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Abstract

In this paper, we generalize the upper bound in Varadhan's Lemma. The standard formulation of Varadhan's Lemma contains two important elements, namely an upper semicontinuous integrand and a rate function with compact sublevel sets. However, motivated by results from queueing theory, in this paper we do not assume that rate functions have compact sublevel sets. Moreover, we drop the assumption that the integrand is upper semicontinuous and replace it by a weaker condition. We prove that the upper bound in Varadhan's Lemma still holds under these weaker conditions. Additionally, we show that only measurability of the integrand is required when the rate function is continuous.

Keywords. Varadhan's Lemma ⋆ exponential integrals ⋆ large deviations principle ⋆ upper bound

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1 Introduction

Exponential integrals often play an important role in the proofs of large deviations principles (LDPs). Varadhan's Lemma is a powerful generalization of Laplace's method to find bounds for the logarithmic asymptotics of exponential integrals. Especially the upper bound provided in this lemma turns out to be a very useful tool for proving large deviations upper bounds. However, the result is stated under somewhat restrictive conditions, which rule out many interesting cases. In particular, certain rate functions arising in queueing theory do not satisfy the conditions of Varadhan's Lemma. Motivated by this observation, we will loosen the conditions under which the upper bound is known to hold.

2 Main result

Let $\mathcal{X}$ be a topological space and let $\mathcal{B}$ be a σ-algebra on $\mathcal{X}$. Equip $[-\infty, \infty]$ with the standard Borel σ-algebra. For $\epsilon > 0$, let μ_ϵ be a probability measure defined on $\mathcal{B}$ and let $\phi\colon \mathcal{X} \to [-\infty, \infty]$ be a measurable function. Additionally, let $\phi^{(\epsilon)}\colon \mathcal{X} \to [-\infty, \infty]$ be a measurable function for each $\epsilon > 0$. We will say that the family of probability measures $\{\mu_\epsilon \mid \epsilon > 0\}$ satisfies a large deviations upper bound (LDUB) with rate function J if

$$\limsup_{\epsilon\to 0} \epsilon \log \mu_\epsilon(F) \le - \inf_{x\in \mathrm{cl}F} J(x) \tag{1}$$

for any $F \in \mathcal{B}$, where $J\colon \mathcal{X} \to [0, \infty]$ is a function and $\mathrm{cl}A$ denotes the closure of a set A.

Varadhan's Lemma provides sufficient conditions such that the inequality

$$\limsup_{\epsilon\to 0} \epsilon \log \int_{\mathcal{X}} \exp\Big[\tfrac{1}{\epsilon}\phi^{(\epsilon)}(x)\Big]\mu_\epsilon(\mathrm{d}x) \le \sup_{x\in\mathcal{X}}[\phi(x) - J(x)] \tag{2}$$

holds. The original conditions for the upper bound in Varadhan's Lemma are given in the following lemma (cf. [6, Th. 3.1] and [7, Th. 2.3]).

Lemma 2.1. *Suppose that μ_ϵ satisfies* (1) *for some function $J\colon \mathcal{X} \to [0, \infty]$ and assume that the following conditions hold.*

a. The space $\mathcal{X}$ is a regular topological space and $\mathcal{B}$ contains its Borel σ-algebra.

b. The function J is lower semicontinuous.

c. The function J has compact sublevel sets.

d. For each $\delta > 0$ and $x \in \{J < \infty\}$ there exists an open neighborhood $\mathcal{O}$ of x and some $\epsilon^* > 0$ such that $\phi^{(\epsilon)}(y) \leq \phi(x) + \delta$ for all $y \in \mathcal{O}$ and all $\epsilon \in (0, \epsilon^*)$.

e. The functions $\phi^{(\epsilon)}$ satisfy

$$\lim_{M\to\infty} \limsup_{\epsilon\to 0} \epsilon \log \int_{\mathcal{X}} \exp\left[\tfrac{1}{\epsilon}\phi^{(\epsilon)}(x)\right] \mathbb{1}_{\{\phi^{(\epsilon)}(x)>M\}}\, \mu_\epsilon(\mathrm{d}x) = -\infty.$$

f. The function ϕ takes values in $[-\infty, \infty)$.

Then (2) *holds.*

Condition d essentially requires that ϕ is upper semicontinuous. Indeed, this is often assumed in advance (cf. [2], [3] and [7]). However, in queueing theory one encounters functions ϕ that are not upper semicontinuous (cf. Example 3.3). Moreover, one also encounters rate functions J that do not have compact level sets (cf. [4, Ex. 4.8]). Therefore, we would like to weaken the assumptions and generalize the upper bound in Varadhan's Lemma.

Our strategy is as follows. In the proof of Lemma 2.2, we will provide some elementary conditions under which Varadhan's upper bound holds. Then we will show that the set of conditions in the original statement imply the elementary conditions. Next, we will show that some other novel sets of conditions also imply the elementary conditions. In particular, these novel sets of conditions do not require compact sublevel sets of J nor upper semicontinuity of ϕ. Importantly, this will shed some light on the role of the original assumptions in Varadhan's Lemma. The proof of Lemma 2.2 is inspired by the proof of Varadhan's Lemma in [3].

As is customary, we define $\exp(-\infty) = 0$, $\log(0) = -\infty$, $\exp(\infty) = \log(\infty) = \infty$ and $0 \cdot \infty = 0$. For $b \in [-\infty, \infty]$, we define $f_b = f \wedge b = \min\{f, b\}$ for a function $f \colon \mathcal{X} \to [-\infty, \infty]$.

Lemma 2.2. *Suppose that μ_ϵ satisfies* (1) *for some function $J \colon \mathcal{X} \to [0, \infty]$ and define $B := \lim_{M\to\infty} \sup_{x\in\mathcal{X}} [\phi_M(x) - J(x)]$.*

Consider the following sets of conditions. We will refer to condition y of C.x by C.x.y. Condition C.2.b and C.3.b are identical.

C.1 *The conditions a, b, c and d of Lemma 2.1 hold.*

C.2 a. *The superlevel set $\phi^{-1}([w,\infty])$ is closed for every $w \in \mathbb{R}$ satisfying the inequality $w \geq B$.*

b. *For each $\delta > 0$ there exists $\epsilon^* > 0$ such that $\phi^{(\epsilon)} \leq \phi + \delta$ on the set $\{\phi > B\}$ and $\phi^{(\epsilon)} \leq B + \delta$ on $\{\phi \leq B\}$ for all $\epsilon \in (0, \epsilon^*)$.*

C.3 a. *The function J is continuous.*

b. *For each $\delta > 0$ there exists $\epsilon^* > 0$ such that $\phi^{(\epsilon)} \leq \phi + \delta$ on the set $\{\phi > B\}$ and $\phi^{(\epsilon)} \leq B + \delta$ on $\{\phi \leq B\}$ for all $\epsilon \in (0, \epsilon^*)$.*

Suppose that the conditions of at least one of the sets C.1, C.2 or C.3 are satisfied. Then it holds that

$$\limsup_{\epsilon \to 0} \epsilon \log \int_{\mathcal{X}} \exp\left[\tfrac{1}{\epsilon}\phi^{(\epsilon)}(x)\right] \mu_\epsilon(\mathrm{d}x) \leq \lim_{M \to \infty} \sup_{x \in \mathcal{X}} [\phi_M(x) - J(x)] \tag{3}$$

if and only if

$$\lim_{M \to \infty} \limsup_{\epsilon \to 0} \epsilon \log \int_{\mathcal{X}} \exp\left[\tfrac{1}{\epsilon}\phi^{(\epsilon)}(x)\right] \mathbb{1}_{\{\phi^{(\epsilon)}(x) > M\}} \mu_\epsilon(\mathrm{d}x) \leq \lim_{M \to \infty} \sup_{x \in \mathcal{X}} [\phi_M(x) - J(x)]. \tag{4}$$

Proof. The statement of the lemma is trivial if $B = \infty$, so in the remainder of this proof we assume that $B < \infty$.

First, observe the following. If the functions ϕ and $\phi^{(\epsilon)}$ satisfy one of the sets of assumptions, then the functions ϕ_b and $\phi_b^{(\epsilon)}$ satisfy the same set of assumptions for each $b \in [-\infty, \infty]$. Moreover, for each

$b > 0$ it holds that

$$\begin{aligned}
&\limsup_{\epsilon\to 0} \epsilon \log \int_{\mathcal{X}} \exp\left[\tfrac{1}{\epsilon}\phi^{(\epsilon)}(x)\right] \mu_\epsilon(\mathrm{d}x) \\
&= \limsup_{\epsilon\to 0} \epsilon \log \Bigg(\int_{\mathcal{X}} \exp\left[\tfrac{1}{\epsilon}\phi_b^{(\epsilon)}(x)\right] \mathbb{1}_{\{\phi^{(\epsilon)}(x)\le b\}}\, \mu_\epsilon(\mathrm{d}x) \\
&\qquad + \int_{\mathcal{X}} \exp\left[\tfrac{1}{\epsilon}\phi^{(\epsilon)}(x)\right] \mathbb{1}_{\{\phi^{(\epsilon)}(x)> b\}} \mu_\epsilon(\mathrm{d}x) \Bigg) \\
&= \max\Bigg\{ \limsup_{\epsilon\to 0} \epsilon \log \int_{\mathcal{X}} \exp\left[\tfrac{1}{\epsilon}\phi_b^{(\epsilon)}(x)\right] \mathbb{1}_{\{\phi^{(\epsilon)}(x)\le b\}}\, \mu_\epsilon(\mathrm{d}x), \\
&\qquad \limsup_{\epsilon\to 0} \epsilon \log \int_{\mathcal{X}} \exp\left[\tfrac{1}{\epsilon}\phi^{(\epsilon)}(x)\right] \mathbb{1}_{\{\phi^{(\epsilon)}(x)> b\}} \mu_\epsilon(\mathrm{d}x) \Bigg\}.
\end{aligned}$$

Suppose that for each fixed $b > 0$ equation (2) holds with $\phi^{(\epsilon)}$ and ϕ replaced by $\phi_b^{(\epsilon)}$ and ϕ_b, respectively. Then the first term in the maximum above is bounded above by $\sup_{x\in\mathcal{X}}[\phi_b(x) - J(x)]$ and thus by $\sup_{x\in\mathcal{X}}[\phi(x) - J(x)]$ for each $b > 0$.

Hence, to prove the lemma, it suffices to show that if the functions ϕ and $\phi^{(\epsilon)}$ satisfy the conditions of C.1, C.2 or C.3, then for each $b > 0$ equation (2) holds with $\phi^{(\epsilon)}$ and ϕ replaced by $\phi_b^{(\epsilon)}$ and ϕ_b, respectively.

Suppose that for each fixed $b > 0$ the following holds. For each $\delta > 0$ and $w \in (-\infty, b]$ such that $w \ge \sup_{x\in\mathcal{X}}[\phi_b(x) - J(x)]$, there exists some $n_\delta \in \mathbb{N}$ and measurable sets $H_{\delta,0}, H_{\delta,1}, \dots, H_{\delta,n_\delta}$ such that $\mathcal{X} = \cup_{k=0}^{n_\delta} H_{\delta,k}$ and

$$\limsup_{\epsilon\to 0} \sup_{x\in H_{\delta,k}} \phi_b^{(\epsilon)}(x) - \inf_{x\in \mathrm{cl} H_{\delta,k}} J(x) \le w + \delta. \tag{5}$$

Then it follows that

$$\begin{aligned}
\limsup_{\epsilon\to 0} \epsilon \log \int_{\mathcal{X}} \exp\left[\tfrac{1}{\epsilon}\phi_b^{(\epsilon)}(x)\right] \mu_\epsilon(\mathrm{d}x) &\le \max_{k=0,\dots,n_\delta} \limsup_{\epsilon\to 0} \epsilon \log \int_{H_{\delta,k}} \exp\left[\tfrac{1}{\epsilon}\phi_b^{(\epsilon)}(x)\right] \mu_\epsilon(\mathrm{d}x) \\
&\le \max_{k=0,\dots,n_\delta} \left\{ \limsup_{\epsilon\to 0} \sup_{x\in H_{\delta,k}} \phi_b^{(\epsilon)}(x) - \inf_{x\in \mathrm{cl} H_{\delta,k}} J(x) \right\} \\
&\le w + \delta,
\end{aligned}$$

where the first inequality is an immediate application of [2, Lem. 1.2.15] and the second inequality follows from an easy estimate of the integral combined with the LDUB (1).

Hence, to prove the lemma, it suffices to construct for each $b > 0$ and each $\delta > 0$ a finite number of measurable sets $H_{\delta,0}, H_{\delta,1}, \ldots, H_{\delta,n_\delta}$ such that (5) holds, given that the functions ϕ and $\phi^{(\epsilon)}$ satisfy one of the sets of assumptions.

From now on, we fix $b > 0$ and $\delta > 0$. The proof is trivial if $\{\phi_b > B\}$ is empty or $\sup_{x\in\mathcal{X}}[\phi_b(x) - J(x)] = b$, so we assume that this is not the case. Fix any $w \in (-\infty, b)$ such that $w > \sup_{x\in\mathcal{X}}[\phi_b(x) - J(x)]$.

- *Proof for C.1.* Suppose that the functions ϕ and $\phi^{(\epsilon)}$ satisfy the conditions of C.1, i.e., the conditions of [6, Th. 3.1]. We will show that the proof of [6, Th. 3.1] fits into the framework described above.

  By assumption C.1.b and C.1.d, for each $x \in \{J < \infty\}$ there exist an open neighborhood $\mathcal{O}_x$ of x and some $\epsilon^*_x > 0$ such that $\phi^{(\epsilon)}(y) \le \phi(x) + \delta/2$ and $J(y) \ge J(x) - \delta/2$ for all $y \in \mathcal{O}_x$ and all $\epsilon \in (0, \epsilon^*_x)$. In addition, for each $x \in \{J < \infty\}$ there exists an open neighborhood $\mathcal{O}^*_x$ of x such that $\mathrm{cl}\mathcal{O}^*_x \subset \mathcal{O}_x$, due to $\mathcal{X}$ being regular (C.1.a).

  Pick any $Z \in \mathbb{R}$ such that $Z \ge b - w$. Clearly, $\{J \le Z\} \subset \cup_{x\in\{J\le Z\}}\mathcal{O}^*_x$. Since $\{J \le Z\}$ is compact by assumption C.1.c, there exists a finite subcover. Then there exist $n_\delta \in \mathbb{N}$ and $x_1, \ldots, x_{n_\delta} \in \{J \le Z\}$ such that $\{J \le Z\} \subset \cup_{k=1}^{n_\delta}\mathcal{O}^*_{x_k}$.

  Take $H_{\delta,0} = \mathcal{X} \setminus \cup_{k=1}^{n_\delta}\mathcal{O}^*_{x_k}$ and $H_{\delta,k} = \mathcal{O}^*_{x_k}$ for $k = 1, \ldots, n_\delta$. These sets are measurable due to C.1.a. Moreover,

$$\begin{aligned}
\limsup_{\epsilon\to 0} \sup_{x\in H_{\delta,k}} \phi_b^{(\epsilon)}(x) - \inf_{x\in \mathrm{cl}H_{\delta,k}} J(x) &\le \limsup_{\epsilon\to 0} \sup_{x\in \mathcal{O}_{x_k}} \phi_b^{(\epsilon)}(x) - \inf_{x\in \mathcal{O}_{x_k}} J(x) \\
&\le \phi_b(x_k) + \delta/2 - J(x_k) + \delta/2 \\
&\le w + \delta
\end{aligned}$$

  for $k = 1, \ldots, n_\delta$. Since $H_{\delta,0}$ is closed and $H_{\delta,0} \subset \{J > Z\}$, we also get

$$\limsup_{\epsilon\to 0} \sup_{x\in H_{\delta,0}} \phi_b^{(\epsilon)}(x) - \inf_{x\in \mathrm{cl}H_{\delta,0}} J(x) \le b - \inf_{x\in H_{\delta,0}} J(x) \le b - (b - w) = w.$$

Hence, $\mathcal{X} = \cup_{k=0}^{n_\delta} H_{\delta,k}$ and (5) is satisfied.

- *Proof for C.2 and C.3.* Suppose that the functions ϕ and $\phi^{(\epsilon)}$ satisfy the conditions of C.2 or C.3. Conditions C.2.b and C.3.b are the same and imply that $\phi^{(\epsilon)} \le B + \delta < w + \delta$ on the set $\{\phi \le B\}$ and $\phi^{(\epsilon)} \le \phi + \delta < w + \delta$ on the set $\{B < \phi < w\}$ for all $\epsilon \in (0, \epsilon^*)$, for some $\epsilon^* > 0$. Hence, we may take $H_{\delta,0} = \{\phi < w\}$, which is clearly measurable.

  For $k \in \mathbb{N}$, define the measurable sets

$$L_i^k = \phi_b^{-1}\big(\big[c_{i-1}^k, c_i^k\big]\big)$$

  for $i = 1, \ldots, k$, where

$$c_i^k = w + \frac{i}{k}(b - w)$$

  for $i = 0, \ldots, k$. Observe that $c_i^k - c_{i-1}^k = \frac{b-w}{k}$ and that $\{\phi \ge w\} = \{\phi_b \ge w\} = \cup_{i=1}^k L_i^k$ for every $k \in \mathbb{N}$.

  C.2. Suppose that second set of assumptions holds. Then we have $\mathrm{cl} L_i^k \subset \mathrm{cl}\phi_b^{-1}\big(\big[c_{i-1}^k, b\big]\big)$ and $\phi_b^{-1}\big(\big[c_{i-1}^k, b\big]\big) = \phi^{-1}\big(\big[c_{i-1}^k, \infty\big]\big)$ is closed by assumption (C.3.a), so $\mathrm{cl} L_i^k \subset \phi_b^{-1}\big(\big[c_{i-1}^k, b\big]\big)$. But $c_{i-1}^k = c_i^k - \frac{b-w}{k}$, so $c_i^k \le \phi_b(x) + \frac{b-w}{k}$ for all $x \in L_i^k$. Hence, we get

$$\begin{aligned} \sup_{x \in \mathrm{cl} L_i^k} \big[c_i^k - J(x)\big] &\le \sup_{x \in \mathrm{cl} L_i^k} \left[\phi_b(x) + \frac{b-w}{k} - J(x)\right] \\ &\le \sup_{x \in \mathcal{X}} [\phi_b(x) - J(x)] + \frac{b-w}{k}. \end{aligned}$$

  C.3. Suppose that the third set of assumptions holds. Then

$$\sup_{x \in \mathrm{cl} L_i^k} \big[c_i^k - J(x)\big] = \sup_{x \in L_i^k} \big[c_i^k - J(x)\big],$$

by continuity of J (C.2.a). We get $c_i^k \le \phi_b(x) + \frac{b-w}{k}$ for all $x \in \mathrm{cl} L_i^k$ and

$$\sup_{x\in L_i^k}\left[c_i^k - J(x)\right] \le \sup_{x\in L_i^k}\left[\phi_b(x) + \frac{b-w}{k} - J(x)\right]$$
$$\le \sup_{x\in\mathcal{X}}[\phi_b(x) - J(x)] + \frac{b-w}{k}.$$

It does not matter which of the two sets of conditions holds: we get the same inequality in both cases. Now take $n_\delta \in \mathbb{N}$ such that $\frac{b-w}{n_\delta} \le \delta$ and define $H_{\delta,i} = L_i^{n_\delta}$ for $i = 1, \dots, n_\delta$. Then

$$\begin{aligned}\limsup_{\epsilon\to 0}\sup_{x\in H_{\delta,i}} \phi_b^{(\epsilon)}(x) - \inf_{x\in \mathrm{cl} H_{\delta,i}} J(x) &\le \sup_{x\in H_{\delta,i}} \phi_b(x) - \inf_{x\in \mathrm{cl} H_{\delta,i}} J(x)\\ &\le c_i^{n_\delta} - \inf_{x\in \mathrm{cl} H_{\delta,i}} J(x)\\ &\le \sup_{x\in\mathcal{X}}[\phi_b(x) - J(x)] + \delta\\ &\le w + \delta\end{aligned}$$

for each $i = 1, \dots, n_\delta$. Hence, $\mathcal{X} = \cup_{i=0}^{n_\delta} H_{\delta,i}$ and (5) is satisfied.

□

Assuming that ϕ is real-valued and taking $\phi^{(\epsilon)} = \phi$ in the previous lemma (so that conditions C.2.b and C.3.b are void), we obtain the following corollary. Note that only measurability of ϕ is required when J is continuous.

Corollary 2.3. *Suppose that ϕ takes values in $\mathbb{R}$ and that μ_ϵ satisfies* (1) *for some function $J\colon \mathcal{X} \to [0,\infty]$. Define $B = \sup_{x\in\mathcal{X}}[\phi(x) - J(x)]$ and assume that at least one of the following sets of conditions holds.*

1. *The conditions a, b, c and d of Lemma 2.1 hold.*

2. *The superlevel set $\phi^{-1}([w,\infty])$ is closed for every $w \in \mathbb{R}$ satisfying the inequality $w \ge B$.*

3. *The function J is continuous.*

Then (2) *holds if and only if*

$$\lim_{M\to\infty}\limsup_{\epsilon\to 0}\epsilon\log\int_{\mathcal{X}}\exp\left[\tfrac{1}{\epsilon}\phi(x)\right]\mathbb{1}_{\{\phi(x)>M\}}\mu_\epsilon(\mathrm{d}x)\le\sup_{x\in\mathcal{X}}[\phi(x)-J(x)]. \tag{6}$$

Note that (6) automatically holds if ϕ is bounded above. Indeed, if ϕ is bounded above by some $C\in\mathbb{R}$, then the integral on the left-hand side of (6) equals 0 for all $M\ge C$.

As remarked before, condition d of Lemma 2.1 essentially requires that ϕ is upper semicontinuous. The corollary shows that this is sufficient to obtain the upper bound if $\phi^{(\epsilon)}=\phi$, so that conditions a, b, c, e and f of Lemma 2.1 are not needed in this case.

In particular, compactness of the sublevel sets of J is not required to obtain the upper bound if $\phi^{(\epsilon)}=\phi$. A better look at the proof of Lemma 2.2 shows that compactness of the sublevel sets of J is used to control the convergence of the functions $\phi^{(\epsilon)}$, which is not necessary if $\phi^{(\epsilon)}=\phi$. Hence, the compactness requirement in [2, Lem. 4.3.6] may be dropped. This observation is useful in queueing theory, because one often encounters rate functions that do not have compact level sets (cf. [4, Ex. 4.8]).

3 Examples

In the upcoming examples we show why our generalization of Varadhan's Lemma is useful. The first example describes a family of functions that does not satisfy the original assumptions of Varadhan's Lemma (cf. Lemma 2.1). Indeed, the functions increase to infinity at some point, so that condition f is not satisfied and condition e of Lemma 2.1 does not necessarily hold. However, the family does satisfy conditions C.1, C.2 and C.3, so Lemma 2.2 is applicable. The example also shows that the tail condition in (4) is nontrivial, in the sense that the left-hand side of (4) may take any value in $[-\infty,\infty]$.

Example 3.1. Let $\mathcal{X}=\{0,1,2\}$ and let its topology be given by the power set of $\mathcal{X}$. For $\epsilon\in(0,1]$, define the probability measure μ_ϵ via $\mu_\epsilon(\{2\})=\frac{1}{2}\exp\big(1-\epsilon^{-2}\big)$, $\mu_\epsilon(\{1\})=\frac{1}{2}\exp(-z/\epsilon)$ for some $z>0$ and $\mu_\epsilon(\{0\})=1-\mu_\epsilon(\{1\})-\mu_\epsilon(\{2\})$. Then μ_ϵ satisfies an LDP with rate function J given by $J(0)=0$, $J(1)=z$ and $J(2)=\infty$.

For $\epsilon\in(0,1]$, define the function $\phi^{(\epsilon)}$ via $\phi^{(\epsilon)}(0)=-\infty$, $\phi^{(\epsilon)}(1)=f(\epsilon)$ and $\phi^{(\epsilon)}(2)=g(\epsilon)$, where $f(\epsilon)\uparrow y$ for some $y\in\mathbb{R}$ and $g(\epsilon)\uparrow\infty$ as $\epsilon\to 0$.

The only reasonable choice for the function ϕ is taking $\phi(0) = -\infty$, $\phi(1) = y$ and $\phi(2) = \infty$. Observe that all three sets of conditions in Lemma 2.2 are satisfied and that

$$\lim_{M\to\infty} \sup_{x\in\mathcal{X}} [\phi_M(x) - J(x)] = y - z \in \mathbb{R}.$$

But it holds that

$$\begin{aligned}\lim_{M\to\infty} \limsup_{\epsilon\to 0} \epsilon \log \int_{\mathcal{X}} \exp\left[\tfrac{1}{\epsilon}\phi^{(\epsilon)}(x)\right] \mathbb{1}_{\{\phi^{(\epsilon)}(x)>M\}} \mu_\epsilon(\mathrm{d}x) \\ = \limsup_{\epsilon\to 0} \epsilon \log\left(\tfrac{1}{2}\exp\left[\tfrac{1}{\epsilon}g(\epsilon)\right]\exp\left[1-\epsilon^{-2}\right]\right) \\ = \limsup_{\epsilon\to 0}\left[g(\epsilon) - \tfrac{1}{\epsilon}\right],\end{aligned}$$

so

$$\limsup_{\epsilon\to 0} \epsilon \log \int_{\mathcal{X}} \exp\left[\tfrac{1}{\epsilon}\phi^{(\epsilon)}(x)\right] \mu_\epsilon(\mathrm{d}x) \le \lim_{M\to\infty} \sup_{x\in\mathcal{X}} [\phi_M(x) - J(x)]$$

if and only if

$$\limsup_{\epsilon\to 0}\left[g(\epsilon) - \tfrac{1}{\epsilon}\right] \le y - z.$$

Note that $\limsup_{\epsilon\to 0}\left[g(\epsilon) - \tfrac{1}{\epsilon}\right]$ may take any value in $[-\infty, \infty]$, depending on how fast $g(\epsilon)$ increases to ∞ as $\epsilon \to 0$.

As illustrated in the previous example, the conditions e and f in Lemma 2.1 are not necessary to obtain an upper bound. Moreover, if J is continuous and the measures μ_ϵ are absolutely continuous, then we may obtain an even tighter upper bound than in the standard formulation of Varadhan's Lemma. This is shown in the next example.

Example 3.2. Let $\mathcal{X} = \mathbb{R}$ and let $\phi\colon \mathcal{X} \to [-\infty, \infty]$ be a bounded Lebesgue measurable function. Let $N \subset \mathcal{X}$ be any measurable null set and define the Lebesgue measurable function $\phi^*\colon \mathcal{X} \to [-\infty, \infty]$ via $\phi^* = \phi$ on $\mathcal{X} \setminus N$ and $\phi^* = -\infty$ on N. Let $\{\mu_\epsilon\}$ be a family of probability measures on $\mathcal{X}$ that are absolutely continuous with respect to Lebesgue measure. Suppose that this family satisfies an LDUB with continuous rate function J. Since N is a null set, we obtain

$$\int_{\mathcal{X}} \exp\big[\tfrac{1}{\epsilon}\phi(x)\big]\mu_\epsilon(\mathrm{d}x) = \int_{\mathcal{X}} \exp\big[\tfrac{1}{\epsilon}\phi^*(x)\big]\mu_\epsilon(\mathrm{d}x)$$

for every $\epsilon > 0$. Since J is continuous and ϕ^* is bounded above, Corollary 2.3 implies that

$$\limsup_{\epsilon\to 0} \epsilon \log \int_{\mathcal{X}} \exp\big[\tfrac{1}{\epsilon}\phi^*(x)\big]\mu_\epsilon(\mathrm{d}x) \le \sup_{x\in\mathcal{X}}[\phi^*(x) - J(x)] = \sup_{x\in\mathcal{X}\setminus N}[\phi(x) - J(x)].$$

Since N was an arbitrary null set, it follows that

$$\limsup_{\epsilon\to 0} \epsilon \log \int_{\mathcal{X}} \exp\big[\tfrac{1}{\epsilon}\phi(x)\big]\mu_\epsilon(\mathrm{d}x) \le \operatorname{ess\,sup}(\phi - J),$$

where the essential supremum is taken with respect to Lebesgue measure.

To see that this is indeed a sharper upper bound, consider the following simple example. Let $\mathcal{X} = [0,1]$, $\phi(x) = \mathbb{1}_{\{x=1/2\}}$ and $\mu_\epsilon = \mu$, where μ is the uniform distribution on $[0,1]$. Then $J \equiv 0$ on $\mathcal{X}$ and

$$\limsup_{\epsilon\to 0} \epsilon \log \int_{\mathcal{X}} \exp\big[\tfrac{1}{\epsilon}\phi(x)\big]\mu_\epsilon(\mathrm{d}x) = \operatorname{ess\,sup}(\phi - J) = 0 < 1 = \sup_{x\in\mathcal{X}}[\phi(x) - J(x)].$$

The last example provides an LDUB for observations of a modulated Poisson process. These processes are important in queueing theory to model arrival processes whose parameters depend on a random environment (see for instance [1] and [5]). The example shows that in this context we may obtain functions that are not upper semicontinuous, so that the original version of Varadhan's Lemma does not apply to this case. However, under the assumption that the rate function is continuous we may invoke Corollary 2.3 to prove an LDUB in this case.

Example 3.3. Let M be a Poisson process with arrival intensity $\lambda > 0$ that is modulated by a stochastic ON/OFF switch, i.e., M is modulated by a stochastic process Z with state space $\{0,1\}$. This means that M has arrival intensity $z\lambda$ while Z is in state $z \in \{0,1\}$. For simplicity, assume that Z has càdlàg paths.

Fix some time $t > 0$. We would like to count the number of jobs in the system at time t (i.e., the number of jobs that have arrived in the time interval $[0, t]$), but we only count them if the switch has been ON for at least t_1 time units but less than t_2 time units, where $0 < t_1 < t_2 < t$. Hence, the number of jobs that we count at time t is given by $L(t) = \mathbb{1}_{\left\{t_1 \leq \int_0^t Z(s)\mathrm{d}s < t_2\right\}} M(t)$.

We scale the arrival intensity via $\lambda \mapsto n\lambda$ and the background process via $Z \mapsto Z_n$ in such a way that the probability measure μ_n induced by $\int_0^t Z_n(s)\mathrm{d}s$ satisfies an LDUB with continuous rate function J. We denote the resulting modulated Poisson process by M_n.

We would like to prove an LDUB for $L_n(t) = \mathbb{1}_{\left\{t_1 \leq \int_0^t Z_n(s)\mathrm{d}s < t_2\right\}} M_n(t)$ using a Chernoff bound under this scaling. It is well known that $M_n(t)$ has a Poisson distribution with random parameter $n\lambda \int_0^t Z_n(s)\mathrm{d}s$ (cf. [4, Lem. A.1]). For $\gamma \geq 0$, we let $P_0(\gamma), P_1(\gamma), \ldots$ denote a sequence of independent random variables with a Poisson distribution with parameter γ. Fix a closed set $F \subset \mathbb{R}$ and write

$$
\begin{aligned}
\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\frac{1}{n} L_n(t) \in F\right) &= \limsup_{n\to\infty} \frac{1}{n} \log \int_{[0,t]} \mathbb{P}\left(\mathbb{1}_{\{t_1 \leq x < t_2\}} \frac{1}{n} P_0(n\lambda x) \in F\right) \mu_n(\mathrm{d}x) \\
&= \limsup_{n\to\infty} \frac{1}{n} \log \int_{[0,t]} \mathbb{P}\left(\mathbb{1}_{\{t_1 \leq x < t_2\}} \frac{1}{n} \sum_{k=1}^{n} P_k(\lambda x) \in F\right) \mu_n(\mathrm{d}x) \\
&\leq \limsup_{n\to\infty} \frac{1}{n} \log \int_{[0,t]} \exp\left(n\left(-\inf_{a\in F} I(x;a)\right)\right) \mu_n(\mathrm{d}x),
\end{aligned}
$$

where $I(x; \cdot) = \ell\big(\mathbb{1}_{\{t_1 \leq x < t_2\}}\lambda x; \cdot\big)$ and $\ell(y; \cdot)$ is the rate function corresponding to a Poisson distribution with parameter $y \geq 0$. More specifically, the function ℓ is given by $\ell(y; a) = \infty$ for $a < 0$, $\ell(y; 0) = y$, $\ell(0; a) = \infty$ for $a > 0$ and $\ell(y; a) = a \log(a/y) - a + y$ for $y > 0$ and $a > 0$.

The map $x \mapsto -\inf_{a\in F} I(x; a)$ fails to be upper semicontinuous for certain closed sets F. Indeed, taking $F = \{0\}$ and using that $\ell(y; 0) = y$, we get

$$
-\inf_{a\in F} I(x;a) = \begin{cases} 0 & 0 \leq x < t_1; \\ -\lambda x & t_1 \leq x < t_2; \\ 0 & t_2 \leq x \leq t. \end{cases}
$$

Since this map is not upper semicontinuous, the standard version of Varadhan's Lemma does not apply to this case. However, it is easy to see that the map $x \mapsto -\inf_{a\in F} I(x; a)$ is measurable for each closed set F. Because the sequence of measures μ_n satisfies an LDUB with continuous rate function J and the map $x \mapsto -\inf_{a\in F} I(x;a)$ is bounded above by 0, we may invoke Corollary 2.3 to obtain

$$\limsup_{n\to\infty} \frac{1}{n} \log \int_{[0,t]} \exp\left(n\left(-\inf_{a\in F} I(x;a)\right)\right) \mu_n(\mathrm{d}x) \leq \sup_{x\in[0,t]} \left[-\inf_{a\in F} I(x;a) - J(x)\right],$$

so that

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\frac{1}{n} L_n(t) \in F\right) \leq -\inf_{a\in F} \inf_{x\in[0,t]} [I(x;a) + J(x)]$$

for every closed set F.

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