Observability properties of the homogeneous wave equation on a closed manifold

Emmanuel Humbert, Yannick Privat, Emmanuel Trélat

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Abstract

We consider the wave equation on a compact Riemannian manifold (Ω, g) without boundary (i.e., a closed manifold). We observe the restriction of the solutions to a measurable subset ω of Ω during a time interval $[0, T]$ with $T > 0$. A well known result by Rauch and Taylor, and by Bardos, Lebeau and Rauch asserts that, if ω is open in Ω and if the pair (ω, T) satisfies the Geometric Control Condition, then an observability inequality holds comparing the total energy of solutions to the energy localized in $\omega \times (0, T)$. The observability constant $C_T(\omega)$ is then defined as the infimum over the set of all solutions of the wave equation of the ratio of localized energy of solutions over their total energy.

In this paper, we provide sharp estimates of the observability constant allowing to derive general geometric conditions guaranteeing that the wave equation is observable on ω.

Using the same approach, we also investigate the asymptotics of the observability constant as the observability time T tends to $+\infty$. Under topological assumptions on ω, we show that the ratio $C_T(\omega)/T$ converges to the minimum of two quantities: the first one is of a spectral nature and involves the Laplacian eigenfunctions; the second one is of a geometric nature and involves the average time spent in ω by Riemannian geodesics propagating over Ω.

Keywords: wave equation, observability inequality.

Contents

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1 Introduction and main results

1.1 Framework

Let (Ω, g) be a compact connected Riemannian manifold of dimension n without boundary. The canonical Riemannian volume on Ω is denoted by v_g, inducing the canonical measure dv_g. Measurable sets[1] are considered with respect to the measure dv_g.

Consider the wave equation in Ω,

$$\partial_{tt} y - \triangle_g y = 0 \qquad \text{in } (0,T)\times\Omega, \tag{1}$$

where $\triangle_g$ stands for the usual Laplace-Beltrami operator on Ω for the metric g. We define the closed subspace $L^2_0(\Omega)$ of $L^2(\Omega)$ by $L^2_0(\Omega) = \left\{y \in L^2(\Omega) \mid \int_\Omega y(x)\, dv_g = 0\right\}$, and we endow it with the topology inherited from the norm $\|\cdot\|_{L^2}$. Let us introduce the Sobolev space

$$H^1(\Omega) = \left\{y \in L^2_0(\Omega) \mid -\triangle_g y \in L^2_0(\Omega)\right\},$$

where $\triangle_g y$ is taken in the sense of distributions, as well as $(H^1)'(\Omega)$, the dual space of $H^1(\Omega)$ with respect to the pivot space $L^2_0(\Omega)$. It is understood that $H^1(\Omega)$ is endowed with the $\|\cdot\|_{H^1}$ norm given by $\|u\|^2_{H^1} = \|u\|^2_{L^2} + \|\nabla u\|^2_{L^2}$, whereas the dual space $(H^1)'$ is endowed with the usual topology on dual spaces.

For every set of initial data $(y(0,\cdot), \partial_t y(0,\cdot)) \in L^2_0(\Omega)\times(H^1)'(\Omega)$, there exists a unique solution $y \in \mathcal{C}^0(0,T; L^2(\Omega)) \cap \mathcal{C}^1(0,T;(H^1)'(\Omega))$ of (1).

Let ω be an arbitrary measurable subset of Ω of positive measure, and let $T>0$. The notation χ_ω stands for the characteristic function of ω, in other words the function equal

[1] If M is the usual Euclidean space $\mathbb{R}^n$ then $dv_g = dx$ is the usual Lebesgue measure.

to 1 on ω and 0 elsewhere. The equation (1) is said to be *observable* on ω in time T if there exists a positive constant C such that

$$C\|(y^0,y^1)\|^2_{L^2\times(H^1)'} \leqslant \int_0^T \int_\omega |y(t,x)|^2\, dv_g(x)\, dt,$$

for all $(y^0,y^1)\in L^2_0(\Omega)\times(H^1)'(\Omega)$ such that $(y(0,\cdot),\partial_t y(0,\cdot))=(y^0,y^1)$. It is well known that if ω is an open set, this observability property holds if the pair (ω,T) satisfies the *Geometric Control Condition* in Ω (see the results by Rauch and Taylor in [21] and the ones by Bardos, Lebeau and Rauch in [1]), according to which every ray of geometric optics that propagates in Ω intersects ω within time T. This classical result will be slightly generalized in this paper (see Section 3).

Definition 1 (Observability constant $C_T(\omega)$)**.** *Let ω be a Lebesgue measurable subset of Ω. The* ***observability constant*** *in time T associated to (1) is defined by*

$$C_T(\omega)=\inf\left\{J_T^\omega(y^0,y^1)\mid (y^0,y^1)\in L^2_0(\Omega)\times(H^1)'(\Omega)\setminus\{(0,0)\}\right\}, \tag{2}$$

where

$$J_T^\omega(y^0,y^1)=\frac{\int_0^T\int_\omega |y(t,x)|^2\, dv_g\, dt}{\|(y^0,y^1)\|^2_{L^2\times(H^1)'}}. \tag{3}$$

This paper is devoted to the investigation of the observability constant and is organized as follows. In Section 1.2, we state the main results of the paper. In a nutshell, under topological assumptions on the observation domain ω, we show that the limit of the quantity $C_T(\omega)/T$ as $T\to+\infty$ exists, is finite and we prove that it is the minimum of two quantities: the first one is of a spectral nature and involves the eigenfunctions of $-\triangle_g$; the second one is of a geometric nature and involves the geodesics of Ω. In a second time, we present a low/high frequencies splitting result (Theorem 1). Section 2 is devoted to give a proof of these results. The low/high frequencies splitting result allows us to obtain a characterization of observability (Corollary 1) which shows how the observability property can be characterized only by highfrequency modes. In turn, this provides a new and simpler proof of the results by Rauch and Taylor [21] and by Bardos, Lebeau and Rauch [1]. The main interest of our approach (which is actually a generalization of the preceeding results to a slightly wider class of subsets ω) is to clarify how observability is related to the so-called Geometric Control Conditions (GCC). This is the purpose of Section 3. In Section 4, we show that, under a spectral gap condition, the estimates on the limit of the quantity $C_T(\omega)/T$ as $T\to+\infty$ can be refined.

1.2 Main results

Let us define several quantities and introduce some notations. Let $(\phi_j)_{j\in\mathbb{N}^*}$ be an arbitrary Hilbert basis of $L^2_0(\Omega)$ consisting of eigenfunctions of $-\triangle_g$, associated with the real eigenvalues $(\lambda_j^2)_{j\in\mathbb{N}^*}$ such that $0<\lambda_1\leqslant\lambda_2\leqslant\cdots\leqslant\lambda_j\to+\infty$. Given $T>0$ and $N\in\mathbb{N}$, we define

$$C_T^{>N}(\omega)=\inf\{J_T^\omega(y^0,y^1)\mid \langle y^i,\phi_j\rangle_{(H^i)',H^i}=0,\quad \forall i=0,1,\quad \forall j=1,\dots,N\} \tag{4}$$

with the convention that $H^0 = L^2$.

This allows to define a notion of "highfrequency" observability constant over initial conditions (y^0, y^1) involving only frequencies of index greater than N.

Definition 2 (Highfrequency observability constant)**.** *Let $T > 0$ and ω be a measurable subset of Ω. The* highfrequency observability constant $\alpha^T(\omega)$ *is defined by*

$$\alpha^T(\omega) = \lim_{N \to +\infty} \frac{1}{T} C_T^{>N}(\omega).$$

Note that this limit exists since the mapping $\mathbb{N} \ni N \mapsto C_T^{>N}(\omega)$ is nondecreasing.

Definition 3 (Spectral quantity $g_1(\omega)$)**.** *Let ω be a Lebesgue measurable subset of Ω. The* ***spectral quantity*** $g_1(\omega)$ *is defined by*

$$g_1(\omega) = \inf_{\phi \in \mathcal{E}} \frac{\int_\omega |\phi(x)|^2 \, dv_g}{\int_\Omega |\phi(x)|^2 \, dv_g},$$

where the infimum runs over the set $\mathcal{E}$ of all nonconstant eigenfunctions ϕ of $-\triangle_g$.

Main results: new characterization of the observability constant $C_T(\omega)$.

Theorem 1 (Finite-time observability)**.** *Let ω be a Lebesgue measurable subset of Ω and let $T > 0$. There holds*

$$\frac{C_T(\omega)}{T} \leqslant \min \left(\frac{1}{2} g_1(\omega), \alpha^T(\omega) \right).$$

Moreover, if

$$\frac{C_T(\omega)}{T} < \alpha^T(\omega),$$

then the infimum in the definition of C_T is reached, i.e., there exists $(y^0, y^1) \in L^2_0(\Omega) \times (H^1)'(\Omega) \setminus \{(0,0)\}$ such that

$$\frac{C_T(\omega)}{T} = J_T^\omega(y^0, y^1).$$

It is interesting to note that Theorem 1 provides an explicit characterization of the positiveness of $C_T(\omega)$.

Corollary 1. *Let ω be a Lebesgue measurable subset of Ω and let $T > 0$. We have $C_T(\omega) > 0$ if and only if $\alpha^T(\omega) > 0$.*

Corollary 1 clarifies the conditions needed to obtain the observability condition on any measurable set: it reduces to showing that $\alpha^T(\omega) > 0$. Together with the explicit computation of $\alpha^T(\omega)$ provided in Theorem 3 below, we get sufficient condition ensuring the positiveness of $C_T(\omega)$ that slightly extend the GCC obtained in [1, 21] for ω open.

Remark 1. Note that in [1], the authors also treat manifolds having a boundary. Corollary 1 still holds true in this context but extending our results to such geometries would require a deeper study of $\alpha^T(\omega)$ on manifolds with boundary, which will be the devoted to a future work.

As a consequence of our proof techniques, which are based on a concentration-compactness argument, we get the following large-time asymptotics of the observability constant $C_T(\omega)$.

Theorem 2 (Large-time observability). *Let ω be a Lebesgue measurable subset of Ω. The limit*

$$\alpha^\infty(\omega) = \lim_{T\to+\infty} \alpha^T(\omega)$$

exists, and we have

$$\lim_{T\to+\infty} \frac{C_T(\omega)}{T} = \min\left(\frac{1}{2}g_1(\omega), \alpha^\infty(\omega)\right). \tag{5}$$

Moreover, if $g_1(\omega) < \alpha^\infty(\omega)$, then $g_1(\omega)$ is reached.

Characterization of the quantities $\alpha^T(\omega)$ and $\alpha^\infty(\omega)$.

Definition 4 (Geometric quantity $g_2(\omega)$). *Let ω be a Lebesgue measurable subset of Ω. Let γ be a Riemannian geodesic, traveling at speed one in Ω. For $T > 0$, we introduce the quantity $m_T^\omega(\gamma)$ as the average time spent by γ in ω:*

$$m_T^\omega(\gamma) = \frac{1}{T}\int_0^T \chi_\omega(\gamma(t))\,dt.$$

*The **geometric quantity** $g_2(\omega)$ is defined by*

$$g_2(\omega) = \lim_{T\to+\infty} g_2^T(\omega) \qquad \text{with} \qquad g_2^T(\omega) = \inf_{\gamma\in\Gamma} m_T^\omega(\gamma), \tag{6}$$

where the infimum in the definition of g_2^T is taken over the set Γ of all geodesics of Ω.

Note that the mapping $G_{\chi_\omega} : T \mapsto \inf_{\gamma\in\Gamma} m_T^\omega(\gamma)$ enjoys the following properties: G_ω is nonnegative, bounded above by 1 and is a subadditive function. Therefore, the limit in the definition of $g_2(\omega)$ is well defined.

In [8], notable properties of the geometric quantity $g_2(\omega)$ have been established in the case where Ω is a square, and $\triangle_g$ the Dirichlet-Laplacian operator on Ω. In particular, an efficient algorithm allowing to compute explicitly $g_2(\omega)$ when ω is a finite union of squares has been given.

Theorem 3 (Computation of $\alpha^T(\omega)$). *Let ω be a Lebesgue measurable subset of Ω and let $T > 0$. We have*

$$\frac{1}{2}g_2^T(\mathring{\omega}) \leqslant \alpha^T(\mathring{\omega}) \leqslant \alpha^T(\omega) \leqslant \alpha^T(\overline{\omega}) \leqslant \frac{1}{2}g_2^T(\overline{\omega}).$$

Remark 2. Assume that ω is Jordan mesurable[2]. Then it follows from the definition of $C_T^{>N}$ that

$$\forall N \in \mathbb{N}, \qquad C_T^{>N}(\omega) = C_T^{>N}(\overline{\omega}). \tag{7}$$

[2]A bounded set E is said Jordan measurable if and only if the Lebesgue measure (or similarly the Jordan measure) of ∂E is 0.

As a consequence, Theorem 3 can be improved in that case by noting that $\frac{1}{2}g_2^T(\mathring{\overline{\omega}}) \leqslant \alpha^T(\omega)$. This remark may have some importance: let γ be the support of a closed geodesic of Ω and set $\omega = \Omega \setminus \gamma$. By (7),

$$\alpha^T(\omega) = \alpha^T(\omega) = 1 \text{ and } g_2^T(\mathring{\omega}) = g_2^T(\omega) = 0.$$

Hence, the estimate given by Theorem 3 is far from being sharp while the estimate (7) is sharp.

Let us now emphasize an important case where the constant $\alpha^T(\omega)$ is known explicitly. For that purpose, let us make the following assumption on the subset ω.

(**H**) (*Regularity assumption on* ω) Assume that

$$g_2^T(\Omega \setminus (\overline{\omega} \setminus \mathring{\omega})) = 1.$$

Many measurable sets ω satisfy Assumption (**H**). Geometrically speaking, it means that ω has no *grazing ray*. We say that a geodesic γ is grazing ω whenever $\gamma(t) \in \partial\omega$ over a set of times of positive measure.

We end this section by recasting all previous results in the case where (**H**) is assumed.

Corollary 2. *Given any $T > 0$ and any measurable set ω satisfying* (**H**)*, we have*

$$2\alpha^T(\omega) = g_2^T(\mathring{\omega}) = g_2^T(\overline{\omega}) = g_2^T(\omega).$$

As a consequence of Corollary 1 and Corollary 2, one has the following simple characterization of observability.

Corollary 3. *Let $T > 0$ and let $\omega \subset \Omega$ be a Lebesgue measurable subset of Ω.*

(i) If $g_2^T(\mathring{\omega}) > 0$ then $C_T(\omega) > 0$.

(ii) Assume that ω satisfies the assumption (**H**)*. Then we have the equivalence*

$$g_2^T(\omega) > 0 \Leftrightarrow C_T(\omega) > 0.$$

The first item above is precisely the main result of [1, 21]. Nevertheless, as already said the authors of [1] also deal with the difficult case of manifolds having a boundary, which is not the case in this article. Recovering the boundary case by the method we present here would require a deeper study of the quantity $\alpha^T(\omega)$, what we do not perform here.

If ω is an open set, they proved that the pair (ω, T) has the observability property (i.e. $C_T(\omega) > 0$) as soon as the Geometric Control Condition is satisfied. Recall that the so-called Geometric Control Condition (GCC) reads as follows: *the pair (ω, T) satisfies (GCC) if every geodesic of Ω, travelling with speed 1 and issued at $t = 0$ enters the open set ω before the time T.* As one can easily check, when ω is open, this condition is equivalent to the fact that $g_2^T(\omega) > 0$.

Note also that as expected, the positiveness of $g_1(\omega)$ is not enough to guarantee that the equation (1) is observable on ω. To illustrate this claim, assume that Ω is the torus $\mathbb{T}^2$, in which we choose ω as being the union of four triangles, each of them being at an edge of the square, whose side length is $1/2$. By construction, there are two "trapped rays" of cartesian equations $x = 1/2$ and $y = 1/2$, that just touch ω without crossing it over a positive duration. It follows that $g_2^T(\omega) = g_2(\omega) = C_T(\omega) = 0$ for every $T > 0$. Moreover, simple computations show that $g_1(\omega) > 0$.

From Theorem 2 and Corollary 2, one gets the following asymptotic result.

Corollary 4. *Let ω be a Lebesgue measurable subset of Ω satisfying Condition* (**H**)*. Then*

$$\lim_{T\to+\infty} \frac{C_T(\omega)}{T} = \frac{1}{2} \min\left(g_1(\omega), g_2(\overline{\omega})\right).$$

Remark 3 (Comparison with a result by Lebeau.)**.** In [12, Theorem 2], the author considers the damped wave equation

$$\partial_{tt} y(t,x) - \triangle_g y(t,x) + 2a(x)\partial_t y(t,x) = 0 \qquad \text{in } (0,T)\times\Omega \tag{8}$$

on a compact Riemannian manifold Ω with a $\mathcal{C}^\infty$ boundary, where the function $a(\cdot)$ is a smooth nonnegative function on the closure of Ω. Let us define $t \mapsto E_{(y^0,y^1)}(t)$, the energy function associated to (8) given by

$$E_{(y^0,y^1)}(t) = \int_\Omega \left(|\nabla y(t,x)|^2 + (\partial_t y(t,x))^2\right) dv_g,$$

where y denotes the unique solution of (8) with initial data $(y(0,\cdot), \partial_t y(0,\cdot)) = (y^0, y^1) \in H_0^1(\Omega) \times L^2(\Omega)$.

It is well known (see, e.g., [6]) that there exist two positive constants τ and C such that the inequality

$$E_{(y^0,y^1)}(t) \leqslant Ce^{-2\tau t} E_{(y^0,y^1)}(0), \tag{9}$$

holds for every initial data $(y^0, y^1) \in H_0^1(\Omega) \times L^2(\Omega)$ provided that the pair (ω, T) satisfy the *Geometric Control Condition* (GCC).

It is proved that the smallest decay rate $\tau(a)$ such that (9) is satisfied is given by

$$\tau(a) = \min\left(-\mu(\mathcal{A}_a), g_2(a)\right),$$

where $\mu(\mathcal{A}_a)$ denotes the spectral abscissa of the damped wave operator

$$\mathcal{A}_a = \begin{pmatrix} 0 & \mathrm{Id} \\ \triangle_g & -2a(\cdot)\,\mathrm{Id} \end{pmatrix},$$

and $g_2(a)$ is the geometric quantity defined by (6), replacing χ_ω by a.

Remark 4 (Probabilistic interpretation of the spectral quantity $g_1(\omega)$)**.** Let us provide another interpretation of the quantity $g_1(\omega)$.

It also corresponds to an averaged version of the observability constant $C_T(\omega)$ defined by (2), over random initial data. More precisely, let $(\beta^\nu_{1,j})_{j\in\mathbb{N}^*}$ and $(\beta^\nu_{2,j})_{j\in\mathbb{N}^*}$ be two sequences of Bernoulli random variables on a probability space $(\mathcal{X}, \mathcal{A}, \mathbb{P})$ such that

- for $m=1,2$, $\beta^\nu_{m,j}=\beta^\nu_{m,k}$ whenever $\lambda_j=\lambda_k$,
- all random variables $\beta^\nu_{m,j}$ and $\beta^\nu_{m',k}$, with $(m,m')\in\{1,2\}^2$, j and k such that $\lambda_j\neq\lambda_k$, are independent,
- there holds $\mathbb{P}(\beta^\nu_{1,j}=\pm 1)=\mathbb{P}(\beta^\nu_{2,j}=\pm 1)=\frac{1}{2}$ and $\mathbb{E}(\beta^\nu_{1,j}\beta^\nu_{2,k})=0$, for every j and k in $\mathbb{N}^*$ and every $\nu\in\mathcal{X}$.

Here, the notation $\mathbb{E}$ stands for the expectation over the space $\mathcal{X}$ with respect to the probability measure $\mathbb{P}$.

Then, $g_1(\omega)$ is the largest constant C for which the inequality

$$C\|(y^0,y^1)\|^2_{L^2\times(H^1)'}\leqslant\mathbb{E}\left(\int_0^T\int_\Omega\chi_\omega(x)|y^\nu(t,x)|^2\,dv_g\,dt\right),$$

holds for all $(y^0,y^1)\in L^2_0(\Omega)\times(H^1)'(\Omega)$, where y^ν is defined by

$$y^\nu(t,x)=\sum_{j=1}^{+\infty}\left(\beta^\nu_{1,j}a_je^{i\lambda_jt}+\beta^\nu_{2,j}b_je^{-i\lambda_jt}\right)\phi_j(x),$$

where the coefficients a_j and b_j are defined by (11) for every $j\in\mathbb{N}^*$.

In other words, y^ν denotes the solution of the wave equation (1) with the random initial data $y^\nu_0(\cdot)$ and $y^\nu_1(\cdot)$ determined by their Fourier coefficients $a^\nu_j=\beta^\nu_{1,j}a_j$ and $b^\nu_j=\beta^\nu_{2,j}b_j$.

In this context, the quantity $g_1(\omega)$ is called *randomized observability constant* and we refer to [17, Section 2.3] and [19, Section 2.1] for further explanations on its use in inverse problems. Moreover, a deterministic interpretation of this quantity is provided in [20].

Remark 5 (Extension of Corollary 4 to manifolds with boundary.)**.** One could expect that a similar asymptotic to the one stated in Corollary 4 holds for the Laplace-Beltrami operator on a manifold Ω such that $\partial\Omega\neq\emptyset$, with homogeneous Dirichlet boundary conditions.

For instance, in the one-dimensional case $\Omega=(0,\pi)$ with Dirichlet boundary conditions, it is showed using Fourier analysis tools in [18, Lemma 1] that for every measurable set ω, one has

$$\lim_{T\to+\infty}\frac{C_T(\omega)}{T}=\inf_{j\in\mathbb{N}^*}\int_\omega\phi_j(x)^2\,dv_g=g_1(\omega),\qquad\text{with}\quad\phi_j(x)=\frac{1}{\sqrt{\pi}}\sin(jx),$$

where $C_T(\omega)$ denotes the usual observability constant of the Dirichlet-Laplacian operator on $(0,\pi)$. In higher dimension, the problem is more difficult because we are unable to compute explicitly $\alpha^T(\omega)$ due to the fact that the Egorov theorem, used in the proof of Theorem 3, does not apply.

In Section 4, we focus on the manifolds for which the consecutive eigenvalues are uniformly bounded away from 0. Such an assumption, valid for example for the sphere, has already been used and one can show that it actually implies that the geodesic flow is periodic (see [5]). Under this assumption, we are able to compute explicitly the limit of $C_T(\omega)/T$ as $T \to +\infty$ for any measurable set (without any additional assumption). Namely, we show that this value is exactly $g_1(\omega)$. As an immediate application, we are able to prove that any geodesic of Ω is the support of a quantum measure, thus giving another proof of a result of Macià [14]. Actually, a slight modification of this method allows to extend this result to a more general setting: this is let for a forthcoming paper.

2 Proofs of Theorems 1, 2, 3 and of Corollary 2

2.1 Preliminary results

In what follows, we use the framework and notations introduced at the beginning of Section 1.2. Given any initial data $(y^0, y^1) \in L^2_0(\Omega) \times (H^1)'(\Omega)$, the solution y of (1) such that $(y(0,\cdot), \partial_t y(0,\cdot)) = (y^0, y^1)$ can be expanded as

$$y(t,x) = \sum_{j=1}^{+\infty} \left(a_j e^{i\lambda_j t} + b_j e^{-i\lambda_j t} \right) \phi_j(x), \tag{10}$$

where the sequences $(a_j)_{j\in\mathbb{N}^*}$ and $(b_j)_{j\in\mathbb{N}^*}$ belong to $\ell^2(\mathbb{C})$ and are determined from the initial data (y^0, y^1) by

$$\begin{aligned} a_j &= \frac{1}{2} \left(\int_\Omega y^0(x)\phi_j(x)\, dv_g - \frac{i}{\lambda_j} \int_\Omega y^1(x)\phi_j(x)\, dv_g \right), \\ b_j &= \frac{1}{2} \left(\int_\Omega y^0(x)\phi_j(x)\, dv_g + \frac{i}{\lambda_j} \int_\Omega y^1(x)\phi_j(x)\, dv_g \right) \end{aligned} \tag{11}$$

for every $j \in \mathbb{N}^*$. Moreover,

$$\|(y^0, y^1)\|^2_{L^2\times(H^1)'} = 2\sum_{j=1}^{+\infty}(|a_j|^2 + |b_j|^2).$$

It follows in particular from (2) that

$$C_T(\omega) = \inf_{\sum_{j=1}^{+\infty}(|a_j|^2+|b_j|^2)=1} \frac{1}{2}\int_0^T \int_\omega \left| \sum_{j=1}^{+\infty} \left(a_j e^{i\lambda_j t} + b_j e^{-i\lambda_j t} \right) \phi_j(x) \right|^2 dv_g\, dt.$$

Let $N \in \mathbb{N}^*$. In accordance with this expression, let us set

$$C_T^{\leq N}(\omega) = \inf_{\sum_{j=1}^{N}(|a_j|^2+|b_j|^2)=1} \frac{1}{2}\int_0^T \int_\omega \left| \sum_{j=1}^{N} \left(a_j e^{i\lambda_j t} + b_j e^{-i\lambda_j t} \right) \phi_j(x) \right|^2 dv_g\, dt.$$

Note also that the quantity $C_T^{>N}(\omega)$ defined by (4) is as well given by

$$C_T^{>N}(\omega) = \inf_{\sum_{j=N+1}^{+\infty}(|a_j|^2+|b_j|^2)=1} \frac{1}{2}\int_0^T \int_\omega \left| \sum_{j=N+1}^{+\infty} \left(a_j e^{i\lambda_j t} + b_j e^{-i\lambda_j t} \right) \phi_j(x) \right|^2 dv_g\, dt.$$

An obvious but crucial observation is that

$$C_T(\omega) \leqslant \min\left(C_T^{\leqslant N}(\omega), C_T^{>N}(\omega)\right), \tag{12}$$

for every $N \in \mathbb{N}^*$.

Remark 6. According to the these considerations and to (10), the quantity $g_1(\omega)$ can be rewritten in terms of the eigenfunctions ϕ_j as

$$g_1(\omega) = \inf_{\sum_{j=1}^{+\infty} |a_j|^2+|b_j|^2=1} \sum_{\lambda\in U_\infty} \sum_{(j,k)\in I_\infty(\lambda)^2} \left(a_j\overline{a_k} + b_j\overline{b_k}\right) \int_\omega \phi_j\phi_k \, dv_g, \tag{13}$$

where U_∞ is the set of all distinct eigenvalues λ_k and $I_\infty(\lambda) = \{j \in \mathbb{N}^* \mid \lambda_j = \lambda\}$.

Let us roughly explain the main lines of the proof of Theorem 2. An important ingredient of the proof is the knowledge of the asymptotic behavior of $C_T^{\leqslant N}(\omega)$ as N and T tend to $+\infty$. More precisely, we will use the following result whose proof is postponed to Section 2.4 for the sake of clarity.

Proposition 1. *Let $N \in \mathbb{N}^*$. For every measurable subset ω of Ω, there holds*

$$\lim_{T\to+\infty} \frac{C_T^{\leqslant N}(\omega)}{T} = \frac{1}{2} \inf_{\substack{\phi\in\mathcal{E}_N\\ \phi\neq 0}} \frac{\int_\omega |\phi(x)|^2 \, dv_g}{\int_\Omega |\phi(x)|^2 \, dv_g}, \tag{14}$$

where $\mathcal{E}_N$ denotes the space of all eigenfunctions of $\triangle_g$ associated to eigenvalues λ such that $\lambda \leqslant \lambda_N$.

Using this result, we show that any minimizing sequence of initial conditions for the observability constant $C_T(\omega)$ defined by (2) converges either to some element of the energy space of initial conditions, namely $L_0^2(\Omega) \times (H^1(\Omega))'$ or must concentrate on high frequencies and, thanks to Theorem 3, must be supported in the neighborhood of a geodesic minimizing the geometric quantity $g_2(\omega)$. In both cases, we compute the asymptotics of the observability constant as $T \to +\infty$.

2.2 Proof of Theorem 2

Let us first show that the quantities $\alpha^\infty(\omega)$ and $\lim_{T\to+\infty} C_T(\chi_\infty)/T$ are well defined, in other words that the limit exists. For that purpose, we use the following result.

Lemma 1. *Let $t, \varepsilon > 0$. There exists $T_0 > 0$ such that for all $T > T_0$, $N \in \mathbb{N}$,*

$$\frac{C_t^{>N}(\omega)}{t} \leqslant \frac{C_T^{>N}(\omega)}{T} + \varepsilon.$$

The proof of this Lemma is postponed to Section 2.3. Let

$$C^- = \liminf_{T\to+\infty} \frac{C_T(\omega)}{T} \quad \text{and} \quad C^+ = \limsup_{T\to+\infty} \frac{C_T(\omega)}{T}.$$

Define $\varepsilon = 4(C^+ - C^-)$ and assume that $\varepsilon > 0$. Let $t > 0$ such that $\frac{C_t(\omega)}{t} \geqslant C^+ - \varepsilon$. Applying Lemma 1 with $N = 0$ yields the existence of $T_0 > 0$ such that

$$\frac{C_t(\omega)}{t} \leqslant \frac{C_T(\omega)}{T} + \varepsilon$$

for all $t > T_0$. Fix now $T > T_0$ satisfying $\frac{C_T(\omega)}{T} \leqslant C^- + \varepsilon$. Combining these inequalities, we obtain

$$C^+ - \varepsilon \leqslant \frac{C_t(\omega)}{t} \leqslant \frac{C_T(\omega)}{T} + \varepsilon \leqslant C^- + 2\varepsilon.$$

The definition of ε leads to a contradiction which proves that $\varepsilon = 0$ and hence $C^+ = C^-$. Now, letting N tend to $+\infty$ in Lemma 1, we obtain that for all $t, \varepsilon > 0$, there exists $T_0 > 0$ such that

$$\alpha^T(\omega) \leqslant \alpha^T(\omega) + \varepsilon$$

for all $t > T_0$. The same argument as for $\lim C_T/T$ then shows that $\alpha^\infty(\omega)$ exists.

We now prove (5). To avoid technicalities, we will denote similarly a sequence and any of its subsequences throughout this proof.

Let us introduce $(T_k)_{k\in\mathbb{N}}$, a sequence of positive numbers tending to $+\infty$ and $(Y_k)_{k\in\mathbb{N}}$ be a minimizing sequence of the functional $J^\omega_{T_k}/T_k$ defined by (3) over $L^2_0(\Omega) \times (H^1)'(\Omega) \setminus \{(0,0)\}$. Writing $Y_k = (y_k^0, y_k^1)$ for all $k \in \mathbb{N}$, we assume without loss of generality that $\|Y_k\|^2_{L^2(\Omega)\times(H^1)'} = 1$ by using an homogeneity argument. Up to a subsequence, one can write

$$\lim_{k\to+\infty} \frac{C_{T_k}(\omega)}{T_k} = \lim_{k\to+\infty} \frac{1}{T_k}\int_0^T |y_k(t,x)|^2 \, dv_g dt$$

where y_k is the solution of (1) with initial data $(y_k(0,\cdot), \partial_t y_k(0,\cdot)) = Y_k$.

With respect to the considerations of Section 2.1, the function y_k can be expanded as

$$y_k(t,x) = \sum_{j=1}^{+\infty} \left(a_j^k e^{i\lambda_j t} + b_j^k e^{-i\lambda_j t} \right) \phi_j(x),$$

where the coefficients a_j^k and b_j^k are determined from the initial data (y_k^0, y_k^1) by (11) and satisfy

$$\sum_{j=1}^{+\infty} |a_j^k|^2 + |b_j^k|^2 = \|Y_k\|^2_{L^2(\Omega)\times(H^1)'} = 1.$$

Since the sequence $(Y_k)_{k\in\mathbb{N}}$ is bounded in the Hilbert space $L^2_0(\Omega) \times (H^1)'(\Omega)$, it converges up to a subsequence weakly in $L^2_0(\Omega) \times (H^1)'(\Omega)$ to some $Y_\infty \in L^2_0(\Omega) \times (H^1)'(\Omega)$ satisfying moreover

$$\|Y_\infty\|^2_{L^2(\Omega)\times(H^1)'} \leqslant \liminf_{k\to+\infty} \|Y_k\|^2_{L^2(\Omega)\times(H^1)'} = 1,$$

by semicontinuity of the $L^2(\Omega) \times (H^1)'$-norm in Ω for the weak topology. Let $(a^\infty, b^\infty) \in (\ell^2(\mathbb{C}))^2$ be the sequences of coefficients determined from the initial data Y_∞ by (11).

Let us define y_∞ as the solution of the wave equation (1) with initial conditions $(y_\infty(0,\cdot), \partial_t y_\infty(0,\cdot)) = Y_\infty$, as well as the function $\widetilde{y}_k$ defined by

$$\widetilde{y}_k = y_k - y_\infty.$$

Note that, by using the linearity of the wave equation (1), the function $\widetilde{y}_k$ is a solution of (1) with initial data $(\widetilde{y}_k(0,\cdot), \partial_t \widetilde{y}_k(0,\cdot)) = \widetilde{Y}_k$, where

$$\widetilde{Y}_k = (\widetilde{y}_k^0, \widetilde{y}_k^1) = Y_k - Y_\infty = (y_k^0 - y_\infty^0, y_k^1 - y_\infty^1)$$

for every $k \in \mathbb{N}$.

We now state two lemmas that are essential ingredients for the rest of the proof. They will be proved in Section 2.3.

Lemma 2. *We have*

$$1 = \|Y_\infty + \widetilde{Y}_k\|^2_{L^2(\Omega)\times(H^1)'} = \|Y_\infty\|^2_{L^2(\Omega)\times(H^1)'} + \|\widetilde{Y}_k\|^2_{L^2(\Omega)\times(H^1)'} + \mathrm{o}(1) \qquad \text{as } k \to +\infty. \tag{15}$$

and

$$\begin{aligned}\frac{1}{T_k}\int_0^{T_k}\!\!\int_\omega |y_k(t,x)|^2\, dv_g\, dt \;&=\; \frac{1}{T_k}\int_0^{T_k}\!\!\int_\omega |y_\infty(t,x)|^2\, dv_g\, dt \\ &\quad + \frac{1}{T_k}\int_0^{T_k}\!\!\int_\omega |\widetilde{y}_k(t,x)|^2\, dv_g\, dt + \mathrm{o}(1) \qquad \text{as } k \to +\infty.\end{aligned} \tag{16}$$

Lemma 3. *Let ω be a Lebesgue measurable subset of Ω. There holds*

$$\lim_{k\to+\infty} \frac{J^\omega_{T_k}(Y_\infty)}{T_k} = \alpha^\infty(\omega).$$

Assuming that $\liminf_{k\to+\infty} \|\widetilde{Y}_k\|_{L^2(\Omega)\times(H^1)'} > 0$, *there holds*

$$\liminf_{k\to+\infty} \frac{J^\omega_{T_k}(\widetilde{Y}_k)}{T_k} \geqslant \alpha^\infty(\omega).$$

According to Lemma 2, one has

$$\begin{aligned}\frac{C_{T_k}(\omega)}{T_k} &= \frac{1}{T_k}\int_0^{T_k}\!\!\int_\omega |y_k(t,x)|^2\, dv_g\, dt \\ &= \frac{1}{T_k}\frac{\int_0^{T_k}\!\int_\omega |y_\infty(t,x)|^2\, dv_g\, dt + \int_0^{T_k}\!\int_\omega |\widetilde{y}_k(t,x)|^2\, dv_g\, dt + \mathrm{o}(1)}{\|Y_\infty\|^2_{L^2(\Omega)\times(H^1)'} + \|\widetilde{Y}_k\|^2_{L^2(\Omega)\times(H^1)'} + \mathrm{o}(1)}.\end{aligned} \tag{17}$$

As a consequence of (17), we have the following alternative: if $\liminf_{k\to+\infty} \|\widetilde{Y}_k\|_{L^2(\Omega)\times(H^1)'} = 0$, then we have

$$\lim_{k\to+\infty} \frac{C_{T_k}(\omega)}{T_k} = \lim_{k\to+\infty} \frac{J^\omega_{T_k}(Y_\infty)}{T_k} \geqslant \frac{1}{2} g_1(\omega)$$

according to Lemma 3, leading to the estimate (5).

At the opposite, if $\liminf_{k\to+\infty}\|\widetilde{Y}_k\|_{L^2(\Omega)\times(H^1)'}>0$, then there holds[3]

$$\lim_{k\to+\infty}\frac{C_{T_k}(\omega)}{T_k}\geqslant\min\left\{\lim_{k\to+\infty}\frac{J^{\omega}_{T_k}(Y_\infty)}{T_k},\liminf_{k\to+\infty}\frac{J^{\omega}_{T_k}(\widetilde{Y}_k)}{T_k}\right\}.$$

Using now the two estimates of Lemma 3 yields

$$\lim_{k\to+\infty}\frac{C_{T_k}(\omega)}{T_k}\geqslant\min\left(\frac{1}{2}g_1(\omega),\alpha^\infty(\omega)\right).$$

It remains to prove the opposite inequality. Combining (12) with the estimate (14) in Proposition 1 and making then N tend to $+\infty$ yields

$$\lim_{T\to+\infty}\frac{C_T(\omega)}{T}\leqslant\lim_{N\to+\infty}\lim_{T\to+\infty}\frac{C_T^{\leqslant N}(\omega)}{T}=\frac{1}{2}g_1(\omega).$$

Now, from (12) and letting N and then T tend to $+\infty$, one has

$$\lim_{T\to+\infty}\frac{C_T(\omega)}{T}=\lim_{T\to+\infty}\frac{C_T(\overline{\omega})}{T}\leqslant\lim_{T\to+\infty}\lim_{N\to+\infty}\frac{C_T^{>N}(\overline{\omega})}{T}=\alpha^\infty(\omega).$$

As a consequence of both previous inequalities, we infer that

$$\lim_{T\to+\infty}\frac{C_T(\omega)}{T}\leqslant\min\left(\frac{1}{2}g_1(\omega),\alpha^\infty(\omega))\right).$$

To sum-up, we have proved that (5) holds for every Lebesgue measurable subset ω of Ω.

2.3 Proof of Lemmas 1, 2 and 3

Proof of Lemma 1. Let $t>0$, $\varepsilon>0$. Choose $T_0>0$ such that $\frac{t}{T_0}\leqslant\varepsilon$. Let $T>T_0$ and $N\in\mathbb{N}$. Define $m=\lfloor T/t\rfloor$, where $\lfloor\cdot\rfloor$ denotes the integer part. Introduce also the functional

$$J^{\omega}_{T_1,T_2}(Y)=\frac{\int_{T_1}^{T_2}\int_\omega|y(t,x)|^2\,dv_g\,dt}{\|Y\|^2_{L^2\times(H^1)'}}$$

where $0<T_1<T_2<+\infty$, $Y\in L^2_0(\Omega)\times(H^1(\Omega))'$ and y is the solution of the wave equation (1) associated to Y. We write for any $Y\in L^2_0(\Omega)\times(H^1(\Omega))'$ involving only frequencies of index higher than N,

$$\frac{J^\omega_T(Y)}{T}\geqslant\frac{J^\omega_{mt}(Y)}{T}\geqslant\frac{J^\omega_{mt}(Y)}{mt}-J^\omega_{mt}(Y)\left|\frac{1}{mT}-\frac{1}{T}\right|.$$

[3]One has

$$\frac{a+b}{A+B}\geqslant\min\left(\frac{a}{A},\frac{b}{B}\right)$$

for every positive real numbers a, b, A, B. Indeed, one has $a+b=\frac{a}{A}A+\frac{b}{B}B\geqslant(A+B)\min\left(\frac{a}{A},\frac{b}{B}\right)$.

Since $J^\omega_{mt}(Y) \leqslant mt$ and $T - mt \leqslant t$, one gets

$$J^\omega_{mt}(Y)\left|\frac{1}{mT} - \frac{1}{T}\right| \leqslant \varepsilon.$$

Therefore, we obtain

$$\begin{aligned}\frac{J^\omega_T(Y)}{T} &\geqslant \frac{J^\omega_{mt}(Y)}{mt} - \varepsilon \geqslant \frac{\sum_{\alpha=0}^{m-1} J^\omega_{\alpha t,(\alpha+1)t}(Y)}{mt} - \varepsilon \\ &\geqslant \min_{\alpha\in\{0,m-1\}} \frac{J^\omega_{\alpha t,(\alpha+1)t}(Y)}{t} - \varepsilon \geqslant \frac{J^\omega_{\alpha_0 t,(\alpha_0+1)t}(Y)}{t} - \varepsilon,\end{aligned}$$

for some $\alpha_0 \in \{0, m-1\}$. Let us notice that one can write

$$\frac{J^\omega_{\alpha_0 t,(\alpha_0+1)t}(Y)}{t} = \frac{J^\omega_t(Y')}{t},$$

where Y' is the initial condition associated to the solution $y' : (x,s) \mapsto y(s+\alpha_0 t, x)$ of (1). In other words, $Y' = \mathbb{S}_{\alpha_0 t} Y$, where $\{\mathbb{S}_t\}_{t\in\mathbb{R}}$ is the unitary wave group associated with (1). As easily checked, Y' also involves only frequencies of index higher that N. As a consequence, we obtain

$$\frac{J^\omega_T(Y)}{T} \geqslant \frac{J^\omega_t(Y')}{t} - \varepsilon \geqslant \frac{C^{>N}_t(\omega)}{t} - \varepsilon.$$

for all $Y \in L^2_0(\Omega) \times (H^1(\Omega))'$ involving only frequencies of index higher than N. Since Y is arbitrary, this proves Lemma 1. $\square$

Proof of Lemma 2. Since $(Y_k)_{k\in\mathbb{N}}$ converges weakly to Y_∞ in $L^2(\Omega)\times(H^1(\Omega))'$, the sequence $(\widetilde{Y}_k)_{k\in\mathbb{N}}$ converges weakly in $L^2(\Omega)\times(H^1(\Omega))'$ to 0 as $k\to+\infty$. Therefore, one has $\langle Y_\infty, \widetilde{Y}_k\rangle_{L^2(\Omega)\times(H^1)'} = \mathrm{o}(1)$ as $k \to +\infty$, which yields directly (15) by expanding $\|Y_\infty + \widetilde{Y}_k\|^2_{L^2(\Omega)\times(H^1)'}$.

Let us now prove (16). One computes

$$\begin{aligned}\frac{1}{T_k}\int_0^{T_k}\int_\omega |y_k(t,x)|^2\,dv_g\,dt &= \frac{1}{T_k}\int_0^{T_k}\int_\omega |y_\infty(t,x)|^2\,dv_g\,dt + \frac{1}{T_k}\int_0^{T_k}\int_\omega |\widetilde{y}_k(t,x)|^2\,dv_g\,dt \\ &\quad + \frac{2}{T_k}\mathrm{Re}\left(\int_0^{T_k}\int_\omega y_\infty(t,x)\overline{\widetilde{y}_k(t,x)}\,dv_g\,dt\right).\end{aligned}$$

To prove (16), it suffices to prove that $I_k = \mathrm{o}(1)$ as $k \to +\infty$, where

$$I_k = \frac{1}{T_k}\int_0^{T_k}\int_\omega y_\infty(t,x)\overline{\widetilde{y}_k(t,x)}\,dv_g\,dt.$$

Fix $k \in \mathbb{N}^*$ and define

$$\begin{aligned}\widetilde{c}^k_j &: \mathbb{R}_+ \ni t \mapsto \widetilde{a}^k_j e^{i\lambda_j t} + \widetilde{b}^k_j j e^{-i\lambda_j t} \\ c^\infty_j &: \mathbb{R}_+ \ni t \mapsto a^\infty_j e^{i\lambda_j t} + b^\infty_j j e^{-i\lambda_j t}\end{aligned}$$

for every $j \in \mathbb{N}$, so that I_k rewrites

$$I_k = \frac{1}{T_k}\int_0^{T_k}\int_\omega \sum_{j,\ell=1}^{+\infty} c_j^\infty(t)\overline{\widetilde{c}_\ell^k(t)}\phi_j(x)\phi_\ell(x)\,dv_g\,dt.$$

Let J and L denote two nonzero integer that will be chosen in an adequate way in the sequel. We decompose I_k as $I_k = I_k^1 + I_k^2 + I_k^3$, where

$$\begin{aligned} I_k^1 &= \frac{1}{T_k}\int_0^{T_k}\int_\omega \sum_{j=J+1}^{+\infty}\sum_{\ell=1}^{+\infty} c_j^\infty(t)\overline{\widetilde{c}_\ell^k(t)}\phi_j(x)\phi_\ell(x)\,dv_g\,dt \\ I_k^2 &= \frac{1}{T_k}\int_0^{T_k}\int_\omega \sum_{j=1}^{J}\sum_{\ell=1}^{L} c_j^\infty(t)\overline{\widetilde{c}_\ell^k(t)}\phi_j(x)\phi_\ell(x)\,dv_g\,dt \\ I_k^3 &= \frac{1}{T_k}\int_0^{T_k}\int_\omega \sum_{j=1}^{J}\sum_{\ell=L+1}^{+\infty} c_j^\infty(t)\overline{\widetilde{c}_\ell^k(t)}\phi_j(x)\phi_\ell(x)\,dv_g\,dt. \end{aligned}$$

Let us now estimate the terms I_k^1 and I_k^3. Using the Cauchy-Schwarz inequality and the fact that $\int_\omega |f|\,dv_g \leqslant \int_\Omega |f|\,dv_g$ for every $f \in L^1(\Omega)$, there holds

$$\begin{aligned} |I_k^1| &\leqslant \frac{1}{T_k}\int_0^{T_k}\int_\omega \left| \sum_{j=J+1}^{+\infty}\sum_{\ell=1}^{+\infty} c_j^\infty(t)\overline{\widetilde{c}_\ell^k(t)}\phi_j(x)\phi_\ell(x)\right| dv_g\,dt \\ &\leqslant \sup_{t\in[0,T_k]} \left(\int_\omega \left| \sum_{j=J+1}^{+\infty} c_j^\infty(t)\phi_j(x)\,dv_g \right|^2 \int_\omega \left| \sum_{\ell=1}^{+\infty} \overline{\widetilde{c}_\ell^k(t)}\phi_\ell(x)\,dv_g \right|^2 \right)^{1/2} \\ &\leqslant \sup_{t\in[0,T_k]} \left(\int_\Omega \left| \sum_{j=J+1}^{+\infty} c_j^\infty(t)\phi_j(x)\,dv_g \right|^2 \int_\Omega \left| \sum_{\ell=1}^{+\infty} \overline{\widetilde{c}_\ell^k(t)}\phi_\ell(x)\,dv_g \right|^2 \right)^{1/2} \\ &\leqslant \sup_{t\in[0,T_k]} \left(\sum_{j=J+1}^{+\infty} |c_j^\infty(t)|^2 \sum_{\ell=1}^{+\infty} |\widetilde{c}_\ell^k(t)|^2 \right)^{1/2} \leqslant 2\left(\sum_{j=J+1}^{+\infty} (|a_j^\infty|^2 + |b_j^\infty|^2) \right)^{1/2}, \quad (18) \end{aligned}$$

by using that $\|\widetilde{Y}_k\|_{L^2\times(H^1)'} \leqslant 1$.

Now, we claim that, since $(\lambda_j)_{j\in\mathbb{N}}$ is non-decreasing and tends to $+\infty$ as $j \to +\infty$, it is possible to choose J and L such that $J < L$ and for every $j \leqslant J$ and $\ell \geqslant L+1$,

$$|\lambda_j - \lambda_\ell| \geqslant 1.$$

For such a choice of integers J and L, one has

$$\begin{aligned} |I_k^3| &= \frac{1}{T_k}\left| \int_0^{T_k}\int_\omega \sum_{j=1}^{J}\sum_{\ell=L+1}^{+\infty} c_j^\infty(t)\overline{\widetilde{c}_\ell^k(t)}\phi_j(x)\phi_\ell(x)\,dv_g\,dt \right| \\ &= \frac{1}{T_k}\left| \sum_{j=1}^{J}\sum_{\ell=L+1}^{+\infty} C_{j\ell}\int_\omega \phi_j(x)\phi_\ell(x)\,dv_g \right| \end{aligned}$$

with

$$
\begin{aligned}
C_{j\ell} &= \int_0^{T_k} c_j^\infty(t)\overline{\widetilde{c}_\ell^k(t)}\,dt \\
&= \frac{2a_j^\infty\overline{\widetilde{a}_\ell^k}}{\lambda_j-\lambda_\ell}\sin\left((\lambda_j-\lambda_\ell)\frac{T_k}{2}\right)e^{i(\lambda_j-\lambda_\ell)\frac{T_k}{2}} - \frac{2a_j^\infty\overline{\widetilde{b}_\ell^k}}{\lambda_j+\lambda_\ell}\sin\left((\lambda_j+\lambda_\ell)\frac{T_k}{2}\right)e^{i(\lambda_j+\lambda_\ell)\frac{T_k}{2}} \\
&\quad -\frac{2b_j^\infty\overline{\widetilde{a}_\ell^k}}{\lambda_j+\lambda_\ell}\sin\left((\lambda_j+\lambda_\ell)\frac{T_k}{2}\right)e^{-i(\lambda_j+\lambda_\ell)\frac{T_k}{2}} + \frac{2b_j^\infty\overline{\widetilde{b}_k^\ell}}{\lambda_j-\lambda_\ell}\sin\left((\lambda_j-\lambda_\ell)\frac{T_k}{2}\right)e^{-i(\lambda_j-\lambda_\ell)\frac{T_k}{2}}
\end{aligned}
$$

It follows that

$$
I_k^3 \leqslant S_{JL}^-(a^\infty,\widetilde{a}^k)+S_{JL}^+(a^\infty,\widetilde{b}^k)+S_{JL}^+(b^\infty,\widetilde{a}^k)+S_{JL}^-(b^\infty,\widetilde{b}^k)
$$

where for $(u,v)\in(\ell^2(\mathbb{C}))^2$, one has

$$
\begin{aligned}
S_{JL}^-(u,v) &= \frac{2}{T_k}\left|\sum_{j=1}^{J}\sum_{\ell=L+1}^{+\infty}\frac{2u_j\overline{v_\ell}}{\lambda_j-\lambda_\ell}\sin\left((\lambda_j-\lambda_\ell)\frac{T_k}{2}\right)e^{i(\lambda_j-\lambda_\ell)\frac{T_k}{2}}\int_\omega \phi_j(x)\phi_\ell(x)\,dv_g\right|, \\
S_{JL}^+(u,v) &= \frac{2}{T_k}\left|\sum_{j=1}^{J}\sum_{\ell=L+1}^{+\infty}\frac{2u_j\overline{v_\ell}}{\lambda_j+\lambda_\ell}\sin\left((\lambda_j+\lambda_\ell)\frac{T_k}{2}\right)e^{i(\lambda_j+\lambda_\ell)\frac{T_k}{2}}\int_\omega \phi_j(x)\phi_\ell(x)\,dv_g\right|.
\end{aligned}
$$

Now, using the Cauchy-Schwarz inequality and the fact that the integral of a nonnegative function over ω is lower than the integral of the same function over Ω, one gets

$$
\begin{aligned}
S_{JL}^-(a^\infty,\widetilde{a}_\ell^k) &\leqslant \sum_{j=1}^{J}|a_j^\infty|\left(\int_\Omega\left|\sum_{\ell=L+1}^{+\infty}\frac{\overline{\widetilde{a}_\ell^k}}{\lambda_j-\lambda_k}e^{i(\lambda_j-\lambda_\ell)\frac{T_k}{2}}\frac{\sin\left((\lambda_j-\lambda_\ell)\frac{T_k}{2}\right)}{T_k/2}\phi_\ell(x)\right|^2 dv_g\right)^{1/2} \\
&= \sum_{j=1}^{J}|a_j^\infty|\left(\sum_{\ell=L+1}^{+\infty}\frac{|\widetilde{a}_\ell^k|^2}{(\lambda_j-\lambda_\ell)^2}\frac{\sin^2\left((\lambda_j-\lambda_\ell)\frac{T_k}{2}\right)}{(T_k/2)^2}\right)^{1/2}. \\
&\leqslant \frac{2\sum_{j=1}^{J}|a_j^\infty|}{T_k}\leqslant\frac{2J}{T_k},
\end{aligned}
$$

by using that $\|\widetilde{a}^k\|_{\ell^2}\leqslant 1$ and $\|a^\infty\|_{\ell^2}\leqslant 1$

Now, using similar reasonings as for estimating the other terms, one gets in the end

$$
|I_k^3|\leqslant\frac{8J}{T_k}. \tag{19}
$$

To conclude, let us fix $\varepsilon>0$. One first choose J large enough (and independent of k) so that $|I_k^1|\leqslant\varepsilon/2$ which is possible according to (18). According to the previous discussion, we fix $L>J$ so that $\inf\{|\lambda_j-\lambda_\ell|,\ j\leqslant J,\ \ell\geqslant L+1\}\geqslant 1$, which guarantees that the estimate (19) holds true. Using now that the sequence $(\widetilde{Y}_k)_{k\in\mathbb{N}}$ converges weakly to 0 in $L^2(\Omega)\times(H^1(\Omega))'$, we claim that there exists K_{JL} such that

$$k \geqslant K_{JL} \Rightarrow |I_k^2| + |I_k^3| \leqslant \frac{\varepsilon}{2},$$

according to the expression of I_k^2 and the estimate (19). The expected conclusion follows by combining the two latter inequalities. □

Proof of Lemma 3. Let $N\in\mathbb{N}^*$ and let $Y_\infty^{\leqslant N}$ be the projection of Y_∞ onto the first N eigenspaces, namely

$$Y_\infty^{\leqslant N} = \left(\sum_{j=1}^N \langle y_\infty^0, \phi_j\rangle_{L^2}\phi_j, \sum_{j=1}^N \langle y_\infty^1, \phi_j\rangle_{(H^1)',H^1}\phi_j\right)$$

and introduce $\Delta_{T_k}^N J = \left|\frac{J_{T_k}(Y_\infty^{\leqslant N})}{T_k} - \frac{J_{T_k}(Y_\infty)}{T_k}\right|$. Using the notations (18), one has

$$\begin{aligned}
\Delta_{T_k}^N J &= \frac{1}{T_k}\left|\int_0^{T_k}\int_\omega \left(-\left|\sum_{j=1}^N c_j^\infty(t)\phi_j(x)\right|^2 + \left|\sum_{j=1}^{+\infty} c_j^\infty(t)\phi_j(x)\right|^2\right) dv_g\, dt\right| \\
&= \frac{1}{T_k}\left|\int_0^{T_k}\int_\omega \left(\left|\sum_{j=N+1}^{+\infty} c_j^\infty(t)\phi_j(x)\right|^2\right.\right. \\
&\qquad \left.\left. +2\mathrm{Re}\left(\sum_{j=1}^N\sum_{\ell=N+1}^{+\infty} c_j^\infty(t)\overline{c_\ell^\infty(t)}\phi_j(x)\phi_\ell(x)\right)\right) dv_g\, dt\right|.
\end{aligned}$$

Using the Cauchy-Schwarz inequality and the fact that $\int_\omega |f|\, dv_g \leqslant \int_\Omega |f|\, dv_g$ for every $f\in L^1(\Omega)$, there holds

$$\begin{aligned}
\frac{1}{T_k}\int_0^{T_k}\int_\omega \left(\left|\sum_{j=N+1}^{+\infty} c_j^\infty(t)\phi_j(x)\right|^2\right) dv_g\, dt &\leqslant \frac{1}{T_k}\int_0^{T_k}\int_\Omega \left(\left|\sum_{j=N+1}^{+\infty} c_j^\infty(t)\phi_j(x)\right|^2\right) dv_g\, dt \\
&\leqslant \frac{1}{T_k}\int_0^{T_k} \sum_{j=N+1}^{+\infty} |c_j^\infty(t)|^2 \leqslant 2\sum_{j=N+1}^{+\infty}(|a_j^\infty|^2 + |b_j^\infty|^2)
\end{aligned}$$

and similarly, introducing

$$R_{T_k}^N = \frac{1}{T_k}\int_0^{T_k}\int_\omega \left|\sum_{j=1}^N\sum_{\ell=N+1}^{+\infty} c_j^\infty(t)\overline{c_\ell^\infty(t)}\phi_j(x)\phi_\ell(x)\, dv_g\, dt\right|,$$

one has

$$\begin{aligned}
R_{T_k}^N &\leqslant \frac{1}{T_k}\int_0^{T_k}\int_\Omega \left| \sum_{j=1}^N \sum_{\ell=N+1}^{+\infty} c_j^\infty(t)\overline{c_\ell^\infty(t)}\phi_j(x)\phi_\ell(x)\,dv_g\,dt\right| \\
&\leqslant \frac{1}{T_k}\int_0^{T_k}\left(\int_\Omega \sum_{j=1}^N |c_j^\infty(t)|^2\phi_j(x)^2\,dv_g \int_\Omega \sum_{\ell=N+1}^{+\infty} |c_\ell^\infty(t)|^2\phi_\ell(x)^2\,dv_g\right)^{1/2} dt \\
&\leqslant \frac{1}{T_k}\int_0^{T_k}\left(\sum_{j=1}^N |c_j^\infty(t)|^2 \sum_{\ell=N+1}^{+\infty} |c_\ell^\infty(t)|^2\right)^{1/2} dt \\
&\leqslant 2\left(\sum_{j=1}^N (|a_j^\infty(t)|^2+|b_j^\infty(t)|^2) \sum_{\ell=N+1}^{+\infty} (|a_\ell^\infty(t)|^2+|b_\ell^\infty(t)|^2)\right)^{1/2} dt \\
&\leqslant 2\left(\sum_{\ell=N+1}^{+\infty} (|a_\ell^\infty(t)|^2+|b_\ell^\infty(t)|^2)\right)^{1/2} dt.
\end{aligned}$$

Combining the two latter estimates, we infer that $(\Delta_{T_k}^N J)_{N\in\mathbb{N}^*}$ converges to 0 as$N\to+\infty$ independently of k.

As a consequence, by noting that $|J_{T_k}(Y_\infty)/T_k| \leqslant |J_{T_k}(Y_\infty^{\leqslant N})/T_k| + \Delta_{T_k}^N J$, to prove the first statement of Lemma 3, it is enough to prove that

$$\lim_{N\to+\infty}\lim_{k\to+\infty}\frac{J_{T_k}^\omega(Y_\infty^N)}{T_k} = \frac{1}{2}g_1(\omega),$$

which is true according to Proposition 1.

Let us now prove the second statement of the lemma. It is based on an argument which is similar to the one used in the proof of Lemma 1. Let $T>0$ and $m_k = \lfloor T_k/T \rfloor$, where $\lfloor\cdot\rfloor$ denotes the integer part. Let us define $t_k = T_k/m_k$ so that t_k converges to T as $k\to+\infty$. As in the proof of Lemma 1, we use the functional

$$J_{T_1,T_2}^\omega(Y) = \frac{\int_{T_1}^{T_2}\int_\omega |y(t,x)|^2\,dv_g\,dt}{\|Y\|_{L^2\times(H^1)'}^2}$$

where $0<T_1<T_2<+\infty$, $Y\in L_0^2(\Omega)\times(H^1(\Omega))'$ and y is the solution of the wave equation (1) associated to Y.

We write

$$\begin{aligned}
\frac{J_{T_k}^\omega(\widetilde{Y}_k)}{T_k} &= \frac{\sum_{\alpha=0}^{m_k-1} J_{\alpha t_k,(\alpha+1)t_k}^\omega(\widetilde{Y}_k)}{m_k t_k} \geqslant \min_{\alpha\in\{0,m_k-1\}} \frac{J_{\alpha t_k,(\alpha+1)t_k}^\omega(\widetilde{Y}_k)}{t_k} \\
&\geqslant \frac{J_{\alpha_k t_k,(\alpha_k+1)t_k}^\omega(\widetilde{Y}_k)}{t_k}, \qquad (20)
\end{aligned}$$

for some $\alpha_k \in \{0, m_k - 1\}$. Let us notice that one can write

$$\frac{J^\omega_{\alpha_k t_k, (\alpha_k+1)t_k}(\widetilde{Y}_k)}{t_k} = \frac{J^\omega_{t_k}(\widetilde{Y}'_k)}{t_k},$$

where $\widetilde{Y}'_k$ is the initial condition associated to the solution $\widetilde{y}'_k : (x,t) \mapsto \widetilde{y}(t + \alpha_k t_k, x)$ of (1). In other words, $\widetilde{Y}'_k = \mathbb{S}_{\alpha_k t_k}\widetilde{Y}_k$. Using that the wave group $\{\mathbb{S}_t\}_{t\in\mathbb{R}}$ is unitary shows that the coefficients $\widetilde{a}^{k'}_j$ and $\widetilde{b}^{k'}_j$ defined from the initial data $\widetilde{Y}'_k$ by the formula (11) satisfy $|\widetilde{a}^{k'}_j| = |\widetilde{a}^k_j|$ and $|\widetilde{b}^{k'}_j| = |\widetilde{b}^k_j|$ for every $j \in \mathbb{N}^*$. Therefore, the sequence $(\widetilde{Y}'_k)_{k\in\mathbb{N}}$ converges weakly to 0 in $L^2(\Omega) \times (H^1(\Omega))'$.

For $N \in \mathbb{N}^*$, let us introduce $\widetilde{Y}'^{>N}_k$ as the projection of $\widetilde{Y}'_k$ onto the subspace of $L^2(\Omega) \times (H^1(\Omega))'$ spanned by $\{(\phi_j)_{j\geqslant N+1}\}$, in other words

$$\widetilde{Y}'^{\leqslant N}_k = \widetilde{Y}_k - \widetilde{Y}'^{>N}_k, \text{ where } \widetilde{Y}'^{\leqslant N} = \left(\sum_{j=1}^N \langle \widetilde{y}'_k(0,\cdot), \phi_j\rangle_{L^2}\phi_j, \sum_{j=1}^N \langle \partial_t \widetilde{y}'_k(0,\cdot), \phi_j\rangle_{(H^1)',H^1}\phi_j\right).$$

Since the integer N is fixed, it follows that the sequence $(\widetilde{Y}'^{\leqslant N}_k)_{k\in\mathbb{N}}$ belongs to a finite dimensional subspace of $L^2(\Omega) \times (H^1(\Omega))'$ and converges therefore to 0 weakly and thus strongly in this space. Noting that the assumption $\liminf_{k\to+\infty} \|\widetilde{Y}_k\|_{L^2(\Omega)\times(H^1)'} > 0$ can be equivalently rewritten $\liminf_{k\to+\infty} \|\widetilde{Y}'_k\|_{L^2(\Omega)\times(H^1)'} > 0$, the previous considerations lead to write

$$J^\omega_{t_k}(\widetilde{Y}_k) = J^\omega_{t_k}(\widetilde{Y}^{>N}_k) + \mathrm{o}(1) \qquad \text{as } k \to +\infty. \tag{21}$$

Since each term $\widetilde{Y}'^{>N}_k$ only involves eigenfunctions of index larger than N, one has

$$\frac{C^{>N}_{t_k}(\omega)}{t_k} \leqslant \frac{J^\omega_{t_k}(Y'^{>N}_k)}{t_k},$$

and according to the estimates (20) and (21), we infer that

$$\liminf_{k\to+\infty} \frac{J^\omega_{T_k}(\widetilde{Y}_k)}{T_k} = \liminf_{k\to+\infty} \frac{J^\omega_{t_k}(\widetilde{Y}'_k)}{t_k} = \liminf_{k\to+\infty} \liminf_{N\to+\infty} \frac{J^\omega_{t_k}(\widetilde{Y}'^{>N}_k)}{t_k} \geqslant \alpha^T(\omega)$$

leading to the desired conclusion by using in particular that such an estimate holds for arbitrary times T. $\square$

2.4 Proof of Proposition 1 (low frequencies)

To avoid technicalities, we first prove this result when the eigenvalues are simple. In other words, we will prove that

$$\lim_{T\to+\infty} \frac{C^{\leqslant N}_T(\omega)}{T} = \frac{1}{2} \min_{1\leqslant j\leqslant N} \int_\omega \phi_j(x)^2 \, dv_g, \tag{22}$$

for every $N \in \mathbb{N}^*$. The required ingredients to generalize this result to (14) are provided at the end of this section.

Consider two sequences a and b in $\ell^2(\mathbb{C})$. One has

$$\begin{aligned}\frac{1}{2T}\int_0^T\int_\omega\left|\sum_{j=1}^N\left(a_je^{i\lambda_jt}+b_je^{-i\lambda_jt}\right)\phi_j(x)\right|^2dx\,dt &= \frac{1}{2T}\sum_{j=1}^N\alpha_{jj}\int_\omega\phi_j(x)^2\,dv_g\\ &+\frac{1}{2T}\sum_{j=1}^N\sum_{\substack{k=1\\k\neq j}}^N\alpha_{jk}\int_\omega\phi_j(x)\phi_k(x)\,dv_g,\end{aligned}$$

where α_{jk} is defined by

$$\begin{aligned}\alpha_{jk}=&\frac{2a_j\overline{a_k}}{\lambda_j-\lambda_k}\sin\left((\lambda_j-\lambda_k)\frac{T}{2}\right)e^{i(\lambda_j-\lambda_k)\frac{T}{2}}-\frac{2a_j\overline{b_k}}{\lambda_j+\lambda_k}\sin\left((\lambda_j+\lambda_k)\frac{T}{2}\right)e^{i(\lambda_j+\lambda_k)\frac{T}{2}}\\ &-\frac{2b_j\overline{a_k}}{\lambda_j+\lambda_k}\sin\left((\lambda_j+\lambda_k)\frac{T}{2}\right)e^{-i(\lambda_j+\lambda_k)\frac{T}{2}}+\frac{2b_j\overline{b_k}}{\lambda_j-\lambda_k}\sin\left((\lambda_j-\lambda_k)\frac{T}{2}\right)e^{-i(\lambda_j-\lambda_k)\frac{T}{2}}\end{aligned}$$

whenever $\lambda_j \neq \lambda_k$, and

$$\alpha_{jk}=T(a_j\overline{a_k}+b_j\overline{b_k})-\frac{\sin(\lambda_jT)}{\lambda_j}(a_j\overline{b_k}e^{i\lambda_jT}+b_j\overline{a_k}e^{-i\lambda_jT})$$

whenever $\lambda_j = \lambda_k$. It follows that

$$\lim_{T\to+\infty}\frac{\alpha_{jj}}{T}=|a_j|^2+|b_j|^2,$$

for every $j \in \mathbb{N}^*$ and, using that the eigenvalues are simple,

$$|\alpha_{jk}|\leqslant\frac{4\max_{1\leqslant j,k\leqslant N}(\lambda_j,\lambda_k)}{|\lambda_j^2-\lambda_k^2|},$$

whenever $j \neq k$. Then by expanding the sum it suffices to note that, when passing to the limit in T, all terms such that $j \neq k$ are equal to 0 and there only remain the diagonal terms.

Hence, we claim that

$$\begin{aligned}\lim_{T\to+\infty}\frac{C_T^{\leqslant N}(\omega)}{T} &= \lim_{T\to+\infty}\inf_{\sum_{j=1}^N(|a_j|^2+|b_j|^2)=1}\frac{1}{2T}\int_0^T\int_\omega\left|\sum_{j=1}^N\left(a_je^{i\lambda_jt}+b_je^{-i\lambda_jt}\right)\phi_j(x)\right|^2dx\,dt\\ &= \inf_{\sum_{j=1}^N(|a_j|^2+|b_j|^2)=1}\lim_{T\to+\infty}\frac{1}{2T}\int_0^T\int_\omega\left|\sum_{j=1}^N\left(a_je^{i\lambda_jt}+b_je^{-i\lambda_jt}\right)\phi_j(x)\right|^2dx\,dt\\ &= \frac{1}{2}\inf_{\sum_{j=1}^N(|a_j|^2+|b_j|^2)=1}\sum_{j=1}^N(|a_j|^2+|b_j|^2)\int_\omega\phi_j(x)^2\,dv_g\\ &= \frac{1}{2}\min_{1\leqslant j\leqslant N}\int_\omega\phi_j(x)^2\,dv_g.\end{aligned}$$

Indeed, since the sum is finite we can invert the minimum and the limit. The equality (22) follows then easily.

Let us now establish (14). For general real eigenvalues sequence (λ_j), we claim that

$$\begin{aligned}
\lim_{T\to+\infty}\frac{C_T^{\leqslant N}(\omega)}{T} &= \lim_{T\to+\infty}\inf_{\sum_{j=1}^N(|a_j|^2+|b_j|^2)=1}\frac{1}{2T}\int_0^T\int_\omega\left|\sum_{j=1}^N\left(a_je^{i\lambda_jt}+b_je^{-i\lambda_jt}\right)\phi_j(x)\right|^2dx\,dt\\
&= \lim_{T\to+\infty}\inf_{\sum_{j=1}^N(|a_j|^2+|b_j|^2)=1}\Big(\frac{1}{T}\sum_{\lambda\in U_N}\sum_{(j,k)\in I_N(\lambda)^2}\alpha_{jk}\int_\omega\phi_j(x)\phi_k(x)\,dv_g\Big)\\
&\quad+\frac{1}{T}\sum_{\substack{(\lambda,\mu)\in U_N^2\\ \lambda\neq\mu}}\sum_{\substack{j\in I_N(\lambda)\\ k\in I_N(\mu)}}\alpha_{jk}\int_\omega\phi_j(x)\phi_k(x)\,dv_g,
\end{aligned}$$

where

$$\lim_{T\to+\infty}\frac{\alpha_{jk}}{T}=\left\{\begin{array}{ll} a_j\overline{a_k}+b_j\overline{b_k} & \text{if } (j,k)\in I_N(\lambda)^2,\\ 0 & \text{if } j\in I_N(\lambda),\ k\in I_N(\mu),\ \text{with } (\lambda,\mu)\in U_N^2 \text{ and } \lambda\neq\mu.\end{array}\right.$$

The conclusion follows, by mimicking the previous reasoning and by inverting the limit and the infimum.

2.5 Proof of Theorem 1

Notice first that every function $y:(t,x)\mapsto e^{i\lambda t}\phi_\lambda(x)$, where λ denotes any nonzero eigenvalue of $-\triangle_g$ and ϕ any associated eigenfunction, is a smooth solution of the wave equation (1). As a consequence and according to (2), there holds

$$\frac{C_T(\omega)}{T}\leqslant\frac{1}{2}\int_\omega\phi_\lambda(x)^2\,dv_g,$$

and we infer that $C_T(\omega)\leqslant\frac{1}{2}g_1(\omega)$ by passing into the infimum over the set of all nonconstant eigenfunctions of $-\triangle_g$.

The estimate

$$\frac{1}{T}C_T(\omega)\leqslant\alpha^T(\omega)$$

follows from the the fact that $C_T(\omega)\leqslant C_T^{>N}(\omega)$ for every $N\in\mathbb{N}^*$.

Let us now prove that the infimum in the definition of C_T is attained as soon as

$$\frac{C_T(\omega)}{T}<\alpha^T(\omega). \tag{23}$$

The proof is a direct consequence of the one of Theorem 2. Nevertheless, for the sake of readability, we provide here a short sketch recalling the main steps.

Let $(Y_k)_{k\in\mathbb{N}}$ be a minimizing sequence of the functional J_T^ω defined by (3) over $L_0^2(\Omega)\times(H^1)'(\Omega)\setminus\{(0,0)\}$ satisfying $\|Y_k\|^2_{L^2(\Omega)\times(H^1)'}=1$. Since the sequence $(Y_k)_{k\in\mathbb{N}}$ is bounded in the Hilbert space $L^2_0(\Omega)\times(H^1)'(\Omega)$, it converges up to a subsequence weakly in $L^2_0(\Omega)\times(H^1)'(\Omega)$ to some $Y_\infty\in L^2_0(\Omega)\times(H^1)'(\Omega)$ satisfying moreover

$$\|Y_\infty\|^2_{L^2(\Omega)\times(H^1)'}\leqslant\liminf_{k\to+\infty}\|Y_k\|^2_{L^2(\Omega)\times(H^1)'}=1,$$

by property of the weak convergence.

We will prove that

$$\frac{C_T(\omega)}{T}=\min\left(J^\omega_T(Y_\infty),\alpha^T(\omega)\right),$$

which implies the expected conclusion when combined with (23).

Let us introduce y_∞ as the solution of the wave equation (1) with initial conditions Y_∞, as well as the function $\widetilde{y}_k$ defined by $\widetilde{y}_k=y_k-y_\infty$. Using the same arguments as in the proof of Lemma 2 yields

$$1=\|Y_\infty+\widetilde{Y}_k\|^2_{L^2(\Omega)\times(H^1)'}=\|Y_\infty\|^2_{L^2(\Omega)\times(H^1)'}+\|\widetilde{Y}_k\|^2_{L^2(\Omega)\times(H^1)'}+\mathrm{o}(1)\qquad\text{as }k\to+\infty,$$

and

$$\begin{aligned}\int_0^T\!\!\int_\omega|y_k(t,x)|^2\,dv_g\,dt\ &=\ \int_0^T\!\!\int_\omega|y_\infty(t,x)|^2\,dv_g\,dt\\&\quad+\int_0^T\!\!\int_\omega|\widetilde{y}_k(t,x)|^2\,dv_g\,dt+\mathrm{o}(1)\qquad\text{as }k\to+\infty.\end{aligned}$$

At this step, there are two possibilities: either $\liminf_{k\to+\infty}\|\widetilde{Y}_k\|_{L^2(\Omega)\times(H^1)'}=0$ and there holds

$$C_T(\omega)=\frac{\int_0^T\!\int_\omega|y_\infty(t,x)|^2\,dv_g\,dt}{\|Y_\infty\|^2_{L^2(\Omega)\times(H^1)'}}+\mathrm{o}(1)\qquad\text{as }k\to+\infty,$$

saying that the infimum is a minimum, or $\liminf_{k\to+\infty}\|\widetilde{Y}_k\|_{L^2(\Omega)\times(H^1)'}>0$. In this case, following exactly the same approach as in the second part of the proof of Lemma 3 (in fact, the proof is even simpler since we do not have to deal with the dependence of the observability time T with respect to the index k anymore) leads to

$$\frac{C_T(\omega)}{T}\geqslant\min\left(J^\omega_T(Y_\infty),\alpha^T(\chi_\omega)\right).$$

Moreover

$$\frac{C_T(\omega)}{T}\leqslant\lim_{N\to+\infty}C_T^{>N}(\omega)=\alpha^T(\omega).$$

By minimality of $C_T(\omega)$, there holds $C_T(\omega)\leqslant J^\omega_T(Y_\infty)$.

Combining all these facts shows that

$$\min\left(J^\omega_T(Y_\infty),\alpha^T(\omega)\right)\leqslant\frac{C_T(\omega)}{T}\leqslant\min\left(J^\omega_T(Y_\infty),\alpha^T(\omega)\right).$$

This proves Theorem 1.

2.6 Proof of Theorem 3 (high frequencies)

For $T>0$ and γ a geodesic of Ω, let us recall (see Definition 4) that $m_T^\omega(\gamma)=\frac{1}{T}\int_0^T\chi_\omega(\gamma(t))dt$ and that $g_2^T(\omega)=\inf_{\gamma\in\Gamma}m_T^\omega(\gamma)$, where the infimum is taken over the set Γ of all geodesics of Ω. Therefore, with the notations of Definition 4, there holds

$$g_2(\omega)=\lim_{T\to+\infty}g_2^T(\omega).$$

This section is devoted to providing an asymptotic estimate of the constant $C_T^{>N}(\omega)$, involving only highfrequency terms: the precise statement is given in Theorem 3. The proof is divided into two steps. In the first step, we establish a first inequality by using Gaussian beams. The converse inequality will then be proved by using the Egorov theorem.

For any $h\in C^0(\overline{\Omega})$, we set

$$C_T^{>N}(h)=\inf_{\sum_{j=1}^N(|a_j|^2+|b_j|^2)=1}\frac{1}{2}\int_0^T\int_\Omega\left|\sum_{j=1}^N\left(a_je^{i\lambda_jt}+b_je^{-i\lambda_jt}\right)\phi_j(x)h(x)\right|^2dv_g\,dt.$$

For any geodesic γ in Ω, we define

$$m_T^h(\gamma)=\frac{1}{T}\int_0^T h(\gamma(t))dt$$

and

$$g_2^T(h)=\inf_{\gamma\in\Gamma}m_T^h(\gamma)$$

where the infimum is taken over the set Γ of all geodesics of Ω.

First step: $\displaystyle\limsup_{N\to+\infty}\frac{C_T^{>N}(\omega)}{T}\leqslant\frac{1}{2}g_2^T(\overline{\omega})$ for every $N\in\mathbb{N}^*$ and every measurable subset ω.

Let $t\mapsto\gamma(t)$ be a geodesic[4] for the Riemannian metric g. For every solution w of (1), we denote by E_w the energy of w defined by

$$E_w:[0,T]\ni t\mapsto\|\partial_tw(t,\cdot)\|^2_{L^2(\Omega)}+\|\nabla w(t,\cdot)\|^2_{L^2(\Omega)}$$

and there holds $E_w(t)=E_w(0)$ for every $t\in[0,T]$.

Following [15, Corollary 11], there exists a function w_j solution of (1) in $\mathcal{C}^0(0,T;H^1(\Omega))\cap\mathcal{C}^1(0,T;L^2_0(\Omega))$ satisfying $E_{w_j}(t)=1$ for every $t\in[0,T]$ and

$$\sup_{t\in(0,T)}\|\chi_{\Omega\setminus B_j(t)}\partial_tw_j(t,\cdot)\|^2_{L^2(\Omega)}+\|\chi_{\Omega\setminus B_j(t)}\nabla w_j(t,\cdot)\|^2_{L^2(\Omega)}=\mathrm{O}\left(\frac{1}{\sqrt{j}}\right),\qquad(24)$$

[4] Recall that such a curve γ is a ray for the operator $\Box=\partial_{tt}-\triangle_g$, in other words a null bicharacteristic of $\Box$ whose hamiltonian is equal to $1/4$ for every t (see, e.g., [22]).

where $B_j(t)$ denotes the ball centered at $\gamma(t)$ of radius $j^{-1/4}$. Note that the results of [15] are established in $\mathbb{R}^d$ but their extension to a manifold is straightforward.

Moreover, an immediate adaptation of the approach in [15] yields that the function $w_j^{>N}$ defined by

$$w_j^{>N} = c_{jN} \sum_{k=N+1}^{+\infty} \langle w_j, \phi_k \rangle_{L^2(\Omega)} \phi_k$$

where c_{jN} is chosen in such a way that $E_{w_j^{>N}}(\cdot) = 1$, inherits the property (24).

The functions $w_j^{>N}$ are referred to as *Gaussian beams*. The next lemma provides an essential argument to compare the observability constant with the geometric quantity g_2.

Lemma 4. *Let us consider a function h in $\mathcal{C}^0(\overline{\Omega})$ The sequence $(\delta_{T,j}(h))_{j\in\mathbb{N}}$ where*

$$\delta_{T,j}(h) = \frac{\int_0^T h(x) \int_\Omega |\partial_t w_j^{>N}(t,x)|^2 \, dv_g \, dt}{E_{w_j^{>N}}(T)} = \int_0^T \int_\Omega h(x) |\partial_t w_j^{>N}(t,x)|^2 \, dv_g \, dt$$

satisfies

$$\lim_{j\to+\infty} \delta_{T,j}(h) = \frac{1}{2} \int_0^T h(\gamma(t)) \, dt.$$

For the sake of clarity, the proof of Lemma 4 is postponed to Section 2.8.

To provide an estimate of $C_T^{>N}(h)$, we define $\widetilde{w}_j^{>N} = \partial_t w_j^{>N}$. Then, the function $\widetilde{w}_j^{>N}$ is a solution of (1) with $(\widetilde{w}_j^{>N}(0,\cdot), \partial_t \widetilde{w}_j^{>N}(0,\cdot)) \in L^2(\Omega) \times (H^1)'(\Omega)$ and we infer that

$$\begin{aligned}
\frac{C_T^{>N}(h)}{T} &\leqslant \frac{1}{T} \lim_{j\to+\infty} \frac{\int_0^T \int_\omega |\widetilde{w}_j^{>N}(t,x)|^2 \, dv_g \, dt}{\|(\widetilde{w}_j^{>N}(0,\cdot), \partial_t \widetilde{w}_j^{>N}(0,\cdot))\|^2_{L^2\times(H^1)'}} \\
&= \lim_{j\to+\infty} \frac{\delta_{T,j}(h)}{T} = \frac{1}{2T} \int_0^T h(\gamma(t)) \, dt.
\end{aligned}$$

To conclude, let us introduce the sequence $(h_k)_{k\in\mathbb{N}^*}$ defined by

$$h_k : x \mapsto \begin{cases} 1 - \frac{\operatorname{dist}(x,\omega)}{k} & \text{if } \operatorname{dist}(x,\omega) \leqslant 1/k \\ 0 & \text{else.} \end{cases}$$

Notice that $(h_k)_{k\in\mathbb{N}^*}$ converges weakly-$\star$ to $\chi_{\overline{\omega}}$ in $L^\infty(\Omega)$, is such that $h_k \in \mathcal{C}^0(\overline{\Omega})$ and satisfies $\chi_\omega \leqslant h_k \leqslant 1$ a.e. in Ω for every $k \in \mathbb{N}^*$. Hence, using Lemma 4, one gets

$$\frac{C_T^{>N}(\omega)}{T} \leqslant \frac{C_T^{>N}(h_k)}{T} \leqslant \frac{1}{2T} \int_0^T h_k(\gamma(t)) \, dt,$$

and the right-hand side term converges to $\frac{1}{2T} \int_0^T \chi_{\overline{\omega}}(\gamma(t)) \, dt$ as $k \to +\infty$.

Now, considering a geodesic γ such that the quantity $\frac{1}{2T} \int_0^T \chi_{\overline{\omega}}(\gamma(t)) \, dt$ is arbitrarily close to $g_2^T(\overline{\omega})$ leads to the desired result.

Second step: $\alpha^T(\omega) \geqslant \frac{1}{2} g_2^T(\mathring{\omega})$.

Without loss of generality, we can assume that ω is an open set since $\alpha^T(\omega) \geqslant \alpha^T(\mathring{\omega})$ for any ω. We extend the definition of $C_T^{>N}$ to all $h \in L^\infty(\Omega, [0,1])$ by setting

$$C_T^{>N}(h) = \inf_{\sum_{j=N+1}^{+\infty}(|a_j|^2+|b_j|^2)=1} \frac{1}{2}\int_0^T \int_\Omega h(x) \left| \sum_{j=N+1}^{+\infty} \left(a_j e^{i\lambda_j t} + b_j e^{-i\lambda_j t}\right) \phi_j(x) \right|^2 dv_g\, dt,$$

As an infimum of continuous linear function for the weak-$\star$ topology of $L^\infty(\Omega, [0,1])$, the mapping $h \mapsto C_T^{>N}(h)$ is concave and upper semicontinuous for this topology.

Let $h \in \mathcal{C}^\infty(\overline{\Omega})$. We are going to prove that, T being a fixed positive number, one has

$$\frac{C_T^{>N}(h)}{T} \geqslant \frac{1}{2} g_2^T(h) + \mathrm{o}(1) \qquad \text{as } N \to +\infty. \tag{25}$$

In particular, letting N tend to ∞, we will obtain that

$$\alpha^T(h) := \lim_{N\to+\infty} \frac{C_T^{>N}(h)}{T} \geqslant \frac{1}{2} g_2^T(h).$$

It is worth noticing that the $\mathrm{o}(1)$ in (25) does actually depend on h, which makes the conclusion not obvious. We proceed in this way: since ω is open, one can construct an increasing sequence of functions (h_k) such $0 \leqslant h_k \leqslant 1$, $h_k(x) = 0$ if $\mathrm{dist}(x, \Omega \setminus \omega) \leqslant \frac{1}{k}$ and $h_k(x) = 1$ if $\mathrm{dist}(x, \Omega \setminus \omega) \geqslant \frac{2}{k}$. Since $k \mapsto h_k$ increases, we have $\alpha^T(\omega) \geqslant \alpha^T(h_k)$ for any k. To conclude, we apply the following lemma whose proof is postponed to Section 2.9.

Lemma 5. *We have*

$$g_2^T(\omega) = \lim_{k\to+\infty} g_2^T(h_k).$$

Hence, it is enough to establish (25) to get the expected conclusion.

Let us establish (25). Let $h \in \mathcal{C}^\infty(\overline{\Omega})$, such that $h \geqslant 0$ in Ω. Denote by $L^2_N(\Omega)$, $(H^1(\Omega))'_N$ the spaces of L^2 and $(H^1)'$ functions of Ω that are L^2-orthogonal to $\mathrm{Span}\{\phi_i\}_{1\leqslant i\leqslant N}$.

For every $N \geqslant 1$, we consider initial data $(y_{0,N}, y_{1,N})$ normalized in $L^2_N(\Omega) \times (H^1(\Omega))'_N$ for the wave equation (1) for some $N \geqslant 1$. Denoting by λ the operator[5] defined by $\lambda = \sqrt{-\triangle_g}$, the solution y_N of (1) associated to $(y_{0,N}, y_{1,N})$ then satisfies

$$y_N = e^{it\lambda} z_{0,N} + e^{-it\lambda} z_{1,N} \tag{26}$$

where $z_{0,N} = \frac{1}{2}(y_{0,N} - i\lambda^{-1} y_{1,N})$ and $z_{1,N} = \frac{1}{2}(y_{0,N} + i\lambda^{-1} y_{1,N})$. We will first find a lower bound of

$$\liminf_{N\to+\infty} \frac{1}{T}\int_0^T \int_\Omega h(x) |y_N(t,x)|^2 \, dv_g\, dt.$$

Introduce the operator $\mathcal{G}_T$ associated to this problem, defined by

$$(\mathcal{G}_T u | v) = \int_0^T \int_\Omega h(x) u(t,x) \overline{v}(t,x) \, dv_g\, dt,$$

[5]Note that since $N \geqslant 1$, λ can be considered as an invertible operator from $L^2_N(\Omega)$ to $(H^1(\Omega))'_N$.

for $u, v \in L^2([0,T] \times \Omega)$.

The identity (26) shows that

$$\big(\mathcal{G}_T y_N | y_N\big) = A_N + B_N + 2C_N$$

where

$$A_N = \big(\mathcal{G}_T e^{it\lambda} z_{0,N} | z_{0,N}\big)\ ,\ B_N = \big(\mathcal{G}_T e^{it\lambda} z_{1,N} | z_{1,N}\big)\ \text{ and } C_N = \mathrm{Re}\big(\mathcal{G}_T e^{it\lambda} z_{0,N} | z_{1,N}\big).$$

Let us study

$$A_N = \int_0^T \int_\Omega e^{-it\lambda} h e^{it\lambda} z_{0,N} \overline{z_{0,N}}\, dx\, dt.$$

Introduce the pseudodifferential operator $a = \mathrm{Op}(h)$ obtained by Weyl quantization. Actually, since h is a function of Ω, a is the multiplication by h. Applying the Egorov theorem[6] shows the existence of a smooth family of operators Q_t of order 0 with principal symbol $h(\gamma_{x,\xi}(t))$ such that

$$e^{-it\lambda} h e^{it\lambda} = Q_t + S_t$$

where S_t is d-smoothing for all $d > 0$. Moreover, since the principal symbol of Q_t is $h(\gamma_{x,\xi}(t))$, there exists a smooth family R_t of 1-smoothing operators such that

$$Q_t = \mathrm{Op}(h(\gamma_{x,\xi}(t))) + R_t.$$

Notice moreover, that $R_t = Q_t - \mathrm{Op}(h(\gamma_{x,\xi}(t))) + S_t$. We obtain

$$A_N = A_N^1 + A_N^2 \tag{27}$$

where

$$A_N^1 = \int_0^T \int_\Omega \mathrm{Op}(h(\gamma_{x,\xi}(t))) z_{0,N} \overline{z_{0,N}}\, dv_g\, dt, \qquad A_N^2 = \int_0^T \int_\Omega R_t z_{0,N} \overline{z_{0,N}}\, dv_g\, dt.$$

By definition of $g_2^T(h)$, one has $\int_0^T h(\gamma_{x,\xi}(t))dt \geqslant T g_2^T(h)$. Hence, it follows from Garding's inequality that

$$\mathrm{Op}\left(\int_0^T h(\gamma_{x,\xi}(t))dt\right) \geqslant a(T g_2^T(h)) = T g_2^T(h),$$

by noting that

$$A_N^1 = \int_\Omega \mathrm{Op}\left(\int_0^T h(\gamma_{x,\xi}(t))\, dt\right) z_{0,N} \overline{z_{0,N}}\, dv_g.$$

[6]**Egorov Theorem ([4, 23]):** *Let P be an operator of order m with principal symbol p. There exists a smooth family of operators Q_t of order m with principal symbol q_t such that*

- *for all $d \in \mathbb{N}$, $e^{-it\lambda} P e^{it\lambda} - Q_t$ is d-smoothing;*
- *$q_t(x,\xi) = p(\gamma_{x,\xi}(t))$ where $\gamma_{x,\xi}$ is the geodesic of Ω starting at $x \in \Omega$ in the direction of $\xi \in T_x^*\Omega$.*

As a consequence, there holds

$$\frac{A_N^1}{T} \geqslant g_2^T(h)\|z_{0,N}\|_{L^2(\Omega)}^2. \tag{28}$$

Let us now prove that $\lim_{N\to\infty} A_N^2 = 0$. Choose any subsequence of (A_N^2), still noted A_N^2. Since R_t is 1-smoothing and since $t \mapsto R_t$ is smooth, the operator $\int_0^T R_t dt$ is also 1-smoothing. As a consequence, there exists a constant $C > 0$ such that for all N,

$$\left\| \int_0^T R_t z_{0,N}(x)dt \right\|_{H^1(\Omega)} \leqslant C.$$

By the Sobolev embedding theorem, there exists a function $v \in L^2(\Omega)$ such that a subsequence of $\left(\int_0^T R_t z_{0,N}(x)dt\right)_{N\geqslant 1}$ (still denoted by $\left(\int_0^T R_t z_{0,N}(x)dt\right)_{N\geqslant 1}$) tends to v in L^2. Hence,

$$\begin{aligned} A_N^2 &= \int_\Omega \left(\int_0^T R_t z_{0,N}(x)dt\right) \overline{z_{0,N}(x)}dx \\ &\leqslant \int_\Omega \left[\left(\int_0^T R_t z_{0,N}(x)dt\right) - v(x)\right] \overline{z_{0,N}(x)}dx + \int_\Omega v(x)\overline{z_{0,N}(x)}dx. \end{aligned}$$

The first integral in the right hand side converges to 0 by convergence of $\left(\int_0^T R_t z_{0,N}(x)dt\right) - v(x)$ to 0 in $L^2(\Omega)$ while using Hölder inequality, the second integral satisfies, using that $z_{0,N} \in L_N^2(\Omega)$,

$$\int_\Omega v(x)z_{0,N}(x)dx = \int_\Omega p_N(v)(x)z_{0,N}(x)dx \leqslant \|p_N(v)\|_{L^2(M)}\|z_{0,N}\|_{L^2(M)}$$

where p_N is the L^2 projection of v onto $L_N^2(\Omega)$. Clearly,

$$\lim_{N\to+\infty} \|p_N(v)\|_{L^2(M)} = 0.$$

Hence, $\lim_{N\to+\infty} A_N^2 = 0$. Since this holds for any subsequences of A_N^2, this proves that the sequence A_N^2 tends to 0 as N tends to $+\infty$. Together with (27) and (28), we obtain

$$\frac{A_N}{T} \geqslant g_2^T(h)\|z_{0,N}\|_{L^2(\Omega)}^2. \tag{29}$$

With the same argument, we also get that

$$\frac{B_N}{T} \geqslant g_2^T(h)\|z_{1,N}\|_{L^2(\Omega)}^2. \tag{30}$$

Now, observe that

$$C_N = \int_\Omega R_t' z_{0,N}\overline{z_{1,N}}dtdx$$

where R'_t is the operator defined by $R'_t = \int_0^T e^{it\lambda} h e^{it\lambda} dt$. By [2, Lemma A.1 p. 45], the operator R'_t is 1-smoothing, and the argument above, showing that $\lim_{N\to+\infty} A_N^2 = 0$, proves that

$$\lim_{N\to+\infty} C_N = 0.$$

Together with (29) and (30), we obtain that

$$\liminf_{N\to\infty} \frac{1}{T}\int_0^T \int_\Omega h(x)|y_N|(t,x)^2 dx dt \geqslant g_2^T(h)\left(\|z_{0,N}\|^2_{L^2(\Omega)} + \|z_{1,N}\|^2_{L^2(\Omega)}\right).$$

Observing that

$$\begin{aligned}\|z_{0,N}\|^2_{L^2(\Omega)} + \|z_{1,N}\|^2_{L^2(\Omega)} &= \frac{1}{2}\left(\|y_{0,N}\|^2_{L^2(\Omega)} + \|\lambda^{-1}y_{1,N}\|^2_{L_2(\Omega)}\right)\\ &= \frac{1}{2}\left(\|y_{0,N}\|^2_{L^2(\Omega)} + \|y_{1,N}\|^2_{(H^1)'(\Omega)}\right),\end{aligned}$$

this ends the proof of Theorem 3.

2.7 Proof of Corollary 2

Using Theorem 3, it is clear that this is enough to prove that Assumption (1.2) implies that $g_2(\mathring{\omega}) = g_2(\overline{\omega})$. To see this fact, consider any geodesic $\gamma : [0,T] \to M$ which is close to minimise the infimum in the definition of $g_2^T(\mathring{\omega})$. Hence, there is some arbitrarily small ε such that

$$\begin{aligned}g_2^T(\mathring{\omega}) + \varepsilon &\geqslant \frac{1}{T}\int_0^T \chi_\omega(\gamma(t))\,dt\\ &= \frac{1}{T}\int_0^T \chi_{\overline{\omega}}(\gamma(t))\,dt - \frac{1}{T}\int_0^T \chi_{\overline{\omega}\setminus\omega}(\gamma(t))\,dt\\ &= \frac{1}{T}\int_0^T \chi_{\overline{\omega}}(\gamma(t))\,dt + \frac{1}{T}\int_0^T \chi_{\Omega\setminus(\overline{\omega}\setminus\omega)}(\gamma(t))\,dt - 1\\ &\geqslant g_2^T(\overline{\omega}) + g_2^T(\Omega\setminus(\overline{\omega}\setminus\omega)) - 1 \geqslant g_2^T(\overline{\omega}).\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, one gets $g_2^T(\mathring{\omega}) \geqslant g_2^T(\overline{\omega})$. Since the converse inequality is obvious, the proof is complete.

2.8 Proof of Lemma 4

According to the Green formula, one has

$$\begin{aligned}\delta_{T,j}(h) &= \int_\Omega h(x)\left[w_j^{>N}(t,x)\partial_t w_j^{>N}(t,x)\right]_{t=0}^{t=T} dv_g - \int_0^T\int_\Omega h(x) w_j^{>N}(t,x)\partial_{tt} w_j^{>N}(t,x)\,dv_g\\ &= \int_\Omega h(x)\left(w_j^{>N}(T,x)\partial_t w_j(T,x) - w_j^{>N}(0,x)\partial_t w_j^{>N}(0,x)\right) dv_g\\ &\quad -\int_0^T\int_\Omega h(x) w_j^{>N}(t,x)\triangle_g w_j^{>N}(t,x)\,dv_g.\end{aligned}$$

Let us show that

$$\int_\Omega h(x)\left(w_j^{>N}(T,x)\partial_t w_j^{>N}(T,x) - w_j^{>N}(0,x)\partial_t w_j^{>N}(0,x)\right) dv_g = \mathrm{o}(1) \quad \text{as } j\to+\infty.$$

Since $E_{w_j}(\cdot)=1$, the sequence $(\partial_t w_j^{>N}(T,\cdot))_{j\in\mathbb{N}^*}$ is bounded in $L^2(\Omega)$ and up to a subsequence, converges therefore to some function u_1^* weakly in $L^2(\Omega)$. Similarly, the sequence $(\nabla w_j^{>N}(T,\cdot))_{j\in\mathbb{N}^*}$ is bounded in $L^2(\Omega)$. Since $\int_\Omega w_j^{>N}(x)\,dv_g = 0$, it follows from the Poincaré-Wirtinger inequality that, up to a subsequence, $(w_j^{>N}(T,\cdot))_{j\in\mathbb{N}^*}$ converges to some function u_2^* weakly in $H^1(\Omega)$. According to Rellich-Kondratov theorem, the aforementioned subsequence of $(w_j^{>N}(T,\cdot))_{j\in\mathbb{N}^*}$ converges also strongly to u_2^* in $L^2(\Omega)$. Combining this result with the fact that $\chi_{\Omega\setminus B_j(T)}$ converges strongly in $L^2(\Omega)$ to the function equal to 1 almost everywhere in Ω leads to

$$\begin{aligned}\int_\Omega |u_2^*|\,dv_g &\leqslant \liminf_{j\to+\infty}\int_\Omega \chi_{\Omega\setminus B_j(T)}|w_j^{>N}(T,x)|\,dv_g \\ &\leqslant |\Omega|^{1/2}\left(\liminf_{j\to+\infty}\int_\Omega \chi_{\Omega\setminus B_j(T)}w_j^{>N}(T,x)^2\,dv_g\right)^{1/2}\end{aligned}$$

by using the Cauchy-Schwarz inequality and the fact that the mapping $L^2(\Omega)\ni u\mapsto |u|\in L^2(\Omega)$ is convex and lower semicontinuous for the strong topology of $L^2(\Omega)$, and thus also for the weak topology of $L^2(\Omega)$.

Now, according to the energy estimate (24) on $w_j^{>N}$, we infer that $|u_2^*|=0$ necessarily. Hence, the sequence $(\int_\Omega h(x)w_j^{>N}(T,x)\partial_t w_j^{>N}(T,x)\,dv_g)_{j\in\mathbb{N}^*}$ converges to 0 as $j\to+\infty$.

One shows in a similar way that the sequence $(\int_\Omega \chi_\omega(x)w_j^{>N}(0,x)\partial_t w_j^{>N}(0,x)\,dv_g)_{j\in\mathbb{N}^*}$ converges to 0 as $j\to+\infty$. The expected result follows.

We also claim that

$$-\int_0^T\int_\Omega h(x)w_j^{>N}(t,x)\triangle_g w_j^{>N}(t,x)\,dv_g = \int_0^T\int_\Omega h(x)|\nabla_g w_j^{>N}(t,x)|^2\,dv_g\,dt + \mathrm{o}(1),$$

as $j\to+\infty$. Indeed, using the Green formula, one has

$$\begin{aligned}-\int_0^T\int_\Omega h(x)w_j^{>N}(t,x)\triangle_g w_j^{>N}(t,x)\,dv_g &= \int_0^T\int_\Omega h(x)|\nabla_g w_j^{>N}(t,x)|^2\,dv_g\,dt \\ &\quad + \int_0^T\int_\Omega w_j^{>N}(t,x)^2\nabla h(x)\,dv_g\,dt,\end{aligned}$$

and using the same reasoning as previously, one shows that the sequence $(w_j^{>N})_{j\in\mathbb{N}^*}$ converges strongly to 0 in $L^2(0,T;L^2(\Omega))$.

We have then proved that

$$\delta_{T,j}(h) = \int_0^T\int_\Omega h(x)|\nabla_g w_j^{>N}(t,x)|^2\,dv_g\,dt + \mathrm{o}(1) \quad \text{as } j\to+\infty. \tag{31}$$

As a result,

$$\begin{aligned}\delta_{T,j}(h) &= \frac{\int_0^T\int_\Omega h(x)|\partial_t w_j^{>N}(t,x)|^2\,dv_g\,dt}{E_{w_j^{>N}}(T)}\\ &= \frac{1}{2}\frac{\int_0^T\int_\Omega h(x)(|\partial_t w_j^{>N}(t,x)|^2+|\nabla_g w_j^{>N}(t,x)|^2)\,dv_g\,dt+\mathrm{o}(1)}{E_{w_j^{>N}}(T)}\\ &= \frac{1}{2}\int_0^T\int_{B_j(t)} h(x)(|\partial_t w_j^{>N}(t,x)|^2+\nabla_g w_j^{>N}(t,x)|^2)\,dv_g\,dt+\mathrm{o}(1),\end{aligned}$$

by using (31).

Now, introduce

$$\theta_j:\mathbb{R}_+\ni t\mapsto \int_{B_j(t)} h(x)(|\partial_t w_j^{>N}(t,x)|^2+\nabla_g w_j^{>N}(t,x)|^2)\,dv_g.$$

According to (24), one has the following alternative: either $\gamma(t)\notin\mathrm{supp}(h)$ and then $(\theta_j(t))_{j\in\mathbb{N}^*}$ converges to 0 as $j\to+\infty$, or $\gamma(t)\in\mathrm{supp}(h)$ and $(\theta_j(t))_{j\in\mathbb{N}^*}$ converges to $h(\gamma(t))$ as $j\to+\infty$. In other words, $(\theta_j(t))_{j\in\mathbb{N}^*}$ converges to $\langle\delta_{\gamma(t)},h\rangle_{\mathcal{M},\mathcal{C}^0}$ as $j\to+\infty$.

According to the Lebesgue dominated convergence theorem, the conclusion follows.

2.9 Proof of Lemma 5

Assume that ω is open and that $(h_k)_{k\in\mathbb{N}}$ is a noncreasing sequence of functions such $0\leqslant h_k\leqslant 1$ in Ω, $h_k(x)=0$ if $\mathrm{dist}(x,\Omega\setminus\omega)\leqslant\frac{1}{k}$ and $h_k(x)=1$ if $\mathrm{dist}(x,\Omega\setminus\omega)\geqslant\frac{2}{k}$. We will prove that

$$g_2^T(\omega)=\lim_{k\to+\infty} g_2^T(h_k). \tag{32}$$

The fact that $g_2^T(\omega)\geqslant\limsup_{k\to+\infty} g_2^T(h_k)$ is obvious since $\chi_\omega\geqslant h_k$ for all $k\in\mathbb{N}$. Consider a sequence of rays $\gamma_k:[0,T]\to\Omega$ such that

$$g_2^T(h_k)\geqslant\frac{1}{T}\int_0^T h_k(\gamma_k(t))dt+\mathrm{o}(1)\quad\text{as } k\to+\infty. \tag{33}$$

The set of rays is compact since each rays is entirely determined by it position $x\in\Omega$ at time 0 and its derivative at time 0 with lies on the unit cotangent bundle of Ω. Hence there exists $\gamma:[0,T]\to\Omega$ such that $\gamma_k\to\gamma$ uniformly on $[0,T]$. For any $t\in[0,T]$, it holds that

$$\liminf_{k\to+\infty} h_k(\gamma_k(t))\geqslant\chi_\omega(\gamma(t)).$$

Indeed, if $\gamma(t)\in\omega$, then since ω is open, $h_k(\gamma_k(t))=1=\chi_\omega(\gamma(t))$ as soon as k is large enough. If $\gamma(t)\notin\omega$, the inequality is obvious since $\chi_\omega(\gamma(t))=0$. Together with (33), Lebesgue dominated convergence theorem implies that

$$g_2^T(h_k)\geqslant\frac{1}{T}\int_0^T h_k(\gamma_k(t))dt+\mathrm{o}(1)\geqslant\frac{1}{T}\int_0\chi_\omega(\gamma(t))dt+\mathrm{o}(1)\geqslant g_2^T(\omega)+\mathrm{o}(1)\quad\text{as } k\to+\infty,$$

which proves (32).

3 Characterization of the observability (proof of Corollary 1)

We first claim that $C_T(\omega) > 0 \Rightarrow \alpha^T(\omega) > 0$. Indeed, one has $C_T(\omega) \leqslant C_T^{>N}(\omega)$ for any $N \geqslant 0$, and it follows from the definition of α^T that $\alpha^T(\omega) = 0 \Rightarrow C_T(\omega) = 0$, whence the claim.

Let us prove the converse. Assume by contradiction that

$$\alpha^T(\omega) > 0 \qquad \text{and} \qquad C_T(\omega) = 0. \tag{34}$$

For any $s > 0$, let us denote by E_s the set of solutions of (1) vanishing identically on $[0, s] \times \omega$. This space is sometimes called "space of invisible solutions". Notice that E_s is a vector space.

For every $k \in \mathbb{N}$, one introduces the following property.

(H_k) (with $k \in \mathbb{N}^*$) for all $\varepsilon > 0$, there exists a non trivial solution $y_{k,\varepsilon} \in E_{T-\varepsilon}$ involving only frequencies of index greater than k, i.e. such that the pair of its initial conditions $(y^0_{k,\varepsilon}, y^1_{k,\varepsilon})$ satisfies

$$\int_\Omega y^i_{k,\varepsilon}(x)\phi_j(x)dv_g = 0, \qquad i = 0, 1, \quad j = 1, \dots, k.$$

If $k = 0$ this property writes: there exists a non trivial solution $y_{0,\varepsilon} \in E_{T-\varepsilon}$.

We will prove by induction that Property (H_k) holds true for every $k \in \mathbb{N}$. As a consequence, if $\varepsilon > 0$ and N are fixed, Property (H_N) yields the existence of a solution $y_{T,\varepsilon} \in E_{T-\varepsilon}$ of (1) involving only frequencies of index higher than N. Using $(y_{T,\varepsilon}(0,\cdot), \partial_t y_{T,\varepsilon}(0,\cdot))$ as test functions in the functional J^ω_T, one infers that $C^{>N}_{T-\varepsilon}(\omega) = 0$. Letting N tend to $+\infty$ yields that $\alpha_{T-\varepsilon}(\omega) = 0$. Finally, noting that for all initial conditions (y^0, y^1), one has

$$\left| J^\omega_{T-\varepsilon}(y^0, y^1) - J^\omega_T(y^0, y^1) \right| \leqslant \frac{\varepsilon}{T - \varepsilon},$$

one infers that

$$\alpha_T(\omega) \leqslant \alpha_{T-\varepsilon}(\omega) + \frac{\varepsilon}{T - \varepsilon}$$

and as a consequence, one has $\alpha^T(\omega) = 0$ whence the contradiction.

It remains now to show Property (H_k) holds true for every $k \in \mathbb{N}$ under the assumption (34).

Let us first prove that (H_0) is true. According to Theorem 1, the infimum defining $C_T(\omega)$ is reached by some initial conditions (y_0, y_1) such that the associated solution y of (1) satisfies $y = 0$ a.e. on $[0, T] \times \omega$. In other words, the dimension of E_T is at least equal to 1, and this holds also for $E_{T-\varepsilon}$ for any ε since $E_T \subset E_{T-\varepsilon}$.

Assume now that (H_k) is true for some $k \in \mathbb{N}$ and let us prove (H_{k+1}). Let $\varepsilon > 0$ and let $y \in E_{T-\varepsilon/2}$ with initial condition (y_0, y_1) satisfying

$$\int_\Omega y^i(x)\phi_j(x)\, dv_g = 0, \qquad \text{for all } i = 0, 1, \quad j = 1, \dots, k.$$

The crucial point is that for all $a \in [0, \varepsilon/2]$, the function $\tau_a(y) : (t,x) \to y(t+a,x)$ belongs to $E_{T-\frac{\varepsilon}{2}-a} \subset E_{T-\varepsilon}$. We now show the existence a function z writing as a nonzero linear combination of functions $(\tau_a(y))_{a\in[0,\varepsilon/2]}$, with initial conditions (z^0, z^1) satisfying the orthogonality conditions

$$\int_\Omega z^i(x)\phi_j(x)\, dv_g = 0, \qquad i = 0, 1, \quad j = 1, \dots k+1.$$

Let (y_a^0, y_a^1) be the initial conditions associated to y_a. We expand the solution $\tau_a(y)$ as

$$\tau_a(y)(t,\cdot) = \sum_{j=k+1}^{+\infty} \left(a_j(a) e^{i\lambda_j t} + b_j(a) e^{-i\lambda_j t} \right) \phi_j(\cdot).$$

where $(a_j(a))_{j\in\mathbb{N}^*}$ and $(b_j(a))_{j\in\mathbb{N}^*}$ belong to $\ell^2(\mathbb{R})$. In particular, one has

$$a_j(a) = e^{ia\lambda_j} a_j(0) \text{ and } b_j(a) = e^{-ia\lambda_j} b_j(0).$$

If $a_{k+1}(0) = b_{k+1}(0) = 0$, then y belongs to $E_{T-\varepsilon}$ and involves only frequencies of index higher than $k+1$ which shows that (H_{k+1}) holds true. For this reason, we assume that $a_{k+1}(a) \neq 0$ or $b_{k+1}(a) \neq 0$. Hence, there exists j such that $\lambda_j > \lambda_{k+1}$, and $a_j(0) \neq 0$ or $b_j(0) \neq 0$. Otherwise, the function y would be a nonzero multiple of an eigenfunction belonging to the eigenspace associated to the eigenvalue λ_k and would vanish on ω. Because of the hypoanalyticity of the Laplacian operator, this is impossible as soon as ω has a positive Lebesgue measure (see [3, 7, 13]), which is the case since $\alpha^T(\omega) > 0$. Hence, let us consider $j > k$ such that $\lambda_j > \lambda_k$ and $a_j(0) \neq 0$ or $b_j(0) \neq 0$. Since $\lambda_j > \lambda_k$, one can find $0 < a < a' \leqslant \varepsilon/2$ such that the vectors $(1, e^{i\lambda_k a}, e^{i\lambda_k a'})$ and $(1, e^{i\lambda_j a}, e^{i\lambda_j a'})$ are linearly independent. In other words, there exist real numbers $c_0, c_a, c_{a'}$ such that

$$c_0 + c_a e^{i\lambda_k a} + c_{a'} e^{i\lambda_k a'} = 0 \tag{35}$$

and

$$c_0 + c_a e^{i\lambda_j a} + c_{a'} e^{i\lambda_j a'} \neq 0. \tag{36}$$

Set $z = c_0 y + c_a y_a + c_{a'} y_{a'}$. Then, z is the desired solution: it belongs to $E_{T-\varepsilon}$ and moreover $z \neq 0$ according to (36). Finally, z involves only frequencies of index higher than $k+1$ according to (35). This shows (H_{k+1}).

4 Spectral gap assumption and consequences

In the case where the eigenvalues are well separated in a sense made precise below, another argument than the one used in the proof of Theorem 2 leads to a more precise result.

Theorem 4. *Assume that the spectrum $(\lambda_j)_{j\in\mathbb{N}^*}$ of Ω satisfies the uniform gap property*

(UG) *There exists $\gamma > 0$ such that, if $\lambda_j \neq \lambda_k$ with $j \neq k$, then $|\lambda_j - \lambda_k| \geqslant \gamma$.*

Then for every measurable subset ω of Ω, there holds

$$\lim_{T\to+\infty}\frac{C_T(\omega)}{T}=\frac{1}{2}g_1(\omega).$$

In other words, together with Theorems 2 and 3, one has

$$g_1(\omega)\leqslant g_2(\overline{\omega}) \tag{37}$$

for every measurable subset ω of Ω. Note that such an inequality obviously does not holds true without specific assumptions on the sets ω and Ω. Indeed, it is enough to consider as a counterexample that Ω is a flat torus and that ω is a rectangle strictly contained in Ω (see [17, 19] for various examples).

Remark 7 (Characterization of manifolds for which $g_1(\omega)\leqslant g_2(\omega)$)**.** A natural issue is to wonder whether one can characterize the manifolds (Ω,g) satisfying (37) for any measurable subset ω of Ω. Such an issue will be answered (among others) in the forthcoming paper [10]. More precisely, it will be stated that this property holds true if and only if Ω is a manifold whose geodesic flow is periodic and such that each geodesic is the support of a quantum limit. We will also establish further information on quantum limits and properties of the geodesic flow by using similar techniques to those developed in the present paper.

Proof. According to Theorem 1, we have $\lim_{T\to+\infty}\frac{C_T(\omega)}{T}\leqslant\frac{1}{2}g_1(\omega)$. We will now show that the converse inequality is still true as soon as (UG) is satisfied. Let us denote by U_∞ the set of all distinct eigenvalues λ_k and let us set $I_\infty(\lambda)=\{j\in\mathbb{N}^*\mid\lambda_j=\lambda\}$. One has

$$\begin{aligned}
\frac{C_T(\omega)}{T} &= \inf_{\sum_{j=1}^{+\infty}|a_j|^2+|b_j|^2=1}\frac{1}{T}\int_0^T\int_\omega\left|\sum_{j=1}^{+\infty}\left(a_je^{i\lambda_jt}+b_je^{-i\lambda_jt}\right)\phi_j(x)\right|^2 dv_g\,dt\\
&= \inf_{\sum_{j=1}^{+\infty}|a_j|^2+|b_j|^2=1}\Bigg(\sum_{\lambda\in U_\infty}\sum_{(j,k)\in I_\infty(\lambda)^2}a_j\overline{a_k}\int_\omega\phi_j(x)\phi_k(x)\,dv_g\\
&\quad+\frac{1}{T}\sum_{\substack{(\lambda,\mu)\in U_\infty^2\\ \lambda\neq\mu}}\sum_{\substack{j\in I_\infty(\lambda)\\ k\in I_\infty(\mu)}}\frac{a_j\overline{a_k}(e^{i(\lambda_j-\lambda_k)T}-1)}{\lambda_j-\lambda_k}\int_\omega\phi_j(x)\phi_k(x)\,dv_g,\\
&\quad+\sum_{\lambda\in U_\infty}\sum_{(j,k)\in I_\infty(\lambda)^2}b_j\overline{b_k}\int_\omega\phi_j(x)\phi_k(x)\,dv_g\\
&\quad+\frac{1}{T}\sum_{\substack{(\lambda,\mu)\in U_\infty^2\\ \lambda\neq\mu}}\sum_{\substack{j\in I_\infty(\lambda)\\ k\in I_\infty(\mu)}}\frac{b_j\overline{b_k}(e^{i(\lambda_k-\lambda_j)T}-1)}{\lambda_j-\lambda_k}\int_\omega\phi_j(x)\phi_k(x)\,dv_g,\\
&\quad+2\mathrm{Re}\left(\frac{1}{T}\sum_{(\lambda,\mu)\in U_\infty^2}\sum_{\substack{j\in I_\infty(\lambda)\\ k\in I_\infty(\mu)}}\frac{a_j\overline{b_k}(e^{i(\lambda_k+\lambda_j)T}-1)}{\lambda_j+\lambda_k}\int_\omega\phi_j(x)\phi_k(x)\,dv_g\right)\Bigg).
\end{aligned} \tag{38}$$

Let us fix $(a_j)_{j\in\mathbb{N}^*}$, $(b_j)_{j\in\mathbb{N}^*}$ in $\ell^2(\mathbb{C})$ such that $\sum_{j=1}^{+\infty}|a_j|^2+|b_j|^2=1$ and let

$$S=\sum_{\substack{(\lambda,\mu)\in U_N^2\\ \lambda\neq\mu}}\sum_{\substack{j\in I_\infty(\lambda)\\ k\in I_\infty(\mu)}}\frac{a_j\overline{a_k}(e^{i(\lambda_j-\lambda_k)T}-1)}{\lambda_j-\lambda_k}\int_\omega \phi_j(x)\phi_k(x)\,dv_g.$$

One has

$$\begin{aligned}|S| &\leqslant \left|\int_\omega \sum_{\substack{(\lambda,\mu)\in U_N^2\\ \lambda\neq\mu}}\sum_{\substack{j\in I_\infty(\lambda)\\ k\in I_\infty(\mu)}}\frac{a_je^{i\lambda_jT}\phi_j(x)\overline{a_k}e^{-i\lambda_kT}\phi_k(x)}{\lambda_j-\lambda_k}\,dv_g\right|\\ &\quad+\left|\int_\omega \sum_{\substack{(\lambda,\mu)\in U_N^2\\ \lambda\neq\mu}}\sum_{\substack{j\in I_\infty(\lambda)\\ k\in I_\infty(\mu)}}\frac{a_j\phi_j(x)\overline{a_k}\phi_k(x)}{\lambda_j-\lambda_k}\,dv_g\right|\end{aligned}$$

According to [16] and since the sums run over distinct eigenvalues enjoying a uniform gap property, there exists a constant $C_\gamma>0$ uniform with respect to the sequence $(a_j)_{j\in\mathbb{N}^*}$ such that

$$|S|\leqslant C_\gamma\sum_{j=1}^{+\infty}|a_j|^2\int_\omega \phi_j(x)^2\,dv_g\leqslant C_\gamma.$$

We thus infer that

$$\frac{1}{T}\left|\sum_{\substack{(\lambda,\mu)\in U_N^2\\ \lambda\neq\mu}}\sum_{\substack{j\in I_\infty(\lambda)\\ k\in I_\infty(\mu)}}\frac{a_j\overline{a_k}(e^{i(\lambda_j-\lambda_k)T}-1)}{\lambda_j-\lambda_k}\int_\omega \phi_j(x)\phi_k(x)\,dv_g\right|\leqslant\frac{C_\gamma}{T}.$$

$$\left|\mathrm{Re}\left(\frac{1}{T}\sum_{(\lambda,\mu)\in U_\infty^2}\sum_{\substack{j\in I_\infty(\lambda)\\ k\in I_\infty(\mu)}}\frac{a_j\overline{b_k}(e^{i(\lambda_k+\lambda_j)T}-1)}{\lambda_j+\lambda_k}\int_\omega \phi_j(x)\phi_k(x)\,dv_g\right)\right|\leqslant\frac{C}{T}.$$

for some $C>0$ independent of the sequences $(a_j)_{j\in\mathbb{N}^*}$, $(b_j)_{j\in\mathbb{N}^*}$ and the real number T. Finally, mimicking such a reasoning for each term of (38) yields that

$$\begin{aligned}\frac{C_T(\omega)}{T} &\geqslant \inf_{\sum_{j=1}^{+\infty}|a_j|^2+|b_j|^2=1}\sum_{\lambda\in U_\infty}\sum_{(j,k)\in I_\infty(\lambda)^2}(a_j\overline{a_k}+b_j\overline{b_k})\int_\omega\phi_j(x)\phi_k(x)\,dv_g+\mathrm{O}\left(\frac{1}{T}\right)\\ &= g_1(\omega)+\mathrm{O}\left(\frac{1}{T}\right).\end{aligned}$$

The theorem is proved. $\square$

Remark 8 (Application of Theorem 4)**.** Theorem 4 applies in particular in the following cases:

- *the one-dimensional torus* $\mathbb{T} = \mathbb{R}/(2\pi)$*.* The operator $\triangle_g = \partial_{xx}$ is defined on the subset of the functions of $H^2(\mathbb{T})$ having zero mean. Its eigenvalues are all double, given by $\lambda_j = j$ for every $j \in \mathbb{N}^*$ and associated to the eigenfunctions $e_j^1 = \sqrt{\frac{1}{\pi}} \sin(j\cdot)$ and $e_j^2 = \sqrt{\frac{1}{\pi}} \cos(j\cdot)$. In this case, the spectral gap is $\gamma = 1$ and one gets

$$\lim_{T \to +\infty} \frac{C_T(\omega)}{T} = \frac{1}{\pi} \inf_{j \in \mathbb{N}^*} \inf_{\alpha \in [0,1]} \int_\omega \left(\sqrt{\alpha} \sin(jx) + \sqrt{1-\alpha} \cos(jx)\right)^2 dx$$

  Using straightforward computations, this last expression simplifies into

$$\lim_{T \to +\infty} \frac{C_T(\omega)}{T} = \frac{1}{\pi} \left(\frac{|\omega|}{2} - \sup_{j \in \mathbb{N}^*} \sqrt{\left(\int_\omega \sin(2jx)\, dx\right)^2 + \left(\int_\omega \cos(2jx)\, dx\right)^2} \right)$$

- *the unit sphere* $\mathbb{S}^n$ *of* $\mathbb{R}^{n+1}$*.* The operator $\triangle_g$ is defined from the usual Laplacian operator on the Euclidean space $\mathbb{R}^{n+1}$ by the formula $\triangle_g = r^2 \triangle_{\mathbb{R}^{n+1}} - \partial_{rr} - \frac{n}{r}\partial_r$ where $r = \|x\|_{\mathbb{R}^{n+1}}$ for every $x \in \mathbb{R}^{n+1}$. Its eigenvalues are $\lambda_k = k(k+n-1)$ where $k \in \mathbb{N}$. Moreover, the multiplicity of λ_k is $k(k+n-1)$ and the space of eigenfunctions is the space of homogeneous harmonic polynomials[7] of degree k. As a result, one gets

$$\lim_{T \to +\infty} \frac{C_T(\omega)}{T} = \inf_{k \in \mathbb{N}} \inf_{\phi \in \mathcal{H}_k} \frac{\int_\omega |\phi(x)|^2\, dx}{\int_{\mathbb{S}^n} |\phi(x)|^2\, dx},$$

  where $\mathcal{H}_k$ denotes the space of homogeneous harmonic polynomials of degree k.

As a byproduct of Theorem 4, we recover a well known result on the existence of quantum measures[8] supported by closed geodesics.

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[7] An orthogonal basis of spherical harmonics is given by

$$Y_{l_1,\dots l_n}(\theta_1, \dots \theta_n) = \frac{1}{\sqrt{2\pi}} e^{i l_1 \theta_1} \prod_{j=2}^{n} \widetilde{P}_{l_j,j}^{l_{n-1}}(\theta_j)$$

where the indices are integers satisfying $|l_1| \leqslant l_2 \leqslant \dots \leqslant l_n$ and the eigenvalue is $-l_n(l_n + n - 1)$. The functions in the product are defined by

$$\widetilde{P}_{L,j}^{l}(\theta) = \sqrt{\frac{2L+j-1}{2} \frac{(L+l+j-2)!}{(L-l)!}} \sin^{\frac{2-j}{2}}(\theta) P_{L+\frac{j-2}{2}}^{-(l+\frac{j-2}{2})}(\cos\theta),$$

where, for two real numbers ν and μ, the function $P_\nu^{-\mu}$ is the associated Legendre function of the first kind defined by

$$P_\nu^{-\mu}(x) = \frac{1}{\Gamma(1+\mu)} \left(\frac{1-x}{1+x}\right)^{\mu/2} F\left(-\nu, \nu+1, 1+\mu, \frac{1-x}{2}\right),$$

where Γ is the Euler's Gamma function and F is the hypergeometric function (see e.g. [9]).

[8] Recall that a quantum limit for $-\triangle_g$ is a weak-limit (in the space of Radon measures) of a sequence of measures $(\phi_{j_k}(x)^2\, dx)_{k \in \mathbb{N}^*}$, where ϕ_{j_k} are nonzero eigenfunctions of $-\triangle_g$ with positive eigenvalues λ_{j_k} such that $\lambda_{j_k} \to +\infty$ as $k \to +\infty$.

Corollary 5. *Assume that the spectrum $(\lambda_j)_{j\in\mathbb{N}^*}$ of Ω satisfies the uniform gap property (UG). Then, for any closed geodesic γ of Ω, there exists a quantum measure supported along γ.*

This is exactly the main result of Macià [14] which extends a result of Jakobson and Zelditch [11] on the sphere. As a consequence also noted in [14], under the additional assumption that Ω is a Zoll manifold with maximally degenerate Laplacian, any invariant measure for the geodesic flow on $T^*\Omega$ is a quantum measure.

In the forthcoming paper [10], we prove that if the spectrum satisfies the uniform gap property (UG), then any invariant measure is a quantum limit. In particular, this shows that the assumption of a maximally degenerate Laplacian done in [14] is not necessary.

Proof. A direct adaptation of the proof of Theorem 4 allows to prove the following generalization: under the assumption *(UG)*, for every $N\in\mathbb{N}^*$ and every measurable subset ω of Ω, there holds

$$\lim_{T\to+\infty}\frac{C_T^{>N}(\omega)}{T}=\frac{1}{2}\inf_{\substack{\phi\in\mathcal{E}_N\\ \phi\neq 0}}\frac{\int_\omega|\phi(x)|^2\,dv_g}{\int_\Omega|\phi(x)|^2\,dv_g}.$$

According to Theorem 3, there exists a subsequence $(\phi_{j_k})_{k\in\mathbb{N}^*}$ of $(\phi_j)_{j\in\mathbb{N}^*}$ whose associated eigenvalues tends to $+\infty$ and such that

$$\limsup_{k\to+\infty}\int_\omega|\phi_{j_k}|^2(x)dx\leqslant g_2(\overline{\omega})$$

i.e., for every measurable subset ω of Ω, there exists a quantum mesure μ satisfying $\mu(\mathring{\omega})\leqslant g_2(\overline{\omega})$. Let $\gamma\subset\Omega$ be a periodic geodesic and let $\varepsilon>0$. We apply the last inequality to $\omega_\varepsilon=\Omega\setminus G_\varepsilon$ where $G_\varepsilon=\{x\in\Omega,\ \mathrm{dist}(x,\gamma)\geqslant\varepsilon\}$. Since $g_2(\omega_\varepsilon)=0$, there exists a quantum measure μ_ε such that $\mu_\varepsilon(\mathring{\omega}_\varepsilon)=0$. We obtain the expected conclusion by noting that $(\omega_\varepsilon)_{\varepsilon>0}$ shrinks to γ as $\varepsilon\searrow 0$ and that the family $(\mu_\varepsilon)_{\varepsilon>0}$ converges, up to a subsequence, in the sense of measures to some quantum limit μ. □

5 Concluding remarks and perspectives

We provide here a list of open problems and issues that we will investigate next.

Manifolds with boundary. The introduction of the so-called *highfrequency observability constant* $\alpha^T(\omega)$ is of interest because it allows to characterize the positiveness of $C_T(\omega)$ in terms of the quantity $\alpha^T(\omega)$. The result stated in Corollary 1 is devoted to this equivalence. It still holds true in the case of a manifold with boundary. Nevertheless, one has to overcome technical difficulties to get a characterization of $\alpha^T(\omega)$, in other words an equivalent statement to the ones of Theorem 3 and Corollary 3, in such a frame.

Schrödinger equation. Little is known on internal observability of the Schrödinger equation. For instance it is known that GCC implies internal observability, but this sufficient condition is far from being sharp. Internal observability has also been established for particular geometries. An issue is to apply the methods developed in this paper to provide sufficient conditions for observability writing in terms of a new geometric quantity.

Shape optimization. A challenging problem is to maximize the functional $\chi_\omega \mapsto C_T(\omega)$ over the set $\mathcal{U}_L = \{\chi_\omega \in L^\infty(\Omega, \{0,1\}), \ |\omega| = L|\Omega|\}$, for some fixed $L \in (0,1)$. A first issue is to investigate the existence of maximizers. We believe that the methods developed in this article will allow to prove non-existence, as well as to capture the behavior of maximizing sequences of domain, at least for particular choices of Ω.

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