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QUASI-OPTIMALITY OF APPROXIMATE SOLUTIONS IN NORMED VECTOR SPACES

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ABSTRACT: We discuss quasi-optimality of approximate solutions to operator equations in normed vector spaces, defined either by Petrov–Galerkin projection or by residual minimization. Examples demonstrate the sharpness of the estimates.

Let \( X \) and \( Y \) be real normed vector spaces. Let \( B : X \to Y \) be a linear operator. Fix \( u \in X \) – the “unknown”. Let \( X_h \times Y_h \subset X \times Y \) be nontrivial finite-dimensional subspaces. Abbreviate

\[
\gamma_h := \inf_{w \in X_h \setminus \{0\}} \frac{\|Bw\|_{Y_h}}{\|w\|_X} \quad \text{and} \quad \|B\| := \sup_{w \in (u+X_h)\setminus\{0\}} \frac{\|Bw\|_{Y_h}}{\|w\|_X}.
\]

Throughout, we assume the “discrete inf-sup condition”: \( \gamma_h > 0 \). We define \( B_h : X_h \to Y_h \) by \( w \mapsto (Bw)|_{Y_h} \). In the first proposition we require \( \dim X_h = \dim Y_h \). In the second we admit \( \dim Y_h \geq \dim X_h \).

**Proposition 1.** Suppose \( \dim X_h = \dim Y_h \). Then there exists a unique \( u_h \in X_h \) such that

\[
\langle Bu_h, v \rangle = \langle Bu, v \rangle \quad \forall v \in Y_h.
\]

The mapping \( u \mapsto u_h \) is linear with \( \|u_h\|_X \leq \gamma_h^{-1}\|Bu\|_{Y_h} \) and satisfies the quasi-optimality estimate:

\[
\|u - u_h\|_X \leq (1 + \gamma_h^{-1}\|B\|) \inf_{w \in X_h} \|u - w\|_X.
\]

**Proof.** The map \( B_h \) is linear and injective by (1). It is bijective due to finite \( \dim X_h = \dim Y_h = \dim Y_h' \). Thus a unique \( u_h := B_h^{-1}(Bu)|_{Y_h} \) exists and \( u \mapsto u_h \) is linear. By (1), \( \gamma_h\|u_h\|_X \leq \|B_h u_h\|_{Y_h'} = \|Bu\|_{Y_h'} \). From \( \|u - u_h\|_X \leq \|u - w_h\|_X + \|w_h - u_h\|_X \) and \( \gamma_h\|w_h - u_h\|_X \leq \|B(u - w_h)\|_{Y_h'} \), we obtain (3). \( \square \)

**Proposition 2.** The set \( U_h := \arg\inf_{w \in X_h} \|Bu - Bw_h\|_{Y_h'} \subset X_h \) of residual minimizers is nonempty, convex and bounded. Any \( u_h \in U_h \) satisfies the quasi-optimality estimate

\[
\|u - u_h\|_X \leq (1 + 2\gamma_h^{-1}\|B\|) \inf_{w \in X_h} \|u - w\|_X.
\]

**Proof.** The first statement is elementary: consider the metric projection of \( (Bu)|_{Y_h} \in Y_h' \) onto \( B_h X_h \subset Y_h' \). Quasi-optimality is obtained as above, except that \( \|B(u-u_h)\|_{Y_h'} \leq \|B(u-w)\|_{Y_h'} + \|B(u-w_h)\|_{Y_h'} \leq 2\|B(u-w_h)\|_{Y_h'} \). \( \square \)

The set \( U_h \) of minimizers is a singleton if the unit ball of \( Y_h' \) is strictly convex. Since \( Y_h \) is finite-dimensional, this is the case if and only if the norm of \( Y_h \) is Gâteaux differentiable.

The constants in (3) and (4) are sharp: Take \( X = Y = \mathbb{R}^2 \) with the \( \|\cdot\|_1 \) norm. Then \( \|\cdot\|_\infty \) is the norm of \( Y' \). Take \( u := (0,1) \) and \( B(w_1, w_2) := (w_1 + w_2, w_2) \). Set \( X_h := \mathbb{R} \times \{0\} \) (→ \( B \) is identity on \( X_h \)). Observe \( \|B\| = 1 \).

• For (3) let \( Y_h := \mathbb{R} \times \{0\} \). Then \( \|Bw_h\|_{Y_h'} = \|w_h\|_X \) for all \( w_h \in X_h \) gives \( \gamma_h = 1 \). Now, \( u_h := (1,0) \in X_h \) solves (2). In the quasi-optimality estimate we have \( \|u - u_h\|_X = 2 \) while \( \|u - w_h\|_X = 1 \) for \( w_h = 0 \).

• For (4) let \( Y_h := Y \). Again, \( \gamma_h = 1 \). Since \( Bu = (1,1) \), the set of minimizers \( U_h \) is the segment \( [0,2] \times \{0\} \).

For \( u := (2,0) \in U_h \) we have \( \|u - u_h\|_X = 3 \) while \( \|u - w_h\|_X = 1 \) for \( w_h = 0 \). With a slight perturbation of the norms, say, we can achieve \( U_h = \{u_h\} \) without essentially changing the distances.

If \( X \) and \( Y \) are Hilbert spaces and \( B : X \to Y' \) is bounded by \( \|B\| \) then in both propositions the mapping \( P_h : X \to Y', u \mapsto u_h, \) is a well-defined bounded linear projection with \( \|P_h\| \leq \gamma_h \|B\| \). The argument of


then improves the quasi-optimality estimate to \( \|u - u_h\|_X \leq \|P_h\| \inf_{w \in X_h} \|u - w\|_X \).

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