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HAMILTONIAN STATIONARY LAGRANGIAN FIBRATIONS

EVELINE LEGENDRE AND YANN ROLLIN

Abstract. Hamiltonian stationary Lagrangian submanifolds (HSLAG) are a natural generalization of special Lagrangian manifolds (SLAG). The latter only make sense on Calabi-Yau manifolds whereas the former are defined for any almost Kähler manifold. Special Lagrangians, and, more specificaly, fibrations by special Lagrangians play an important role in the context of the geometric mirror symmetry conjecture. However, these objects are rather scarce in nature. On the contrary, we show that HSLAG submanifolds, or fibrations, arise quite often. Many examples of HSLAG fibrations are provided by toric Kähler geometry. In this paper, we obtain a large class of examples by deforming the toric metrics into non toric almost Kähler metrics, together with HSLAG submanifolds.

1. Introduction

Let M be a closed smooth manifold endowed with a symplectic form ω and an almost complex structure J. If the bilinear form defined on each tangent space at $m \in M$

$$g_J(v,w) = \omega(v, Jw), \quad \forall v, w \in T_m M$$

is a Riemannian metric, we say that J is a *compatible* almost complex structure on the symplectic manifolds (M,ω). Such a triplet (M,ω,J) is then called an *almost Kähler manifold*. If in addition J is integrable, (M,ω,J) is a *Kähler manifold*. The space of all compatible complex structures on a given symplectic manifold will be denoted $\mathcal{AC}_\omega$.

Let L be a closed manifold and $\ell : L \to M$ a Lagrangian embedding. In other words, $\dim M = 2\dim L$ and $\ell^*\omega = 0$. Hamiltonian transformations $u \in \mathrm{Ham}_\omega$ of (M,ω) act on such Lagrangian maps by composition on the left

$$u \cdot \ell = u \circ \ell.$$

Sometimes we shall use a different, yet equivalent, point of view where Ham_ω leaves $\ell : L \to M$ fixed and acts instead on the space $\mathcal{AC}_\omega$ by

$$u \cdot J = u^* J.$$

Given an almost Kähler manifolds (M,ω,J), a Lagrangian embedding is called *Hamiltonian stationary* if it is a critical point of the volume functional under Hamiltonian deformations. As a short hand, such an embedding will be called a *HSLAG embedding* and its image a *HSLAG submanifold* of (M,ω,J). These Lagrangians were introduced by Oh [9] and may be understood as a natural generalization of special Lagrangian manifolds (SLAG), defined in the case of a Calabi-Yau manifolds.

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More concretely, for $J \in \mathcal{AC}_\omega$ and a Lagrangian embedding $\ell : L \to M$, the volume $\mathrm{vol}(\ell, J) := \mathrm{vol}(\ell(L), g_J)$ is the volume of L endowed with the pull-back metric $\ell^* g_J$. By definition of the the action of Ham_ω, we have

$$\mathrm{vol}(u \cdot \ell, J) = \mathrm{vol}(\ell, u \cdot J),$$

so that finding critical points of the volume functional under the action of Ham_ω on either $\mathcal{AC}_\omega$ or the space of Lagrangian embeddings are equivalent problems.

The group $\mathrm{Diff}(L)$ of diffeomorphisms of L also acts on the space of Lagrangian embeddings on the right by

$$\ell \cdot v = \ell \circ v, \text{ where } v \in \mathrm{Diff}(L).$$

However the volume is invariant under this action so that the problem of finding a stationary Lagrangian embeddind is greatly underdetermined. Sometimes we may think of Lagrangian submanifold rather than embeddings to avoid this infinite dimensional degree of freedom in the equation.

1.1. **The case of a single Lagrangian.** We shall prove the following existence theorem for *rigid* HSLAG embeddings upto small deformations of the compatible almost complex structure:

Theorem A. *Let (M, ω, J_0) be a closed Kähler manifold and L a closed manifold. Let G be the group of Hamiltonian isometries of (M, ω, J_0) and $\ell : L \to M$ be a rigid HSLAG embedding.*

There exists a G-invariant neighborhood W of J_0 in $\mathcal{AC}_\omega$ and a map $\psi : W \to \mathrm{Ham}_\omega$ such that $\psi(J_0) = \mathrm{id}$, with the property that every G-orbit in W contains a representative J, such that the Lagrangian embedding $\psi(J) \cdot \ell : L \to M$ is HSLAG with respect to J.

Remarks 1.1.1. (1) As we shall see, the map ψ is in fact continuous once suitable Hölder topologies are introduced.
(2) The rigidity assumption of Theorem A means that infinitesimal deformation of the given HSLAG can only come from Hamiltonian isometries. This property will be introduced precisely at Definition 2.2.1.
(3) If the group of Hamiltonian isometries G is trivial, the above theorem, in particular the construction of the map ψ, follows from the implicit function theorem. If G is not trivial, the problem becomes obstructed and this is more tricky. In this case the solution J in a given G-orbit comes from the minimization of a finite dimensional problem in a G-orbit.

A large pool of examples is provided by toric Kähler manifolds which typically have a non trivial group G. It turns out that Lagrangian tori of standard toric Kähler manifolds are automatically rigid and stable HSLAG. More precisely, we have the following result:

Theorem B. *Let (M, ω, J_0) be a closed toric Kähler manifold endowed with the Guillemin metric. Let $\mu : M \to P$ be the moment map with image the moment polytope P. Then for any interior point p of P, the Lagrangian torus $L_p = \pi^{-1}(p)$ is HSLAG, rigid and stable in (M, ω, J_0).*

A special case of this result concerns $\mathbb{CP}^n$ endowed with its standard toric action and the Fubini-Study metric, was proved by Ono in [11]. Theorem A applies in this context for each regular fiber. Notice that in the one dimensional case, $\mathbb{CP}^1$

is the round sphere and complex dimension one HSLAG submanifolds are just curves of constant curvature.

Remark 1.1.2. Few constructions of HSLAG submanifolds are known. By variationnal methods, Schoen and Wolfson provided an existence theorem for HSLAG submanifolds with singularities, on 4-dimensional almost Kähler manifolds (cf. [12, 13]). Joyce, Schoen and Lee constructed microscopic HSLAG tori in almost Kähler manifolds (cf. [6, 7]). More recently, Biquard and Rollin constructed HSLAG representative of vanishing cycles in the smoothing of extremal Kähler surfaces with $\mathbb{Q}$-Gorenstein singularities in [2].

We point out that the above constructions all deal with *small* or possibly *singular* Lagrangian submanifolds, whereas the construction proposed in the current paper addresses the case of *large* and *smooth* Lagrangian submanifolds.

1.2. **Toric fibrations.** SLAG fibrations are a conerstone of geometric mirror symmetry but seem to be rather scarce in nature. Our initial motivation for this work was to exhibit various fibrations by HSLAG submanifolds and show that contrarily to the SLAG case, they arise quite often.

A *local Lagrangian toric fibration* of a Kähler manifold (M, ω, J_0) is a smooth family of Lagrangian maps $\ell_t : L \to M$, with parameter $t \in B(0, r) \subset \mathbb{R}^n$, for some $r > 0$, where $n = \dim_{\mathbb{C}} M$ and $L = \mathbb{T}^n$ is a real n-dimensional torus, such that the map $B(0, r) \times L \to M$ defined by $(t, x) \mapsto \ell_t(x)$ is a smooth embedding. If every torus $\ell_t : L \to M$ is HSLAG, we say that the local fibration is HSLAG.

Example 1.2.1. Toric Kähler manifolds provide natural examples of local toric HSLAG fibrations according to Theorem B. The fibration does not extend globally since tori collapse at the boundary of the moment polytope.

Other interesting (singular) HSLAG toric fibrations also occur as part of the SYZ mirror symmetry conjecture. In this case, tori are SLAG hence HSLAG.

Remark 1.2.2. The reader may wonder why we restric to Lagrangian fibrations with toric fibers as above: let (M, ω) be a $2n$-dimensional manifold endowed with a submersion $\pi : M \to B^n$, where B^n is an open ball in $\mathbb{R}^n$, such that the fibers are Lagrangian compact submanifolds of M. An elementary argument shows that in such a situation, the fibers must be n-dimensional manifolds diffeomorphic to the real torus $\mathbb{T}^n$. Indeed, let $x_1, \cdots, x_n$ be the standard coordinate functions on $\mathbb{R}^n$ understood as functions on M. These functions induce Hamiltonian vector fields $Y_1, \cdots, Y_n$ on M. Since the fibers are Lagrangian, the vector fields Y_j must be tangent to the fibers of π. The fact that the functions x_i are invariant along the fibers and that Y_j leave the symplectic form invariant imply that the Lie brackets $[Y_i, Y_j]$ vanish. We have n globally defined linearly independent vector fields Y_j along each Lagrangian fibers. As they are compact, any connected component of the fibers must be diffeomorphic the real torus.

Definition 1.2.3. Given a HSLAG embedding $\ell : L \to M$ into a Kähler manifold (M, J_0, ω), where L and M are closed, let G_ℓ be the subgroup of Hamiltonian isometries G preserving the image of $\ell : L \to M$. In other words $u \in G_\ell$ if and only if $u \circ \ell(L) = \ell(L)$. We denote by G_ℓ^o the identity component of G_ℓ.

Let $\ell_t : L \to M$ be a local Lagrangian toric fibration such that $\ell_0 = \ell$. Such fibration is said to be G_ℓ^o*-invariant* if the action of G_ℓ^o preserves the image of each embedding $\ell_t : L \to M$.

Notice that with the above definition, G^o_ℓ acts trivially on the parameter space $t \in B(0,r)$.

Our next result is an existence theorem for local HSLAG toric fibrations:

Theorem C. *Let (M,ω,J_0) be a closed Kähler manifold and $\ell_t : L \to M$ a $G^o_{\ell_0}$-invariant local toric HSLAG fibration for $t \in B(0,r) \subset \mathbb{R}^n$ such that ℓ_0 is rigid.*

Then for all sufficiently small positive perturbation of J, there exists a Hamiltonian transformation v of M and $\delta \in (0,r)$ such that $\tilde{\ell}_t = v \circ \ell_t$ defines a local HSLAG fibration in (M,ω,J) for $t \in B(0,\delta)$.

The definition of *positive perturbations* of J_0 shall be given at §6.2. Although we need this technical assumption for the proof of Theorem C, we conjecture that every generic almost complex structure is positive. By definition, the set of positive almost complex structures is open, once an appropriate topology is introduced on $\mathcal{AC}_\omega$. In Theorem D and Theorem 6.2.3, we manage to prove that if $\ell_0 : M \to L$ is rigid and stable, then the space of positive almost complex structure form a non-empty open set of $\mathcal{AC}$ with J_0 in its closure. To avoid technical aspects of the statement, we state the result as follows:

Theorem D. *Let (M,ω,J_0) be a Kähler manifold and $\ell : L \to M$ be a rigid and stable HSLAG, where M and L are closed.*

Then, the open set of positive deformations is not empty and has J_0 in its closure. More precisely, there exists a smooth path of almost complex structures $\tilde{J}_s \in \mathcal{AC}_\omega$ defined for $s \in [0,\varepsilon)$ with $\varepsilon > 0$, such that $\tilde{J}_0 = J_0$ and J_s is positive for all $s > 0$.

In particular, Theorem D shows that the statement in Theorem C, with the additional assumption of stability of ℓ_0, is not empty. This applies to the case of toric HSLAG fibrations coming from toric Kähler manifolds given by Theorem B.

Remark 1.2.4. Considering G^o_ℓ-invariant HSLAG toric fibration is not really restrictive. Indeed, we shall prove in Theorem 6.3.2 that given a rigid HSLAG embedding $\ell : L \to M$ into a Kähler manifold (M,ω,J_0), where L is a real torus, it is always possible to find a G^o_ℓ-invariant HSLAG local fibration $\ell_t : L \to M$ such that $\ell_0 = \ell$, under some mild assumptions. In particular, these assumptions apply in the context of toric Kähler manifolds (cf. Proposition 6.3.1), which constitute our main class of examples and applications.

Remark 1.2.5. The Hamiltonian transformation v provided by Theorem C may not be close to the identity as it comes from an auxiliary finite dimensional minimization problem in the spirit of Theorem A. Some evidence of this fact are illustrated with a particular example (cf. Example 6.1.1).

Theorem C applies to the case of $\mathbb{CP}^n$ with its Fubini-Study metric. The result only deals with positive pertubation of the metric, but we expect it to hold under softer genericity assumptions.

More generally, we would expect that the standard singular HSLAG fibration by Lagrangian tori of $\mathbb{CP}^n$ can be globally deformed for a generic choice of Kähler metric close to the Fubini-Study metric on $\mathbb{CP}^n$.

We gather our results in the case of toric manifolds in the following corollary:

Corollary E. *Let (M, ω, J_0) be a closed toric Kähler manifolds endowed with the Guillemin metric and $\mu : M \to P$ the corresponding moment map. Let t_0 be a point in the interior of the moment polytope P. Then*

(1) *for every compatible almost complex structure J sufficiently close to J_0, there exists a Hamiltonian transformation v of (M, ω) such that $v(\mu^{-1}(t_0))$ is a HSLAG submanifold of (M, ω, J).*
(2) *The space of positive deformations J of the almost complex structure of (M, ω, J_0) with respect to $\mu^{-1}(t_0)$ is a non empty open set with J_0 in its closure. Is J is positive and sufficiently close to J_0, there exists a Hamiltonian transformation v with the property that each submanifold $v(\mu^{-1}(t))$ is HSLAG in (M, ω, J) for t sufficiently close to t_0 in P.*

2. Basic theory of HSLAG

In the rest of this paper, M and L are always a closed (i.e. compact without boundary) manifolds, unless specified otherwise. In addition $\dim M = 2 \dim L$, ω is a given symplectic form on M and $\ell : L \to M$ is a Lagrangian embedding, that is an embedding such that $\ell^*\omega = 0$.

2.1. **The Euler-Lagrange equation of HSLAG.** HSLAG manifolds are defined by a variational problem. The corresponding Euler-Lagrange equation is easily recovered as we shall explain now. Let H be the mean curvature vector field along $\ell : L \to M$ defined using the metric g_J. The *Maslov form* is the 1-form $\alpha_H \in \Gamma(\Lambda^1 L)$ defined by

$$\alpha_H = \ell^*\omega(H, \cdot).$$

Let $f_t \in \mathrm{Ham}_\omega$ be a family of Hamiltonian transformations such that $f_0 = \mathrm{id}|_M$. Then

$$V = \frac{d}{dt} f_t \Big|_{t=0}$$

is a Hamiltonian vector field on (M, ω). Let $v : M \to \mathbb{R}$ be a Hamiltonian function for V (i.e. such that $dv = \iota_V \omega$). The family f_t induces a Hamiltonian deformation of the map $\ell : L \to M$ defined by $\ell_t = f_t \circ \ell$. The resctriction of the vector field V along $\ell : L \to M$ is precisely the infinitesimal variation of the family ℓ_t at $t = 0$. The usual first variation formula of the volume gives

$$\begin{aligned}
\frac{d}{dt}\mathrm{vol}(\ell_t, g_J)\Big|_{t=0} &= -\int_L \langle H, V \rangle \mathrm{vol}^{g_L} \\
&= -\int_L \langle \alpha_H, \alpha_V \rangle \mathrm{vol}^{g_L} \\
&= -\int_L \langle \alpha_H, \ell^* dv \rangle \mathrm{vol}^{g_L} \\
&= -\int_L \langle d^*\alpha_H, \ell^* v \rangle \mathrm{vol}^{g_L}
\end{aligned}$$

where $g_L = \ell^* g_J$.

A critical point is obviously given by the equation

$$d^*\alpha_H = 0,$$

where d^* is the operator defined on the space of differential forms on L, using the metric g_L.

The second variation of the volume is obtained by differentiating the t-dependent quantity $d^*\alpha_H$ along the deformation ℓ_t. It turns out that the second variation is of the form

$$\Box \ell^* v$$

where $\Box$ is an elliptic operator of order 4 acting functions on L. More precisely, we have the following formula (cf. [6]), which holds for any Lagrangian embedding (not necessarily stationary) in a Kähler manifold

$$\Box u = \Delta^2 u + d^*\alpha_{\mathrm{Ric}^\perp(J\nabla u)} - 2d^*\alpha_{B(JH,\nabla u)} - JH \cdot JH \cdot u. \tag{2.1}$$

Here Δ is the Riemannian Laplacian on (L, g_L), B is the second fundamental form of $\ell : L \to M$ and $\mathrm{Ric}^\perp$ is defined by $\mathrm{Ric}(x, y) = \langle \mathrm{Ric}^\perp(x), y\rangle$ for x, y normal vectors to L. If the almost complex structure is not integrable, the formula is more complicated, but the leading term Δ^2 remains unchanged.

We introduce the Kernel of this operator

$$\mathcal{K}_L = \ker \Box.$$

Remark 2.1.1. If $\ell : L \to M$ is a minimal submanifold (i.e. $H = 0$), the operator $\Box$ is simpler. In particular, we have

$$\langle \Box u, u\rangle = \int_L |\Delta u|^2 - \mathrm{Ric}(J\nabla u, J\nabla u)\mathrm{vol}^{g_L}$$

and it follows that $\mathcal{K}_L = \mathbb{R}$ for any Kähler manifold of non positive Ricci curvature. These conditions are fullfield in the case of

- *special Lagrangians* in Calabi-Yau manifolds,
- standard Lagrangian tori in the flat torus $\mathbb{C}^2/\Gamma$.

Generally, $\mathcal{K}_L$ is not reduced to $\mathbb{R}$ and $\Box$ need not be selfadjoint. However, we have the following lemma:

Lemma 2.1.2. *Given a Kähler manifold (M, ω, J) and a Lagrangian embedding $\ell : L \to M$, the operator $\Box$ on L can be written*

$$\Box v = Dv - JH \cdot JH \cdot v$$

where D is selfajoint.

In particular, if $\ell : L \to M$ is HSLAG, the vector field JH is divergence free on L and $\Box$ is selfadjoint.

Proof. Using L^2 inner product on L with g_L the induced metric, we see that $\langle \Delta^2 v, w\rangle = \langle v, \Delta^2 w\rangle$. Using the fact that $\alpha_{J\nabla w} = dw$ on L, we find

$$\langle d^*\alpha_{\mathrm{Ric}^\perp(J\nabla v)}, w\rangle = \langle \alpha_{\mathrm{Ric}^\perp(J\nabla v)}, \alpha_{J\nabla w}\rangle = \int_L \mathrm{Ric}(J\nabla v, J\nabla w)\mathrm{vol}^{g_L}$$

Since the Ricci tensor is symmetric, we deduce that

$$\langle d^*\alpha_{\mathrm{Ric}^\perp(J\nabla v)}, w\rangle = \langle v, d^*\alpha_{\mathrm{Ric}^\perp(J\nabla w)}\rangle$$

Similarly, we compute

$$\langle d^*\alpha_{B(JH,\nabla v)}, w\rangle = \langle \alpha_{B(JH,\nabla v)}, \alpha_{J\nabla w}\rangle = \langle \alpha_{B(JH,\nabla v)}, J\nabla v\rangle.$$

The tensor $S(u, v, w) = \langle JB(u, v), w\rangle$ is symmetric. We deduce that

$$\langle d^*\alpha_{B(JH,\nabla v)}, w\rangle = \langle v, d^*\alpha_{B(JH,\nabla w)}\rangle.$$

This proves the first statement of the lemma.

Integration by part gives

$$\langle -JH\cdot JH\cdot v, w\rangle = \langle JH\cdot v, JH\cdot w\rangle + \int vwJH\cdot \mathrm{vol}^{g_L}.$$

If the embedding is HSLAG, then $d^*\alpha_H = 0$. But this equation is equivalent to the fact that $\mathrm{div}(JH) = 0$ and in turns $JH\cdot \mathrm{vol}^{g_L} = \mathrm{div}(JH)\mathrm{vol}^{g_L} = 0$. It follows that the operator $v \mapsto -JH\cdot JH\cdot v$ is also selfadjoint. □

2.2. **Hamiltonian isometries, rigidity and stability.** Suppose that (M,ω,J_0) is a Kähler manifold and and $\ell : L \to M$ is a HSLAG embedding. Let G be the group of Hamiltonian isometries of the Kähler manifold. Any one parameter subgroup $f_t : M \to M$ of G gives rise to a deformation $\ell_t := f_t \circ \ell$ of ℓ. However

$$g_{L,t} = \ell_t^* g_{J_0} = \ell^* f_t^* g_{J_0} = \ell^* g_{J_0} = g_L$$

since f_t is an isometry. In particular the quantity $d^*\alpha_H$ is independent of t, therefore $\square\ell^* v = 0$, for a Hamiltonian function $v : M \to \mathbb{R}$ given by the variation f_t.

The space of Hamiltonian functions induced by 1-parameter families of Hamiltonian isometries of (M,ω,J_0) is the space of Killing potentials denoted $\mathcal{K}_M$. Our observation shows that there is a canonical *restriction map*

$$\begin{aligned} \ell^* : \mathcal{K}_M &\to \mathcal{K}_L \\ v &\mapsto v\circ\ell \end{aligned}$$

where $\mathcal{K}_L$ is the kernel of the operator $\square$.

Definition 2.2.1. For a Kähler manifold (M,ω,J_0), a Hamiltonian stationary Lagrangian embedding $\ell : L \to M$ is called *rigid* if the associated restriction map $\mathcal{K}_M \to \mathcal{K}_L$ is surjective.

If in addition, the operator $\square$ is positive on a complement of the kernel $\mathcal{K}_L$, we says that $\ell : L \to M$ is *stable*.

According to Definition 2.2.1, a Hamiltonian Lagrangian embedding is rigid if, and only if, infinitesimal Hamiltonian stationary deformations of the Lagrangian can only come from of (globally defined) infinitesimal Hamiltonian isometry of the Kähler manifold (M,ω,J_0).

Examples 2.2.2.

(1) If $\mathcal{K}_L = \mathbb{R}$, the Lagrangian embedding $\ell : L \to M$ must be rigid.
(2) In particular all examples given by Remark 2.1.1 (SLAG in Calabi-Yau manifolds, Lagrangian tori in the flat torus, HSLAG with $H = 0$ in non-positive Ricci-curvature Kähler manifolds) are rigid.
(3) It is also known that the Clifford torus in $\mathbb{CP}^n$ is rigid and stable [9, 10].

3. Examples of Lagrangians toric fibrations

Fibrations of Calabi-Yau manifolds by special Lagrangian (SLAG) tori play a central role in mirror symmetry. Unfortunately, these fibrations seem to be scarce in nature. Hamiltonian stationary Lagrangians (HSLAG) are a natural generalization of SLAG and they make sense on every (almost) Kähler manifold. We then turn to the question of existence of HSLAG fibrations for Kähler manifolds.

A large pool of examples of such fibrations arise from toric Kähler geometry. A toric Kähler manifold is endowed with a Hamiltonian isometric toric action. Generic orbits of the torus action are Lagrangian tori. Since the metric is invariant under the torus action, it follows that the mean curvature of the orbits is also invariant and, in turn, that the Maslov form α_H must be parallel. In particular $d^*\alpha_H = 0$ and we have the following lemma:

Lemma 3.0.1. *The fibration by Lagrangian tori of a toric Kähler manifold has the property that each smooth fiber (i.e. corresponding to an interior point of the polytope) is Hamiltonian stationary with respect to the toric metric.*

3.1. **Kähler reductions.** Let $(\tilde{M}, \tilde{\omega})$ be a symplectic manifold and G a connected compact subgroup of Hamiltonnian diffeomorphisms. Assume that $\mu : \tilde{M} \to \mathfrak{g}^*$ is a G–equivariant momentum map and that $0 \in \mathfrak{g}^*$ is a regular value of μ. Then $N = \mu^{-1}(0)$ is a G–invariant submanifold and G–orbits are coisotropic, so the quotient $\pi : \tilde{M} \to M = \mu^{-1}(0)/G$ inherits of a symplectic structure ω defined as $\pi^*\omega = \tilde{\omega}_{|TN}$. The symplectic orbifold obtained this way is called the *symplectic reduction* of $(\tilde{M}, \tilde{\omega})$.

Whenever there is a G–invariant compatible Kähler metric $\tilde{g}$ on $(\tilde{M}, \tilde{\omega})$ then the whole Kähler structure descends on M making the quotient map

$$\pi : \mu^{-1}(0) \longrightarrow M$$

a Riemannian submersion. The resulting structure (M, ω, g) is called the *Kähler reduction* of $(\tilde{M}, \tilde{\omega}, \tilde{g})$.

It would be interesting to see if the statonnary properties (being HSLAG, stable, rigid..) of a G–invariant Lagrangian lying in $\mu^{-1}(0) \subset \tilde{M}$ are preserved under this operation. We prove below it is the case for toric manifold.

Lemma 3.1.1. *$\tilde{L}$ is a G–invariant Lagrangian of $(\tilde{M}, \tilde{\omega})$ lying in $\mu^{-1}(c)$ if and only if $\pi(\tilde{L}) = L$ is a Lagrangian in (M, ω). Suppose that $\tilde{g}$ is a G–invariant compatible Kähler metric on $(\tilde{M}, \tilde{\omega})$ such that there exists a positive constant κ depending only on $(\tilde{M}, \tilde{\omega}, \tilde{g})$ and G such that $vol(\pi^{-1}(L)) = \kappa\, vol(L)$ for every Lagrangian L in M. Then L is HSLAG whenever $\tilde{L}$ is and is stable whenever $\tilde{L}$ is.*

Proof. Put $N = \mu^{-1}(c)$. The first statement follows the fact that for every $p \in N$, the kernel of ω_p in T_pN (i.e those $v \in T_pN$ such that $\omega_p(v, w) = 0$ for all $w \in T_pN$)) coincides with the tangent space of the orbit of G.

Given a function $f \in C^\infty(M)$, take any $\tilde{f} \in C^\infty(\tilde{M})$ which is a G–invariant extension of the function $\pi^*f \in C^\infty(N)^G$. Let $\tilde{X}$ be the Hamiltonian vector field associated to $\tilde{f}$. Observe that $\tilde{X}$ is tangent to N since for each $a \in \mathfrak{g}$

$$\langle d\mu(\tilde{X}), a\rangle = -\tilde{\omega}(X_a, \tilde{X}) = d\tilde{f}(X_a) = 0.$$

Hence, for any Lagrangian $L \subset M$ and $\tilde{L} = \pi^{-1}(L)$ the variation $\tilde{\phi}_t(\tilde{L})$ induced by $t\tilde{X}$, stays in N and $\tilde{\phi}_t(\tilde{L}) = \pi^{-1}(\phi_t(L))$. Thus, $\mathrm{vol}(\tilde{\phi}_t(\tilde{L})) = \kappa \mathrm{vol}(\phi_t(L))$. So that L is HSLAG if $\tilde{L}$ is.

Moreover, for any Lagrangian $L \subset M$, the G–invariant submanifold $\tilde{L} = \pi^{-1}(L)$ is Lagrangian in $(\tilde{M}, \tilde{\omega})$ via the composition of inclusions $\tilde{\iota} : \tilde{L} \hookrightarrow N \hookrightarrow \tilde{M}$ and

$$\tilde{\iota}^*\tilde{\omega}(\tilde{X}, \cdot) = \pi^*(\iota^*\omega(X_f, \cdot))$$

where $df = -\omega(X_f, \cdot)$. Hence, $\kappa\Box(f_{|_L}) = \Box(\pi^*(f_{|_L}))$. $\square$

Corollary 3.1.2. *Assume, in addition to the hypothesis of Lemma 3.1.1, that $\tilde{L}$ is rigid then, for any function $f \in C^\infty(L)$ such that $\Box f = 0$, there exists a Killing potential $\tilde{f} \in C^\infty(\tilde{M})$ such that $\tilde{f}_{|_{\tilde{L}}} = \pi^* f$.*

3.2. **Case of $(\mathbb{CP}^n, \omega_{FS})$ and Kähler toric manifolds.** Oh observed that each Lagrangian torus of $\mathbb{C}^{n+1}$ of the form

$$T_{r_0,\dots,r_n} = \{z \in \mathbb{C}^{n+1} \,|\, |z_i| = r_i\} \tag{3.1}$$

is a critical point of the volume under Hamiltonian deformations (HSLAG). He proved also that it is rigid and stable by direct computations. One way to state his result is the following

Proposition 3.2.1 (Oh, [10])**.** *Let $\phi_t \in Ham(\mathbb{C}^{n+1}, \omega_{FS})$ be a path of Hamiltonian diffeomorphisms with $\pi_0 = Id$ and $\tilde{X} = (\frac{d}{dt}\phi_t)_{t=0} \in \mathfrak{ham}(\mathbb{C}^{n+1}, \omega_{FS})$. Let $\tilde{f}$ be a smooth Hamiltonian function for $\tilde{X}$. Then, $L = T_{r_0,\dots,r_n}$ is a HSLAG and*

$$\frac{d^2}{dt^2} vol(\phi_t(L)) \geq 0$$

Moreover, $\frac{d^2}{dt^2} vol(\phi_t(L)) = 0$ if and only if $\tilde{f}_{|_L}$ lies in the span of

$$\{\sin\theta_i,\, \cos\theta_i,\, \sin(\theta_i - \theta_j),\, \cos(\theta_i - \theta_j)\}_{i,j=0}^n.$$

Remarks 3.2.2. The hypothesis of Oh's result can be replaced by "given a function $f \in C^\infty(L)$, or equivalently an exact 1–form $\alpha_V = \iota^*\omega(V, \cdot) \in \Omega^1(L)$,...".

By Delzant–Lerman–Tolman theory, see [3, 8], any compact toric symplectic orbifold (M, ω, T) is the symplectic reduction of $(\mathbb{C}^d, \omega_{std})$ with respect to a subtorus $G \subset \mathbb{T}^d$. Via this construction the manifold inherits of a Kähler metric ω_G, called the Guillemin metric [5].

We recall the Delzant construction which is determined by the rational labelled polytope (P, ν, Λ) associated to (M, ω, T). Denote the Lie algebra $\mathfrak{t} = \mathrm{Lie}\, T$ and the momentum map $x : M \to P \subset \mathfrak{t}^*$. We use the convenient convention that $\nu = \{\nu_1, \dots, \nu_d\}$ is a set of vectors in $\mathfrak{t}$ so that if $F_1, \dots, F_d$ are the codimension 1 faces (*the facets*) then ν_k is normal to F_k and inward to P. The lattice Λ defines the torus as $T = \mathfrak{t}/\Lambda$. Being *rational*[1] means $\nu \subset \Lambda$. From the rational pair (P, ν), the torus G is determined by its Lie algebra $\mathfrak{g} = \ker \mathbf{q}$ where $\mathbf{q} : \mathbb{R}^d \to \mathfrak{t}$ is

$$\mathbf{q}(x) := \sum_{i=1}^d x_i \nu_i.$$

Hence, the compactness of P implies that $\mathbf{q}$ is surjective and that $T = \mathbb{R}^d/G$.

The defining affine functions of (P, ν) are the $\ell_1, \dots, \ell_d \in \mathrm{Aff}(\mathfrak{t}^*, \mathbb{R})$ such that $P = \{x \in \mathfrak{t}^* \,|\, \ell_k(x) \geq 0\}$ and $d_k = \nu_k$. Denote the inclusion $\iota : \mathfrak{g} \hookrightarrow \mathbb{R}^d$ and $\mu_o : \mathbb{C}^d \to (\mathbb{R}^d)^*$ the (homogenous of degree 2) momentum map of the action of $\mathbb{T}^d$ on $\mathbb{C}^d$ so that $\iota^*\mu_o$ is a momentum map for the action of G on $\mathbb{C}^d$. One side of the Delzant–Lerman–Tolman correspondence states that (M, ω, T) is T–equivariently symplectomorphic to the symplectic reduction of $(\mathbb{C}^d, \omega_{std})$ at the level $c = \iota^*(\ell_1(0), \dots, \ell_d(0)) \in \mathfrak{g}^*$. In the following, we identify (M, ω, T) with this reduction.

[1] To recover the original convention introduced by Lerman and Tolman in the rational case, take $m_k \in \mathbb{Z}$ such that $\frac{1}{m_k}\nu_k$ is primitive in Λ so $(P, m_1, \dots m_d, \Lambda)$ is a rational labelled polytope.

Note that since $\mu_o(z) = \frac{1}{2}(|z_0|^2, \dots, |z_n|^2)$ the defining equations of $N = (\iota^*\mu_o)^{-1}(c)$ involve only the square radii $r_i^2 = |z_i|^2$ and, thus, N is foliated by tori (of various dimension between $\dim G$ and d) of the form (3.1). Moreover, for each $x \in \mathring{P}$ the interior of P, $L_x = \mu^{-1}(x) \subset M$ is a Lagrangian torus such that $\pi^{-1}(L_x) = \tilde{L}_x$ is a d–dimensional torus of the form (3.1). Finally, observe that if $\tilde{f}$ is a Killing potential of $(\mathbb{C}^d, \omega_{std})$ which is G–invariant on *some* d–dimensional torus of the form (3.1) then $\tilde{f}$ is G–invariant on *any* d–dimensional torus of the form (3.1).

Lemma 3.2.3. *Let G be a subtorus of $\mathbb{T}^d$ and $\iota : G \hookrightarrow \mathbb{T}^d$ the inclusion. The volume, with respect to the standard flat metric of $\mathbb{C}^d$, of the orbits of G is constant on the regular level set of the momentum map $\iota^*\mu_o : \mathbb{C}^d \to \mathfrak{g}^*$.*

Proof. Let g be the standard metric on $\mathbb{C}^d$. We have

$$g = \sum_{i=1}^{d} \frac{d\mu_i^2}{2\mu_i} + 2\mu_i d\theta_i^2$$

where $\mu_i = \frac{1}{2}|z_i|^2$ and θ_i is the angle coordinates. Hence, for $z \in \mathbb{C}^d$ such that $G \cdot z$ is of full dimension, the action identifies $G \cdot z$ with G and the metric induced on the orbit is $\iota^* h$ where $h = 2\mu_i d\theta_i^2$. Clearly $\iota^* h$ only depends on $\iota^*(\mu_o(z))$. □

This last lemma implies that the present situation fulfill the hypothesis of Lemma 3.1.1 which we combine with Corollary 3.1.2 and Proposition 3.2.1, to get the following Proposition.

Proposition 3.2.4. *Any $L_x \subset M$ Lagrangian torus obtained as level set of the momentum map $\mu : M \to \mathfrak{t}^*$ is HSLAG, stable and rigid for the Guillemin metric ω_G.*

Remark 3.2.5. Lemma 3.2.3 applies more generally to *toric rigid metrics* in the sense of [1]. Therefore there is a bigger class of metric on which Proposition 3.2.4 extends.

The complex projective space with its Fubini-Study metric $(\mathbb{CP}^n, \omega_{FS})$ is a case of that last Proposition. Indeed, one way to see $\mathbb{CP}^n$ is as a Kähler reduction of $\mathbb{C}^{n+1}$ with respect to the diagonal Hamiltonian action of S^1. This coicides with the Hodge fibration $\pi : S^{2n+1} \to \mathbb{CP}^n$ and fits with the toric structure of $(\mathbb{CP}^n, \omega_{FS})$ obtained by quotient of the isometric toric action of $\mathbb{T}^{n+1}$ on $(\mathbb{C}^{n+1}, \omega_{std})$. We denote this quotient $T = \mathbb{T}^{n+1}/S^1$, $\mathfrak{t} = \mathrm{Lie}\, T$ and the momentum map $\mu : \mathbb{CP}^n \to \mathfrak{t}^*$. For any $L_x \subset \mathbb{CP}^n$ Lagrangian torus obtained as level set of the moment map, $\pi^{-1}(L_x)$ is a torus of the form (3.1). It may be more convenient to take the (more natural) momentum map

$$[Z_0 : \dots : Z_n] \mapsto \left(\frac{|Z_0|^2}{\sum_{i=0}^n |Z_i|^2}, \cdots, \frac{|Z_n|^2}{\sum_{i=0}^n |Z_i|^2} \right) \in \mathbb{R}^{n+1}$$

so that the r_i's in (3.1) are just the $|Z_i|$'s.
The S^1 invariant functions on $T_{r_0,\dots,r_n}$ in the kernel of $\Box$ are $\{\sin(\theta_i - \theta_j), \cos(\theta_i - \theta_j)\}_{i,j=0}^n$ so Proposition 3.2.4 holds in this case and reads as follow.

Proposition 3.2.6 ([11]). *Any $L_x \subset \mathbb{CP}^n$ Lagrangian torus obtained as level set of the moment map $\mu : \mathbb{CP}^n \to \mathfrak{t}^*$ is HSLAG, stable and rigid for the Fubini-Study metric.*

4. Deformation theory

In this section, we are assuming that $\ell : L \to M$ is a Hamiltonian stationary Lagrangian embedding, where (M, ω, J_0) is a Kähler manifold. For each almost complex structure $J \in \mathcal{AC}_\omega$ sufficiently close to J_0, we would like to find a Hamiltonian deformation $\tilde{\ell} : L \to M$ of the Lagrangian embedding $\ell : L \to M$ which is Hamiltonian stationary with respect to (M, ω, J). It turns out that this problem can not be solved directly by the implicit function theorem since the equations are generally overdetermined.

4.1. **Diffeomorphisms of the source space.** The group of diffeomorphisms $\mathrm{Diff}(L)$ acts on Lagrangian embeddings $\ell : L \to M$ by composition on the right. Such an infinite dimensional group action gives a huge group of symmetries which preserves the equation for HSLAG embeddings. This indeterminacy of the equations is easily removed by seeking Hamiltonian deformations of the image

$$\mathcal{L} = \ell(L)$$

as a Lagrangian submanifold of M instead. This boils down to consider the space of Lagrangian embeddings upto reparametrizations.

4.2. **Group of isometries.** From now on, we are assuming that $\ell : L \to M$ is a HSLAG embedding with respect to (M, ω, J_0) and that it is rigid.

The group G of Hamiltonian isometries of (M, ω, J_0) has a corresponding space of Hamiltonian potentials $\mathcal{K}_M \subset C^\infty(M)$. Their restriction to L is $\ell^*(\mathcal{K}_M) \subset C^\infty(L)$ and agrees with $\mathcal{K}_L = \ker \square_{\ell, J_0}$ by the rigidity assumption. An issue when trying to apply directly the implicit function theorem is that the linearization of the HSLAG equations given by the operator $\square_{\ell, J_0}$ is generally neither injective nor surjective.

4.3. **Lagrangian neighborhood theorem.** A sufficiently small tubular neighborhood $\mathcal{V}$ of $\mathcal{L} = \ell(L)$ in M is symplectomorphic to a neigborhood $\mathcal{U}$ of the zero section $\ell_0 : L \to T^*L$ endowed with its canonical symplectic form. In addition ℓ is identified to ℓ_0 via the symplectomorphism. Small Lagrangian deformations of $\mathcal{L}$ are given by graphs of sections α of $T^*L \to L$, where α is a sufficiently small closed 1-form on L. Furthermore, Hamiltonian deformations are given by exact 1-forms. Thus, every smooth function f on L, defines a Lagrangian submanifold of T^*L which is the graph of $df : L \to T^*L$ and this graph is a Hamiltonian deformation of the zero section $\ell_0 : L \to T^*L$. If f is sufficiently small (in C^1-norm), using the symplectomorphism between the tubular neighborhoods $\mathcal{U}$ and $\mathcal{V}$, each section df defines a Lagrangian embedding $\ell_f : L \to M$ which is a Hamiltonian deformation of $\ell : L \to M$.

4.4. **Vector bundles.** The image of the map $\ell^* : \mathcal{K}_M \to C^\infty(L)$ is $\mathcal{K}_L = \ker \square_{\ell, J_0}$ by rigidity, but it may not be injective. After passing to a subspace $\mathcal{K}_M^o \subset \mathcal{K}_M$, we obtain an isomorphism

$$\ell^* : \mathcal{K}_M^o \to \mathcal{K}_L.$$

For any embedding $\tilde{\ell} : L \to M$ sufficiently close to $\ell : L \to M$, the map $\tilde{\ell}^* : \mathcal{K}_M^o \to C^\infty(L)$ remains injective. Using this observation, we are going to introduce vector bundles of finite rank over the space of functions.

However, the space of smooth functions $C^\infty(L)$ is not suited to apply the implicit function theorem. Instead, we shall work with Hölder spaces $C^{k,\eta}(L)$, where

$0 < \eta < 1$ is the Hölder parameter and k is the number of derivatives accounted for. For any function $f \in C^{4,\eta}(L)$ sufficiently small, the map $\ell_f^* : \mathcal{K}_M^o \to C^{3,\eta}(L)$ remains injective. Its image is a finite dimensional vector space denoted

$$\mathfrak{K}_f \subset C^{3,\eta}(L).$$

The spaces $\mathfrak{K}_f$ are the fibers of a smooth vector bundle $\mathfrak{K}$ over a neighborhood of the origin in $C^{4,\eta}(L)$.

Let $\mathcal{H}'$ be the orthogonal complement of $\mathcal{K}_L = \mathfrak{K}_0$ in $C^{3,\eta}(L)$, where the L^2 inner product is induced by ℓ and J_0. For any $f \in C^{4,\eta}(L)$ sufficiently small, we have a splitting of vector bundles

$$C^{3,\eta}(L) = \mathfrak{K}_f \oplus \mathcal{H}',$$

and the projection on the first factor parallel to $\mathcal{H}'$ is denoted

$$\pi_f : C^{3,\eta}(L) \to \mathfrak{K}_f.$$

Similarly, we will need to consider the L^2-orthogonal complement $\mathcal{H}$ of $\mathcal{K}_L$ in $C^{4,\eta}(L)$ which gives the splitting

$$C^{4,\eta}(L) = \mathcal{K}_L \oplus \mathcal{H}. \tag{4.1}$$

4.5. **Implicit function theorem.** We introduce the map

$$\Psi : C^{4,\eta}(L) \times C^{4,\eta}(L) \times \mathcal{AC}^{2,\eta}_\omega \to C^{0,\eta}(L) \tag{4.2}$$

defined as follows: let (f, k, J) be an element of $C^{4,\eta}(L) \times C^{4,\eta}(L) \times \mathcal{AC}^{2,\eta}_\omega$. The function f admits a decomposition $f = f_L + h$ where $f_L \in \mathcal{K}_L$ and $h \in \mathcal{H}$ according to the splitting (4.1). For f and k sufficiently small, we may use the Lagrangian embedding $\ell_{k+h} : L \to M$. We define

$$\Psi(f, k, J) = d^*\alpha_H + \pi_{h+k}(f)$$

where $d^*\alpha_H$ is computed with respect to the Lagrangian embedding $\ell_{h+k} : L \to M$ and the almost complex structure J.

The differential of Ψ at $(0, 0, J_0)$ is given by

$$\left.\frac{\partial \Psi}{\partial f}\right|_{(0,0,J_0)} \cdot \dot{f} = \dot{f}_L + \Box_{\ell,J_0}\dot{h}$$

where we used the decomposition $\dot{f} = \dot{f}_L + \dot{h} \in \mathcal{K}_L \oplus \mathcal{H}$. This operator is clearly an isomorphism.

By the implicit function function theorem, we deduce the following proposition

Proposition 4.5.1. *There are open neighborhoods U, V of 0 in $C^{4,\eta}(L)$ and a G-invariant open neigborhood W of the almost complex structure J_0 in $\mathcal{AC}^{2,\eta}_\omega$ together with a smooth map*

$$\phi : V \times W \to U$$

such that

$$\Psi(\phi(k, J), k, J) = 0$$

for all $(k, J) \in V \times W$. Furthermore $\phi(k, J)$ is the only solution $f \in U$ of the equation $\Psi(f, k, J) = 0$ where $(k, J) \in V \times W$.

By definition, a solution of the equation $\Psi(f,k,J)=0$ provides a Lagrangian embedding $\ell_{h+k}:L\to M$ satisfying the equation

$$d^*\alpha_H \in \mathfrak{K}_{h+k}. \tag{4.3}$$

Thus, our problem of finding a HSLAG embedding is solved up to a finite dimensional obstruction. The solution of this type are called *relatively HSLAG* embeddings.

Definition 4.5.2. A Lagrangian embedding $\ell:L\to M$ into an almost Kähler manifold (M,ω,J) such that $d^*\alpha_H$ is the restriction of a Hamiltonian potential in $\mathcal{K}_M$ is called *a relatively HSLAG embedding.*

More information on the regularity of $(k,J)\in V\times W$ provide more regularity on $f=\phi(k,J)$ by standard bootstrapping argument for elliptic equations. In particular, we have the following lemma:

Lemma 4.5.3. *If $(k,J)\in V\times W$ are smooth, then $f=\phi(k,J)$ is smooth and so is $d^*\alpha_H$ for the corresponding relatively HSLAG embedding.*

It is worth pointing out that many cases are dealt with using the following corollary:

Corollary 4.5.4. *Let $\ell:L\to M$ be a rigid HSLAG embedding into (M,ω,J_0) with $\mathcal{K}_L=\mathbb{R}$. Then for all compatible almost complex structure J sufficiently close to J_0 in $C^{2,\eta}$-norm, there exists a Hamiltonian deformations $\tilde{l}:L\to M$ of ℓ which is a HSLAG embedding with respect to (M,ω,J).*

Proof. For J sufficiently close to J_0, we use the decomposition $\phi(0,J)=f_L+h$ given by the splitting (4.1). By assumption f_L must be a constant. The embedding $\ell_h:L\to M$ satisfies the equation $d^*\alpha_H=c$, for some constant $c\in\mathbb{R}$, with respect to the almost Kähler structure (M,ω,J). Since $d^*\alpha_H$ is L^2-orthogonal to constants, we deduce that $c=0$. □

4.6. **Residual isometry group action and killing the obstruction.** The next step is to look for solutions of the equation $\Psi=0$ such that the finite dimensional obstruction (4.3) vanishes. The residual group action of G on W is the key to achieve this goal. Indeed, we have a *modified volume functional*

$$\widetilde{\mathrm{vol}}:W\to\mathbb{R}$$

defined by

$$\widetilde{\mathrm{vol}}(J)=\mathrm{vol}(L,\ell_h^*g_J). \tag{4.4}$$

where $\phi(0,J)=f=f_L+h\in\mathcal{K}_L\oplus\mathcal{H}$ is given by Proposition 4.5.1.

Notice that $\widetilde{\mathrm{vol}}$ is a perturbation of the volume functional

$$\mathrm{vol}:W\to\mathbb{R}$$

of $\ell:L\to M$ defined by $\mathrm{vol}(J)=\mathrm{vol}(L,\ell^*g_J)$. Nevertheless, $\widetilde{\mathrm{vol}}$ and vol need not to agree generally.

Each G-orbit of almost complex structure in W, is compact, since G is. Thus the modified volume functional $\widetilde{\mathrm{vol}}$ restricted to a G-orbit admits critical points, for instance a minimum. Let J be such a point in a given orbit. Then we have the following result.

Proposition 4.6.1. *Let J be a smooth almost complex structure, which is a critical point of the modified volume functional $\widetilde{\mathrm{vol}}$ given by (4.4), restricted to a G-orbit of W. Then, the Lagrangian embedding $\ell_h : L \to M$ deduced from $\phi(0, J) = f_L + h$ via Proposition 4.5.1 is HSLAG.*

As a direct consequence, we obtain a proof of one of our main results:

Proof of Theorem A. We have a map $W \to U$ given by $J \mapsto \phi(0, J)$. For each J the decomposition $\phi(0, J) = f_L + h \in \mathcal{K}_L \oplus \mathcal{H}$ provides a Lagrangian embedding ℓ_h, which is a Hamiltonian deformation of ℓ. An easy exercice of symplectic geometry shows that one can define a smooth map ψ on W such that $\psi(J)$ is a Hamiltonian transformation of (M, ω) with the property that $\psi(J) \circ \ell = \ell_h$ and $\psi(J_0) = \mathrm{id}$ as in Theorem A. We give an outline of the argument to keep this paper self-contained. The function h is a priori defined on L. However a tubular neighborhood of $\mathcal{L} = \ell(L)$ is identified with a neighborhood $\mathcal{V}$ of the 0-section of the contangent bundle $T^*L \to L$. The function h can be understood as a function on $\mathcal{V}$ by pull back. We fix a suitable smooth compactly supported cut-off function φ on $\mathcal{V}$ equal to 1 in a neighborhood of $\mathcal{L}$. Then φh makes sense as a globally defined function on M. The corresponding Hamiltonian vector field $X_{\varphi h}$ is well defined on (M, ω). By integrating upto time 1, the flow of the vector field defines Hamiltonian transformation $\psi(J)$ of (M, ω), with regularity a $C^{4,\eta}$. If J is sufficiently close to J_0, the function h is very close to 0 in $C^{3,\eta}$-norm. In particular we have $\psi(J) \circ \ell = \ell_h$, by construction.

By assumption J is smooth here, hence by Lemma 4.5.3 so is h. Thus $\psi(J)$ is also a smooth Hamiltonian transformation so that we can avoid the complication of introducing a group of Hamiltonian transformations with suitable Hölder topology.

For the second part of the theorem, given $J \in W$, it suffices to find $u \in G$ such that $u \cdot J$ is a critical point of the functional $\widetilde{\mathrm{vol}}$ (such u exists by compactness of G). Then $u \cdot J$ satisfies the claim of Theorem A thanks to Proposition 4.6.1. ☐

The rest of this section is devoted to the proof of the proposition.

Proof of Proposition 4.6.1. Let $f = \phi(0, J)$. Notice that by Lemma 4.5.3, the function f is smooth and so is h. The smooth embedding, $\ell_h : L \to M$, satisfies the equation $d^*\alpha_H = \psi$ for some $\psi \in \mathfrak{K}_h$, which is also smooth. By definition, $\psi = v \circ \ell_h$ for some function $v \in \mathcal{K}_M$, by rigidity.

Let $u_t \in G$ be a 1-parameter subgroup of G such that $u_0 = id$ and $\frac{du_t(x)}{dt}|_{t=0} = X_v(x)$. In other words, the tangent vector to the 1-parameter subgroup is the Hamiltonian vector field X_v associated to v.

We consider the orbit $\tilde{J}_t = u_t^* J$ under the 1-parameter subgroup action. Since J is a critical point of the modified volume functional on its G-orbit, we have

$$\left.\frac{d}{dt}\right|_{t=0} \widetilde{\mathrm{vol}}(\tilde{J}_t) = 0.$$

Looking more closely at the modified volume functional, this means that the volume is critical for the 1-parameter family of Lagrangian embeddings $\ell_{\tilde{h}_t} : L \to M$ with respect to the almost complex structures $\tilde{J}_t$, where $\phi(0, \tilde{J}_t) = \tilde{f}_t = \tilde{f}_{L,t} + \tilde{h}_t \in \mathcal{K}_L \oplus \mathcal{H}$. By Lemma 4.5.3, the functions $\tilde{f}_t$, $\tilde{h}_t$ and $\tilde{f}_{L,t}$ are all smooth. Furthermore, they depend smoothly on t.

Changing the point of view, this means that the volume of the Lagrangian embeddings $\tilde{\ell}_t := u_t \circ \ell_{\tilde{h}_t} : L \to M$ is critical with respect to the fixed almost complex structure J.

Since the volume of a Lagrangian embedding depends only on the volume of its image, we may work upto the action $\mathrm{Diff}(L)$. Hence, we may assume after composing $\tilde{\ell}_t$ on the right by a suitable family of diffeomorphism that they are given by $\ell_{\tilde{h}_t + k_t} : L \to M$, where k_t is a smooth family of functions on L such that $\frac{\partial k_t}{\partial t}|_{t=0} = \psi$.

The solution of the relative HSLAG equations are invariant under the action of $\mathrm{Diff}(L)$, therefore we have a one parameter family of solutions of the equation

$$\Psi(\tilde{f}_t, k_t, J) = 0,$$

with critical volume at $t = 0$.

Differentiating at $t = 0$ gives the identity

$$\frac{\partial \Psi}{\partial f}|_{(f,0,J)} \cdot \dot{f} = -\frac{\partial \Psi}{\partial k}|_{(f,0,J)} \cdot \dot{k} \tag{4.5}$$

where

$$\dot{f} = \left.\frac{\partial \tilde{f}_t}{\partial t}\right|_{t=0} \quad \text{and} \quad \dot{k} = \left.\frac{\partial k_t}{\partial t}\right|_{t=0} = \psi.$$

The computation of the operator $\frac{\partial \Psi}{\partial k}$ is similar to $\frac{\partial \Psi}{\partial h}$ and we find

$$\left.\frac{\partial \Psi}{\partial k}\right|_{(f,0,J)} \cdot \dot{k} = L_{f,J} \cdot \dot{k} + \Box_{\ell_h, J} \dot{k}.$$

where $L_{f,J} \cdot \dot{k} = \frac{d}{dt}\pi_{h+k_t}(f)|_{t=0}$. We choose $\varepsilon_0 > 0$ sufficiently small, to be fixed afterward. According to Lemma 4.6.3, upto passing to sufficiently small open sets U and W, we have the estimates

$$\|L_{f,J} \cdot \dot{k}\|_{L^2} \leq \varepsilon_0 \|\dot{k}\|_{L^2}$$

and since $\dot{k} = \psi \in \mathfrak{K}_h$, by Lemma 4.6.4

$$\|\Box_{\ell_h, J} \dot{k}\|_{L^2} \leq \varepsilon_0 \|\dot{k}\|_{L^2}.$$

We deduce that the L^2-norm of the LHS of (4.5) is bounded by $2\varepsilon_0 \|\psi\|_{L^2}$. Applying Lemma 4.6.2, we obtain the estimate

$$\|\dot{f}\|_{L^2} \leq 2C\varepsilon_0 \|\psi\|_{L^2}.$$

In particular we have an estimate

$$\|\dot{h}\|_{L^2} \leq 2C\varepsilon_0 \|\psi\|_{L^2}.$$

If we choose $\varepsilon_0 = \frac{1}{4C}$, we have

$$\|\dot{h}\|_{L^2} \leq \frac{1}{2}\|\psi\|_{L^2}.$$

By the first variation formula, we have

$$\dot{\mathrm{vol}} = \langle \alpha_H, d\dot{h} + d\psi \rangle = \langle d^* \alpha_H, \dot{h} + \psi \rangle = \langle \psi, \dot{h} + \psi \rangle = \|\psi\|^2 + \langle \psi, \dot{h} \rangle.$$

By Cauchy-Schwartz inequality, it follows that

$$\dot{\mathrm{vol}} \geq \|\psi\|^2 - \|\psi\| \|\dot{h}\| \geq \frac{1}{2}\|\psi\|^2.$$

This is a contradiction unless $\psi = 0$. □

Here are the technicals lemmata used in the proof of Proposition 4.5.1. These results are standard and some proofs may be omited.

The operator $\frac{\partial \Psi}{\partial h}|_{0,0,J_0}$ is an isomorphism. The first eigenvalue of small perturbations of this operator remain uniformly bounded away from 0 in the sense of the following lemma:

Lemma 4.6.2. *For all neighborhood U and W of Proposition 4.5.1 chosen sufficiently small, there exists a constant $C > 0$ such that all $(f, J) \in U \times W$ and $F \in C^{4,\eta}(L)$*

$$\|F\|_{L^2} \leq C \left\| \left. \frac{\partial \Psi}{\partial h} \right|_{(f,0,J)} \cdot F \right\|_{L^2}.$$

Lemma 4.6.3. *For all $\varepsilon > 0$ there are sufficiently small open sets U and W in Proposition 4.5.1 such that for all $(f, J) \in U \times W$ and all function $F \in \mathfrak{K}_h$*

$$\|L_{f,J} F\|_{L^2} \leq \varepsilon \|F\|_{L^2}.$$

Proof. The proof is easily done by contradiction. If the lemma is not true, there exists $\varepsilon > 0$ and a family $(f_j, J_j) \in U \times W$ converging toward $(0, J_0)$ and $F_j \in \mathfrak{K}_{h_j}$ such that $\|F_j\|_{L_2} = 1$ while $\|L_{f_j,J_j} F_j\|_{L^2} > \varepsilon$.

Using the fact that $\mathfrak{K}$ is a finite rank bundle over $U \times W$ with a smoothly varying L^2-inner product, we may assume, after passing to a subsequence, that F_j converge to $F \in \mathfrak{K}_0 = \mathfrak{K}_L$ with the property that $\|F\|_{L^2} = 1$ and $L_{0,J_0} F \neq 0$. But this is impossible since $\pi_h(0)$ is identically 0, so that $L_{0,J_0} \equiv 0$. □

Lemma 4.6.4. *For all $\varepsilon > 0$ there are sufficiently small open sets U and W, as in Proposition 4.5.1, such that for all function $F \in \mathfrak{K}_h$, we have*

$$\|\Box_{\ell_h,J} F\| \leq \varepsilon \|F\|_{L^2}.$$

5. The volume functional

This section gathers some results and observations on the volume functional that will be needed to show that the space of positive perturbations, introduced at §6.2 in not empty, under mild assumption, in particular for the proof of Theorem D and Theorem 6.2.3.

5.1. **Main technical result.** We start with a Kähler manifold (M, ω, J_0). Let G be its group of Hamiltonian isometries. We consider a rigid HSLAG embedding $\ell : L \to M$ as in Theorem A.

Let G_ℓ be the subgroup of isometries of G preserving the image of $\ell : L \to M$. In other words $u \in G_\ell$ if and only if $u \circ \ell(L) = \ell(L)$. We denote by G_ℓ^o the identity component of G_ℓ.

We consider the subspace of compactible almost complex structures which differ from J_0 by a diffeomorphism of M. The connected component of J_0 in this space is denoted $\mathcal{J}_\omega \subset \mathcal{AC}_\omega$.

The standard volume functional here refers to the map

$$\mathrm{vol}_J : G/G_\ell^o \to \mathbb{R}$$

defined by $\mathrm{vol}_J([u]) = \mathrm{vol}(L, \ell^* g_{u \cdot J})$.

The variational formulas for the volume functional are much easier to carry out assuming the base complex structure is integrable but the result is likely to hold for almost Kähler metrics. Of course, it holds for any almost Kähler metric (J, ω) in a sufficiently small neighborhood of the Kähler metric (J_0, ω). Then main technical result of this section is the following theorem:

Theorem 5.1.1. *There exists a smooth* 1*-parameter family of complex structures* $J_s \in W$ *defined for* $s \geq 0$ *such that*

(1) J_0 *is our given complex structure*
(2) $\ell : L \to M$ *is HSLAG with respect to* J_s *for all* $s \geq 0$
(3) $\mathrm{vol}_{J_s} : G/G^o_\ell \to \mathbb{R}$ *admits a non-degenerate local minimum at* $u = \mathrm{id}$ *for all* $s > 0$.

5.2. **Some variational formulae.** First, we have to carry out some variational formulas. In order to do that we identify a neighborhood of L with a neighborhood of the zero section in T^*L and a set of coordinates $x_1, \dots, x_n$ on L is completed on T^*L to give Darboux coordinates $\omega = \sum_{i=1}^n dx_i \wedge dy_i$. In these coordinates, dy_i vanishes on TL, so the volume form $\mathrm{d}v_J$ on L is

$$\mathrm{d}v_J = \sqrt{|g_L|} dx_1 \wedge \cdots \wedge dx_n$$

where $g_L = g_{ij} dx_i \otimes dx_j$ is the restriction of the metric g on L, $|g_L|$ the determinant of (g_{ij}). The variation of Φ along a path $\{\mathrm{id}\} \times J_t$ starting at J is then given by

$$\frac{d}{dt}_{t=0} \mathrm{vol}_{J_t}(\mathrm{id}) = \int_L \mathrm{tr}(g_L^{-1} \dot{g_L}) \mathrm{d}v_J = \int_L \mathrm{tr}(g_L^{-1} \dot{g}) \mathrm{d}v_J. \tag{5.1}$$

More generally, for a given $u \in G$,

$$d_J \Phi(\dot{J})\,(u) = \frac{d}{dt}_{t=0} \mathrm{vol}_{J_t}(u) = \int_{u(L)} \mathrm{tr}(g_{u(L)}^{-1} \dot{g}) \mathrm{d}v_J.$$

For $J \in \mathcal{J}_\omega$, the tangent $T_J \mathcal{J}_\omega$ is a subspace of endormorphism of TM and has a natural complex structure $\mathbb{J}$ given by

$$\mathbb{J} A = J \circ A$$

for $A \in T_J \mathcal{J}_\omega$. Denoting the orbit of J by under Symp_0 by $\mathcal{G}.J = \mathrm{Symp}_0(\omega) \cdot J$, one can prove, see [4], that

$$T_J \mathcal{J}_\omega = \{ -\mathcal{L}_Z J \,|\, Z \in \mathfrak{symp}(\omega) + J \mathfrak{ham}(\omega) \} \simeq T_J \mathcal{G}.J + \mathbb{J} T_J \mathcal{G}.J. \tag{5.2}$$

The last equality uses $\mathcal{L}_{JZ} J = J \mathcal{L}_Z J$ which holds thanks to the integrability of J. Note that the right hand side of (5.2) is not a direct sum in general.

Let J_t be a path in $\mathcal{J}_\omega$ defined for small t such that $g_t = \omega(J_t \cdot, \cdot)$ be the corresponding variation of Riemannian metrics, so that $g_0 = g$.

For $\phi \in C^\infty(M)$, let $X_\phi \in \mathfrak{ham}(\omega)$ be the corresponding Hamiltonian vector field that is $d\phi = -\omega(X_\phi, \cdot)$. We denote by D the Levi-Civita connection and $d^c \phi = -d\phi \circ J$. Recall that for a 1–form $\alpha \in \Omega^1(M)$, the Levi-Civita is defined as $D\alpha\,(X, Y) = X.\alpha(Y) - \alpha(D_X Y)$ and the Hessian of ϕ is $Dd\phi$, a symmetric tensor since D has no torsion. The J–invariant and anti-invariant parts of $D\alpha$ are

$$D^{\pm} \alpha\,(X, Y) = \frac{1}{2}(D\alpha\,(X, Y) \pm D\alpha\,(JX, JY)).$$

Lemma 5.2.1. *Let* $\phi \in C^\infty(M)$,

a) *if* $\dot{J} = -\mathcal{L}_{X_\phi} J$, *then* $\dot{g} = -2D^- d^c \phi$.
b) *if* $\dot{J} = -J\mathcal{L}_{X_\phi} J$, *then* $\dot{g} = 2D^- d\phi$.

Proof. Since ω does not vary along the path (ω, g_t, J_t) and that $\mathcal{L}_{X_\phi}\omega = 0$, in the case a), we have, using [4, Lemma 1.20.2], that

$$\dot{g} = -\omega(\dot{J},) = \omega(\mathcal{L}_{X_\phi}J(\cdot), \cdot) = -2D^-(X_\phi^\flat) = -2D^-d^c\phi \tag{5.3}$$

since $g(X_\phi, \cdot) = d^c\phi$. In case b), we have

$$\begin{aligned}\dot{g} = -\omega(\dot{J},) = \omega(J\mathcal{L}_{X_\phi}J(\cdot), \cdot) &= -g(\mathcal{L}_{X_\phi}J\cdot, \cdot)\\ &= -(\mathcal{L}_{X_\phi}\omega) + \mathcal{L}_{X_\phi}g(J\cdot, \cdot) = \mathcal{L}_{X_\phi}g(J\cdot, \cdot).\end{aligned}$$

Since (g, J) is Kähler, $DJ = 0$ and so, for $Z, Y \in \Gamma(TM)$, we have

$$\begin{aligned}\dot{g}(Y,Z) &= \mathcal{L}_{X_\phi}g(JY, Z) = g(D_{JY}X_\phi, Z) + g(JY, D_Z X_\phi)\\ &= (JY).g(X_\phi, Z) - g(X_\phi, D_{JY}Z) + (Z).g(JY, X_\phi) - g(D_Z JY, X_\phi)\\ &= (JY).d^c\phi(Z) - d^c\phi(D_{JY}Z) + Z.d^c\phi(JY) - d^c\phi(D_Z(JY))\\ &= -(JY).d\phi(JZ) + d\phi(D_{JY}JZ) + Z.d\phi(Y) - d\phi(D_Z(Y))\\ &= -Dd\phi\,(JY, JZ) + Dd\phi\,(Z, Y)\\ &= -Dd\phi\,(JY, JZ) + Dd\phi\,(Y, Z) = 2D^-d\phi(Y, Z)\end{aligned} \tag{5.4}$$

□

Consequently, taking a variation $\dot{J} = -\mathcal{L}_{X_\phi}J - J\mathcal{L}_{X_\psi}J$ we have

$$d_J\Phi(\dot{J})(u) = -2\int_{u(L)} \mathrm{tr}\left(g^{-1}_{u(L)}(D^-d^c\phi - D^-d\psi)\right) \mathrm{d}v_J. \tag{5.5}$$

Lemma 5.2.2. *Let $\phi \in C^\infty(M)$ and $X, Y \in T_pM$ then*
a) $D^-d^c\phi(X,Y) = -D^-d\phi(JX, Y)$,
b) $2D^+d\phi(X,Y) = dd^c\phi(X, JY)$,
c) $2(D^-d\phi(JX, X) + D^-d\phi(X, X)) = dd^c\phi(X, JX)$.

Proof. For a), recall that $d^c\phi = -d\phi \circ J$. We use that $DJ = 0$ as follow

$$\begin{aligned}2D^-d^c\phi(X,Y) &= Dd^c\phi(X,Y) - Dd^c\phi(JX, JY)\\ &= X.d^c\phi(Y) - d^c\phi(D_XY) - (JX).d^c\phi(JY) + d^c\phi(D_{JX}JY)\\ &= -X.d\phi(JY) + d\phi(D_XJY) - (JX).d\phi(Y) + d\phi(D_{JX}Y)\\ &= -Dd\phi(X, JY) - Dd\phi(JX, Y)\\ &= Dd\phi(J^2X, JY) - Dd\phi(JX, Y)\\ &= -2D^-d\phi(JX, Y)\end{aligned} \tag{5.6}$$

For b), note that $Dd\phi$ is the Hessian, thus symmetric, hence

$$\begin{aligned}2D^+d^c\phi(X,Y) &= Dd\phi(X,Y) + Dd\phi(JX, JY)\\ &= X.d\phi(Y) - d\phi(D_XY) + (JY).d\phi(JX) - d\phi(D_{JY}JX).\end{aligned} \tag{5.7}$$

Using that D has no torsion, we see it coincides with

$$\begin{aligned}dd^c\phi(X, JY) &= X.d^c\phi(JY) - (JY).d^c\phi(X) - d^c\phi(D_XJY - D_{JY}X)\\ &= X.d\phi(Y) + (JY).d\phi(JX) - d\phi(D_XY) - d\phi(D_{JY}JX).\end{aligned} \tag{5.8}$$

For c), using again that $Dd\phi$ is symmetric we have

$$\begin{aligned} 2(D^- d\phi(JX,X) + D^- d\phi(X,X)) &= Dd\phi(JX,X) + Dd\phi(X,JX) \\ &\quad + Dd\phi(X,X) - Dd\phi(JX,JX) \\ &= Dd\phi(X,X+JX) + Dd\phi(JX,X-JX) \\ &= 2D^+ d\phi(X,X-JX) \end{aligned} \tag{5.9}$$

Now, using formula b) just above, we have $2D^+d\phi(X,X-JX) = dd^c\phi(X,J(X-JX)) = dd^c\phi(X,JX) + dd^c\phi(X,X) = dd^c\phi(X,JX)$. □

5.3. **Particular variations.** Let φ be a real smooth function on M. We consider a family of almost complex structures $J_s \in \mathcal{J}_\omega$, defined for $s \in \mathbb{R}$ sufficiently close to zero and such that

$$\frac{\partial}{\partial s} J_s = -\mathcal{L}_{X_\varphi} J_s + J_s \mathcal{L}_{X_\varphi} J_s \tag{5.10}$$

for all s sufficiently small, where X_φ is the Hamiltonian vector field deduced from φ.

We have the following variation formula for the volume:

Proposition 5.3.1. *Let φ_s be a smooth family of real smooth functions on M for small $s \in \mathbb{R}$ and consider the corresponding of $J_s \in \mathcal{J}_\omega$ satisfying (5.10). Then*

$$\frac{\partial}{\partial s}\mathrm{vol}_{J_s}(u) = \int_L ((\Delta^{g_s}\varphi) \circ u \circ \ell)\ \mathrm{vol}((u\circ\ell)^* g_{J_s}).$$

Proof. Using the first and last formulas of Lemma 5.2.2, we get that

$$d_J\Phi(\dot{J})(u) = -2\int_{u(L)} \mathrm{tr}(g^{-1}_{u(L)} dd^c\phi(\cdot, J\cdot))\mathrm{d}v_J = -2\int_{u(L)} \Delta^g\phi \ \mathrm{d}v_J \tag{5.11}$$

where $\Delta^g\phi$ is the Laplacian of ϕ on M with respect to the Kähler metric g. Indeed, taking $\{v_i\}_{i=1}^n$ an orthonormal basis of T_pL, the trace

$$\mathrm{tr}(g^{-1}_{u(L)} dd^c\phi(\cdot, J\cdot)) = \sum_{i=1}^n dd^c\phi(v_i, Jv_i)$$

is the symplectic trace of $dd^c\phi$ which turns out to be the Laplacian of ϕ at p up to a sign, see for e.g. [4, p.33]. Observe that to obtain (5.11) we didn't use the fact that G fixes J so the formula holds at any point of $\mathcal{J}_\omega$. □

The variation of J_s depends on first derivatives of ϕ. If ϕ vanishes upto order 1 along the image of $\ell : L \to M$, the almost complex structures J_s are independent of s along ℓ as well. The metric g_s deduced from ω and g_s must also be independent of s along ℓ. Is φ vanishes to a higher order, say upto order 2 along ℓ, the mean curvature along ℓ will also be independent of s. Since the metric, the mean curvature are independent of s we have the following lemma

Lemma 5.3.2. *Let $\varphi : M \to \mathbb{R}$ be a smooth function vanishing upto order 2 along the image of $\ell : L \to M$, and J_s be a family of almost complex structures defined from φ as above. Then $\ell : L \to M$ is HSLAG with respect to J_s for all s sufficiently small.*

For $s=0$, since u acts by isometry on g_{J_0} we deduce from Proposition 5.3.1 that

$$\frac{\partial}{\partial s}\mathrm{vol}_{J_s}(u)|_{s=0}=\int_L \left((\Delta^{g_0}\varphi)\circ u\circ\ell\right)\ \mathrm{vol}(g_L).$$

Lemma 5.3.3. *Let (N^k,g) be a smooth compact Riemannian manifold of dimension k and $p\in N$. Let $r:N\to\mathbb{R}$ be the Riemannian distance to p in N. The function r^4 is a smooth function in the neigborhood of $p\in N$ with the property that*

$$\Delta(r^4)=-4r^2(k+2)+\mathbf{o}(r^3),$$

where Δ is the Laplacian associated to the metric g.

Proof. It is well-known that r^2 is smooth, so does r^4. Taking normal coordinates (r,θ) centered at p we know the metric tend up to order 1 to the Euclidean metric at p. Then $|\nabla r|^2\to 1$ and $\Delta r^2\to -2k$ when $r\to 0$. It suffices to use the formula $\Delta r^4=2r^2\Delta r^2-2g(\nabla r^2,\nabla r^2)=2r^2(\Delta r^2-4g(\nabla r,\nabla r))$ to get the result. □

All the ingredients above can be used to give a proof of the main result of this section.

Proof of Theorem 5.1.1. Returning to our setup, still identifying a neigborhood of $\ell(L)$ in M with a neigborhood $\mathcal{V}$ of the 0–section in T^*L and considering the Kähler metric induced by g on $\mathcal{V}$. Denote the natural projection $T^*L\to L$. We consider the function $\varphi:\mathcal{V}\to\mathbb{R}$ defined by $\varphi(\alpha)=\frac{1}{-4(n+2)}d^g(\pi(\alpha),\alpha)^4$ where d^g is the distance induced by g. Then, when restricted on a fiber $T^*_pL\cap\mathcal{V}$, φ is the 4-th power of the distance function to p up to a constant factor and satisfies the last lemma. If (r_p,θ_p) are normal coordinates of $T^*_pL\cap\mathcal{V}$ centered at p then $\varphi(p,r_p,\theta_p)=\frac{r_p^4}{-4(n+2)}$. The lemma above (see the proof to be convinced it works as well) gives then

$$\Delta^g\varphi=r_p^2+\mathcal{O}(r_p^3).$$

Let $u_t\in G$ be a one parameter subgroup of G such that $u_0=\mathrm{id}$. Since φ vanishes upto order 3 along $\ell:L\to M$ we have $\Delta^{g_0}\varphi=0$ upto order 1 along ℓ, where g_0 is the Riemannian metric deduced from J_0 and ω. Therefore

$$\frac{\partial}{\partial t}\left((\Delta^{g_0}\varphi)\circ u_t\circ\ell\right)|_{t=0}=0$$

on L. In particular

$$\frac{\partial}{\partial t}\frac{\partial}{\partial s}\mathrm{vol}_{J_s}(u_t)|_{(s,t)=0}=0.$$

The second order t-derivative is given by

$$\frac{\partial^2}{\partial t^2}\frac{\partial}{\partial s}\mathrm{vol}_{J_s}(u_t)|_{(s,t)=0}=\frac{1}{2}\int_L Q(X,X)\mathrm{vol}(g_L)$$

where Q is the Hessian quadratic form of $\Delta^{g_0}\varphi$ and X is the Hamiltonian vector field tangent to u_t at $t=0$ along ℓ. If $\Delta^{g_0}\varphi=r_p^2+\mathcal{O}(r_p^3)$, we deduce that Q is definite positive in directions transverse to ℓ. If u_t is transverse to G^0_ℓ, then X is not everywhere tangent to ℓ. Hence we have proved that

$$\frac{\partial^2}{\partial t^2}\frac{\partial}{\partial s}\mathrm{vol}_{J_s}(u_t)|_{(s,t)=0}>0$$

unless u_t is tangent to G^o_ℓ at $t=0$.

We deduce that if u_t is transverse to G^0_ℓ at $t=0$

$$\mathrm{vol}_{J_s}(u_t) = c + b_s t^2 + o(t^2)$$

where the constant b_s is strictly positive for $s>0$. This completes the proof of Theorem 5.1.1. □

5.4. **Other properties of the volume functional.** Our variationnal formulas can be used to show that the standard volume functional has Morse properties for generic almost complex structures. Unfortunately we are interested in the modified volume functional (4.4) in this paper, and this is why we relied on Theorem 5.1.1 instead. For the interested reader, we state the following result, which is in the spirit of the proof of existence of Morse function in the finite dimensional setting, although it is not used in the rest of the paper:

Proposition 5.4.1. *Given a compact Kähler manifold (M,ω,J_0) and a compact Lagrangian $\ell : L \hookrightarrow M$. The map $\Phi : \mathcal{J}_\omega \longrightarrow C^\infty({}^G\!/_{G_\ell})$, defined as*

$$\Phi(J) := \mathrm{vol}_J \tag{5.12}$$

is a submersion in a neigborhood of J_0.

Proof. The map $u \mapsto u(L)$ is injective on ${}^G\!/_{G_\ell}$ so there exists a point $p\in L$ such that there is a neighborhood V of $id \in G$ satisfying the condition:

$$\forall u \in V,\ u(p)\in L \text{ if and only if } u \in G_\ell. \tag{5.13}$$

The neighborhood V depends on p but since it is an open condition on p we can choose a neighborhood U of p in L such that condition (5.13) holds. Then the orbit map $\psi(q,\gamma)=\gamma(q)$ induces a smooth foliation of the image $\mathcal{W} := \psi(U\times V)$, in M. The leafs of W are $u(L)\cap W \simeq U$ for $u\in G/G_\ell$. Actually, we have a diffeomorphism

$$\tau : \mathcal{W} \xrightarrow{\sim} U \times V_\ell \tag{5.14}$$

where $V_\ell \subset G/G_\ell$ denotes the image of V via the quotient map $G\to G/G_\ell$.

We can easily pushforward any bump function ψ_U from U to the whole $\mathcal{W}$ via the action of G and any function f on V_ℓ defines a G_ℓ–invariant function on V. The pull-back $\tau^*(\psi_U\times f)$ on $\mathcal{W}$ may be extended to M so that it integrates to 0. Taking the Green function of this extension to be the variation $\dot J$ as above, we get that (5.11) becomes

$$d_J\Phi(\dot J)(u) = f(u)$$

for all $u\in V_\ell$. From which we conclude that $d_J\Phi$ is surjective on $C^\infty(V_\ell)$. □

6. Deformation theory for local HSLAG toric fibrations

6.1. **Obstacles to overcome.** So far, we only considered the case of a single HSLAG embedding $\ell : L \to M$. We would like to extend the theory to the case of a local HSLAG toric fibration $\ell_t : L \to M$ as defined in §1.2. There are several issues for extending Theorem A to a Lagrangian fibration:

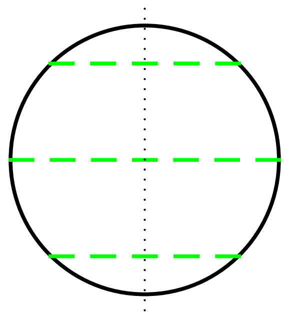
(1) The fibration becomes singular at the boundary of the polytope in the case of the standard fibration by Lagrangian tori of a toric Kähler manifold. In the case of a SLAG fibration of a K3 surface, certain fibers have several irreducible components. Such issues related involve complicated analytical problems that we shall not tackle at this paper. This is why we restrict our attention to local fibrations by smooth Lagrangian tori as in §1.2.

(2) Proposition 4.6.1 involves the choice a local minimum of the volume functional seen as a function on G. One cannot make a consistent choice a minimum for a family of tori, unless the volume has some special properties. We shall prove that for a generic choice of metric the volume functional is non degenerate, which allows to get around this issue.

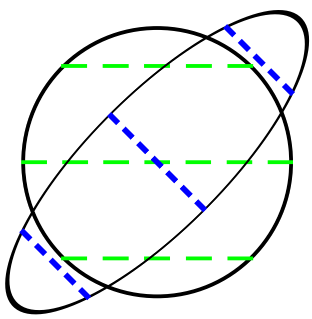
Example 6.1.1. The fact that the Hamiltonian transformation v of Theorem C is not necessarily small can be readily observed in an example.

We consider the unit 2-dimensional sphere in $\mathbb{R}^3$ with a given axis going from the north to the south pole and its standard complex structure obtained by rotating each tangent plane by an angle $\pi/2$.

Every embedded circle C is automatically Lagrangian in such a low dimensional case. Furthermore, the HSLAG property is equivalent to the fact that the circle has constant curvature. There is a standard fibration $\ell_t : S^1 \to S^2$, for $t \in (-1, 1)$ of the sphere by circles of constant curvature, known as the parallels, obtained by rotating the sphere about its axis



We pick an axis (D) going through the shere center, wich is distinct from the given axis, and does not belong to the equator plane. We consider a deformation of the sphere, with prescriped area, into an ellipsoïd of revolution, with axis (D). As shown by the picture below, there is an obvious fibration by circle of constant curvature. However this fibration is far from the original fibration which suggests that a jump must occur.



However, this only a heuristic argument since we do not have a proof that this HSLAG fibration is the only one on an ellipsoïd of revolution.

We now give a correct argument. The equator $\ell_0 : S^1 \to S^2$ is a geodesic, hence its Maslov form α_0 vanishes. On the other hand, the sign of the integral $\int_{S^1} \alpha_t$ of the Maslov form α_t of ℓ_t changes when t goes through 0.

We consider a variation J_s of the standard complex structure J_0 on the sphere S^2, compatible with the symplectic form. The Maslov form of ℓ_t now depends on s as well, and we denote it by $\alpha_{t,s}$. By continuity, there exists t_s for each sufficiently small deformation J_s such that $\int_{S^1} \alpha_{t_s,s} = 0$. If there is a Hamiltonian transformation v_s of the sphere such that $v \circ \ell$ is a HSLAG fibration with respect to J_s, then $v \circ \ell_{t_s}$ has constant curvature with exact Maslov form and it must be a geodesic.

Now, we pick a particular variation J_s, where each J_s induces the metric of an ellipsoïd of revolution with three distinct axis unless $s = 0$. Such surfaces are known to have only three closed geodesics. Upto a rotation, we may assume that none of the three geodesics agree with the equator of the starting round sphere. This shows that v_s is not close to the identity for s close to 0.

6.2. **Positive perturbations.** We start with a Kähler manifold (M, ω, J_0). Let G be its group of Hamiltonian isometries. We consider a rigid HSLAG embedding $\ell : L \to M$ as in Theorem A.

Let G_ℓ be the subgroup of isometries of G preserving the image of $\ell : L \to M$. In other words $u \in G_\ell$ if and only if $u \circ \ell(L) = \ell(L)$. We denote by G^o_ℓ the identity component of G_ℓ.

Using the notation of §4.6, we consider almost complex structures $J \in W$ sufficiently close to J_0 and the corresponding relatively HSLAG embedding $\ell_h : L \to M$ on (M, ω, J).

The modified volume functional $\widetilde{\mathrm{vol}} : W \to \mathbb{R}$ is generally not G-invariant. However G_ℓ leaves $\mathcal{L} = \ell(L)$ invariant by definition. It follows that the map Ψ defined at (4.2) is G_ℓ-equivariant, and so is the map ϕ defined in Proposition 4.5.1. In turn, the modified volume functional $\widetilde{\mathrm{vol}} : W \to \mathbb{R}$ is G_ℓ-invariant. Hence, the modified volume functional may be understood as a map

$$\widetilde{\mathrm{vol}}_J : G/G^o_\ell \to \mathbb{R}$$

defined by $\widetilde{\mathrm{vol}}_J([u]) = \widetilde{\mathrm{vol}}(u \cdot J)$. By Proposition 4.6.1, critical points of this functional correspond to HSLAG embeddings. Such critical points are generally not non-degenerate. For instance, the choice $J = J_0$ provides a constant function $\widetilde{\mathrm{vol}}_{J_0}$ since J_0 is G-invariant and all critical points are degenerate.

Definition 6.2.1. An almost complex structure $J \in W$ is called a *positive deformation of J_0* with respect to the rigid HSLAG $\ell : L \to M$ if the corresponding functional $\widetilde{\mathrm{vol}} : G/G^o_\ell \to \mathbb{R}$ admits a non degenerate local minimum. The subset of W that consists of positive deformations is denoted W^+.

We have the following obvious result, by stability of non-degenerate local minimum:

Lemma 6.2.2. *The set of positive deformations W^+ of J_0 is an open subset of W, endowed with its $C^{2,\eta}$-topology.*

The openness result does not insure that W^+ is non-empty. Although we suspect that it is never empty, we shall prove it under some reasonnable technical assumptions:

Theorem 6.2.3. *Let (M, ω, J_0) be a Kähler manifold and $\ell : L \to M$ be a rigid and stable HSLAG.*

Then, the open set of positive deformations $W^+ \subset W$ of J_0 is not empty. Furthermore, there exists a smooth family of complex structure $J_s \in W$, defined for $s \geq 0$, such that $J_s \in W^+$ for all $s > 0$. In other words, J_0 is in the closure of W^+.

Notice that Theorem D is a less technical restatement of the above theorem.

Proof. Let J_s be a family of complex structures as in the above proposition.

Lemma 6.2.4. *Under the stability assumption of Theorem 6.2.3, the functional*

$$\widetilde{\mathrm{vol}}_{J_s} : G/G^o_\ell \to \mathbb{R}$$

admits non-degenerate local minimum at the identity for every $s > 0$ sufficiently small.

Proof. The fact that $\widetilde{\mathrm{vol}}$ admits a critical point at the identity is clear, since $\ell : L \to M$ is a HSLAG in (M, ω, J_s).

The only thing to be proved is the fact that it is non-degenerate. Let $u_t \in G$ be a one parameter subgroup of G transverse to G_ℓ, with $u_0 = \mathrm{id}$ and $\dot{k} \in \mathcal{K}_M$ a corresponding Hamiltonian function such that $\frac{d}{dt} u_t|_{t=0} = X_{\dot{k}}$.

Let $J_{s,t} = u_t^* J_s$ and $\ell_{t,s} = \ell_{h_{s,t}} : L \to M$ be the corresponding solution, a relatively HSLAG embedding, provided by the implicit function theorem as in Proposition 4.5.1.

By definition $\ell_{s,0} = \ell$. We can switch our point of view using $\tilde{\ell}_{s,t} = u_t \circ \ell_{s,t}$ with a fixed complex structure J_s. By the second variation formula, we have

$$\frac{d^2}{dt^2} \mathrm{vol}(\tilde{\ell}_{s,t}, g_{J_s})|_{t=0} = \langle \square_{J_s}(\dot{k} + \dot{h}), \dot{k} + \dot{h} \rangle.$$

where $\dot{h} \in \mathcal{H}$ is the projection of $\frac{\partial}{\partial k} \phi|_{(0,J_s)} \cdot \dot{k}$ on $\mathcal{H}$. Expanding the above inner product and integrating by part we obtain

$$\langle \square_{J_s} \dot{k}, \dot{k} \rangle + \langle \square_{J_s} \dot{h}, \dot{h} \rangle + 2 \langle \square_{J_s} \dot{k}, \dot{h} \rangle$$

For $s > 0$, since J_s is provided by Theorem 5.1.1 and we have the non-degeneracy property (3). The second term is non negative for s sufficiently small by the stability assumption of $\ell : L \to M$ into (M, ω, J_0). The third term is controlled by the first two terms since $\square_{J_s} \dot{k}$ converges to 0 as s goes to 0 and the eigenvalues of $\square_{J_s}$ are uniformly bounded from below in the direction $\dot{h}$. Hence the second variation of the volume must be positive for $s > 0$ sufficiently small. ∎

In conclusion, the complex structures J_s belong to W^+ for $s > 0$ sufficiently small. This completes the proof of Theorem 6.2.3. ∎

6.3. **Invariant Lagrangian fibrations.** We would like to extend the deformation theory of §4 to the case of a fibration. For this purpose, we choose to restrict to the case of G^o_ℓ-invariant fibrations in the sense of Definition 1.2.3. This technical assumption is not too demanding as it is satisfied by examples provided by toric Kähler geometry. We state few observations on G^o_ℓ-invariant fibrations in the following Proposition.

Proposition 6.3.1. *Let $\ell_t : L \simeq M^{2n}$ be a Lagrangian fibration with $\ell_0 = \ell$ and $t \in B(0, \varepsilon)$ and such that L is compact.*

1. *If G^o_ℓ acts effectively on $\ell(L)$ then G^o_ℓ is a torus of dimension at most n.*
2. *If ℓ_t is a G^o_ℓ-invariant fibration then G^o_ℓ is a torus of dimension at most n.*
3. *If $G^o_\ell = \mathbb{T}^n$ then there exists a G^o_ℓ-invariant fibration $\tilde{\ell}_t : L \to M$ such that $\tilde{\ell}_0 = \ell$ is a neigborhood of $\ell(L)$.*
4. *If (M, g, ω, J) is a toric Kähler manifold with momentum map $\mu : M \to P \subset \mathbb{R}^n$, then $\{\mu^{-1}(p) \,|\, p \in \mathring{P}\}$ is a $\mathbb{T}^n$–invariant fibration.*

Proof. The first affirmation follows the observation that the orbit of G^o_ℓ in $\ell(L)$ must be isotropic, thus a torus, thanks to the formulas

$$d\omega(X_a, X_b) = -\omega([X_a, X_b], \cdot) = -\omega(X_{[a,b]}, \cdot) \tag{6.1}$$

where X_a is the vector field induced on M by $a \in \mathrm{Lie}\, G^o_\ell$. The second affirmation is a consequence of the observation that in the case of G^o_ℓ-invariant fibration we have $G^o_\ell \subset G^o_{\ell_t}$ and there is an open and dense subset of $t \in B(0, \varepsilon)$ such that $G^o_{\ell_t}$ acts effectively on $\ell_t(L)$. For the third one, we consider the generic orbits of $G^o_\ell = \mathbb{T}^n$, which must be Lagrangian by the formula (6.1) above. The fourth affirmation is obvious. □

Theorem 6.3.2. *Let (M, ω, J_0) be a Kähler manifold and $\ell : L \to M$ a rigid HSLAG embedding, where L is a real torus.*

Assume that non-trivial harmonic forms on L for the induced metric do not vanish at any point. Then there exists a G^o_ℓ-invariant HSLAG fibration $\ell_t : L \to M$ such that $\ell_0 = \ell$.

Proof of Theorem 6.3.2. The compact group G_ℓ preserves the image of $\ell : L \to M$. Thus G_ℓ has an induced action on L by diffeomorphism. This action also induces a symplectic G_ℓ-action on T^*L. The starting point of our setup to apply the implicit function Theorem (cf. §4) requires the choice of a symplectic diffeomorphism between a neighborhood of the image of ℓ in M and a neighborhood of the zero-section in T^*L. This symplectomorphism can be chosen to be G_ℓ-equivariant.

We have a Riemannian metric g_L on L induced by g_{J_0}. Since G_ℓ acts isometrically on (M, g_{J_0}), the induced action on (L, g_L) is also isometric. In particular, G_ℓ acts on the space $\mathcal{H}^1(L, g_L)$ of harmonic 1-forms of (L, g_L). Since elements of G^o_ℓ are homotopic to the identity in $\mathrm{Diff}(L)$, they act trivially on the cohomology of L, hence on the space of harmonic 1-forms $\mathcal{H}^1(L, g_L)$.

One can construct a standard Lagrangian toric fibration

$$\mathcal{H}^1(L, g_L) \times L \to T^*L$$

given by $(\alpha, x) \mapsto \alpha_x$. This construction is G_ℓ equivariant, by definition. Using the G_ℓ-equivariant indentification between a neighborhood of the 0-section of T^*L and a neighborhood of the image of ℓ, we deduce a local lagrangian toric fibration

$$\hat{\ell} : K \times L \to M$$

where K is a G_ℓ-invariant neighborhood of the origin in $\mathcal{H}^1(L, g_L)$. For K sufficiently small, this map is indeed an embedding since by assumption, harmonic 1-forms do not vanish at any point. We use the notation $\hat{\ell}_\alpha = \hat{\ell}(\alpha, \cdot)$ in the sequel.

By definition $\hat{\ell}_0 = \ell$ so it must be HSLAG. However $\hat{\ell}_\alpha$ may not be HSLAG for $\alpha \in K$. By construction the fibration is G_ℓ-equivariant and the action induced by G^o_ℓ is trivial on the parameter space $\alpha \in K$.

Using a version of the implicit function theorem with parameter as in Proposition 4.5.1, one can perturb each map $\hat{\ell}_\alpha$ for $\alpha \in K$ by a Hamiltonian deformation, provided K is sufficiently small, in order to get a relatively HSLAG Lagrangian embedding. More precisely, there exists a smooth map

$$\phi : K \to U,$$

with the notations of Proposition 4.5.1, such that the lagrangian embedding ℓ_α defined by the 1-form $\alpha + dh_\alpha$, where $h_\alpha = \phi(\alpha)$ is relatively HSLAG.

By uniqueness of the solution of the IFT and the fact that G acts by isometries on g_{J_0} we obtain that ϕ is a G_ℓ equivariant map. In particular, the Lagrangian fibration $\ell_\alpha : L \to M$ is also G^o_ℓ-invariant.

The invariance of the metric also implies that the volume of ℓ_α is invariant under the action of G. This forces the equation $d^*\alpha_H = 0$ by the first variation formula for the volume. Therefore, each ℓ_α must be HSLAG. □

6.4. **Fibrations and positive perturbations.** At this stage, all the tools necessary to handle the case of HSLAG fibrations have been introduced.

Let $\ell_t : L \to M$ be a HSLAG toric fibrations into a Kähler manifold (M, ω, J_0), with G its the group of Hamiltonian isometries. We are assuming that ℓ_t is $G^o_{\ell_0}$-invariant.

We are assuming that $J \in W^+$ is a positive perturbation with respect to $\ell = \ell_0$. By stability of non-degenerate minimum, we deduce that J is positive with respect to every ℓ_t, for t sufficiently small.

Provided ℓ_0 is rigid, using the implicit function theorem, we deduce a family of Hamiltonian deformations $\ell_{t,h_{t,u}}$ of ℓ_t which are relatively HSLAG with respect to the complex structures $u \cdot J$ for all $u \in G$. Equivalently, $u \cdot \ell_{t,h_{t,u}}$ is relatively HSLAG with respect to J. For each t, there exists a non-degenerate local minimum of the modified volume functional $\widetilde{\mathrm{vol}}$. Since it is non degenerate, we may choose u_t, depending smoothly on t such that $u_t \cdot \ell_{t,h_{t,u_t}}$ achieve such a local minimum of $\widetilde{\mathrm{vol}}$.

By Proposition 4.6.1, $u_t \cdot \ell_{t,h_{t,u_t}} : L \to M$ must be HSLAG with respect to J. We deduce the following proposition:

Proposition 6.4.1. *Let $\ell_t : L \to M$ be a G^o_ℓ-invariant HSLAG toric fibration in a Kähler manifold (M, ω, J_0) such that ℓ_0 is rigid.*

For each positive almost complex structure J compatible with ω and sufficiently close to J_0, there exists a smoothly varying family of Hamiltonian transformations v_t such that $v_t \circ \ell_t$ is HSLAG with respect to (M, ω, J).

Since we have a local smooth fibration, we readily deduce

Corollary 6.4.2. *Under the assumptions of Proposition 6.4.1, there exists Hamiltonian transformations v such that $v \circ \ell_t$ is a HSLAG toric fibration with respect to (M, ω, J).*

This proves Theorem C.

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