = -(\psi_n^-, \psi_{n-1}^+)_J = -\lim_{\epsilon \to 0} \psi_n^- (t_{n-1} + |\epsilon|) = a_n (-1)^{1+p_n} (1 + p_n),
\]

and similarly

\[
\beta_n^o := (\psi_n^o, \psi_n^o)_J = \|\psi_n^o\|^2 + \|\psi_n^+\|^2 = \beta_n^- + \beta_n^+.
\]

Set \( \delta_n := \gamma_{n,n-1}^o / \sqrt{\beta_n^- \beta_n^+} \). Then the Gramian of \( \Psi \) is the tridiagonal symmetric \( N \times N \) matrix

\[
M_\Psi := \begin{pmatrix}
1 & \delta_2 & \delta_3 & \cdots \\
\delta_2 & 1 & \delta_3 & \cdots \\
\delta_3 & \delta_3 & 1 & \ddots \\
\vdots & \vdots & \ddots & \ddots
\end{pmatrix}
\]

Note \( c^T M_\Psi c = |c|^2 + \sum_{n=2}^N 2 \delta_n c_n c_{n-1} \) for any \( c \in \mathbb{R}^N \), so we wish to compare the magnitude of the second term to \( |c|^2 \). Observe

\[
|2 \delta_n c_n c_{n-1}| = |\gamma_{n,n-1}^o| \times 2 \frac{|c_n| |c_{n-1}|}{\sqrt{\beta_n^-} \sqrt{\beta_n^+}} \leq \frac{\beta_n}{1 + p_{\text{min}}} \times \left( \frac{c_n^2}{\beta_n^-} + \frac{c_{n-1}^2}{\beta_n^+} \right),
\]

so that \( \sum_{n=2}^N |2 \delta_n c_n c_{n-1}| \leq \frac{1}{1 + p_{\text{min}}} |c|^2 \). Therefore, \( C_- |c|^2 \leq c^T M_\Psi c \leq C_+ |c|^2 \), with \( C_\pm = 1 \pm \frac{1}{1 + p_{\text{min}}} \). \( \square \)

On a uniform mesh \( \mathcal{T} \) and with uniform polynomial degree \( p \) one obtains \( \delta_n = -\frac{1}{2} (1+p) \) for each \( n \) in the Gramian (8). It is a tridiagonal symmetric Toeplitz matrix with eigenvalues \( \{1 + \frac{1}{1+p} \cos \frac{k\pi}{N+1} : k = 1, \ldots, N\} \). Letting \( N \to \infty \) shows that the constants in (5) cannot be improved in general.

The fact that the Gramian (8) is well-conditioned (5), together with Lemma 2.1, allows for fast computation of the discontinuous part of a function \( f \in F \). Let \( \Delta \in \mathbb{R}^N \) denote the vector of jumps of \( f \), whose components are \( \Delta_n = \lim_{\epsilon \to 0} \{ f(t_n - |\epsilon|) - f(t_n + |\epsilon|) \} \) for \( n = 1, \ldots, N \). Then Lemma
2.1 and definition (4) of $\psi_n$ imply $b_n := (\psi_n, f)_J = \Delta_n / \sqrt{\beta_n^{+} + \beta_n^{-}}$. The coefficients $c \in \mathbb{R}^N$ of the discontinuous part of $f$ with respect to the basis (4) satisfy the linear system $M_{\theta} c = b$, which can be quickly solved approximatively by the conjugate gradient method. This may be cheaper than projecting onto $E$ and then taking the difference because the dimension of $E$ is significantly larger than $N$ when the polynomial degrees $p_n$ in (2) are large.

3. Application: adaptive approximation

As an application of the above construction we describe an algorithm for the adaptive approximation of a given function $g \in L_2(J)$. Suppose $\mathcal{T}_i$, $i = 0, 1, 2, \ldots$, is a sequence of meshes as in the introduction, which are nested, $\mathcal{T}_i \subset \mathcal{T}_{i+1}$. Let $F_i \subset Y$ be the space of piecewise affine splines on with respect to $\mathcal{T}_i$, and set $E_i := F_i \cap H^1(J)$, which is then the space of continuous piecewise affine splines. Note that $F_i = E_i + E_i^\perp$ where the prime denotes the derivative. Let $E_i^\perp$ denote the $L_2$-orthogonal complement of $E_i$ in $F_i$, and let $Q_i^\perp : L_2(J) \to E_i^\perp$ be the surjective $L_2$-orthogonal projection.

Suppose $g_i \in E_i^\perp$ is an approximation of $g$. Then we consider $g_i^+ := Q_i^\perp g_i \in E_i^\perp$, and use the coefficients of $g_i^+$ with respect to the Riesz basis (4) for $E_i^\perp$ as error indicators for the marking of subintervals to be adaptively refined. The adaptive algorithm is as follows. Let $\mathcal{T}_0$ be given. Fix a threshold parameter $\theta \in (0, 1]$. For each $i = 0, 1, 2, \ldots$, do: 1) Compute the $L_2$-orthogonal projection $g_i$ of $g$ onto $E_i$, or an approximation thereof. 2) Compute the projection $g_i^+ := Q_i^\perp g_i$. Set $N_i = \#(\mathcal{T}_i \cap (0, T))$ for the number of interior nodes in $\mathcal{T}_i$, and $[N_i] := \{ 1, \ldots, N_i \}$. Let $c \in \mathbb{R}^{N_i}$ be the vector of the coefficients of $g_i^+ \in E_i^\perp$ with respect to the Riesz basis (4) for $E_i^\perp$. Select $M_i \subset [N_i]$ of minimal size such that $\sum_{n \in M_i} c_n^2 \geq \theta^2 |c|^2$. 3) If $\mathcal{T}_i = \{ 0 = t_0 < t_1 < \ldots < t_N < t_{N+1} = T \}$, let the new mesh $\mathcal{T}_{i+1}$ contain $\mathcal{T}_i$, and the new nodes $\frac{1}{2}(t_{n-1} + t_n)$ and $\frac{1}{2}(t_n + t_{n+1})$ for all $n \in M_i$.

The Riesz basis property guarantees that the part of the indicator $g_i^+$ corresponding to the marked subset $M_i$ carries a fraction of its total $L_2(J)$ norm that is comparable to $\theta$.

As an example we take $g : t \mapsto t^{-1/3}$ on $J := [0, T]$ with $T := 1$. We set $\mathcal{T}_0 := \{ 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1 \}$ for the initial mesh. Instead of the $L_2$-orthogonal projection we use the 4-point Gauss–Legendre quadrature rule on each subinterval to obtain $g_i$. The error $\| g - g_i \|_J$ for the above adaptive is shown in Figure 2. Due to the lack of smoothness of the given function $g$, it is approximated by piecewise constant functions on a uniform mesh with a rate of $\approx 1/6$ with respect to the mesh size $\# \mathcal{T}_i$, while adaptivity recovers the asymptotic rate of one.

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