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JUMPLETS

ROMAN ANDREEV

ABSTRACT. In this note we describe a locally supported Riesz basis consisting of “jumplets” for the orthogonal complement of continuous splines on an interval in the space of discontinuous ones.

1. Introduction

Let $T > 0$ and write $J := [0, T]$. For $N \geq 1$ let a finite partition of $J$ be given by

$$F = \{0: t_0 < t_1 < \ldots < t_N < t_{N+1} := T\}.$$  

Let $\mathbb{P}_d$ denote the space of real-valued polynomials on $\mathbb{R}$ of degree at most $d$. We write

$$F := \{f \in L^2(J) : f|_{(t_{n-1}, t_n)} \in \mathbb{P}_d\} \quad \text{and} \quad E := F \cap H^1(J)$$

for the space $F$ of piecewise polynomials on the interval $[0, T]$ of given polynomial degrees $p_n \geq 0$ and for its subset $E \subset F$ of the continuous ones. Here and throughout, $L^2$ and $H^1$ refer to the usual Lebesgue and Sobolev spaces of real-valued functions on $J$.

In the next Section 2 we construct a locally supported Riesz basis for the orthogonal complement of $E$ in $F$. Its dimension is obviously $N$, which is the number of interior nodes of $F$. We refer to $E$ as the continuous part of $F$, and to its $L^2$-orthogonal complement in $F$ as the discontinuous part of $F$. An application to adaptive approximation in $L^2$ is discussed in Section 3.

2. Construction of the basis

If $I \subset \mathbb{R}$ is an interval, we write $(\cdot, \cdot)$ for the $L^2(I)$ scalar product, and $\| \cdot \|_I$ for the norm. The euclidean norm is denoted by $| \cdot |$. For integer $d \geq 0$ let $\ell_d \in \mathbb{P}_d$ be the Legendre polynomial on the unit interval $[0, 1]$ characterized by $\ell_d(1) = \sqrt{2d + 1}$ and $\int_0^1 \ell_d(t)q(t)dt = 0$ for any $q \in \mathbb{P}_{d-1}$. Then $\{\ell_k : 0 \leq k \leq d\}$ is an orthonormal basis for $\mathbb{P}_d \cap \mathbb{P}_1([0, 1])$.

For integer $d \geq 0$ define the polynomial $R_d := \sum_{k=0}^d \ell_k(x)\ell_k$. Its key property is that $(R_d, q)_{[0,1]} = q(1)$ for any polynomial $q \in \mathbb{P}_d$, as can be verified by expanding $q$ into the Legendre basis. Further, $\|R_d\|_{[0,1]} = 1 + d$. Together, these observations imply $R_d(1) = (R_d, R_d)_{[0,1]} = (1 + d)^2$. Using $\ell_k(0) = (-1)^k\ell_k(1)$ in the definition of $R_d$ we also obtain $R_d(0) = (-1)^k(1 + d)$.

For each $n = 1, \ldots, N + 1$, let $J_n := (t_{n-1}, t_n)$ be the $n$-th subinterval in the partition $F$ from (1). Write $\alpha_n := |t_n - t_{n-1}|^{-1}$ for the inverse of its length, and define the constants $\beta_n := \alpha_n(1 + p_n)^2$, where $p_n$ is the polynomial degree from (2). Define the functions $\psi_n^-$ and $\psi_{n-1}^+$ in $F$ having the values

$$\psi_n^-(t) := \alpha_n p_n(\alpha_n(t - t_{n-1})) \quad \text{and} \quad \psi_{n-1}^+(t) := \alpha_n p_n(\alpha_n(t_n - t))$$

if $t \in J_n$, and zero else. Note that $\psi_n^+$ is the reflection of $\psi_n^-$ about the vertical at the middle of the subinterval $J_n$. Since $\beta_n$ is associated with both, $\psi_n^-$ and $\psi_n^+$, we also write $\beta_{n-1}^+ := \beta_n^- := \beta_n$. These functions are designed towards the following observation.

**Lemma 2.1.** $(\psi_n^+, f)_I = \lim_{\epsilon \to 0} f(t_n \pm |\epsilon|)$ for all $f \in F$ and $n = 1, \ldots, N$.

**Proof.** If $f \in F$ then $q : t \mapsto f(t_n + s/\alpha_n)$ is a polynomial of degree $p_n$ on the unit interval $[0, 1]$. Since $\psi_n^-$ is supported on $J_n$, we employ the definition of $\psi_n^-$ and the properties of $R_d$ to obtain $(\psi_n^-, f)_I = (\psi_n^-, q)_{[0,1]} = q(1) = \lim_{\epsilon \to 0} f(t_n - |\epsilon|)$. The proof for $\psi_n^+$ is analogous. \(\square\)
The announced basis for the discontinuous part of $F$ is now defined by
\begin{equation}
\psi_n := \frac{\psi_n^- - \psi_n^+}{\sqrt{\beta_n^- + \beta_n^+}}, \quad n = 1, \ldots, N.
\end{equation}

**Proposition 2.2.** Assume that $p_n \geq p_{\min} \geq 1$ in (2). Then, $\Psi := \{\psi_n : n = 1, \ldots, N\}$ is a Riesz basis for the $L_2$-orthogonal complement of $E$ in $F$. More precisely, for all vectors $c \in \mathbb{R}^N$,
\begin{equation}
C_-|c|^2 \leq \|\Psi^Tc\|_2^2 \leq C_+|c|^2 \quad \text{with} \quad C_\pm = 1 \pm \frac{1}{1 + p_{\min}}.
\end{equation}

**Proof.** Set $\psi_n := \psi_n^- - \psi_n^+$. Then $(\psi_n^0, c_j) = \lim_{e \to 0} e(t_n - |e|) - \lim_{e \to 0} e(t_n + |e|) = 0$ for any $e \in E$ by continuity of $e$. Hence, each $\psi_n$ is $L_2$-orthogonal to $E$. By definition, $\psi_n \in F$. Therefore, once (5) has been established, the fact that the $L_2$-orthogonal complement of $E$ in $F$ is $N$-dimensional shows that $\Psi$ is a basis for it. Now we compute
\begin{equation}
\gamma_{n,n-1}^0 := (\psi_n^0, \psi_{n-1}^0)_J = - (\psi_n^-, \psi_{n-1}^+) = - \lim_{e \to 0} \psi_n^-(t_{n-1} + |e|) = a_n(-1)^{1+p}(1+p_n),
\end{equation}
and similarly
\begin{equation}
\beta_n^0 := (\psi_n^0, \psi_n^0)_J = \|\psi_n^-\|_2^2 + \|\psi_n^+\|_2^2 = \beta_n^- + \beta_n^+.
\end{equation}

Set $\delta_n := \gamma_{n,n-1}^0/\sqrt{\beta_n^- \beta_{n-1}^+}$. Then the Gramian of $\Psi$ is the tridiagonal symmetric $N \times N$ matrix
\begin{equation}
M_\Psi := \begin{pmatrix}
1 & \delta_2 & 0 & \cdots \\
\delta_2 & 1 & \delta_3 & 0 & \cdots \\
0 & \delta_3 & 1 & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & \delta_N & 1
\end{pmatrix}.
\end{equation}

Note $c^T M_\Psi c = |c|^2 + \sum_{n=2}^N 2 \delta_n c_n c_{n-1}$ for any $c \in \mathbb{R}^N$, so we wish to compare the magnitude of the second term to $|c|^2$. Observe
\begin{equation}
|2 \delta_n c_n c_{n-1}| \leq |\gamma_{n,n-1}^0| \times 2 \frac{|c_n| |c_{n-1}|}{\sqrt{\beta_n^- \beta_{n-1}^+}} \leq \frac{\beta_n}{1 + p_{\min}} \times \left(\frac{c_n^2}{\beta_n^+} + \frac{c_{n-1}^2}{\beta_{n-1}^-}\right),
\end{equation}
so that $\sum_{n=2}^N |2 \delta_n c_n c_{n-1}| \leq \frac{1}{1 + p_{\min}} |c|^2$. Therefore, $C_-|c|^2 \leq c^T M_\Psi c \leq C_+|c|^2$, with $C_\pm = 1 \pm \frac{1}{1 + p_{\min}}$. \hfill \Box

On a uniform mesh $\mathcal{T}$ and with uniform polynomial degree $p$ one obtains $\delta_n = -\frac{1}{2} \left(\frac{1}{1+p}\right) p_{n-1}$ for each $n$ in the Gramian (8). It is a tridiagonal symmetric Toeplitz matrix with eigenvalues $\{1 + \frac{1}{1+p} \cos \frac{k\pi}{N+1} : k = 1, \ldots, N\}$. Letting $N \to \infty$ shows that the constants in (5) cannot be improved in general.

The fact that the Gramian (8) is well-conditioned (5), together with Lemma 2.1, allows for fast computation of the discontinuous part of a function $f \in F$. Let $\Delta \in \mathbb{R}^N$ denote the vector of jumps of $f$, whose components are $\Delta_n = \lim_{e \to 0} \{f(t_n - |e|) - f(t_n + |e|)\}$ for $n = 1, \ldots, N$. Then Lemma

![Figure 1. The basis function $\psi_2$ from (4) for polynomial degrees $p = 1$ (left) / $p = 2$ (right).](image-url)
2.1 and definition (4) of \( \psi \) imply \( b_n := (\psi_n, f) = \Delta_n / \sqrt{\beta_n^+ + \beta_n^-} \). The coefficients \( c \in \mathbb{R}^N \) of the discontinuous part of \( f \) with respect to the basis (4) satisfy the linear system \( M_g c = b \), which can be quickly solved approximatively by the conjugate gradient method. This may be cheaper than projecting onto \( E \) and then taking the difference because the dimension of \( E \) is significantly larger than \( N \) when the polynomial degrees \( p_n \) in (2) are large.

3. Application: adaptive approximation

As an application of the above construction we describe an algorithm for the adaptive approximation of a given function \( g \in L_2(J) \). Suppose \( \mathcal{T}_i, i = 0, 1, 2, \ldots \), is a sequence of meshes as in the introduction, which are nested, \( \mathcal{T}_i \subset \mathcal{T}_{i+1} \). Let \( F_i \subset Y \) be the space of piecewise affine splines on with respect to \( \mathcal{T}_i \), and set \( E_i := F_i \cap H^1(J) \), which is then the space of continuous piecewise affine splines. Note that \( F_i = E_i^i + E_i^e \), where the prime denotes the derivative. Let \( E_i^e \) denote the \( L_2 \)-orthogonal complement of \( E_i \), in \( F_i \), and let \( Q_i^e : L_2(J) \to E_i^e \) be the surjective \( L_2 \)-orthogonal projection.

Suppose \( g_i \in E_i^i \) is an approximation of \( g \). Then we consider \( g_i^+ := Q_i^e g_i \in E_i^e \), and use the coefficients of \( g_i^+ \) with respect to the Riesz basis (4) for \( E_i^e \) as error indicators for the marking of subintervals to be adaptively refined. The adaptive algorithm is as follows. Let \( \mathcal{T}_0 \) be given. Fix a threshold parameter \( \theta \in (0, 1] \). For each \( i = 0, 1, 2, \ldots \), do: 1) Compute the \( L_2 \)-orthogonal projection \( g_i \) of \( g \) onto \( E_i^i \), or an approximation thereof. 2) Compute the projection \( g_i^+ := Q_i^e g_i \). Set \( N_i = \#(\mathcal{T}_i \cap (0, T)) \) for the number of interior nodes in \( \mathcal{T}_i \), and \( [N_i] := \{1, \ldots, N_i\} \). Let \( c \in \mathbb{R}^{N_i} \) be the vector of the coefficients of \( g_i^+ \in E_i^e \) with respect to the Riesz basis (4) for \( E_i^e \). Select \( M_i \subset [N_i] \) of minimal size such that \( \sum_{n \in M_i} c_n^2 \geq \theta^2 |c|^2 \). 3) If \( \mathcal{T}_i = \{0 = t_0 < t_1 < \ldots < t_{N_i} < T_{N_i+1} = T\} \), let the new mesh \( \mathcal{T}_{i+1} \) contain \( \mathcal{T}_i \), and the new nodes \( \frac{1}{2}(t_{n-1} + t_n) \) and \( \frac{1}{2}(t_n + t_{n+1}) \) for all \( n \in M_i \).

The Riesz basis property guarantees that the part of the indicator \( g_i^+ \) corresponding to the marked subset \( M_i \) carries a fraction of its total \( L_2(J) \) norm that is comparable to \( \theta \).

As an example we take \( g : t \mapsto t^{-1/3} \) on \( J := [0, T] \) with \( T := 1 \). We set \( \mathcal{T}_0 := \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\} \) for the initial mesh. Instead of the \( L_2 \)-orthogonal projection we use the 4-point Gauss–Legendre quadrature rule on each subinterval to obtain \( g_i \). The error \( \|g - g_i\|_J \) for the above adaptive is shown in Figure 2. Due to the lack of smoothness of the given function \( g \), it is approximated by piecewise constant functions on a uniform mesh with a rate of \( \approx 1/6 \) with respect to the mesh size \( \#(\mathcal{T}) \), while adaptivity recovers the asymptotic rate of one.

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FIGURE 2. Left: \( L_2 \) error achieved by the adaptive approximation algorithm for \( g : t \mapsto t^{-1/3} \) on \( J = (0, 1) \) by piecewise constant functions, and the error of the \( L_2 \) best approximation of \( g \) on a uniform mesh, as a function of the mesh size. Right: \( g \) and its adaptive approximation \( g_4 \) for \( \theta = 3/4 \).

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