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Abstract

Recent concepts of quasi-static MHD turbulence between plane walls are presented. It is shown that the dimensionality of this type of flow is governed by the ratio of the diffusion length associated to the Lorentz force to the channel width. Depending on turbulence dimensionality, three different dissipation mechanisms are activated that correspond to three different scalings for the intensity of turbulent fluctuations. In all three regimes, the relative turbulent intensity is found to *increase* with the applied magnetic field, in apparent contrast to commonly accepted ideas.

Keywords: MHD, turbulence dimensionality, two-dimensional turbulence, turbulence promoters

1 Introduction

Turbulence under an externally imposed, static magnetic field has a well-known tendency to two-dimensionality [5]. This effect is driven by the diffusive nature of the Lorentz force at low magnetic Reynolds number [9] which smoothes out velocity gradients along the magnetic field. The final dimensionality of the flow, is however strongly determined by the nature of the boundaries of the fluid domain. In the generic case of a channel perpendicular to the field and bounded by two walls, Hartmann boundary layers that develop along the walls preclude a fully quasi-two dimensional state, so the flow can be at best two-dimensional in its bulk, or *quasi-two-dimensional* [9, 6]. With non dissipative boundaries (periodic or slip-free boundary conditions), the transition between strictly two-dimensional and three-dimensional states is mainly governed by the stability properties of large two-dimensional structures [3]. When walls are present, by contrast, the flow states span a continuous spectrum of states involving different types of three-dimensional effects (presence of transversal velocity, weak velocity gradients preserving topological equivalent between planes perpendicular to the magnetic field, full three-dimensionality) [4, 7, 2].
Considering a generic channel configuration, we show that these states of dimensionality are solely determined by the ratio l_z/h of the diffusion length associated to the Lorentz force to the channel width. We show that three different states can be distinguished, each of which characterised by a how the intensity of turbulent fluctuations scales with the externally applied forcing. These scalings imply that in the process of making the flow quasi-two dimensional, the effect of the magnetic field is to lower Joule dissipation to the point where it actually increases the intensity of turbulence. These theoretical concepts are verified experimentally on the FLOWCUBE experimental platform where turbulence is driven in a cubic container by injecting electric current at one of the Hartmann walls of the vessel. In FLOWCUBE, the intensity of the forcing is measured directly by the quantity of current that is injected in the flow.

2 Dimensionality and turbulence intensity

Consider a horizontal channel of width h, filled with liquid metal (density ρ, viscosity ν, electric conductivity σ) and pervaded by a homogeneous static magnetic field $B\mathbf{e}_z$. Both the tendency to two-dimensionality and the sources of three-dimensionality of MHD flows can be seen from the curl of the Navier-Stokes equation and the cur l of Ohm's law within the quasi-static approximation, which read:

$$\partial_z J_z = -\nabla_\perp \cdot \mathbf{J}_\perp = \frac{\rho}{B}(\mathbf{u}\cdot\nabla\omega_z + \omega\cdot\nabla u_z) - \frac{\rho\nu}{B}\nabla^2\omega_z, \quad (1)$$

$$\nabla\times\mathbf{J} = \sigma B\partial_z\mathbf{u}. \quad (2)$$

These equations express that horizontal layers of fluid can only be fed by electric current if either inertia or viscous friction exist there to balance the Lorentz force. Furthermore, any electric current "leak" pulled into the core by either of these mechanisms results in the presence of velocity gradients along the magnetic field, and hence in three-dimensionality. Conversely, quasi-two-dimensionality is only possible if none of these forces acts, and $\partial_z\mathbf{u} = 0$ in the core. Velocity gradients would then still exist in the

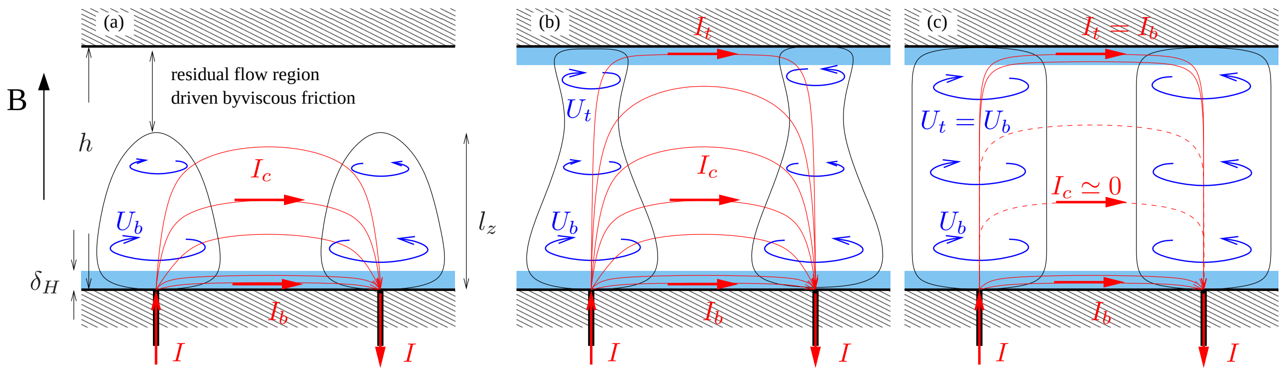

Figure 1: Schematic representation of generic flow configurations in a channel in an external magnetic field. For illustration purposes, the flow is driven electrically by injecting electric current at the bottom wall. Hartmann layers are represented in light blue, paths of electric current in red, and fluid flow in blue. (a) $l_z << h$ (b) $l_z \gtrsim h$ (c) $l_z >> h$.

Hartmann layers because of viscous friction, and accordingly, the entire horizontally divergent electric current flows there. The amount of current available to flow into horizontal planes is determined by the intensity of the forcing driving turbulence (or equivalently, by the total current directly injected into the flow, if it is electrically driven, as on figure 1). The height l_z of the region where the the forcing current is consumed by this mechanism determines the distance over which the Lorentz force is able to diffuse momentum. Depending on whether in a structure of size $l_\perp$ and typical velocity $U(l_\perp)$, the mechanism is predominantly viscous or inertial, l_z respectively scales as:

$$l_z^\nu \sim \frac{l_\perp^2}{h} Ha \qquad \text{or} \qquad l_z^{(N)} \sim l_\perp N^{1/2}, \tag{3}$$

where $Ha = Bh(\rho/\sigma\nu)^{1/2}$ and $N = \sigma B^2 l_\perp/(\rho U)$ are the Hartmann number and interaction parameter. The scaling for $l_z^{(N)}$ was first proposed by [9] and experimentally verified by [7]. The ratio of the diffusion length l_z to the height of the channel determines the dimensionality of the flow, leading to three different cases illustrated on figure 1.

If $l_z \ll h$, momentum diffusion by the Lorentz force does not reach out to both boundaries of the channel, electric eddy currents spread over $l_z \ll h$ and the flow is three-dimensional, with regions near one or both channel walls (depending on the geometry of the forcing) where only a weak, residual flow exists.

If $l_z \sim h$, structures extend across the whole channel but three-dimensionality persists in the bulk.

If $l_z \gg h$, the flow is quasi-two dimensional and electric current flows almost exclusively in the Hartmann boundary layers.

It is important to notice that $l_z^{(N)}$ is scale-dependent and so strictly speaking, for a turbulent flow to be quasi-two dimensional, all scales have to satisfy $l_z(l_\perp) \gg h$. Hence, an intermediate state exists where large scales are quasi-two-dimensional while smaller scales are three-dimensional [9, 4]. The three different regimes of flow dimensionality correspond to different electric current paths and therefore different levels of Joule dissipation. We shall now characterise them through the relation between forcing and the measured Reynolds number. For this, we start by noticing that the total current induced by the forcing, or directly injected into the flow, I spreads between the bulk (I_c), and through each of the Hartmann layer (I_b and I_t), so that $I = I_c + I_b + I_t$. Since I_c diverges into the core over the height of the structure, $I_c \sim 2\pi l_\perp \max\{l_z, h\} J_\perp^c$. The horizontal current density $J_\perp^c$ is then estimated from (1) using only either the first or the second term in the RHS of the equation, depending on whether viscous effects or inertia dominate, respectively. Using $Re_b = U_b l_\perp/\nu$ and $Re_0 = (I/2\pi\nu\sqrt{\rho\sigma})$ to express these scalings non dimensionally (indices b and t refer to bottom and top plates), it comes that if $l_z \gg 1$ then the flow is quasi-two-dimensional[8, 1] and

$$Re_b \sim Re_0. \tag{4}$$

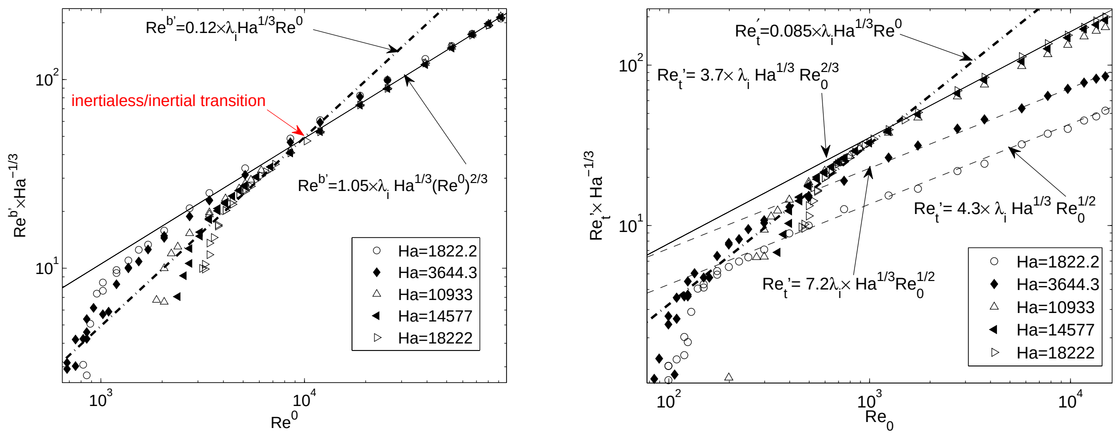

Figure 2: $Re'_b Ha^{-1/3}$ and $Re'_t Ha^{-1/3}$ *vs.* Re^0, representing the RMS of velocity fluctuations near the bottom wall. Solid, dashed-dotted and dashed lines respectively represent scaling laws (4), (5) and (6).

If $l_z \ll 1$ or $l_z \sim 1$, the core current is pulled in by inertia [7], and

$$Re_b \sim Re_0^{3/2}. \tag{5}$$

Furthermore, (4) remains valid for any value of l_z/h if the current in the core is pulled by viscous effects. Regimes where this scaling holds shall therefore be called *inertialess*.

While scalings near the bottom wall give a measure of turbulent intensity where it is forced, scalings near the top wall give a measure of its intensity away from where the forcing is applied. It turns out that if $l_z \sim h$ or $l_z \gg h$, then scalings for Re_t are essentially the same as for Re_b, albeit for a small correcting factor [7]. If $l_z \ll h$, on the other hand, the top wall is outside of the region where turbulence diffuses under the action of the Lorentz force. Any residual flow there is viscously entrained by the neighbouring turbulence, and damped by the Lorentz force. This balance provides a scaling of the form

$$Re_t \sim Re_0^{1/2}, \tag{6}$$

which characterises the residual flow in this region. All three scalings are observed experimentally to a great precision, over a wide range of values of Re_0 and Ha (see figure 2). For the purpose of the experiment, Re_b and Re_t were built on the RMS of velocity fluctuations and half of the scale at which energy was injected into the flow (materialised by the spacing between current injection electrodes L_i). For low forcing, the inertialess regime dominates both near the wall where turbulence is forced and the wall where it isn't. For high magnetic fields (hence $l_z^{(N)}(l_\perp)/h \gg 1$), a transition takes places to the inertial regime where (5) holds in both regions. At lower fields, where $l_z^{(N)}(l_\perp)/h < 1$, this transition is only visible where turbulence is forced. The large scales of turbulence do not reach the top wall and (6) becomes valid in this region.

3 Can magnetic fields enhance turbulence ?

An important feature of the measurements of turbulent intensity on figure 2 is that the prefactor in all scalings laws linking Re_b to Re_0 depends on Ha. This wasn't foreseen in theory and must be attributed to the fact that while the injected current I induces an average flow, we diagnosed turbulent fluctuations, which only receive a fraction of the energy of the average flow. It appears that the ratio of the RMS of velocity fluctuations to the average velocity not only depends on Ha but monotonically increases with

it. In other words, the relative turbulent intensity *increases* with the magnetic field, in apparent contradiction to the widely accepted idea that the magnetic field should suppress turbulence !
The paradox can be resolved by considering the energy balance integrated across the channel and observing that most of the power injected by the forcing $\mathcal{P}$ is dissipated ohmically. Using the expression of the Lorentz force put forward by [9], ohmic dissipation integrated over the channel is expressed as:

$$\int_0^h \epsilon_J dz = -\sigma B^2 \langle \int_0^h \mathbf{u} \cdot \Delta^{-1} \partial_{zz} \mathbf{u} \rangle dz \sim -\sigma B^2 l_\perp^2 h \left[\langle \mathbf{u} \rangle \cdot \partial_{zz}^2 \langle \mathbf{u} \rangle + \langle \mathbf{u}' \cdot \partial_{zz}^2 \mathbf{u}' \rangle \right],$$

where brackets and prime respectively refer to average and fluctuating quantities. Approximating the derivatives as $\partial_{zz}^2 \langle \mathbf{u} \rangle \sim \langle U_b \rangle^2 / l_z^2$ and $\partial_{zz}^2 \mathbf{u}' \sim \langle U_b' \rangle / l_z'^2$, and using (3) with reference velocities $\langle U_b \rangle$ and $\langle U_b'^2 \rangle^{1/2}$ to evaluate respectively l_z and l_z', it comes that:

$$\int_0^h \epsilon_J dz \sim -\rho \frac{h}{l_\perp} \langle U_b \rangle^3 \left(1 + \alpha^3\right), \tag{7}$$

where $\alpha = \langle U_b'^2 \rangle^{1/2} / \langle U_b \rangle$ is the relative intensity of turbulent fluctuations. The total power driving the flow across the channel is determined by the injected current as:

$$\int_0^h \mathcal{P}_I dz \sim \frac{IB}{2\pi l_\perp} \langle U_b \rangle, \tag{8}$$

and by virtue of (5), equating (8) and (7) for $Ha >> 1$, leads to an estimate for α:

$$\alpha \sim \lambda Ha^{1/3} Re_b^{-1/6}. \tag{9}$$

Experimental data support the scaling $Ha^{1/3}$ and confirm that fluctuations become more intense as the field is increased. It must be noted that this scaling applies to fully developed turbulence, for which $\alpha \gtrsim 1$. The underlying mechanism is that while the magnetic field does suppress turbulence, it does so by elongating structures. In the process, velocity gradients are reduced, and eddy currents responsible for the ohmic dissipation are suppressed. The resulting structures are closer quasi-two-dimensionality and dissipate much less energy than their three-dimensional counterpart. As a result, turbulence retains more kinetic energy. In this sense, the application of a static magnetic field can indeed enhance turbulence.

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