The cost of the control in the case of a minimal time of control: the example of the one-dimensional heat equation

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Abstract

In this article, we consider the controllability of the one-dimensional heat equation with an internal control depending only on the time variable and an imposed profile depending on the space variable. It is well-known that in this context, there might exist a minimal time of null-controllability T_0, depending on the behavior of the Fourier coefficients of the profile. We prove two different results. The first one, which is surprising, is that the cost of the controllability in time $T > T_0$ close to T_0 may explode in an arbitrary way. On the other hand, we prove as a second result that for a large class of profiles, the cost of controllability at time $T > T_0$ is bounded from above by $\exp(C(T_0)/(T - T_0))$ for some constant $C(T_0) > 0$ depending on T_0. The main method used here is the moment method and some tools coming from complex analysis.

Keyworlds: null controllability; parabolic equations; minimal time; controllability cost; non-harmonic Fourier series; moment method.

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1 Introduction

Let $T > 0$. In what follows, we will consider the following controlled heat equation on $(0, T) \times (0, \pi)$:

$$\begin{cases} y_t - y_{xx} = f(x)u(t) & \text{in } (0,T) \times (0,\pi), \\ y(0,\cdot) = y^0 & \text{in } (0,\pi), \end{cases} \tag{1}$$

where $y^0 \in L^2(0, \pi)$, $u \in L^2(0, T)$ is the control and $f \in H^{-1}(0, \pi)$ is an imposed profile for this control.

It is well-known that equation (1) is well-posed in the sense that there exists a unique solution $y \in C^0([0, T], L^2(0, \pi)) \cap L^2([(0, T), H^0_1(0, \pi))$ verifying moreover that there exists a constant $C > 0$ such that for every $y^0 \in L^2(0, \pi)$, every $f \in H^{-1}(0, \pi)$ and every $v \in L^2(0, T)$, we have

$$||y||_{C^0([0,T],L^2(0,\pi))} + ||y||_{L^2((0,T),H^0_1(0,\pi))} \leqslant C(||y^0||_{L^2(0,\pi)} + ||f||_{H^{-1}(0,\pi)}||v||_{L^2(0,T)}),$$

which implies notably that the control operator $u \in \mathbb{R} \mapsto f(\cdot)u$ is admissible for the semigroup $e^{t\Delta}$ with domain $D(\Delta)$. The controllability properties of this equation has been widely studied

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(see notably [11], [8] and [2]). The approximate controllability can be easily characterized by the condition

$$f_k \neq 0, \forall k \in \mathbb{N}^*, \tag{2}$$

where

$$f_k := \langle f, e_k \rangle_{H^{-1}(0,\pi), H_0^1(0,\pi)}. \tag{3}$$

Assume from now on and until the end of the article that the condition (2) is satisfied. Concerning the study of the null-controllability of (1), one very efficient tool in the one-dimensional case is the celebrated *moment method* introduced in [10]. Let us consider the 1-D Laplace operator ∂_{xx} with domain $D(\Delta) := H^2(0,\pi) \cap H_0^1(0,\pi)$ and state space $H := L^2(0,\pi)$. It is well-known that $-\Delta : D(\Delta) \to L^2(0,\pi)$ is a positive definite operator with compact resolvent, the k-th eigenvalue is $\lambda_k = k^2$, an associated normalized eigenvector is

$$e_k(x) := \frac{\sqrt{2}}{\sqrt{\pi}} \sin(k\pi x).$$

Let us decompose the initial condition y^0 on the Hilbert basis e_k:

$$y^0(x) = \sum_{k=1}^{\infty} a_k e_k(x), \tag{4}$$

where $(a_k)_{k\in\mathbb{N}^*} \in l^2(\mathbb{N}^*)$. Then, it is classical that imposing $y(T,\cdot) = 0$ is equivalent to saying that for every $k \in \mathbb{N}^*$ we have

$$\int_0^T e^{\lambda_k t} u(t)dt = -\frac{a_k}{f_k}. \tag{5}$$

u is then solution of a moment problem which can be solved by finding a *bi-orthogonal family* to the family of exponentials $\{\exp(\lambda_k t)\}$ on $[0,T]$. Let us introduce the following quantities:

$$I_k(f) := -\frac{\log(|f_k|)}{k^2} \tag{6}$$

and

$$T_0 := \limsup_{k\to\infty} I_k(f) \in [0,\infty]. \tag{7}$$

It is then proved in [2] that:

1. System (1) is null-controllable at any time $T > T_0$.
2. System (1) is not null-controllable at any time $T < T_0$.

Hence, there might exist a minimal time of controllability, depending on the action of the control through the profile f.

Let us mention that in [8], a comprehensive study of (1) is performed in the particular case where $f(x) := \delta_{x_0} \in H^{-1}(0,\pi)$, where $x_0 \in (0,\pi)$. In this particular case, one readily obtain that the minimal time of controllability is given by

$$T_0(x_0) := \limsup_{k\to\infty} -\frac{\log(|\sin(kx_0)|)}{k^2}.$$

The dependance with respect to x_0 of T_0 is then carefully studied and it can notably be proved that:

1. For almost all $x_0 \in (0,\pi)$, $T_0(x_0) = 0$,
2. For every $\tau \in [0,\infty]$, $\{x_0 \in (0,\pi) | T_0(x_0) = \tau\}$ is dense in $(0,\pi)$.

This notably means that the minimal time of controllability can take any value between 0 and ∞. Let us mention that the existence of a minimal time of control for parabolic equations or systems may occur in many other situations, for other examples see notably [1] and [2].

In the case where $T > T_0$, one can easily prove (see for example [7, Chapter 2, Section 2.3]) that for every $y^0 \in L^2(0,\pi)$, there exists a unique optimal (for the $L^2(0,T)$-norm) control $u_{opt} \in L^2(0,T)$ bringing y^0 to 0, the map $y^0 \mapsto u_{opt}$ is then linear continuous. The norm of this operator is called the *optimal null control cost* at time T (or in a more concise form the cost of the control), denoted by $C_H(T)$. By definition, $C_H(T)$ is the smallest constant amongst the constants $C > 0$ such that for every $y^0 \in L^2(0,\pi)$, there exists some control u driving y^0 to 0 at time T with

$$||u||_{L^2(0,T)} \leqslant C||y^0||_{L^2(0,\pi)}.$$

In the case where there is no minimal time of control and under appropriate conditions on the f_k that should not decrease too fast (what we will call "the usual case" from now on), the cost of fast controls for linear parabolic equations or systems with distributed (or equivalently in this case boundary) control has been widely studied (see notably [14], [19], [22], [9], [16], [3] and [17]) and is now quite well-understood in the one-dimensional case, even if there are still some open problems. We mention that understanding the behavior of the cost of controllability is crucial because it can be used to deduce some results in higher dimension (see [3]) and to obtain nonlinear results (see notably [12]). It is shown that in the context of the usual case, the cost of fast controls is roughly of the form $\exp(C/T)$ as $T \to 0$, were C is some appropriate constant depending on the geometry. Let us mention that for all the cases studied up to now, the cost of fast controls was a purely high-frequency phenomena, depending only on the asymptotic behavior of the eigenvalues at infinity (for a precise study, see notably [16], [17] and [18]).

Let us also mention that the multi-dimensional case seems to be out of reach for the moment: there are only few results of existence of minimal time, always in particular geometries (see notably for the case of hypoelliptic diffusions [4] or [5]) and sometimes this minimal time is not even known precisely. Concerning the cost of the control, even in the usual case of the heat equation with boundary or distributed control, there are only some partial results coming from [19], the upper bound being obtained under very strong geometric restrictions. Hence, taking into account the lack of comprehension of the multi-dimensional case, the author thinks that it is very reasonable and interesting to have a look first at the one-dimensional case, which is maybe simpler from the multi-dimensional case but cannot be considered as trivial though.

In this context, a natural question arising is the following: *for equation* (1)*, what is the behavior of $C_H(T)$ when $T \to T_0^+$?* Up to our knowledge, this question has never been investigated in the context of the existence of a minimal time on control in the parabolic case. One could expect that the cost is of the form $\exp(C(T_0)/(T - T_0))$ as $T \to T_0^+$, by analogy with the usual case. However, it is not always the case, as highlighted by the following Theorem:

Theorem 1.1 *For every function $g : \mathbb{R}^{+*} \to \mathbb{R}^{+*}$ supposed to be increasing and verifying $g(x) \to +\infty$ for $x \to +\infty$, for every $T_0 \in [0,\infty)$, there exists $f \in H^{-1}(0,\pi)$ such that for any T close enough to T_0, one has*

$$C_H(T) \geqslant \frac{1}{\sqrt{T}} g\left(\frac{1}{T - T_0}\right).$$

This theorem means that the cost of the control *can increase arbitrarily fast* as T goes to the minimal time T_0, which is very surprising. This can be explained by the fact that contrary to the usual case, the cost of the control depends not only on the behavior of $I_k(f)$ (defined in (6)) at infinity but also on *how it differs from its limit superior* T_0. The proof relies on tools coming from complex analysis in the spirit of [6] (see also [17] and [13]). The main idea (that differs from what was done in the previous references) is to consider the optimal control associated to an initial condition with one pure eigenmode which is not necessarily the first one and that is adapted to the time of control T. f is then chosen is such a way that the sequence f_k decreases "very slowly" to the minimal time T_0, the rate of convergence depending on the choice of the function g. One serious consequence of this result is that it may be hard to obtain sharp results in the semi-linear case in this context.

Our second theorem provides an upper bound on $C_H(T)$ under some adequate assumption on the sequence $(f_k)_{k\in\mathbb{N}}$.

Theorem 1.2 *Assume that f is chosen such that*

$$I_k(f) \leqslant T_0, \forall k \in \mathbb{N}^*. \tag{8}$$

Then there exists a constant $C(T_0) > 0$, depending only on T_0, such that

$$C_H(T) \leqslant \exp\left(\frac{C(T_0)}{T-T_0}\right).$$

Here we use the moment method in the spirit of what was done in [22]. The idea is to use the Paley-Wiener strategy, the main difficulty is to "catch" precisely the minimal time of control, which requires a careful study and a new estimate on the multiplier of [22]. Let us mention that in [2], the strategy used by the authors was slightly different since it was based on an idea coming from [21], the main drawback of the argument being that we cannot estimate precisely the constants appearing and hence Schwartz's strategy is useless in order to estimate precisely the cost of the control. Let us mention that condition (8) is likely far from being sharp to obtain the conclusion of Theorem 1.2 since for $T_0 = 0$, we obtain a different (and in general more restrictive) condition than in the usual case, where the cost of the control is always of the form $\exp(C/T)$.

Concerning some extensions and open problems arising after this study, we can mention the following:

— For a given profile f, can we obtain precise estimates on $C_H(T)$ for T close to T_0? Notably, is it possible to give a lower bound that is true for every profile f? If this lower bound exists, is it of the form $\exp(C/(T-T_0))$?

— We chose in this paper to study a very particular, which is of interest because of the unexpected behavior it highlights. An interesting question would be: can we generalize the study to other cases where there exists a minimal time of control, notably in the case where the minimal time of control occurs because of the condensation of the eigenvalues as in the system presented in [2, Section 6.2]? More generally, can we extend the study in the abstract case given in [2]? (this is more challenging since even in the case where there is no minimal time of control, it seems to the author that there is no general results concerning the cost of fast controls)

2 Proofs of Theorems 1.1 and 1.2

2.1 Proof of Theorem 1.1

In all what follows, C will always be a numerical constant independent of the parameter T. Let us fix some $T > T_0$, where $T_0 \in [0, \infty)$ is given by (7).

Let $n \in \mathbb{N}^*$ to be chosen later (depending on T). We define $y^0 \in L^2(0,\pi)$ as follows:

$$y^0(x) := \sin(nx). \tag{9}$$

One readily verifies that there exists some numerical constant C (independent on n) such that

$$||y^0||_{L^2(0,\pi)} \leqslant C. \tag{10}$$

We consider u the optimal control associated to this initial condition, which verifies by definition and thanks to estimate (10)

$$||u||_{L^2(0,\pi)} \leqslant C_H(T)||y^0||_{L^2(0,\pi)} \leqslant CC_H(T). \tag{11}$$

Thanks to the moment problem verified by any control, we obtain that for any $k \in \mathbb{N}$ we have

$$f_k \int_0^T u(t) \exp\left(k^2 t\right) dt = -\int_0^\pi \sin(nx)\sin(kx)\, dx. \tag{12}$$

Let us define the complex function v by

$$v(z) := \int_{-T/2}^{T/2} u\left(t + \frac{T}{2}\right) \exp(-izt) dt. \tag{13}$$

Using (12) and (13), we deduce that

$$v(in^2) = -\frac{\pi}{2f_n} \exp\left(-\frac{n^2 T}{2}\right), \tag{14}$$

and for every $k \in \mathbb{N}$ with $k \neq n$ we have

$$v(ik^2) = 0. \tag{15}$$

We deduce, using (13), (11) and the Cauchy-Schwarz inequality, that

$$\begin{aligned} |v(z)| &\leqslant \exp\left(\frac{T|Im(z)|}{2}\right) \int_0^T |u(t)| dt \\ &\leqslant C_H(T)\sqrt{T} \exp\left(\frac{T|Im(z)|}{2}\right) ||y_0||_{L^2(0,\pi)} \\ &\leqslant CC_H(T)\sqrt{T} \exp\left(\frac{T|Im(z)|}{2}\right). \end{aligned} \tag{16}$$

In what follows, we call

$$b_k := ik^2. \tag{17}$$

Using the usual representation of the functions of exponential type given for example in [15, Theorem p.56], we have, for every z such that $Im(z) > 0$,

$$\ln(|v(z)|) = \sum_1^\infty \ln\left(\frac{|z - a_l|}{|z - \overline{a_l}|}\right) + \sigma x_2 + \frac{x_2}{\pi} \int_{\mathbb{R}} \frac{\ln(|v(\tau)|)}{|\tau - z|^2} d\tau,$$

where the a_k are all the roots of v of positive imaginary part and σ is the type of v, which verifies thanks to (16) that

$$\sigma \leqslant \frac{T}{2}. \tag{18}$$

We apply this equality at point $b_n = in^2$, then we use (17) (remark that b_n is a pure imaginary number) and (18) to obtain

$$\ln(|v(b_n)|) \leqslant \sum_{l=1}^\infty \ln\left(\frac{|b_n - a_l|}{|b_n - \overline{a_l}|}\right) + \frac{Tn^2}{2} + \frac{|b_n|}{\pi} \int_{\mathbb{R}} \frac{\ln(|f(\tau)|)}{\tau^2 + |b_n|^2} d\tau. \tag{19}$$

Let us study the right-hand side of this equality.

1. First term of the right-hand side: We study
$$\sum_{l=1}^{\infty} \ln\left(\frac{|b_n - a_l|}{|b_n - \overline{a_l}|}\right).$$
It is easy to prove that, if $(z_1, z_2) \in \mathbb{C}^2$, then
$$\frac{|z_1 - z_2|}{|z_1 - \overline{z_2}|} \leqslant 1 \text{ if and only if } Im(z_1)Im(z_2) \geqslant 0.$$
Hence, we readily infer that
$$\sum_{l=1}^{\infty} \ln\left(\frac{|b_n - a_l|}{|b_n - \overline{a_l}|}\right) \leqslant 0. \tag{20}$$
2. Concerning the last term of the right-hand-side, an easy change of variables gives
$$|b_n| \int_{\mathbb{R}} \frac{d\tau}{\tau^2 + |b_n|^2} = \pi.$$
Hence, using the fact that τ is real and (16), we deduce that
$$\frac{|b_n|}{\pi} \int_{\mathbb{R}} \frac{\ln|v(\tau)|}{\tau^2 + b_n^2} d\tau \leqslant \ln(CC_H(T)\sqrt{T}). \tag{21}$$

Using (14), (19), (20) and (21), we deduce that
$$\ln\left(\frac{\pi}{2|f_n|}\right) - \frac{n^2 T}{2} \leqslant \frac{n^2 T}{2} + \ln(CC_H(T)\sqrt{T}), \tag{22}$$
hence there exists a numerical constant $C > 0$ such that
$$C_H(T) \geqslant \frac{C}{|f_n|\sqrt{T}} \exp\left(-n^2 T\right). \tag{23}$$

Now, let us consider any positive and increasing function $h : \mathbb{R}^{+*} \to \mathbb{R}^{+*}$ such that $h(x) \to \infty$ as $x \to \infty$. Such a function is necessarily bijective and we call h^{-1} its reciprocal function. Let us consider $(f_n)_{n\in\mathbb{N}^*} \in l^2(\mathbb{N}^*)$ defined in such a way that
$$I_n(f) = T_0 + \frac{1}{h^{-1}(n^2)}.$$
One can for example consider
$$f_n := \exp\left(-n^2\left(T_0 + \frac{1}{h^{-1}(n^2)}\right)\right) \in l^2(\mathbb{N}^*).$$

In this case it is clear that $\limsup_{n\to\infty} I_n(f) = \lim_{n\to\infty} \frac{\log\left(\frac{1}{|f_n|}\right)}{n^2} = T_0$. Then we have thanks to (23)
$$C_H(T) \geqslant \frac{C}{\sqrt{T}} \exp\left(n^2\left(T_0 - T + \frac{1}{h^{-1}(n^2)}\right)\right). \tag{24}$$
Let us now explain how to choose n. We assume that T is close enough to T_0. Now, we choose n such a way that (for example)
$$\frac{1}{2(T - T_0)} \geqslant h^{-1}(n^2) \geqslant \frac{1}{4(T - T_0)}, \tag{25}$$

which is always possible (at least for T close enough to T_0) since h^{-1} is increasing and goes to ∞ at ∞.

Hence, we deduce using (24) and (25) that

$$C_H(T) \geqslant \frac{C}{\sqrt{T}} \exp\left((T-T_0)h\left(\frac{1}{4(T-T_0)}\right)\right).$$

One then easily obtain the desired result by choosing h in such a way that

$$g(x) = C\exp\left(\frac{1}{x}h\left(\frac{x}{4}\right)\right), \text{i.e. } h(x) = 4x\log\left(\frac{g(4x)}{C}\right),$$

because it is clear that if g is positive, increasing and goes to ∞ at ∞ then h is well-defined at least for large enough x (which is sufficient for our purpose), is increasing and goes to ∞ at ∞. ■

2.2 Proof of Theorem 1.2

In all what follows, C will always be a numerical constant independent of all parameters. We consider some time $T > T_0$. We will construct our bi-orthogonal family by using the celebrated Paley-Wiener Theorem. Let us recall that T_0 is given by (7).

First of all, we define

$$F(z) := \prod_{k=1}^{\infty}(1+\frac{iz}{k^2}) = \frac{\sin(\pi\sqrt{-iz})}{\pi\sqrt{-iz}}. \tag{26}$$

F will be used in what follows for the construction of the biorthogonal family to $(e^{\lambda_k t})_{k\in\mathbb{N}^*}$.

Now, we introduce the multiplier, which is very similar to the one studied in [22]. Let $\nu > 0$ and $\delta \in (0,1)$ some parameters. From now on we call

$$\beta := \frac{T(1-\delta)}{2}. \tag{27}$$

We introduce

$$\sigma_\nu(t) := \exp\left(-\frac{\nu}{1-t^2}\right)$$

prolonged by 0 outside $(-1;1)$. We call

$$H_\beta(z) := C_\nu \int_{-1}^{1} \sigma_\nu(t)e^{-it\beta z}dt, \tag{28}$$

where

$$C_\nu := 1/||\sigma_\nu||_1.$$

Looking carefully at the proof of [22, Lemma 4.3], we can easily deduce

$$\frac{1}{2}e^\nu \leqslant C_\nu \leqslant \frac{3}{2}\sqrt{\nu+1}e^\nu, \tag{29}$$

$$|H_\beta(z)| \leqslant e^{\frac{T}{2}|Im(z)|}, \tag{30}$$

$$H_\beta(x) \leqslant C\sqrt{\nu\beta|x|}|\sqrt{\nu+1}e^{3\nu/4-\sqrt{\nu\beta|x|}}. \tag{31}$$

The main new estimate that will interest us (and that differs from what is done in [22]) is the following:

Lemma 2.1 *For any $x \in \mathbb{R}^+$ and any $r \in (1/2,1)$, we have*

$$H_\beta(ix) \geqslant C(1-\sqrt{r})e^{-\frac{\nu}{1-r}+\beta r x}. \tag{32}$$

Proof of Lemma 2.1 Let $r \in (1/2, 1)$ be some parameter destined to tend to 1, let $\eta \in (0,1)$ and $\mu \in (0,1)$ some other parameters that will be linked to r afterwards. Then, using the expression of H_β given in (28), we obtain by restricting the integral over $((1-\mu)\eta, \eta)$ that

$$|H_\beta(ix)| \geqslant \mu\eta C_\nu e^{\frac{-\nu}{1-\eta^2}+\beta x\eta(1-\mu)}. \tag{33}$$

Now, we choose $\eta = \sqrt{r} \in (0,1)$ and $\mu = 1-\sqrt{r} \in (0,1)$, so that $\eta(1-\mu) = r$.

We obtain thanks to (33) and (29) that

$$H_\beta(ix) \geqslant C\sqrt{r}(1-\sqrt{r})e^{-\frac{\nu}{1-r}+\beta rx} \geqslant C(1-\sqrt{r})e^{-\frac{\nu}{1-r}+\beta rx}.$$

■

Let us know define what will be the Fourier transform of our bi-orthogonal family. We set

$$\Phi_k(z) := \frac{F(z)}{(z+ik^2)F'(ik^2)} \frac{H_\beta(z)}{H_\beta(ik^2)}. \tag{34}$$

Using the definition of F given in (26), it is clear that

$$\Phi_k(i\lambda_n) = \delta_{k,n}. \tag{35}$$

Let us prove that Φ_k is of exponential type $T/2$. This is the purpose of the next lemma.

Lemma 2.2 *There exists some constant $C_k > 0$ such that for every $z \in \mathbb{C}$, one has*

$$\frac{F(z)}{(z+ik^2)F'(ik^2)} \leqslant C_k e^{\pi\sqrt{|z|}}. \tag{36}$$

Consequently, Φ_k is of exponential type $T/2$.

Proof of Lemma 2.2 Since

$$z \mapsto \frac{F(z)}{(z+ik^2)F'(ik^2)} \text{ is continuous on } \mathbb{C},$$

it is enough to prove inequality (36) for $|z|$ large enough. For instance, for $|z| \geqslant 2k^2$, we have

$$\frac{|F(z)|}{|z+ik^2||F'(ik^2)|} \leqslant C|F(z)| \leqslant Ce^{\pi\sqrt{|z|}}.$$

Using (36) together with (30) and the definition of β given in (27), we deduce that Φ_k is of exponential type $T/2$, and the proof is complete.

■

Let us know give a precise estimate of $z \mapsto \frac{F(z)}{(z+ik^2)F'(ik^2)}$ on the real axis.

Lemma 2.3 *For $x \in \mathbb{R}$, one has*

$$\frac{|F(x)|}{|(z+ik^2)F'(ik^2)|} \leqslant Ce^{\pi\sqrt{\frac{|x|}{2}}}. \tag{37}$$

Proof of Lemma 2.3 Since $x \in \mathbb{R}$, we have

$$\frac{|F(x)|}{|(x+ik^2)F'(i\lambda_k)|} \leqslant \frac{|F(x)|}{k^2|F'(ik^2)|}. \tag{38}$$

Let us estimate $|F(x)|$. One more time we use that we know explicitly the form of F and the fact that Re $\sqrt{i} = 1/\sqrt{2}$ to deduce that

$$|F(x)| \leqslant e^{\pi\sqrt{\frac{|x|}{2}}}.$$

It remains us to estimate $|F'(i\lambda_k)|$.

From (26), one easily infer that

$$|F'(ik^2)| = \frac{1}{2k^2}$$

which enables us to conclude as wished thanks to (38).

■

Let us know give the final estimate of our multiplier.

Proposition 2.1 *For some well-chosen ν (depending on δ and T), $\Phi_k \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and*

$$||\Phi_k||_{L^1(\mathbb{R})} \leqslant C\frac{\sqrt{\nu+1}}{1-\sqrt{r}}e^{(\frac{3}{4}+\frac{1}{1-r})\nu - r\beta\lambda_k}. \tag{39}$$

Proof of Proposition 2.1 Putting together (37), (31) and (32), we obtain that for $x \in \mathbb{R}$, we have

$$|\Phi_k(x)| \leqslant C\frac{\sqrt{\nu+1}\sqrt{\nu\beta|x|}|}{1-\sqrt{r}}e^{\pi\sqrt{\frac{|x|}{2}}+(\frac{3}{4}+\frac{1}{1-r})\nu-\sqrt{\nu\beta|x|}-r\beta\lambda_k}.$$

Let us now choose ν. We choose ν in such a way that

$$\sqrt{\beta\nu} = \frac{1}{\sqrt{2}} + 1,$$

i.e.

$$\nu := \frac{(1+\sqrt{2})^2}{(1-\delta)T}. \tag{40}$$

We deduce that

$$|\Phi_k(x)| \leqslant C\frac{\sqrt{\nu+1}\sqrt{|x|}|}{1-\sqrt{r}}e^{(\frac{3}{4}+\frac{1}{1-r})\nu-\sqrt{|x|}-r\beta\lambda_k}.$$

Hence, we have that $\Phi_k \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and

$$||\Phi_k||_{L^1(\mathbb{R})} \leqslant C\frac{\sqrt{\nu+1}}{1-\sqrt{r}}e^{(\frac{3}{4}+\frac{1}{1-r})\nu - r\beta\lambda_k}.$$

■

Proof of Theorem 1.2. We are now able to construct the control. Using the version of the Paley-Wiener Theorem given in [20, Th. 19.3, p. 370], we can state that for every $k \in \mathbb{N}^*$, Φ_k is the Fourier transform of a function $w_k \in L^2(\mathbb{R})$ with compact support $[-T/2, T/2]$. Moreover, by construction $\{w_k\}$ is biorthogonal to the family $\{e^{-k^2 t}\}$ on $[-T/2, T/2]$. Then, one can create the control thanks to the family $\{h_k\}$. Going back to expression (5), we consider a control u defined by

$$u(t) := -\sum_{k=1}^{\infty} \frac{a_k}{f_k} \exp\left(-\frac{Tk^2}{2}\right) w_k\left(t - \frac{T}{2}\right). \tag{41}$$

Let us remark that going back to the expression of T_0 given in (7), the expression of u is meaningful as soon as δ is small enough and r is close enough to 1(depending on $T-T_0$), thanks to (27) and (39), which will be assumed from now on.

By construction, the corresponding solution y of (1) verifies $y(T,\cdot)\equiv 0$. Moreover, one easily verifies that $u\in C^0([0,T],\mathbb{R})$. Using (41), (27) and inequality (39), we obtain

$$||u(t)||_{L^\infty(0,T)}\leqslant C\frac{\sqrt{\nu+1}}{1-\sqrt{r}}e^{(\frac{3}{4}+\frac{1}{1-r})\nu}\sum_k\frac{|a_k|}{|f_k|}e^{-k^2(\frac{rT(1-\delta)}{2}+\frac{T}{2})},$$

that we rewrite as

$$||u(t)||_{L^\infty(0,T)}\leqslant C\frac{\sqrt{\nu+1}}{1-\sqrt{r}}e^{(\frac{3}{4}+\frac{1}{1-r})\nu}\sum_k|a_k|e^{k^2\left(I_k(f)-\frac{rT(1-\delta)}{2}-\frac{T}{2}\right)}.$$

Hence, using condition (8), we deduce

$$||u(t)||_{L^\infty(0,T)}\leqslant C\frac{\sqrt{\nu+1}}{1-\sqrt{r}}e^{(\frac{3}{4}+\frac{1}{1-r})\nu}\sum_k|a_k|e^{k^2\left(T_0-\frac{rT(1-\delta)}{2}-\frac{T}{2}\right)}. \tag{42}$$

The equation (in the variable r)

$$T_0-\frac{rT(1-\delta)}{2}-\frac{T}{2}=-\frac{T-T_0}{2}$$

has a unique solution $r_0\in(0,1)$ given by

$$r_0:=\frac{T_0}{T(1-\delta)} \tag{43}$$

as soon as

$$0<\delta<\frac{T-T_0}{T}. \tag{44}$$

Going back to (42) and using the particular value of r given in (43) together with the Cauchy-Schwarz inequality we obtain that

$$||u(t)||_{L^\infty(0,T)}\leqslant\frac{C}{\sqrt{T-T_0}}\frac{\sqrt{\nu+1}}{1-\sqrt{r_0}}e^{(\frac{3}{4}+\frac{1}{1-r_0})\nu}(\sum_k|a_k|^2)^{1/2}. \tag{45}$$

We now choose $\delta=\frac{T-T_0}{2T}$, which verifies condition (44), so that by (43) we obtain

$$r_0=\frac{2T_0}{T+T_0}. \tag{46}$$

Hence, for T close enough to T_0, taking into account (45), the definition of ν given in (40), (46), and the fact that all the terms appearing in the right-hand side of (45) in front of the exponential are at most powers of $T-T_0$, we obtain

$$||u(t)||_{L^\infty(0,T)}\leqslant e^{\frac{C(T_0)}{T-T_0}}||y^0||_H.$$

■

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