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Comments on ‘The computation of wavelet-Galerkin approximation on a bounded interval’

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In this letter, we identify and correct the errors in (Int. J. Numer. Meth. Eng. 1996; 39:2921–2944). And we also develop clearer procedures for the computation of the connection coefficients from the wavelet-Galerkin scheme.

KEY WORDS: connection coefficient; wavelet-Galerkin method; numerical solution

1. INTRODUCTION

In wavelet-based methods for solving ordinary and partial differential equations, the connection coefficients representing the integrals of the product of the scaling function \( \phi(x) \) and its \( n \)th derivative \( \phi^{(n)}(x-k) \) play an important role [1, 2]. Chen, Hwang and Shin introduced an algorithm to compute the connection coefficients for wavelet-Galerkin approximation in 1996 [1], and the paper has been cited many times by others. However, we have recently found some fundamental errors in the paper; and corrections to those errors have not been found in open literature since the publication of the paper. Some descriptions in the paper are also confusing or inappropriate.
In this letter, we aim to
1. identify these errors;
2. correct the identified errors; and
3. develop clearer procedures for the computation of the connection coefficients.

We will briefly discuss some fundamental knowledge and properties of wavelets in Section 2. Section 3 shows how to compute the values of connection coefficients. Section 4 summarizes the letter.

2. PREPARATION

In 1992, Daubechies [3] constructed a family of compactly supported orthonormal wavelets with members ranging from highly localized to highly smooth ones. Each wavelet number is governed by a set of $L$ coefficients \( \{ p_k : k = 0, \ldots, L - 1 \} \) through the two-scale relations

\[
\phi(x) = \sum_{k=0}^{L-1} p_k \phi(2x - k)
\]

and

\[
\psi(x) = \sum_{k=0}^{L-1} (-1)^k p_{1-k} \phi(2x - k)
\]

where \( \phi(x) \) is called the scaling function with \( \phi(0) = \phi(L - 1) = 0 \), and \( \psi(x) \) is named mother wavelet, respectively. The fundamental support is the finite interval \([0, L - 1]\) for \( \phi(x) \) and the finite interval \([1 - L/2, L/2]\) for \( \psi(x) \), respectively. By [4], the scaling function \( \phi(x) \) has the following property:

\[
\sum_{l=-\infty}^{\infty} l^n \phi(x - l) = \sum_{j=0}^{n} (-1)^j \binom{n}{j} M_j^\phi x^{n-j}, \quad n = 0, 1, \ldots, L/2 - 1
\]

where \( M_j^\phi \) is the \( j \)th moment of \( \phi(x) \), which is defined by the following equation

\[
M_j^\phi = \int_{-\infty}^{\infty} x^j \phi(x) \, dx
\]

with the initial condition \( M_0^\phi = 1 \).

Denote by \( \phi^{(n)}(x) \) the \( n \)th derivative of the scaling function \( \phi(x) \). We have

\[
\phi^{(n)}(x) = \frac{d^n \phi(x)}{dx^n} = \frac{d}{dx} \phi^{(n-1)}, \quad \phi^{(0)} = \phi(x)
\]

Obviously, the compact support of \( \phi^{(n)}(x) \) is \([0, L - 1]\). It follows from Equations (1)–(4) that

\[
\phi^{(n)} = 2^n \sum_{k=0}^{L-1} p_k \phi^{(n)}(2x - k), \quad n = 0, 1, \ldots, L/2 - 1
\]
3. COMPUTATION OF THE CONNECTION COEFFICIENTS

3.1. Computation of $\Gamma_n^x(x)$

As in [1], let us define the connection coefficient

$$\Gamma_n^x(x) = \int_0^x \phi^{(n)}(y - k)\phi(y) \, dy$$  \hspace{1cm} (6)

We are going to derive an easier-to-understand algorithm for computing the connection coefficients in our own way. Errors in the paper [1] are also clearly identified through our derivation.

We start from some simple properties of $\Gamma_n^x(x)$. By the properties of the scaling function $\phi(x)$, as discussed in Section 2, and straightforward computation, it is easy to verify the following relationships for $n = 0, 1, \ldots, L/2 - 1$ and all integers $k$:

$$\Gamma_n^x(x) = \Gamma_n^x(L - 1) \quad \text{for} \ x \geq L - 1$$  \hspace{1cm} (7)

$$\Gamma_n^x(x) = 0 \quad \text{for} \ |k| \geq L - 1 \ \text{or} \ x \leq 0 \ \text{or} \ x \leq k$$  \hspace{1cm} (8)

$$\Gamma_{-k}^n(L - 1) = (-1)^n \Gamma_k^n(L - 1)$$  \hspace{1cm} (9)

$$\Gamma_{-k}^n(x) = (-1)^n \Gamma_k^n(L - 1) \quad \text{for} \ x + k \geq L - 1$$  \hspace{1cm} (10)

Here the condition for Equation (10), which is given in Appendix A, corrects an error in the condition for Equation (56) in [1].

Equations (1), (5) and (6) give

$$\Gamma_n^x(x) = 2^{n-1} \sum_{i,j=0}^{L-1} p_i p_j \Gamma_{2k+i-j}^n(2x - j)$$  \hspace{1cm} (11)

If letting

$$\Gamma^n(L - 1) = [\Gamma_0^n(L - 1), \Gamma_1^n(L - 1), \ldots, \Gamma_{L-2}^n(L - 1)]^T$$

and taking $x = L - 1$ in (11), then from [1], we can easily obtain the values of $\Gamma^n_k(L - 1)$ through the following algorithm:

$$\Gamma^n(L - 1) = D\Gamma^n(L - 1)$$

with normalization condition

$$\sum_{k=0}^{L-2} k^n \Gamma_k^n(L - 1) = \frac{n!}{2}$$

where $D = (d_{l,m})$ for $l, m = 1, 2, \ldots, L - 1$

$$d_{l,m} = 2^{n-1} \left( \sum_{\mu_1(l,m)} p_i p_j + (-1)^n \sum_{\mu_2(l,m)} p_i p_j \right)$$

and

$$\mu_\lambda(l,m) = \{(i,j) : 0 \leq i, j \leq L - 1 \ \& \ 2(l-1) + i - j = (-1)^{\lambda + 1}(m - 1)\}, \quad \lambda = 1, 2$$
After getting the values of $\Gamma_k^n(x)$ for $x = L - 1$, we can design an algorithm to compute the values of $\Gamma_k^n(x)$ for $x$ and $k$ at integers. Actually, we only need to determine the values for $x = 1, 2, \ldots, L - 2$ and $k = x + 2 - L, x + 3 - L, \ldots, x - 1$ since Equations (7)–(10) hold. Let

$$\Gamma^n = [\Gamma^n(1), \ldots, \Gamma^n(L - 2)]^T$$

where

$$\Gamma^n(i) = [\Gamma^n_{i-L+2}(i), \ldots, \Gamma^n_{i-1}(i)]^T, \quad i = 1, 2, \ldots, L - 2$$

By Equations (11)–(13), for the $k$th ($k = 1, \ldots, L - 2$) component of $\Gamma^n(i)$, which is the $i$th ($i = 1, \ldots, L - 2$) component of $\Gamma^n$, we have

$$\Gamma^n_{i-(L-2)+(k-1)}(i) = 2^{n-1} \sum_{i_1, j_1 = 0}^{L-1} p_{i_1, j_1} \Gamma^n_{2[i-(L-2)+(k-1)]+i_1-j_1}(2i - j_1)$$

By Equations (7)–(10), Equation (14) gives

$$2^{1-n} \Gamma^n_{i-(L-2)+(k-1)}(i) = \sum_{\mu_1(i, k, L)} p_{i_1, j_1} \Gamma^n_{2[i-(L-2)+(k-1)]+i_1-j_1}(2i - j_1) + \sum_{\mu_2(i, k, L)} p_{i_1, j_1} \Gamma^n_{2[i-(L-2)+(k-1)]+i_1-j_1}(2i - j_1)$$

with

$$\mu_1(i, k, L) = \mu_1(i, k, L) \cup \mu_2(i, k, L) = \{i_1, j_1 : 0 \leq i_1, j_1 \leq L - 1\}$$

and

$$\mu_2(i, k, L) = \{i_1, j_1 \in \mu(i, k, L) : 2i - j_1 \geq L - 1 \text{ or } 2k + i_1 \leq L - 1\}$$

Here Equation (17) corrects an error in Equation (74) in [1].

If denoting by $d((i - 1)(L - 2) + k)$ the second term on the right-hand side of Equation (15), then from Equations (9) and (10)

$$d((i - 1)(L - 2) + k) = \sum_{\mu_2(i, k, L)} p_{i_1, j_1} \Gamma^n_{2[i-(L-2)+(k-1)]+i_1-j_1}(L - 1)$$

Here Equation (18) corrects an error in Equation (73) in [1]. Packing Equation (18) into the vector, we have

$$d^i = [d((i - 1)(L - 2) + 1), \ldots, d((i - 1)(L - 2) + k), \ldots, d(i(L - 2))]^T$$

Then we have the following relation from Equation (15):

$$(2^{1-n}I - Q_{i,i})\Gamma^n(i) - \sum_{j=1, j \neq i}^{L-2} Q_{i,j}\Gamma^n(j) = d^i \quad \text{for } i = 1, 2, \ldots, L - 2$$

where $Q_{i,j} = (q_{i,j,k,m})$ is a $(L - 2) \times (L - 2)$ matrix with $q_{i,j,k,m} = p_{2i-j}p_{L-1-2k+m}$, and $I$ is a square unit matrix of order $L - 2$. Finally, we get the following linear system for $\Gamma_k^n(x)$ for $x = 1, \ldots, L - 2$ and $k = x - L + 2, \ldots, x - 1$ from Equation (20):

$$\tilde{Q}\Gamma^n = (2^{1-n}I - Q)\Gamma^n = d$$
where $\widetilde{T}$ is a square unit matrix of order $(L-2)^2$, $Q = (Q_{i,j})$ is a square matrix of order $(L-2)^2$, and

$$d = [d^1, d^2, \ldots, d^{L-2}]^T$$

Noticing that $Q$ has eigenvalues $2^{-\lambda} (\lambda = 0, 1, \ldots, L-2)$, we can easily obtain the value of $\Gamma^n$ for $n = 0$ from Equation (21).

To end the study for $n > 0$, we need the following relations. From Equation (3), we have

$$\sum_{l=-\infty}^{\infty} l^n \Gamma^n_l (x) = n! \theta_1(x)$$

(22)

where $\theta_1(x) = \int_0^x \phi(y) dy$ that can be computed by algorithm of [1] for $x$ at integers. Applying Equations (7)–(10) to (22) and rearranging it gives that

$$\sum_{l=x-L+2}^{x-1} l^n \Gamma^n_l (x) = n! \theta_1(x) - \sum_{l=L-1-x}^{L-2} l^n \Gamma^n_l (L-1)$$

(23)

Here Equation (23) corrects an error in Equation (77) in [1]. Equation (23) represents the following relation in vector equation form:

$$[(x-L+2)^n, \ldots, (x-1)^n] \Gamma^n(x) = n! \theta_1(x) - \sum_{l=L-1-x}^{L-2} l^n \Gamma^n_l (L-1)$$

(24)

Here Equation (24) corrects an error in Equation (78) in [1]. Combining Equations (21) and (24) gives the value of $\Gamma^n$ for $n > 0$. Precisely, for $i = 1, 2, \ldots, n$

1. replace the $i$th row of $\tilde{Q}_{i,i} = 2^{1-n} I - Q_{i,i}$ and $\tilde{Q}_{i,j} = -Q_{i,j}$ by $[(i-L+2)^n, \ldots, (i-1)^n]$ and a zero row vector of order $L - 2$, respectively;
2. replace $d(i-1) (L-2) + i$, the $i$th element of $d^j$, by $n! \theta_1(i) - \sum_{l=L-1-i}^{L-2} l^n \Gamma^n_l (L-1)$.

Here steps (1) and (2) correct errors in the part between Equations (78) and (79) in [1].

With similar arguments, we can get the algorithms for the following two connection coefficients. Here, we are going to give out the difference only. To avoid any confusion, we denote Equation (*) in [1] by (C*).

3.2. Computation of $\Lambda_k^{m,n}(x)$

From [1], the values of $\Lambda_k^{m,n}(L-1)$ can be computed by equation

$$(2^{1+m-n} I - A) \Lambda_k^{m,n}(L-1) = b$$

(C84)

for $n < m$. However, we need an additional equation

$$\sum_{k=2-L}^{L-2} k^{n-m} \Lambda_k^{m,n}(L-1) = (-1)^n n! - \sum_{j=1}^{m} \sum_{k=2-L}^{L-2} k^{n-m+j} \binom{m}{j} \Lambda_k^{m-j,n}(L-1)$$

(C93)

for $n \geq m$ since the matrix $2^{1+m-n} I - A$ is singular. After getting the value for $x = L - 1$, using the similar arguments as in Section 3.1, we have the following equations for $x = 1, \ldots, L - 2$, $k = x - L + 2, \ldots, x - 1$

$$(2^{1+m-n} I - Q) \Lambda_k^{m,n} = e$$

(C96)
where \( Q, \Lambda_{m,n} \) and \( e \) are defined by Equations (C69), (C94) and (C97), respectively. And the elements of vector \( e \) are defined by the following equation, which corrects the errors in the original definition (C98, C99) in [1].

\[
e((x-1)(L-2)+k) = \sum_{i,j \in \mu} p_i p_j \Lambda_{2x-2(L-2)+2(k-1)-j+i}^{(m,n)}(L-1) + \sum_{i=0}^{L-1} \sum_{j=0}^{L-1} \sum_{i=1}^{m} p_i p_j \binom{m}{j} j! \Lambda_{2x-2(L-2)+2(k-1)-j+i}^{(m-l,n)}(2x-j)
\]  

(25)

where the index set \( \mu \) is given by

\[
\mu(x,k) = \{(i,j) : 2k+i \leq L-1 \text{ or } 2x-j \geq L-1 \text{ for } 0 \leq i, j \leq L-1\}
\]  

(26)

for \( x, k = 1, 2, \ldots, L-2 \). Here Equation (26) corrects an error in Equation (99) in [1].

If \( n \leq m \), we can obtain the values of \( \Lambda_{m,n} \) by directly solving Equation (C96). If \( n > m \), once again we need additional equation

\[
\sum_{k=x-L+2}^{x-1} k^n \Lambda_k^{m,n}(x) = n! M_0^m(x) - \sum_{k=2-L}^{x+1-L} k^n \Lambda_k^{m,n}(L-1)
\]  

(27)

which corrects an error in (C101) in [1].

3.3. Computation of \( \Upsilon_k^{m,n}(x) \)

From the definition of \( \Upsilon_k^{m,n}(x) \) and the two-scale relation (C34) and (C8), we have

\[
\Upsilon_k^{m,n}(x) = 2^{-(m+n+1)} \sum_{i=0}^{L-1} \sum_{j=0}^{m} p_i p_j \binom{m}{j} j! \Upsilon_{2k+2(j+i)}^{m-l,n}(2x-j)
\]  

(28)

and for \( k \leq 1-L \)

\[
\Upsilon_k^{m,n}(x) = \sum_{j=0}^{n-1} \theta_{n-j}(L-1) \left[ \sum_{l=0}^{j} \binom{j}{l} \frac{(1-k-L)^l}{l!} M_0^{m+j-l}(x) \right]
\]  

(29)

Here Equation (29) corrects an error in Equation (106) in [1].

Then from the properties of \( \Upsilon_k^{m,n}(x) \) and fixed \( m, n \), there are \((3L-4)(L-1)/2\) unknowns for \( x = 1, 2, \ldots, L-1 \) to be determined, which are packed in the following vector

\[
\Upsilon^{m,n} = [\Upsilon^{m,n}(1), \ldots, \Upsilon^{m,n}(L-1)]^T
\]  

(30)

with

\[
\Upsilon^{m,n}(x) = [\Upsilon_{2-L}^{m,n}(x), \Upsilon_{3-L}^{m,n}(x), \ldots, \Upsilon_{x-1}^{m,n}(x)]^T, \quad x = 1, 2, \ldots, L-1
\]  

(31)

which corrects an error in Equation (C108) in [1], and is also very different to the case of study for \( \Upsilon_k^{m}(x) \) and \( \Lambda_k^{m,n}(x) \) for \( x = 1, 2, \ldots, L-2 \).
Table I. Identified errors in [1] and corrections in this letter.

<table>
<thead>
<tr>
<th>Error in [1]</th>
<th>Correction in this letter</th>
</tr>
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<tbody>
<tr>
<td>Condition for Equation (56)</td>
<td>Condition for Equation (10)</td>
</tr>
<tr>
<td>Equation (74)</td>
<td>Equation (17)</td>
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<tr>
<td>Equation (73)</td>
<td>Equation (18)</td>
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<td>Equation (78)</td>
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<tr>
<td>The part between Equations (78) and (79)</td>
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<td>Equation (29)</td>
</tr>
<tr>
<td>Equation (108)</td>
<td>Equation (31)</td>
</tr>
</tbody>
</table>

4. SUMMARY

The paper [1] has many fundamental errors. The identified errors in [1] and corrections in this letter are summarized in Table I. When fixing those errors, we also developed an algorithm for computing the wavelets connection coefficients.

APPENDIX A

A.1. The derivation of Equation (10)

Equation (4) implies that the compact support of \( \phi^{(i)}, \ i = 0, \ldots, n \) is \([0, L - 1]\). Then from the definition of \( \Gamma^n_k(x) \), we have

\[
\Gamma^n_{-k}(x) = \int_0^x \phi^{(n)}(y + k)\phi(y) \, dy = \int_0^x \phi(y) \, d\phi^{(n-1)}(y + k) = \phi^{(n-1)}(y + k)\phi(y)|_0^x - \int_0^x \phi^{(n-1)}(y + k)\phi^{(1)}(y) \, dy
\]

The integration by parts for \( n \) times for \( x + k \geq L - 1 \) and taking \( y + k = \tilde{y} \) give that

\[
\Gamma^n_{-k}(x) = (-1)^n \int_0^x \phi^{(n)}(y)\phi(y + k) \, dy
\]

\[
= (-1)^n \int_k^{x+k} \phi^{(n)}(\tilde{y} - k)\phi(\tilde{y}) \, d\tilde{y} = (-1)^n \int_k^{L-1} \phi^{(n)}(\tilde{y} - k)\phi(\tilde{y}) \, d\tilde{y}
\]

\[
= (-1)^n \int_0^{L-1} \phi^{(n)}(\tilde{y} - k)\phi(\tilde{y}) \, d\tilde{y} = (-1)^n \Gamma^n_k(L - 1)
\]

since \( \int_0^k \phi^{(n)}(y - k)\phi(y) \, dy = 0 \).
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