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Abstract

We consider a quantum particle in a potential $V(x)$ ($x \in \mathbb{R}^N$) in a time-dependent electric field $E(t)$ (the control). Boscain, Caponigro, Chambrion and Sigalotti proved in [2] that, under generic assumptions on V, this system is approximately controllable on the $L^2(\mathbb{R}^N, \mathbb{C})$-sphere, in sufficiently large time T. In the present article we show that approximate controllability does not hold in arbitrarily small time, no matter what the initial state is. This generalizes our previous result for Gaussian initial conditions. Moreover, we prove that the minimal time can in fact be arbitrarily large.

Keywords: Schrödinger equation, quantum control, minimal time.

1. Introduction and main result

In this article, we consider quantum systems described by the linear Schrödinger equation

$$\begin{cases} i\partial_t \psi(t,x) = \left(-\frac{1}{2}\Delta + V(x) - \langle E(t), x\rangle\right)\psi(t,x), & (t,x) \in (0,T)\times\mathbb{R}^N, \\ \psi(0,x) = \psi_0(x), & x \in \mathbb{R}^N. \end{cases} \tag{1}$$

Here, $N \in \mathbb{N}^*$ is the space dimension, $\langle .,. \rangle$ is the euclidian scalar product on $\mathbb{R}^N$, $V : x \in \mathbb{R}^N \to \mathbb{R}$, $E : t \in (0,T) \to \mathbb{R}^N$ and $\psi : (t,x) \in (0,T)\times\mathbb{R}^N \to \mathbb{C}$ are a static potential, a time-dependent electric field, and the wave function, respectively. This equation represents a quantum particle "trapped" by the potential V and under the influence of the electric field E. Planck's constant and the particle mass have been set to one (the dependence on the physical constants is discussed briefly in Section 3.4).

System (1) is a control system in which the control is the electric field E and the state is the wave function ψ, which belongs to the unit sphere $\mathbb{S}$ of $L^2(\mathbb{R}^N, \mathbb{C})$. The expression "bilinear control" refers to the bilinear nature of the term $\langle E(t), x\rangle\psi$ with respect to (E, ψ).

We are interested in the minimal time required to achieve approximate controllability of the system (1). Since in (1) decoherence is neglected, in realistic scenarios the model may only be applicable for small times t (typically on the order of several periods of the ground state). Since, to be practically relevant, controllability results need to be valid for time intervals in which equation (1) remains a reasonable model, quantification of the minimal control time is an important issue.

We consider potentials V that are smooth and subquadratic, i.e.

$$V \in C^\infty(\mathbb{R}^N) \text{ and, } \forall \alpha \in \mathbb{N}^N \text{ such that } |\alpha| \geqslant 2,\ \partial_x^\alpha V \in L^\infty(\mathbb{R}^N)\,. \tag{2}$$

For this class of potentials there is a classical well-posedness result [8], which we quote from [6].

Proposition 1. *Consider V satisfying assumption (2) and $E \in L^\infty_{loc}(\mathbb{R}, \mathbb{R}^N)$. There exists a strongly continuous map $(t, s) \in \mathbb{R}^2 \mapsto U(t, s)$, with values in the set of unitary operators on $L^2(\mathbb{R}^N, \mathbb{C})$, such that*

$$U(t,t) = Id\,, \quad U(t,\tau)U(\tau,s) = U(t,s)\,, \quad U(t,s)^* = U(s,t)^{-1}\,, \quad \forall t,\tau,s \in \mathbb{R}$$

and for every $t, s \in \mathbb{R}$, $\varphi \in L^2(\mathbb{R}^N, \mathbb{C})$, the function $\psi(t, x) := U(t, s)\varphi(x)$ is a weak solution in $C^0([0,T], L^2(\mathbb{R}^N, \mathbb{C}))$ of the first equation of (1) with initial condition $\psi(s, x) = \varphi(x)$.

For V satisfying (2), we introduce the operator

$$\begin{aligned} D(A_V) &:= \left\{\varphi \in L^2(\mathbb{R}^N); -\tfrac{1}{2}\Delta\varphi + V(x)\varphi \in L^2(\mathbb{R}^N)\right\}, \\ A_V\varphi &:= -\tfrac{1}{2}\Delta\varphi + V(x)\varphi\,. \end{aligned}$$

For appropriate potentials V, approximate controllability of (1) in $\mathbb{S}$ (possibly in large time) is a corollary of a general result by Boscain, Caponigro, Chambrion, Mason and Sigalotti (the original proof of [7] is generalized in [2]; inequality (4) below is proved in [7, Proposition 4.6]; an analogous statement for vector valued controls is given in [3, Theorem 2.6]; see also [4] for a survey of results in this area).

Theorem 1. *Let $m \in \{1, ..., N\}$ and assume that*

- *there exists a Hilbert basis $(\phi_k)_{k\in\mathbb{N}}$ of $L^2(\mathbb{R}^N, \mathbb{C})$ composed of eigenvectors of A_V: $A_V\phi_k = \lambda_k\phi_k$ and $x_m\phi_k \in L^2(\mathbb{R}^N)$, $\forall k \in \mathbb{N}$,*
- *$\int_{\mathbb{R}^N} x_m\phi_j(x)\phi_k(x)dx = 0$ for every $j, k \in \mathbb{N}$ such that $\lambda_j = \lambda_k$ and $j \neq k$,*
- *for every $j, k \in \mathbb{N}$, there exists a finite number of integers $p_1, ..., p_r \in \mathbb{N}$ such that*

$$\begin{aligned} &p_1 = j, \quad p_r = k, \quad \textstyle\int_{\mathbb{R}^N} x_m\phi_{p_l}(x)\phi_{p_{l+1}}(x)dx \neq 0, \forall l = 1, ..., r-1\,, \\ &|\lambda_L - \lambda_M| \neq |\lambda_{p_l} - \lambda_{p_{l+1}}|, \forall 1 \leqslant l \leqslant r-1, L, M \in \mathbb{N} \\ &\ \textit{with } \{L, M\} \neq \{p_l, p_{l+1}\}. \end{aligned}$$

Then, for every $\epsilon > 0$ and $\psi_0, \psi_f \in \mathbb{S}$, there exist a time $T > 0$ and a piecewise constant function $u : [0, T] \to \mathbb{R}$ such that the solution of (1) with $E(t) = u(t)e_m$ satisfies

$$\|\psi(T) - \psi_f\|_{L^2(\mathbb{R}^N)} < \epsilon\,. \tag{3}$$

Moreover, for every $\delta > 0$, the existence of a piecewise constant function $u : [0, T] \to (-\delta, \delta)$ such that the solution of (1) with $E(t) = u(t)e_m$ satisfies (3) *implies that*

$$T \geqslant \frac{1}{\delta} \sup_{k\in\mathbb{N}} \frac{|\,|\langle\phi_k, \psi_0\rangle| - |\langle\phi_k, \psi_f\rangle|\,| - \epsilon}{\|x_m\phi_k\|_{L^2(\mathbb{R}^N)}}. \tag{4}$$

In Theorem 1, the time T is not known a priori and may be large. Note that the lower bound on the control time in (4) goes to zero when $\delta \to +\infty$. As a result, Theorem 1 gives no information about the control time if the controls are allowed to be arbitrarily large; in particular, it does not preclude the possibility of approximate controllability *in arbitrarily small time*. In our previous work [1], we proved that, for potentials V satisfying (2), and for particular (Gaussian) initial conditions, approximate controllability does not hold in arbitrarily small time – even with large controls. Specifically, we proved the following result.

Theorem 2. *Assume that V satisfies assumption (2). Let $b > 0$, $x_0, \dot{x}_0 \in \mathbb{R}^N$ and $\psi_0 \in \mathbb{S}$ be defined by*

$$\psi_0(x) := \frac{b^{N/4}}{C_N} e^{-\frac{b}{2}\|x-x_0\|^2 + i\langle \dot{x}_0, x-x_0\rangle} \text{ where } C_N := \left(\int_{\mathbb{R}^N} e^{-\|y\|^2} dy \right)^{1/2} .$$

Moreover, let $\psi_f \in \mathbb{S}$ be a state that does not have a Gaussian profile in the sense that

$$|\psi_f(.)| \neq \frac{\det(S)^{1/4}}{C_N} e^{-\frac{1}{2}\|\sqrt{S}(.-\gamma)\|^2}, \quad \forall \gamma \in \mathbb{R}^N, S \in \mathcal{M}_N(\mathbb{R}) \text{ symmetric positive.}$$

Then there exist positive numbers $T^{**} = T^{**}(\|V''\|_\infty, \|V^{(3)}\|_\infty, b, \psi_f)$ and $\delta = \delta(\|V''\|_\infty, b, \psi_f)$ such that, for every $E \in C^0_{pw}([0, T^{**}], \mathbb{R}^N)$ (piecewise continuous functions $[0, T^{**}] \to \mathbb{R}^N$), the solution ψ of (1) satisfies

$$\|\psi(t) - \psi_f\|_{L^2(\mathbb{R}^N)} > \delta\,, \quad \forall t \in [0, T^{**}]\,.$$

The goal of the present article is to generalize this result to arbitrary initial conditions ψ_0, and to demonstrate that the minimal control time can in fact become arbitrarily large. Specifically, we prove the following

Theorem 3. *Assume that V satisfies assumption (2) and let $\psi_0 \in H^1(\mathbb{R}^N) \cap L^2(\|x\|dx) \cap \mathbb{S}$.*

1. *There exists $\psi_f \in \mathbb{S}$, $T^{**} > 0$ and $\delta > 0$ such that, for every $E \in L^\infty((0, T^{**}), \mathbb{R}^N)$, the solution ψ of (1) satisfies*

$$\|\psi(t) - \psi_f\|_{L^2(\mathbb{R}^N)} > \delta\,, \quad \forall t \in [0, T^{**}]\,.$$

2. *Moreover, if V is of the form*

$$V(x) = W(\epsilon x)\,, \quad \forall x \in \mathbb{R}^N\,, \tag{5}$$

then $T^{**} \geqslant \frac{C}{\epsilon}$ for every $\epsilon \in (0, 1)$, for some positive constant $C = C(\psi_0, \psi_f, W)$.

Remark 1. *If V satisfies (2) and the assumptions of Theorem 1 (which hold generically, this fact may be proved as in [10]), then system (1) is approximately controllable in $\mathbb{S}$ in large time but not in small time $T < T^{**}$. A characterization of the minimal time required for ϵ-approximate controllability is an open problem.*

Remark 2. *In part 2 of the theorem the demonstration that the minimal control time can become infinitely large is accomplished by a particular choice (rescaling) of the potential. In a forthcoming paper we will investigate the conditions on V under which this can also be accomplished by a suitable choice of initial and/or target states. (See also Section 3.4)*

The remainder of this paper is devoted to the proof of Theorem 3. The next section contains some notation and auxiliary results, whereas Section 3 contains the proof proper. There is also a brief discussion of the dependence on Planck's constant (Section 3.4) and an appendix containing the proof of a functional-analytic lemma needed in the argument. We refer to our previous article [1] for bibliographical comments.

2. Notation and auxiliary results

Denote by $\mathcal{M}_N(\mathbb{R})$ the set of $N \times N$ matrices with coefficients in $\mathbb{R}$, $GL_N(\mathbb{R})$ the group of its invertible matrices and I_N its identity element; $\mathrm{Tr}(M)$ the trace and M^* the transposition of a matrix $M \in \mathcal{M}_N(\mathbb{R})$; $\mathcal{S}_N(\mathbb{R})$ the set of symmetric matrices in $\mathcal{M}_N(\mathbb{R})$; $\|.\|$ the Euclidean norm on $\mathbb{R}^N$ and the associated operator norm on $\mathcal{M}_N(\mathbb{R})$; $\dot{x}(t) := \frac{dx}{dt}(t)$, $\ddot{x}(t) := \frac{d^2x}{dt^2}(t)$, for a function x of the scalar variable t and $D_y^2\chi$ the Hessian matrix of a function $\chi : \mathbb{R}^N \to \mathbb{C}$, $D_y^2\chi(x) = \left(\frac{\partial^2\chi}{\partial y_i \partial y_j}\right)_{1\leqslant i,j\leqslant N}$.

The goal of this section is to prove the following result, which will be used in the proof of Theorem 3.

Proposition 2. *Let $T, L, R > 0$,*

$$B := \left\{\sigma \in \mathcal{M}_N(\mathbb{R}); \|\sigma - I_N\| \leqslant \tfrac{1}{2}\right\}, \tag{6}$$

$$\mathcal{K} := \Big\{M \in C^0\left([0,T], \mathcal{M}_N(\mathbb{R})\right) \text{ } L\text{-Lipschitz}; \|M(t)\| \leqslant R,\ \forall t \in [0,T]\Big\}. \tag{7}$$

For $\phi_0 \in \mathbb{S}$ and $M \in \mathcal{K}$, let the function $\chi^M_{\phi_0} : [0,T] \times \mathbb{R}^N \to \mathbb{C}$ be defined as the unique solution of

$$\begin{cases} i\partial_\tau \chi(\tau,y) + \frac{1}{2} Tr\left[M(\tau)^* M(\tau) D_y^2\chi(\tau,y)\right] = 0, & (\tau,y) \in [0,T] \times \mathbb{R}^N, \\ \chi(0,y) = \phi_0(y), & y \in \mathbb{R}^N\,. \end{cases} \tag{8}$$

1. *For every* $\phi_0 \in \mathbb{S}$*, the set*

$$\mathcal{V}(\phi_0) \;:=\; \Big\{ f \in \mathbb{S}; \exists (\tau, \sigma, M, \alpha) \in [0,T] \times B \times \mathcal{K} \times \mathbb{R}^N \text{ such that } \\ |f(x)| = \tfrac{1}{\sqrt{\det(\sigma)}} \left| \chi^M_{\phi_0}(\tau, \sigma^{-1}[x-\alpha]) \right| \text{ for a.e. } x \in \mathbb{R}^N \Big\}$$

is a strict and closed subset of $\mathbb{S}$ *(w.r.t. the strong* $L^2(\mathbb{R}^N)$*-topology).*

2. *For* $\phi_0, \phi_1 \in \mathbb{S}$ *then* $\mathcal{V}(\phi_1) \subset \mathcal{V}(\phi_0) + B_{L^2(\mathbb{R}^N)}\left(0, \|\phi_0 - \phi_1\|_{L^2(\mathbb{R}^N)}\right)$.

Remark 3. *It is clear that the unique solution* $\chi^M_{\phi_0} \in C^1([0,T], L^2(\mathbb{R}^N))$ *of (8) satisfies* $\|\chi^M_{\phi_0}(\tau,.)\|_{L^2(\mathbb{R}^N)} \equiv 1$ *and is given by*

$$\widehat{\chi^M_{\phi_0}}(\tau, \eta) = \widehat{\phi}_0(\eta) e^{-\frac{i}{2}\int_0^\tau \|M(s)\eta\|^2 ds}, \qquad \text{for a.e. } \eta \in \mathbb{R}^N, \forall \tau \in [0,T], \tag{9}$$

where the hat denotes the Fourier transform, defined by

$$\hat{f}(\eta) = \int_{\mathbb{R}^N} e^{-i\langle y, \eta\rangle} f(y) dy, \quad \forall f \in L^1(\mathbb{R}^N).$$

The proof of Proposition 2 will use the following facts, proved in the appendix.

Lemma 1. *Let* $(f_n)_{n\in\mathbb{N}} \in L^2(\mathbb{R}^N)^{\mathbb{N}}$ *that converges to a function* f *in* $L^2(\mathbb{R}^N)$.

1. *If* $(\alpha_n)_{n\in\mathbb{N}} \in \left(\mathbb{R}^N\right)^{\mathbb{N}}$ *satisfies* $\|\alpha_n\| \underset{n\to\infty}{\longrightarrow} +\infty$*, then* $\tau_{\alpha_n} f_n \underset{n\to\infty}{\longrightarrow} 0$ *in* $\mathcal{D}'(\mathbb{R}^N)$.
2. *If* $(\alpha_n)_{n\in\mathbb{N}}$ *converges to* α *in* $\mathbb{R}^N$ *then* $\tau_{\alpha_n} f_n \underset{n\to\infty}{\longrightarrow} \tau_\alpha f$ *in* $L^2(\mathbb{R}^N)$.
3. *If* $(M_n)_{n\in\mathbb{N}}$ *converges toward* M *in* $GL_N(\mathbb{R})$ *then* $f_n \circ M_n \underset{n\to\infty}{\longrightarrow} f \circ M$ *in* $L^2(\mathbb{R}^N)$.

Proof of Proposition 2:

Step 1: Let $(f_n)_{n\in\mathbb{N}} \in \mathcal{V}(\phi_0)^{\mathbb{N}}$ **and** $(\tau_n, \sigma_n, M_n, \alpha_n)_{n\in\mathbb{N}}$ **associated parameters in** $[0,T] \times B \times \mathcal{K} \times \mathbb{R}^N$**. We prove that a subsequence of** $(\tau_{-\alpha_n}|f_n|)_{n\in\mathbb{N}}$ **converges in** $\mathbb{S}$ (in the strong $L^2(\mathbb{R}^N)$-topology). By Ascoli's theorem, there exists $(\tau_\infty, \sigma_\infty, M_\infty)$ in $[0,T] \times B \times \mathcal{K}$ such that, up to extracting a subsequence, $(\tau_n, \sigma_n) \underset{n\to\infty}{\longrightarrow} (\tau_\infty, \sigma_\infty)$ in $[0,T] \times B$ and $M_n(\tau) \underset{n\to\infty}{\longrightarrow} M_\infty(\tau)$ uniformly with respect to $\tau \in [0,T]$. Let

$$k_n(y) := \frac{1}{\sqrt{\det(\sigma_n)}} \chi^{M_n}_{\phi_0}(\tau_n, y), \qquad \text{for a.e. } y \in \mathbb{R}^N, \forall n \in \mathbb{N} \cup \{\infty\}.$$

By (9), we have

$$\widehat{k_n}(\eta) = \frac{1}{\sqrt{\det(\sigma_n)}} \widehat{\phi}_0(\eta) e^{-\frac{i}{2}\int_0^{\tau_n} \|M_n(s)\eta\|^2 ds} \quad \text{for a.e } \eta \in \mathbb{R}^N, \forall n \in \mathbb{N} \cup \{\infty\},$$

and by the dominated convergence theorem, $\widehat{k_n} \underset{n\to\infty}{\longrightarrow} \widehat{k_\infty}$ in $L^2(\mathbb{R}^N)$. Thus, Plancherel's theorem shows that $k_n \underset{n\to\infty}{\longrightarrow} k_\infty$ in $L^2(\mathbb{R}^N)$, which gives $|k_n| \underset{n\to\infty}{\longrightarrow} |k_\infty|$ in $L^2(\mathbb{R}^N)$, and finally $\tau_{-\alpha_n}|f_n| = |k_n| \circ \sigma_n^{-1} \underset{n\to\infty}{\longrightarrow} |k_\infty| \circ \sigma_\infty^{-1}$ in $L^2(\mathbb{R}^N)$, by Lemma 1.3.

Step 2: We prove that $\mathcal{V}(\phi_0)$ is a strict subset of $\mathbb{S}$. Working by contradiction, we assume that $\mathbb{S} = \mathcal{V}(\phi_0)$ and consider the sequence $(f_n)_{n\in\mathbb{N}} \subset \mathbb{S}$, defined by $f_n(x) := \sqrt{n}\theta(nx)$ where $\theta(x) := \pi^{-\frac{N}{4}} e^{-\frac{\|x\|^2}{2}}$. By Step 1, there exist a subsequence $(n_k)_{k\in\mathbb{N}}$, a sequence $(\alpha_k)_{k\in\mathbb{N}}$ of $\mathbb{R}^N$ and $h \in \mathbb{S}$ such that $\tau_{\alpha_k} f_{n_k} \underset{n\to\infty}{\longrightarrow} h$ in $L^2(\mathbb{R}^N)$, and thus in $\mathcal{D}'(\mathbb{R}^N)$. However, for every $\varphi \in C_c^\infty(\mathbb{R}^N)$, we have

$$\begin{aligned} \left|\int_{\mathbb{R}^N} \tau_{\alpha_k} f_{n_k}(x)\varphi(x)dx\right| &= \left|\int_{\mathbb{R}^N} \sqrt{n_k}\theta(n_k y)\varphi(y+\alpha_k)dy\right| \\ &\leqslant \frac{1}{\sqrt{n_k}} \|\theta\|_{L^1(\mathbb{R}^N)} \|\varphi\|_{L^\infty(\mathbb{R}^N)}, \end{aligned}$$

thus $\tau_{\alpha_k} f_{n_k} \underset{n\to\infty}{\longrightarrow} 0$ in $\mathcal{D}'(\mathbb{R}^N)$. Therefore $h = 0$, which is impossible, since $h \in \mathbb{S}$.

Step 3: We prove that $\mathcal{V}(\phi_0)$ is closed in $\mathbb{S}$. Let $(f_n)_{n\in\mathbb{N}} \in \mathcal{V}(\phi_0)^{\mathbb{N}}$ and $f \in \mathbb{S}$ be such that $f_n \underset{n\to\infty}{\longrightarrow} f$ in $L^2(\mathbb{R}^N)$. We use the same notation as in Step 1.

Step 3.1: We prove that $(\alpha_n)_{n\in\mathbb{N}}$ is bounded in $\mathbb{R}^N$. Working by contradiction, we may assume w.l.o.g. that $\|\alpha_n\| \underset{n\to\infty}{\longrightarrow} \infty$. Since $|f_n| \underset{n\to\infty}{\longrightarrow} |f|$ in $L^2(\mathbb{R}^N)$, Lemma 1.1 implies $\tau_{-\alpha_n}|f_n| \underset{n\to\infty}{\longrightarrow} 0$ in $\mathcal{D}'(\mathbb{R}^N)$, which contradicts Step 1.

Step 3.2: We prove that $f \in \mathcal{V}(\phi_0)$. Since $(\alpha_n)_{n\in\mathbb{N}}$ is bounded, some subsequence converges to some $\alpha_\infty \in \mathbb{R}^N$; w.l.o.g. $\alpha_n \underset{n\to\infty}{\longrightarrow} \alpha_\infty$. From Step 1, we know that, up to potentially extracting a subsequence, $\tau_{-\alpha_n}|f_n| = |k_n| \circ \sigma_n^{-1} \underset{n\to\infty}{\longrightarrow} |k_\infty| \circ \sigma_\infty^{-1}$ in $L^2(\mathbb{R}^N)$. Therefore, by Lemma 1.2, $|f_n| =$

$\tau_{\alpha_n}[\tau_{-\alpha_n}|f_n|] \underset{n\to\infty}{\longrightarrow} \tau_{\alpha_\infty}[|k_\infty| \circ \sigma_\infty^{-1}]$ in $L^2(\mathbb{R}^N)$. By uniqueness of the limit, $|f| = \tau_{\alpha_\infty}[|k_\infty| \circ \sigma_\infty^{-1}]$, i.e. $f \in \mathcal{V}(\phi_0)$.

Step 4: We prove that $\mathcal{V}(\phi_1) \subset \mathcal{V}(\phi_0) + B_{L^2(\mathbb{R}^N)}\left(0, \|\phi_0 - \phi_1\|_{L^2(\mathbb{R}^N)}\right)$. Let $f_1 \in \mathcal{V}(\phi_1)$. Then, there exists $(\tau, \sigma, M, \alpha) \in [0,T] \times B \times \mathcal{K} \times \mathbb{R}^N$ and a measurable function $\theta : \mathbb{R}^N \to \mathbb{R}$ such that

$$f_1(x) = \frac{e^{i\theta(x)}}{\sqrt{\det(\sigma)}} \chi^M_{\phi_1}\left(\tau, \sigma^{-1}[x-\alpha]\right) \quad \text{for a.e. } x \in \mathbb{R}^N \,.$$

Let

$$f_0(x) := \frac{e^{i\theta(x)}}{\sqrt{\det(\sigma)}} \chi^M_{\phi_0}\left(\tau, \sigma^{-1}[x-\alpha]\right) \quad \text{for a.e. } x \in \mathbb{R}^N \,.$$

Then, $f_0 \in \mathcal{V}(\phi_0)$ and, by (9) and Plancherel's theorem,

$$\begin{aligned}
\int_{\mathbb{R}^N} |(f_1 - f_0)(x)|^2 dx &= \int_{\mathbb{R}^N} |\chi^M_{\phi_1}(\tau, y) - \chi^M_{\phi_0}(\tau, y)|^2 dy \\
&= (2\pi)^{-N} \int_{\mathbb{R}^N} |\widehat{\chi^M_{\phi_1}}(\tau, \eta) - \widehat{\chi^M_{\phi_0}}(\tau, \eta)|^2 d\eta \\
&= (2\pi)^{-N} \int_{\mathbb{R}^N} |(\widehat{\phi_1} - \widehat{\phi_0})(\eta)|^2 d\eta \\
&= \int_{\mathbb{R}^N} |(\phi_1 - \phi_0)(x)|^2 dx \,,
\end{aligned}$$

which gives the conclusion. □

3. Proof of Theorem 3

In the whole section, the following quantities are kept fixed.

- V satisfying (2),
- $\psi_0 \in \mathbb{S} \cap H^1(\mathbb{R}^N) \cap L^2(\|x\| dx)$,
- $x_0 \,, \dot{x}_0 \in \mathbb{R}^N$ defined by

$$x_0 := \int_{\mathbb{R}^N} x |\psi_0(x)| dx \,, \quad \dot{x}_0 := -i \int_{\mathbb{R}^N} \nabla_x \psi_0(x) \overline{\psi_0(x)} dx$$

- $\phi_0 \in \mathbb{S}$ defined by

$$\phi_0(x) := \psi_0(x + x_0) e^{-i\langle \dot{x}_0, x\rangle} \,.$$

Our strategy to prove Theorem 3 relies on approximate solutions, which are centred at the classical (Newtonian) trajectories. Accordingly, these approximate solutions $\widetilde{\psi}^E$ (defined in eq. (20) below) depend on the classical trajectories $x_c^E : \mathbb{R} \to \mathbb{R}^N$ and certain functions $Q^E, \sigma^E : \mathbb{R} \to \mathcal{M}_N(\mathbb{C})$, which satisfy the ODEs (10) below. The remainder of this section is organized as follows. In Section 3.1, we prove a preliminary result for the solutions of ODEs (10). In Section 3.2, we introduce the explicit approximate solution $\widetilde{\psi}^E$ and prove that the error $\|\psi^E - \widetilde{\psi}^E\|_{L^\infty((0,T),L^2(\mathbb{R}^N))}$ can be bounded uniformly with respect to $E \in L^\infty_{loc}(\mathbb{R}, \mathbb{R}^N)$. Finally, Section 3.3 contains the proof of Theorem 3.

3.1. ODEs for x_c^E, Q^E and σ^E

For $E \in L^\infty_{loc}(\mathbb{R}, \mathbb{R}^N)$, let $x_c^E \in C^1(\mathbb{R}, \mathbb{R}^N)$, $Q^E, \sigma^E \in C^1((T^E_{min}, T^E_{max}), \mathcal{M}_N(\mathbb{R}))$ and $\tau^E \in C^1((T^E_{min}, T^E_{max}), \mathbb{R})$ be the maximal solutions of

$$\begin{cases} \frac{d^2 x_c^E}{dt^2}(t) + \nabla V[x_c^E(t)] = E(t), \\ x_c^E(0) = x_0, \\ \frac{dx_c^E}{dt}(0) = \dot{x}_0, \end{cases} \qquad \begin{cases} \frac{dQ^E}{dt}(t) + Q^E(t)^2 + V''[x_c^E(t)] = 0, \\ Q^E(0) = 0, \end{cases}$$
$$\begin{cases} \frac{d\sigma^E}{dt}(t) = Q^E(t)\sigma^E(t), \\ \sigma^E(0) = I_N, \end{cases} \qquad \begin{cases} \frac{d\tau^E}{dt}(t) = \frac{1}{\det[\sigma^E(t)]^2}, \\ \tau^E(0) = 0, \end{cases} \tag{10}$$

where ∇V and V'' denote the gradient and Hessian matrix of V, respectively. Note that

- $x_c^E(t)$ is defined for every $t \in \mathbb{R}$ because ∇V is globally Lipschitz by assumption (2);
- x_c^E is twice derivable almost everywhere and satisfies the first equality of (10) for almost every $t \in \mathbb{R}$;
- $Q^E(t) \in S_N(\mathbb{R})$ and $\sigma^E(t) \in GL_N(\mathbb{R})$ for every $t \in (T^E_{min}, T^E_{max})$.

A priori, the maximal interval (T^E_{min}, T^E_{max}) may depend on E.

Proposition 3. 1. *There exists $T^* = T^*(\|V''\|_\infty) > 0$ such that, for every $E \in L^\infty_{loc}(\mathbb{R}, \mathbb{R}^N)$,*

$$T^E_{max} > T^*, \tag{11}$$

$$\|Q^E(t)\| \leqslant \frac{1}{2}\,, \quad \forall t \in [0, T^*]\,, \tag{12}$$

$$\|\sigma^E(t) - I_N\| \leqslant \frac{1}{2}\,, \quad \forall t \in [0, T^*]\,, \tag{13}$$

$$|\tau^E(t) - t| \leqslant \frac{t}{2}\,, \quad \forall t \in [0, T^*]\,. \tag{14}$$

2. *Moreover, if V is of the form (5), then $T^* \geqslant \frac{C}{\epsilon}$ for every $\epsilon \in (0,1)$, for some positive constant $C = C(W)$.*

Proof of Proposition 3: Fix $\delta \in \left(0, \frac{1}{2}\right)$ and choose $T^* = T^*(\|V''\|_\infty) > 0$ such that

$$T^*(\delta^2 + \|V''\|_\infty) < \delta, \quad 2NT^*\delta \leqslant \frac{1}{4}, \quad e^{\delta T^*} - 1 \leqslant \frac{1}{2} \tag{15}$$

(the third inequality actually follows from the second). Let $E \in L^\infty_{loc}(\mathbb{R}, \mathbb{R}^N)$.

Step 1: We prove (11) and (12). Let

$$T^E_\sharp := \sup\left\{t \in [0, T^E_{max}); \|Q^E(s)\| \leqslant \delta\,, \forall s \in [0,t]\right\}\,.$$

Working by contradiction, we assume that $T^E_\sharp < T^*$. Then,

$$\begin{aligned} \delta = \|Q^E(T^E_\sharp)\| &= \left\| \int_0^{T^E_\sharp} \Big(Q^E(s)^2 + V''[x^E_c(s)]\Big) ds \right\| \\ &\leqslant T^E_\sharp\Big(\delta^2 + \|V''\|_\infty\Big) < \delta \quad \text{ by (15)}, \end{aligned}$$

which is impossible. Therefore, $T^E_\sharp \geqslant T^*$ and

$$\|Q^E(t)\| \leqslant \delta \quad \text{for every } t \in [0, T^*], \tag{16}$$

which proves (11) and (12).

Step 2: We prove (13). We have

$$\|\sigma^E(t)\| = \left\| I_N + \int_0^t Q^E(s)\sigma^E(s)ds \right\| \overset{(16)}{\leqslant} 1 + \int_0^t \delta\|\sigma^E(s)\|ds\,, \quad \forall t \in [0, T^*]\,,$$

thus, by Gronwall Lemma, $\|\sigma^E(t)\| \leqslant e^{\delta t}$ for every $t \in [0,T^*]$ and

$$\|\sigma^E(t) - I_N\| = \left\| \int_0^t Q^E(s)\sigma^E(s)ds \right\| \overset{(16)}{\leqslant} \int_0^t \delta e^{\delta s} \leqslant e^{\delta T^*} - 1\,, \quad \forall t \in [0,T^*]\,,$$

which, together with (15) implies (13).

Step 3: We prove (14). By Liouville's formula, we have

$$\frac{1}{\det[\sigma^E(t)]^2} = \exp\left(-2\int_0^t \mathrm{Tr}[Q^E(s)]ds \right)\,, \quad \forall t \in [0,T^*]\,. \tag{17}$$

Moreover,

$$\left| 2\int_0^t \mathrm{Tr}[Q^E(s)]ds \right| \leqslant 2\int_0^t N\|Q^E(s)\|ds \leqslant 2NT^*\delta \leqslant \frac{1}{4}, \tag{18}$$

by (16) and (15). Thus, by (17), and (18),

$$\left| \frac{1}{\det[\sigma^E(t)]^2} - 1 \right| \leqslant \frac{1}{4}e^{\frac{1}{4}} < \frac{1}{2}\,, \quad \forall t \in [0,T^*]$$

and so

$$|\tau^E(t) - t| = \left| \int_0^t \left(\frac{1}{\det[\sigma^E(s)]^2} - 1 \right) ds \right| \leqslant \frac{t}{2}, \quad \forall t \in [0,T^*].$$

Step 4: We prove Statement 2. For $\epsilon \in (0,\delta)$, the argument of Step 1 works with δ replaced by ϵ and then $T^* = \frac{1}{\epsilon[1+\|W''\|_\infty]}$ for ϵ small enough. $\square$

Proposition 3 implies that, for every $E \in L^\infty_{loc}(\mathbb{R},\mathbb{R}^N)$, the function τ^E is C^1 and increasing, i.e. a bijection from $[0,T^*]$ to $[0,\tau^{E*}]$, where $\tau^{E*} := \tau^E(T^*) \in \left[\frac{T^*}{2}, \frac{3T^*}{2}\right]$. Denoting the inverse function by $t^E : [0,\tau^{E*}] \to [0,T^*]$, we can now define the C^1 map

$$\left| \begin{array}{llll} M^E: & [0,\tau^{E*}] & \to & SL_N(\mathbb{R}) \\ & \tau & \mapsto & \det[(\sigma^E \circ t^E)(\tau)] \left((\sigma^E \circ t^E)(\tau) \right)^{-1*}\,. \end{array} \right. \tag{19}$$

Thanks to Proposition 3, M^E has the following properties.

Proposition 4. *There exists $R, L > 0$ such that, for every $E \in L^\infty_{loc}(\mathbb{R},\mathbb{R}^N)$,*

$$\|M^E(\tau)\| \leqslant R \quad \textit{and} \quad \left\| \frac{dM^E}{d\tau}(\tau) \right\| \leqslant L\,, \quad \forall \tau \in [0,\tau^{E*}]\,.$$

3.2. Approximate solution

Let $S^E, \Phi^E : (t,x) \in [0, T^E_{max}) \times \mathbb{R}^N \to \mathbb{R}$ be defined by

$$\begin{array}{rcl} S^E(t,x) & := & \int_0^t \left(\frac{1}{2}\|\dot{x}^E_c(s)\|^2 - V[x^E_c(s)] + \langle x^E_c(s), E(s)\rangle \right) ds + \langle \dot{x}^E_c(t), x\rangle \\ \Phi^E(t,x) & := & S^E(t, x - x_c(t)) + \frac{1}{2}\langle Q^E(t)[x - x^E_c(t)], x - x^E_c(t)\rangle \end{array}$$

and let $\chi^E := \chi^{M^E}_{\phi_0}$ (see (19) and (8)). Then we define an approximate solution to (1) by

$$\widetilde{\psi}^E(t,x) := \frac{e^{i\Phi^E(t,x)}}{\sqrt{\det[\sigma^E(t)]}} \chi^E \left(\tau^E(t), \sigma^E(t)^{-1}[x - x^E_c(t)] \right) , \qquad (20)$$

for every $(t,x) \in [0, T^E_{max}) \times \mathbb{R}^N$. Note that $\widetilde{\psi}^E(t,.) \in \mathbb{S}$ for every $t \in [0, T^E_{max})$ because (see Remark 3)

$$\begin{array}{rl} \|\widetilde{\psi}^E(t)\|^2_{L^2(\mathbb{R}^N)} & = \int_{\mathbb{R}^N} \left| \chi^E \left(\tau^E(t), \sigma^E(t)^{-1}[x - x^E_c(t)] \right) \right|^2 \frac{dx}{\det[\sigma^E(t)]} \\ & = \int_{\mathbb{R}^N} \left| \chi^E \left(\tau^E(t), y \right) \right|^2 dy = 1 \,. \end{array}$$

Remark 4. *For background information on the approximate solutions $\widetilde{\psi}^E$, see the literature cited in [1]. Their derivation may roughly be described as proceeding in two steps: one first applies a well-known transformation (see e.g. [9]) to remove the control term; then the Schrödinger equation (arising by Taylor expansion) with the time-dependent quadratic potential $\langle V''[x_c(t)]x, x\rangle$ is solved explicitly (up to solutions of* (8)*). The second step is related to the (generalized) Mehler formula for time-dependent quadratic Hamiltonians; see e.g. Section 3 of [6].*

Proposition 5. *If $\phi_0 \in \mathcal{S}(\mathbb{R}^N)$, then there exists a constant $C(\phi_0, T^*) > 0$ such that, for every $E \in L^\infty_{loc}(\mathbb{R}, \mathbb{R}^N)$, the solution ψ^E of (1) and the function $\widetilde{\psi}^E$ defined by (20) satisfy*

$$\|(\psi^E - \widetilde{\psi}^E)(t)\|_{L^2(\mathbb{R}^N)} \leqslant C(\phi_0, T^*)\|V^{(3)}\|_\infty t \,, \quad \forall t \in [0, T^*] \,,$$

where T^* is defined in Proposition 3.

Proof of Proposition 5: For simplicity, we write Q, σ, M, τ, χ, S, ψ, $\widetilde{\psi}$ for Q^E, σ^E, M^E, τ^E, $\chi^E_{\phi_0}$, S^E, ψ^E and $\widetilde{\psi}^E$.

Step 1: Equation satisfied by $\widetilde{\psi}$. For a function $g(x) := f(Ax)$ we have $\nabla_x g(x) = A^* \nabla_y f(Ax)$ and $\Delta_x g(x) = \mathrm{Tr}[AA^* D_y^2 f(Ax)]$. Thus, for every $(t,x) \in [0, T_{max}^E) \times \mathbb{R}^N$,

$$\begin{aligned}\nabla_x \widetilde{\psi}(t,x) = \ & i\Big(\dot{x}_c(t) + Q(t)[x - x_c(t)]\Big)\widetilde{\psi}(t,x) \\ & + \sigma(t)^{-1*} \nabla_y \chi\left(\tau(t), \sigma(t)^{-1}[x - x_c(t)]\right) \tfrac{e^{i\Phi(t,x)}}{\sqrt{\det[\sigma(t)]}}\,,\end{aligned}$$

and

$$\begin{aligned}&\Delta_x \widetilde{\psi}(t,x) = \Big(i\mathrm{Tr}(Q) - \|\dot{x}_c(t) + Q(t)[x - x_c(t)]\|^2\Big)\widetilde{\psi}(t,x) \\ &+ 2i\left\langle \dot{x}_c(t) + Q(t)[x - x_c(t)], \sigma(t)^{-1*}\nabla_y \chi\left(\tau(t), \sigma(t)^{-1}[x - x_c(t)]\right)\right\rangle \frac{e^{i\Phi(t,x)}}{\sqrt{\det[\sigma(t)]}} \\ &+ \mathrm{Tr}\Big[\sigma(t)^{-1}\sigma(t)^{-1*} D_y^2 \chi\left(\tau(t), \sigma(t)^{-1}[x - x_c(t)]\right)\Big] \frac{e^{i\Phi(t,x)}}{\sqrt{\det[\sigma(t)]}}\,.\end{aligned}$$

Developing the square in the first line and using (19) gives, for every $(t,x) \in [0, T_{max}^E) \times \mathbb{R}^N$,

$$\begin{aligned}&\frac{1}{2}\Delta_x \widetilde{\psi}(t,x) = \\ &\Big(\frac{i}{2}\mathrm{Tr}[Q(t)] - \frac{1}{2}\|\dot{x}_c(t)\|^2 - \frac{1}{2}\|Q(t)[x - x_c(t)]\|^2 - \langle \dot{x}_c(t), Q(t)[x - x_c(t)]\rangle\Big)\widetilde{\psi}(t,x) \\ &+ i\left\langle \sigma(t)^{-1}\left(\dot{x}_c(t) + Q(t)[x - x_c(t)]\right), \nabla_y \chi\left(\tau(t), \sigma(t)^{-1}[x - x_c(t)]\right)\right\rangle \frac{e^{i\Phi(t,x)}}{\sqrt{\det[\sigma(t)]}} \\ &+ \frac{1}{2}\mathrm{Tr}\Big[M[\tau(t)]^* M[\tau(t)] D_y^2 \chi\left(\tau(t), \sigma(t)^{-1}[x - x_c(t)]\right)\Big] \frac{e^{i\Phi(t,x)}}{\det[\sigma(t)]^{5/2}}\,.\end{aligned} \tag{21}$$

Moreover, using the relations $\dot{\tau}(t) = \frac{1}{\det[\sigma(t)]^2}$ and

$$\frac{d}{dt}\Big[\det[\sigma(t)]\Big] = \det[\sigma(t)]\mathrm{Tr}[\sigma(t)\dot{\sigma}(t)] = \det[\sigma(t)]\mathrm{Tr}[Q(t)]\,,$$

$$\frac{d}{dt}\left[\frac{1}{\sqrt{\det[\sigma(t)]}}\right] = -\frac{1}{2\det[\sigma(t)]^{3/2}}\frac{d}{dt}\Big[\det[\sigma(t)]\Big] = -\frac{\mathrm{Tr}[Q(t)]}{2\sqrt{\det[\sigma(t)]}}\,,$$

$$\begin{aligned}\tfrac{d}{dt}\left[\sigma(t)^{-1}[x - x_c(t)]\right] &= -\sigma(t)^{-1}\dot{\sigma}(t)\sigma(t)^{-1}[x - x_c(t)] - \sigma(t)^{-1}\dot{x}_c(t) \\ &= -\sigma(t)^{-1}\Big[Q(t)[x - x_c(t)] + \dot{x}_c(t)\Big]\,,\end{aligned}$$

that hold for every $t \in [0, T^E_{max})$, we obtain, for every $x \in \mathbb{R}^N$ and almost every $t \in (0, T^E_{max})$,

$$
\begin{aligned}
&i\partial_t \widetilde{\psi}(t,x) = \\
&\Big(-\frac{1}{2}\|\dot{x}_c(t)\|^2 + V[x_c(t)] - \langle x_c(t), E(t)\rangle - \langle \ddot{x}_c(t), x - x_c(t)\rangle + \|\dot{x}_c(t)\|^2 \\
&-\frac{1}{2}\langle \dot{Q}(t)[x - x_c(t)], x - x_c(t)\rangle + \langle Q(t)[x - x_c(t)], \dot{x}_c(t)\rangle \\
&-\frac{i}{2}\mathrm{Tr}[Q(t)]\Big)\widetilde{\psi}(t,x) \\
&+\frac{i}{\det[\sigma(t)]^{5/2}}\partial_\tau \chi\left(\tau(t), \sigma(t)^{-1}[x - x_c(t)]\right) e^{i\Phi(t,x)} \\
&-i\left\langle \nabla_y \chi\left(\tau(t), \sigma(t)^{-1}[x - x_c(t)]\right), \sigma(t)^{-1}\left(Q(t)[x - x_c(t)] + \dot{x}_c(t)\right)\right\rangle \frac{e^{i\Phi(t,x)}}{\sqrt{\det[\sigma(t)]}}\,.
\end{aligned}
$$

And finally, by (10), for every $x \in \mathbb{R}^N$ and almost every $t \in (0, T^E_{max})$,

$$
\begin{aligned}
&i\partial_t \widetilde{\psi}(t,x) = \\
&\Big(\frac{1}{2}\|\dot{x}_c(t)\|^2 + V[x_c(t)] - \langle x_c(t), E(t)\rangle - \langle \nabla V[x_c(t)], x - x_c(t)\rangle \\
&-\langle E(t), x - x_c(t)\rangle + \frac{1}{2}\|Q(t)[x - x_c(t)]\| + \frac{1}{2}\langle V''[x_c(t)][x - x_c(t)], x - x_c(t)\rangle \\
&+\langle Q(t)[x - x_c(t)], \dot{x}_c(t)\rangle - \frac{i}{2}\mathrm{Tr}[Q(t)]\Big)\widetilde{\psi}(t,x) \\
&+i\partial_\tau \chi\left(\tau(t), \sigma(t)^{-1}[x - x_c(t)]\right) \frac{e^{i\Phi(t,x)}}{\det[\sigma(t)]^{5/2}} \\
&-i\left\langle \nabla_y \chi\left(\tau(t), \sigma(t)^{-1}[x - x_c(t)]\right), \sigma(t)^{-1}\left(Q(t)[x - x_c(t)] + \dot{x}_c(t)\right)\right\rangle \frac{e^{i\Phi(t,x)}}{\sqrt{\det[\sigma(t)]}}\,.
\end{aligned}
\tag{22}
$$

Combining (21), (22) and (8) gives for every $x \in \mathbb{R}^N$ and almost every $t \in (0, T^E_{max})$,

$$
i\partial_t \widetilde{\psi}(t,x) + \frac{1}{2}\Delta\widetilde{\psi}(t,x) - V(x)\widetilde{\psi}(t,x) + \langle E(t), x\rangle\widetilde{\psi}(t,x) = R(t,x)\widetilde{\psi}(t,x) \tag{23}
$$

where

$$
\begin{aligned}
R(t,x) := \;& -V(x) + V[x_c(t)] + \langle \nabla V[x_c(t)], x - x_c(t)\rangle \\
&+\tfrac{1}{2}\langle V''[x_c(t)][x - x_c(t)], x - x_c(t)\rangle\,.
\end{aligned}
\tag{24}
$$

Step 2: Conclusion. Using Taylor's formula, we get

$$
|R(t,x)| \leqslant \frac{\|V^{(3)}\|_\infty}{3!}\|x - x_c(t)\|^3\,, \quad \forall (t,x) \in [0, T^E_{max}) \times \mathbb{R}^N\,.
$$

Thus, for every $t \in (0, T^*)$,

$$
\begin{aligned}
& \|R(t)\widetilde{\psi}(t)\|^2_{L^2(\mathbb{R}^N)} \\
\leqslant\ & \left(\frac{\|V^{(3)}\|_\infty}{3!}\right)^2 \int_{\mathbb{R}^N} \|x - x_c(t)\|^6 \left|\chi\left(\tau(t), \sigma(t)^{-1}[x - x_c(t)]\right)\right|^2 \frac{dx}{\det[\sigma(t)]} \\
=\ & \left(\frac{\|V^{(3)}\|_\infty}{3!}\right)^2 \int_{\mathbb{R}^N} \|\sigma(t)y\|^6 \left|\chi\left(\tau(t), y\right)\right|^2 dy \\
\leqslant\ & \left(\frac{\|V^{(3)}\|_\infty}{3!}\right)^2 \left(\frac{3}{2}\right)^6 \int_{\mathbb{R}^N} \|y\|^6 \left|\chi\left(\tau(t), y\right)\right|^2 dy \text{ by (13)} \\
\leqslant\ & C\|V^{(3)}\|^2_\infty \int_{\mathbb{R}^N} \left|D^3_\eta \hat{\chi}\left(\tau(t), \eta\right)\right|^2 d\eta \\
\leqslant\ & C\|V^{(3)}\|^2_\infty \int_{\mathbb{R}^N} \left|D^3_\eta \left[\hat{\phi}_0(\eta) e^{-\frac{i}{2}\int_0^{\tau(t)} \|M(s)\eta\|^2 ds}\right]\right|^2 d\eta \text{ by (9)}
\end{aligned}
$$

for some positive constant C that does not depend on E, V and ϕ_0. We deduce from Leibniz formula, (14) and Proposition 4 that

$$
\|R(t)\widetilde{\psi}(t)\|^2_{L^2(\mathbb{R}^N)} \leqslant C(\phi_0, T^*)^2 \|V^{(3)}\|^2_\infty, \quad \forall t \in [0, T^*]
$$

for some positive constant $C(\phi_0, T^*) > 0$ that is finite because $\phi_0 \in \mathcal{S}(\mathbb{R}^N)$. Note that $C(\phi_0, T^*)^2$ is a polynomial function of degree 6 of T^*, which will become relevant in Section 3.4. Let $U(t, s)$ be the evolution operator for equation (1) (see Proposition 1). Then,

$$
(\psi - \widetilde{\psi})(t) = \int_0^t U(t, s)[R(s)\widetilde{\psi}(s)] ds \text{ in } L^2(\mathbb{R}^N), \forall t \in (0, T^*),
$$

and $U(t, s)$ is an isometry of $L^2(\mathbb{R}^N)$ for every $t \geqslant s \geqslant 0$, thus

$$
\|(\psi - \widetilde{\psi})(t)\|_{L^2(\mathbb{R}^N)} \leqslant \int_0^t \|R(s)\widetilde{\psi}(s)\|_{L^2(\mathbb{R}^N)} ds \leqslant C(\phi_0, T^*)\|V^{(3)}\|_\infty t, \quad \forall t \in [0, T^*].
$$

□

3.3. Conclusion

Let T^* be as in Proposition 3, $R, L > 0$ be as in Proposition 4 and $T := \frac{3T^*}{2}$. By Proposition 2, there exists $\psi_f \in \mathbb{S} \backslash \mathcal{V}(\phi_0)$ and $\delta_0 := \text{dist}_{L^2(\mathbb{R}^N)}(\psi_f, \mathcal{V}(\phi_0))$ is positive.

Step 1: We prove the existence of $\psi_1 \in \mathbb{S} \cap \mathcal{S}(\mathbb{R}^N)$ such that

$$
\|\psi_0 - \psi_1\|_{L^2(\mathbb{R}^N)} < \frac{\delta_0}{4}, \quad \int_{\mathbb{R}^N} x|\psi_1(x)| dx = x_0, \quad -i \int_{\mathbb{R}^N} \nabla_x \psi_1(x) \overline{\psi_1(x)} dx = \dot{x}_0.
$$

There exists a sequence $(\xi_\epsilon)_{\epsilon\in(0,1)}$ in $\mathbb{S}\cap\mathcal{S}(\mathbb{R}^N)$ such that

$$\|\psi_0-\xi_\epsilon\|_{H^1(\mathbb{R}^N)}+\|\psi_0-\xi_\epsilon\|_{L^2(\|x\|dx)}\underset{\epsilon\to 0}{\longrightarrow}0\,.$$

Then

$$x_\epsilon:=\int_{\mathbb{R}^N}x|\xi_\epsilon(x)|dx\,,\quad \dot{x}_\epsilon:=-i\int_{\mathbb{R}^N}\nabla_x\xi_\epsilon(x)\overline{\xi_\epsilon(x)}dx$$

converge, respectively, to x_0 and $\dot{x}_0$. Thus, the sequence of functions

$$x\mapsto \xi_\epsilon(x-x_0+x_\epsilon)e^{i\langle \dot{x}_0-\dot{x}_\epsilon,x\rangle}$$

converges to ψ_0 in $L^2(\mathbb{R}^N)$ and gives the conclusion.

Step 2: Distance between the approximate solutions associated to ψ_0 and ψ_1. Step 1 implies that

- for every $E\in L^\infty_{loc}(\mathbb{R},\mathbb{R}^N)$, the quantities T^*, x_c^E, Q^E, σ^E, τ^E, t^E, S^E associated with ψ_0 and ψ_1 are the same;
- the map ϕ_1 defined by the formula

$$\phi_1(x):=\psi_1(x+x_0)e^{-i\langle \dot{x}_0,x\rangle}\qquad\text{for a.e. } x\in\mathbb{R}^N\,,$$

  satisfies

$$\|\phi_1-\phi_0\|_{L^2(\mathbb{R}^N)}=\|\psi_1-\psi_0\|_{L^2(\mathbb{R}^N)}<\frac{\delta_0}{4};$$

- for every $M\in C^0([0,T],\mathcal{M}_N(\mathbb{R}))$, the functions $\chi^M_{\phi_0}$ and $\chi^M_{\phi_1}$ (see (8)) satisfy

$$\|\chi^M_{\phi_1}(\tau)-\chi^M_{\phi_0}(\tau)\|_{L^2(\mathbb{R}^N)}=\|\phi_1-\phi_0\|_{L^2(\mathbb{R}^N)}<\frac{\delta_0}{4}\,,\quad\forall\tau\in[0,T];$$

- for every $E\in L^\infty_{loc}(\mathbb{R},\mathbb{R}^N)$, the approximate solution $\widetilde{\psi}^E$ (resp. $\widetilde{\psi}_1^E$) defined by (20) (resp. defined by (20) with ϕ_0 replaced by ϕ_1) satisfy

$$\|\widetilde{\psi}_1^E(t)-\widetilde{\psi}^E(t)\|_{L^2(\mathbb{R}^N)}=\|\chi^{M^E}_{\phi_1}(\tau(t))-\chi^{M^E}_{\phi_0}(\tau(t))\|_{L^2(\mathbb{R}^N)}<\frac{\delta_0}{4}\,,\quad\forall t\in[0,T^*].$$

Step 3: Conclusion of Statement 1. By Proposition 2 (part 2) we have that

$$\|\psi_f - \widetilde{\psi}_1^E(t)\|_{L^2(\mathbb{R}^N)} \geqslant \frac{3\delta_0}{4}\,, \quad \forall t \in [0, T^*]\,. \tag{25}$$

Let $C(\phi_1, T^*)$ be as in Proposition 5 and

$$T^{**} := \min\left\{T^*; \frac{\delta_0}{4C(\phi_1, T^*)\|V^{(3)}\|_\infty}\right\}\,. \tag{26}$$

Proposition 5 and (25) imply that, for every $t \in [0, T^{**}]$,

$$\begin{aligned}
& \|\psi_f - \psi^E(t)\|_{L^2(\mathbb{R}^N)} \\
\geqslant\ & \Big|\|\psi_f - \widetilde{\psi}_1^E(t)\|_{L^2(\mathbb{R}^N)} - \|\widetilde{\psi}_1^E(t) - \psi_1^E(t)\|_{L^2(\mathbb{R}^N)} - \|\psi_1^E(t) - \psi^E(t)\|_{L^2(\mathbb{R}^N)}\Big| \\
\geqslant\ & \tfrac{3\delta_0}{4} - C(\phi_1, T^*)\|V^{(3)}\|_\infty t - \|\psi_1 - \psi_0\|_{L^2(\mathbb{R}^N)} \\
\geqslant\ & \tfrac{\delta_0}{4} > 0\,.
\end{aligned}$$

Step 4: Proof of Statement 2. If $V(x) = W(\epsilon x)$, we obtain

$$T^{**} := \min\left\{\frac{C(W)}{\epsilon}; \frac{\delta_0}{4C(\phi_1, T^*)\epsilon^3\|W^{(3)}\|_\infty}\right\}\,,$$

which behaves like $\frac{C}{\epsilon}$ as $\epsilon \to 0$. □

3.4. Dependence on Planck's constant

The Schrödinger equation for a quantum particle in a (static) potential V_0, which is subjected to a time-dependent (and spatially homogeneous) electric field E_0, may, after appropriate rescaling, be written in dimension–less form:

$$\begin{cases} i\varepsilon\partial_\tau\Psi(\tau, y) = \Big(-\dfrac{\varepsilon^2}{2}\Delta_y + V_0(y) - \langle E_0(\tau), y\rangle\Big)\Psi(\tau, y)\,, & (\tau, y) \in (0, \Theta) \times \mathbb{R}^N\,, \\ \Psi(0, y) = \Psi_0(y) & y \in \mathbb{R}^N. \end{cases} \tag{27}$$

Here the parameter ε is proportional to the Planck constant $\hbar$; so it is natural to assume that $\varepsilon \ll 1$. Obviously, equation (1) arises from equation (27) by the change of variables

$$\begin{gathered} t = \tfrac{\tau}{\varepsilon}, \quad x = \tfrac{y}{\varepsilon}, \quad t \in [0, T] \Leftrightarrow \tau \in [0, \varepsilon T], \quad x, y \in \mathbb{R}^N \\ \psi(t, x) = \varepsilon^{N/2}\Psi(\varepsilon t, \varepsilon x), \quad V(x) = V_0\left(\varepsilon x\right)\,, \quad E(t) = \varepsilon E_0(\varepsilon t) \end{gathered}$$

(the factor $\varepsilon^{N/2}$ could be omitted; it ensures that $\|\Psi\|_{L^2} = \|\psi\|_{L^2} = 1$). Therefore, the lower bound T^{**} of the minimal time for approximate controllability of system (1) provides a lower bound $\Theta^{**}(\varepsilon)$ of the minimal time for system (27). By the change of variables, we have $\Theta^{**}(\varepsilon) = \varepsilon T^{**}$, where T^{**}, in general, depends on ε (when fixing V_0 first).

Letting $\delta = \varepsilon$ in (15) shows that T^* may be chosen as $T^* = C(\|V_0''\|_\infty)\varepsilon^{-1}$ (note that $\|V''\|_\infty = \varepsilon^2\|V_0''\|_\infty$). So, by (26),

$$T^{**} = \min\left\{\frac{C(\|V_0''\|_\infty)}{\varepsilon}; \frac{\delta_0}{4C(\phi_1, T^*)\varepsilon^3\|V_0^{(3)}\|_\infty}\right\} \tag{28}$$

This seems to imply $T^{**} \sim \varepsilon^{-1}$ (and hence $\Theta^{**}(\epsilon) \sim$ const.), but this is not correct, since $C(\phi_1, T^*)$ also depends on ε. Indeed, $C(\phi_1, T^*)^2$ is a degree–six polynomial in T^* with certain ϕ_1–dependent coefficients $C_j(\phi_1)$; i.e.,

$$\varepsilon^3 C^*(\phi_1, T^*) = \varepsilon^3\left(\sum_{j=0}^{6} C_j(\phi_1)(T^*)^j\right)^{1/2} \sim \left(\sum_{j=0}^{6} C_j'(\phi_1)\varepsilon^{6-j}\right)^{1/2} \sim C_6(\phi_1)$$

as $\varepsilon \to 0$. So for small $\varepsilon > 0$, T^{**} is independent of ε, which implies that $\Theta^{**}(\varepsilon) \sim \varepsilon \ll 1$.

One may wonder whether the reasoning of the present paper could be refined to obtain stronger estimates on the control time, including, potentially, bounds satisfying $\Theta^{**}(\varepsilon) \to \infty$ as $\varepsilon \to 0$. Two observations suggest that this may be possible in certain cases:

(a) the length of time interval $[0, T^*]$ on which the construction of the approximate solutions $\widetilde{\psi}^E$ is valid behaves like ε^{-1} and is independent of ε for eq. (27). It may therefore be possible to iterate the construction to enlarge the relevant time intervals; (b) the appearance of the quantity $C(\phi_1, T^*) \overset{w.l.o.g.}{\approx} C(\phi_0, T^*)$ in the denominator of the second term of (28) suggests that for certain initial conditions ψ_0 the second term, and hence Θ^{**}, may become large.

This circle of ideas will be the subject of a forthcoming paper.

4. Appendix: proof of Lemma 1

1. We have $\tau_{\alpha_n} f_n = \tau_{\alpha_n} f + \tau_{\alpha_n}(f - f_n)$ where $\tau_{\alpha_n}(f - f_n)$ converges strongly to 0 in $L^2(\mathbb{R}^N)$ and thus in $\mathcal{D}'(\mathbb{R}^N)$. Therefore, it suffices to prove that $\tau_{\alpha_n} f \underset{n\to\infty}{\longrightarrow} 0$ in $\mathcal{D}'(\mathbb{R}^N)$.

Let $\varphi \in C_c^\infty(\mathbb{R}^N) \setminus \{0\}$ and $\epsilon > 0$. There exists $A > 0$ such that

$$\int_{\|y\|>A} |f(y)|^2 dy < \frac{\epsilon}{\|\varphi\|_{L^2(\mathbb{R}^N)}} .$$

Let $R > 0$ be such that $\mathrm{Supp}(\varphi) \subset B(0, R)$. We deduce from $\|\alpha_n\| \underset{n\to\infty}{\longrightarrow} +\infty$ the existence of $n_0 \in \mathbb{N}$ such that $B(-\alpha_n, R) \cap B(0, A) = \emptyset \, , \forall n \geqslant n_0$. Then, for every $n \geqslant n_0$,

$$\begin{aligned} \left| \int_{\mathbb{R}^N} \tau_{\alpha_n} f(x)\varphi(x)dx \right| &= \left| \int_{B(-\alpha_n,R)} f(y)\varphi(y+\alpha_n)dy \right| \\ &\leqslant \|f\|_{L^2(B(-\alpha_n,R))}\|\varphi\|_{L^2(\mathbb{R}^N)} < \epsilon \, .\square \end{aligned}$$

2. See e.g. [5], Lemma 4.3.
3. We may assume that $M = I_N$. We have $f_n \circ M_n = (f_n - f) \circ M_n + f \circ M_n$ where $(f_n - f) \circ M_n$ converges to 0 in $L^2(\mathbb{R}^N)$. Thus, it suffices to prove that $f \circ M_n \underset{n\to\infty}{\longrightarrow} f$ in $L^2(\mathbb{R}^N)$. Let $\epsilon > 0$.

   Case 1: $f \in C_c^0(\mathbb{R}^N)$**.** There exists $R > 0$ such that $\mathrm{Supp}(f) \subset B(0, R)$ and $\mathrm{Supp}(f \circ M_n) = M_n^{-1}\mathrm{Supp}(f) \subset B(0, R)$ for every $n \in \mathbb{N}$. By Heine theorem, there exists $\eta > 0$ such that

   $$|f(y) - f(z)| < \frac{\epsilon}{\sqrt{R^N \mathrm{vol}[B(0, 1)]}} , \quad \forall y, z \in \mathbb{R}^N \text{ such that } \|y - z\| < \eta \, .$$

   We chose n_0 large enough so that $\|M_n - I_N\| < \frac{\eta}{R}$ for every $n > n_0$. Then,

   $$\|M_n x - x\| \leqslant \|M_n - I_N\|\|x\| < \eta \, , \quad \forall x \in B(0, R) \, , n > n_0 \, .$$

   Thus, for $n > n_0$,

   $$\|f \circ M_n - f\|_{L^2(\mathbb{R}^N)} = \left(\int_{B(0,R)} |f[M_n(x)] - f(x)|^2 \, dx \right)^{1/2} < \epsilon \, .$$

   Case 2: $f \in L^2(\mathbb{R}^N)$**.** There exists $\widetilde{f} \in C_c^0(\mathbb{R}^N)$ such that $\|f - \widetilde{f}\|_{L^2(\mathbb{R}^N)} < \frac{\epsilon}{4}$. By Case 1, there exists $n_0 \in \mathbb{N}$ such that $\|\widetilde{f} \circ M_n - \widetilde{f}\|_{L^2(\mathbb{R}^N)} < \frac{\epsilon}{4}$ for every $n > n_0$. One may assume that $\sqrt{\det(M_n)} \geqslant \frac{1}{2}$ for every $n > n_0$. Then, for $n > n_0$,

   $$\begin{aligned} \|f \circ M_n - f\|_{L^2(\mathbb{R}^N)} &\leqslant \|f \circ M_n - \widetilde{f} \circ M_n\|_{L^2(\mathbb{R}^N)} \\ &\quad + \|\widetilde{f} \circ M_n - \widetilde{f}\|_{L^2(\mathbb{R}^N)} + \|\widetilde{f} - f\|_{L^2(\mathbb{R}^N)} \\ &\leqslant \left(\frac{1}{\sqrt{\det(M_n)}} + 1 \right) \|\widetilde{f} - f\|_{L^2(\mathbb{R}^N)} + \frac{\epsilon}{4} < \epsilon \, . \end{aligned}$$

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