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Abstract

We introduce an equivalence relation on the space $W^{1,1}(\Omega;\mathbb{S}^1)$ which classifies maps according to their "topological singularities". We establish sharp bounds for the distances (in the usual sense and in the Hausdorff sense) between the equivalence classes. Similar questions are examined for the space $W^{1,p}(\Omega;\mathbb{S}^1)$ when $p>1$.

1 Introduction

Let Ω be a smooth bounded domain in $\mathbb{R}^N$, $N\geq 2$. (Many of the results in this paper remain valid if Ω is replaced by a manifold $\mathcal{M}$, with or without boundary, and the case $\mathcal{M}=\mathbb{S}^1$ is already of interest (see [14, 15]).) In some places we will assume in addition that Ω is simply connected (and this will be mentioned explicitly). Our basic setting is

$$W^{1,1}(\Omega;\mathbb{S}^1)=\{u\in W^{1,1}(\Omega;\mathbb{R}^2)\simeq W^{1,1}(\Omega;\mathbb{C});\ |u|=1 \text{ a.e.}\}.$$

It is clear that if $u,v\in W^{1,1}(\Omega;\mathbb{S}^1)$ then $uv\in W^{1,1}(\Omega;\mathbb{S}^1)$; moreover

$$\text{if } u_n\to u \text{ and } v_n\to v \text{ in } W^{1,1}(\Omega;\mathbb{S}^1) \text{ then } u_nv_n\to uv \text{ in } W^{1,1}(\Omega;\mathbb{S}^1). \tag{1.1}$$

In particular, $W^{1,1}(\Omega;\mathbb{S}^1)$ is a topological group. We call the attention of the reader that maps u of the form $u=e^{\imath\varphi}$ with $\varphi\in W^{1,1}(\Omega;\mathbb{R})$ belong to $W^{1,1}(\Omega;\mathbb{S}^1)$. However they do not exhaust $W^{1,1}(\Omega;\mathbb{S}^1)$: there exist maps in $W^{1,1}(\Omega;\mathbb{S}^1)$ which *cannot* be written as $u=e^{\imath\varphi}$ for some $\varphi\in W^{1,1}(\Omega;\mathbb{R})$. A typical example is the map $u(x)=x/|x|$ in $\Omega=$unit disc in $\mathbb{R}^2$; This was originally observed in [4] (with roots in [29]) and is based on degree theory; see also [9, 12]. Set

$$\mathcal{E}=\{u\in W^{1,1}(\Omega;\mathbb{S}^1);\ u=e^{\imath\varphi} \text{ for some } \varphi\in W^{1,1}(\Omega;\mathbb{R})\}. \tag{1.2}$$

We claim that $\mathcal{E}$ is closed in $W^{1,1}(\Omega;\mathbb{S}^1)$. Indeed, let $u_n=e^{\imath\varphi_n}$ with $u_n\to u$ in $W^{1,1}$. Then $\nabla\varphi_n=-\imath\overline{u}_n\nabla u_n$ converges in L^1 to $-\imath\overline{u}\nabla u$. By adding an integer multiple of 2π to φ_n we may assume that $\left|\int_\Omega\varphi_n\right|\leq 2\pi|\Omega|$. Thus, a subsequence of $\{\varphi_n\}$ converges in $W^{1,1}$ to some φ and $u=e^{\imath\varphi}$.

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Clearly

$$\mathcal{E} \subset \overline{C^\infty(\overline{\Omega}; \mathbb{S}^1)}^{W^{1,1}}. \tag{1.3}$$

Indeed, if $u \in \mathcal{E}$, write $u = e^{\imath\varphi}$ for some $\varphi \in W^{1,1}(\Omega; \mathbb{R})$; let $\varphi_n \in C^\infty(\overline{\Omega}; \mathbb{R})$ be such that $\varphi_n \to \varphi$ in $W^{1,1}$. Then, $u_n = e^{\imath\varphi_n} \in C^\infty(\overline{\Omega}; \mathbb{S}^1)$ and converges to u in $W^{1,1}$. However equality in (1.3) fails in general. For example when $\Omega = \{x \in \mathbb{R}^2;\ 1 < |x| < 2\}$, the map $u(x) = x/|x|$ is smooth, but $u \notin \mathcal{E}$; as above the nonexistence of φ is an easy consequence of degree theory. On the other hand, if Ω is simply connected, equality in (1.3) does hold since $C^\infty(\overline{\Omega}; \mathbb{S}^1) \subset \mathcal{E}$ (recall that any $u \in C^\infty(\overline{\Omega}; \mathbb{S}^1)$ can be written as $u = e^{\imath\varphi}$ with $\varphi \in C^\infty(\overline{\Omega}; \mathbb{R})$) and $\mathcal{E}$ is closed in $W^{1,1}(\Omega; \mathbb{S}^1)$).

To each $u \in W^{1,1}(\Omega; \mathbb{S}^1)$ we associate a number $\Sigma(u) \geq 0$ defined by

$$\Sigma(u) = \inf_{\varphi \in W^{1,1}(\Omega;\mathbb{R})} \int_\Omega |\nabla(ue^{-\imath\varphi})| = \inf_{v \in \mathcal{E}} \int_\Omega |\nabla(u\overline{v})|. \tag{1.4}$$

An immediate consequence of the definition is the relation $\Sigma(\overline{u}) = \Sigma(u)$. As explained in Section 2 the quantity $\Sigma(u)$ plays an extremely important role in many questions involving $W^{1,1}(\Omega; \mathbb{S}^1)$; it has also an interesting geometric interpretation. Note that

$$u \in \mathcal{E} \Longleftrightarrow \Sigma(u) = 0, \tag{1.5}$$

and in particular $\Sigma(1) = 0$. The implication $\Longrightarrow$ is clear. For the reverse implication, assume that $\Sigma(u) = 0$, i.e., there exists a sequence $v_n \in \mathcal{E}$ such that $\int_\Omega |\nabla(u\overline{v}_n)| \to 0$. Then (modulo a subsequence) $u\overline{v}_n \to C$ in $W^{1,1}$, for some constant $C \in \mathbb{S}^1$; therefore $v_n \to \overline{C}u$ in $W^{1,1}$ and thus $u \in \mathcal{E}$ (since $\mathcal{E}$ is closed).

In some sense $\Sigma(u)$ measures how much a general $u \in W^{1,1}(\Omega; \mathbb{S}^1)$ "deviates" from $\mathcal{E}$. More precisely we will prove that

$$\frac{2}{\pi}\,\Sigma(u) \leq \inf_{v \in \mathcal{E}} \int_\Omega |\nabla(u - v)| \leq \Sigma(u), \tag{1.6}$$

with optimal constants. This will be derived as a very special case of our main result Theorem 1.1. In order to state it we need to describe a decomposition of the space $W^{1,1}(\Omega; \mathbb{S}^1)$ according to the following equivalence relation in $W^{1,1}(\Omega; \mathbb{S}^1)$:

$$u \sim v \text{ if and only if } u = e^{\imath\varphi}\, v \text{ for some } \varphi \in W^{1,1}(\Omega; \mathbb{R}); \tag{1.7}$$

in other words, $u \sim v$ if and only if $\Sigma(u\overline{v}) = 0$. We denote by $\mathcal{E}(u)$ the equivalence class of an element $u \in W^{1,1}(\Omega; \mathbb{S}^1)$, that is

$$\mathcal{E}(u) = \{ue^{-\imath\varphi};\ \varphi \in W^{1,1}(\Omega; \mathbb{R})\}.$$

In particular, $\mathcal{E}(1) = \mathcal{E}$. It is easy to see that for every $u \in W^{1,1}(\Omega; \mathbb{S}^1)$, $\mathcal{E}(u)$ is closed (it suffices to apply (1.1) and the fact that $\mathcal{E}$ is closed). In Section 2 we will give an interpretation of the equivalence relation $u \sim v$ in terms of the "topological singularities" of u and v. We may rewrite (1.4) as

$$\Sigma(u) = \inf_{v \in \mathcal{E}(u)} \int_\Omega |\nabla v|. \tag{1.8}$$

Given $u_0, v_0 \in W^{1,1}(\Omega;\mathbb{S}^1)$ the following quantities will play a crucial role throughout the paper:

$$d_{W^{1,1}}(u_0,\mathcal{E}(v_0)) := \inf_{v\sim v_0}\int_\Omega |\nabla(u_0-v)|, \tag{1.9}$$

$$\mathrm{dist}_{W^{1,1}}(\mathcal{E}(u_0),\mathcal{E}(v_0)) := \inf_{u\sim u_0} d_{W^{1,1}}(u,\mathcal{E}(v_0)) = \inf_{u\sim u_0}\inf_{v\sim v_0}\int_\Omega |\nabla(u-v)|, \tag{1.10}$$

$$\mathrm{Dist}_{W^{1,1}}(\mathcal{E}(u_0),\mathcal{E}(v_0)) := \sup_{u\sim u_0} d_{W^{1,1}}(u,\mathcal{E}(v_0)) = \sup_{u\sim u_0}\inf_{v\sim v_0}\int_\Omega |\nabla(u-v)|, \tag{1.11}$$

so that $\mathrm{dist}_{W^{1,1}}(\mathcal{E}(u_0),\mathcal{E}(v_0))$ is precisely the distance between the classes $\mathcal{E}(u_0)$ and $\mathcal{E}(v_0)$. On the other hand we will see below, as a consequence of (1.13), that $\mathrm{Dist}_{W^{1,1}}$ is symmetric, a fact which is not clear from its definition. This implies that $\mathrm{Dist}_{W^{1,1}}$ coincides with the Hausdorff distance

$$H-\mathrm{dist}_{W^{1,1}}(\mathcal{E}(u_0),\mathcal{E}(v_0)) := \max\left(\mathrm{Dist}_{W^{1,1}}(\mathcal{E}(u_0),\mathcal{E}(v_0)),\, \mathrm{Dist}_{W^{1,1}}(\mathcal{E}(v_0),\mathcal{E}(u_0))\right)$$

between $\mathcal{E}(u_0)$ and $\mathcal{E}(v_0)$. Our main result is

Theorem 1.1. *For every $u_0, v_0 \in W^{1,1}(\Omega;\mathbb{S}^1)$ we have*

$$\mathrm{dist}_{W^{1,1}}(\mathcal{E}(u_0),\mathcal{E}(v_0)) = \frac{2}{\pi}\Sigma(u_0\overline{v}_0) \tag{1.12}$$

and

$$\mathrm{Dist}_{W^{1,1}}(\mathcal{E}(u_0),\mathcal{E}(v_0)) = \Sigma(u_0\overline{v}_0). \tag{1.13}$$

The two assertions in Theorem 1.1 look very simple but the proofs are quite tricky; they are presented in Sections 4 and 5. The factor $2/\pi$ in (1.12) represents the ratio of two diameters of $\mathbb{S}^1$, each corresponding to a different metric: the first one computed using the Euclidean metric and the second one using the geodesic distance. This interpretation will become clear in the proof of Lemma 4.2 below.

A useful device for constructing maps in the same equivalence class is the following (see Lemma 4.1 below). Let $T \in \mathrm{Lip}(\mathbb{S}^1;\mathbb{S}^1)$ be a map of degree one. Then

$$T\circ u \sim u,\ \forall\, u \in W^{1,1}(\Omega;\mathbb{S}^1). \tag{1.14}$$

It turns out that this simple device plays a very significant role in the proofs of most of our main results. It allows us to work on the target space only, thus avoiding difficulties due to the possibly complicated geometry and/or topology of the domain (or manifold) Ω. A first example of an application of this technique is given by the proof of the following version of the "dipole construction"; it is the main ingredient in the proof of inequality "$\le$" in (1.13).

Proposition 1.2. *(H. Brezis and P. Mironescu [12, Proposition 2.1]) Let $u \in W^{1,1}(\Omega;\mathbb{S}^1)$. Then there exists a sequence $\{u_n\}\subset\mathcal{E}(u)$ satisfying*

$$u_n \to 1 \text{ a.e., and } \lim_{n\to\infty}\int_\Omega |\nabla u_n| = \Sigma(u). \tag{1.15}$$

For completeness we present the proof of Proposition 1.2 in the Appendix.

A basic ingredient in the proof of inequality "$\ge$" in (1.13) is the following proposition which provides an explicit recipe for constructing "maximizing sequences" for $\mathrm{Dist}_{W^{1,1}}$. In order to describe it we first introduce, for each $n\ge 3$, a map $T_n \in \mathrm{Lip}(\mathbb{S}^1;\mathbb{S}^1)$ with $\deg T_n = 1$ by $T_n(e^{\imath\theta}) = e^{\imath\tau_n(\theta)}$, with τ_n defined on $[0,2\pi]$ by setting $\tau_n(0)=0$ and

$$\tau_n'(\theta) = \begin{cases} n, & \theta\in(2j\,\pi/n^2,(2j+1)\,\pi/n^2] \\ -(n-2), & \theta\in((2j+1)\,\pi/n^2,(2j+2)\,\pi/n^2] \end{cases},\ j=0,1,\dots,n^2-1. \tag{1.16}$$

Proposition 1.3. *For every $u_0, v_0 \in W^{1,1}(\Omega;\mathbb{S}^1)$ such that $u_0 \not\sim v_0$ we have*

$$\lim_{n\to\infty} \frac{d_{W^{1,1}}(T_n \circ u_0, \mathcal{E}(v_0))}{\Sigma(u_0\overline{v}_0)} = 1 \tag{1.17}$$

and the limit is uniform *over all such u_0 and v_0. Consequently*

$$\mathrm{Dist}_{W^{1,1}}(\mathcal{E}(u_0), \mathcal{E}(v_0)) \geq \Sigma(u_0\overline{v}_0). \tag{1.18}$$

As mentioned above, a special case of interest is the distance of a given $u \in W^{1,1}(\Omega;\mathbb{S}^1)$ to the class $\mathcal{E}$. An immediate consequence of Theorem 1.1 is that for every $u \in W^{1,1}(\Omega;\mathbb{S}^1)$ we have

$$\frac{2}{\pi}\Sigma(u) \leq d_{W^{1,1}}(u, \mathcal{E}) \leq \Sigma(u), \tag{1.19}$$

and the bounds are optimal in the sense that

$$\sup_{u\notin\mathcal{E}} \frac{d_{W^{1,1}}(u,\mathcal{E})}{\Sigma(u)} = 1, \tag{1.20}$$

and

$$\inf_{u\notin\mathcal{E}} \frac{d_{W^{1,1}}(u,\mathcal{E})}{\Sigma(u)} = \frac{2}{\pi}. \tag{1.21}$$

There are challenging problems concerning the question whether the supremum and the infimum in the above formulas are achieved (see §5.3).

Remark 1.4. Formulas (1.19)–(1.21) provide a sharp improvement of the inequality

$$\frac{1}{2}\Sigma(u) \leq d_{W^{1,1}}(u,\mathcal{E}) \leq \Sigma(u),\ \forall\, u \in W^{1,1}(\Omega;\mathbb{S}^1), \tag{1.22}$$

established in [12, Sec. 11.6].

Finally, we turn in Section 6 to the classes in $W^{1,p}(\Omega;\mathbb{S}^1)$, $1 < p < \infty$, defined in an analogous way to the $W^{1,1}$-case, i.e., using the equivalence relation

$$u \sim v \text{ if and only if } u = e^{\imath\varphi}\, v \text{ for some } \varphi \in W^{1,p}(\Omega;\mathbb{R}). \tag{1.23}$$

We point out that if $u, v \in W^{1,p}(\Omega;\mathbb{S}^1)$ are equivalent according to the equivalence relation in (1.7), then from the relation $e^{\imath\varphi} = u\overline{v}$ we deduce that

$$\nabla\varphi = -\imath\overline{u}v\nabla(u\overline{v}) \in L^p(\Omega;\mathbb{R}^N); \tag{1.24}$$

whence $u \sim v$ according to (1.23) as well. When $p \geq 2$ and Ω is simply connected we have $W^{1,p}(\Omega;\mathbb{S}^1) = \{u \in W^{1,1}(\Omega;\mathbb{S}^1);\ u = e^{\imath\varphi} \text{ for some } \varphi \in W^{1,p}(\Omega;\mathbb{R})\}$, see Remark 1.10 below. Therefore, the only cases of interest are:

(a) general Ω and $1 < p < 2$,
(b) multiply connected Ω and $p \geq 2$.
In all the theorems below we assume that we are in one of these situations. The distances between the classes are defined analogously to (1.10)–(1.11) by

$$\mathrm{dist}_{W^{1,p}}(\mathcal{E}(u_0), \mathcal{E}(v_0)) := \inf_{u\sim u_0} \inf_{v\sim v_0} \|\nabla(u-v)\|_{L^p(\Omega)}. \tag{1.25}$$

and

$$\mathrm{Dist}_{W^{1,p}}(\mathcal{E}(u_0), \mathcal{E}(v_0)) := \sup_{u\sim u_0} \inf_{v\sim v_0} \|\nabla(u-v)\|_{L^p(\Omega)}. \tag{1.26}$$

The next result establishes a lower bound for $\mathrm{dist}_{W^{1,p}}$:

Theorem 1.5. *For every $u_0, v_0 \in W^{1,p}(\Omega;\mathbb{S}^1)$, $1 \le p < \infty$, we have*

$$\operatorname{dist}_{W^{1,p}}(\mathcal{E}(u_0), \mathcal{E}(v_0)) \ge \left(\frac{2}{\pi}\right) \inf_{w \sim u_0 \bar{v}_0} \|\nabla w\|_{L^p(\Omega)}. \tag{1.27}$$

Remark 1.6. For $p > 1$ the infimum on the R.H.S. of (1.27) is actually a minimum; this follows easily from (1.24) and the fact that $W^{1,p}$ is reflexive.

Note that equality in (1.27) holds for $p = 1$ by (1.12). An example in [27, Section 4] shows that strict inequality ">" may occur in (1.27) for a multiply connected domain in dimension two and $p = 2$. We will show in §6.4 that strict inequality may also occur for simply connected domains when $1 < p < 2$. On the positive side, we prove equality in (1.27) in the case of the distance to $\mathcal{E}$:

Theorem 1.7. *For every $u_0 \in W^{1,p}(\Omega;\mathbb{S}^1)$, $1 < p < \infty$, we have*

$$\operatorname{dist}_{W^{1,p}}(\mathcal{E}(u_0), \mathcal{E}) = \left(\frac{2}{\pi}\right) \inf_{w \sim u_0} \|\nabla w\|_{L^p(\Omega)}. \tag{1.28}$$

Remark 1.8. When $p > 1$ we do not know general conditions on $u_0, v_0 \in W^{1,p}(\Omega;\mathbb{S}^1)$ that guarantee equality in (1.27) (a sufficient condition in the case of multiply connected two dimensional domain and $p = 2$ is given in [27, Th. 4]).

On the other hand, when $p > 1$, $\operatorname{Dist}_{W^{1,p}}$ between distinct classes is infinite:

Theorem 1.9. *For every $u_0, v_0 \in W^{1,p}(\Omega;\mathbb{S}^1)$, $1 < p < \infty$, such that $u_0 \not\sim v_0$ we have*

$$\operatorname{Dist}_{W^{1,p}}(\mathcal{E}(u_0), \mathcal{E}(v_0)) = \infty. \tag{1.29}$$

Remark 1.10. There is another natural equivalence relation in $W^{1,p}(\Omega;\mathbb{S}^1)$, $1 \le p < \infty$, defined by the homotopy classes, i.e.,

$$u \overset{\mathcal{H}}{\sim} v \text{ if and only if } u = h(0) \text{ and } v = h(1) \text{ for some } h \in C\left([0,1]; W^{1,p}(\Omega;\mathbb{S}^1)\right).$$

Homotopy classes have been well-studied (see [10, 11, 22, 28, 30]). Clearly $u \sim v \Longrightarrow u \overset{\mathcal{H}}{\sim} v$ (use the homotopy $h(t) = e^{\imath(1-t)\varphi} v$). Note however that when $1 \le p < 2$ the equivalence relation $u \sim v$ is *much more restrictive* than $u \overset{\mathcal{H}}{\sim} v$; for example let $\Omega =$unit disc in $\mathbb{R}^2$, $u(x) = x/|x|$ and $v(x) = (x-a)/|x-a|$ with $0 \ne a \in \Omega$, then $u \not\sim v$ (in fact, $\operatorname{dist}_{W^{1,1}}(\mathcal{E}(u), \mathcal{E}(v)) = 4|a| > 0$ by (3.17) below) while $u \overset{\mathcal{H}}{\sim} v$, e.g., via the homotopy $h(t) = (x-ta)/|x-ta|$, $0 \le t \le 1$.

Part of the results were announced in [15].

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2 Further comments on $\Sigma(u)$ and $\mathcal{E}(u)$

Given $a, b \in \mathbb{C}$, write as usual $a = a_1 + \imath a_2$, $b = b_1 + \imath b_2$; we also identify a, b with the vectors $a = (a_1, a_2)^T, b = (b_1, b_2)^T \in \mathbb{R}^2$ and set

$$a \wedge b = a_1 b_2 - a_2 b_1 = \operatorname{Im}(\overline{a} b) \in \mathbb{R}. \tag{2.1}$$

2.1 The distributional Jacobian Ju

For every $u \in W^{1,1}(\Omega; \mathbb{S}^1)$ we consider $u \wedge \nabla u \in L^1(\Omega; \mathbb{R}^N)$ defined by its components

$$(u \wedge \nabla u)_j = u \wedge \frac{\partial u}{\partial x_j} = u_1 \frac{\partial u_2}{\partial x_j} - u_2 \frac{\partial u_1}{\partial x_j}, \; j = 1, \dots, N. \tag{2.2}$$

Since $|u|^2 = 1$ on Ω we have

$$u_1 \frac{\partial u_1}{\partial x_j} + u_2 \frac{\partial u_2}{\partial x_j} = 0 \text{ in } \Omega, \tag{2.3}$$

and thus

$$u \wedge \frac{\partial u}{\partial x_j} = -\imath \overline{u} \frac{\partial u}{\partial x_j} \text{ in } \Omega; \tag{2.4}$$

in particular,

$$|u \wedge \nabla u| = |\nabla u| \text{ in } \Omega. \tag{2.5}$$

The following identities are elementary:

$$(uv) \wedge \nabla(uv) = u \wedge \nabla u + v \wedge \nabla v, \qquad \forall\, u, v \in W^{1,1}(\Omega; \mathbb{S}^1), \tag{2.6}$$

$$e^{\imath\varphi} \wedge \nabla(e^{\imath\varphi}) = \nabla\varphi, \qquad \forall\, \varphi \in W^{1,1}(\Omega; \mathbb{R}), \tag{2.7}$$

$$\overline{u} \wedge \nabla\overline{u} = -u \wedge \nabla u, \qquad \forall\, u \in W^{1,1}(\Omega; \mathbb{S}^1). \tag{2.8}$$

Finally we introduce, for every $u \in W^{1,1}(\Omega; \mathbb{S}^1)$, its *distributional* Jacobian Ju, which is an antisymmetric matrix with coefficients in $\mathcal{D}'(\Omega; \mathbb{R})$ defined by

$$(Ju)_{i,j} := \frac{1}{2} \left[\frac{\partial}{\partial x_i} \left(u \wedge \frac{\partial u}{\partial x_j} \right) - \frac{\partial}{\partial x_j} \left(u \wedge \frac{\partial u}{\partial x_i} \right) \right]. \tag{2.9}$$

When $N = 2$, Ju is identified with the *scalar* distribution

$$Ju = \frac{1}{2} \left[\frac{\partial}{\partial x_1} \left(u \wedge \frac{\partial u}{\partial x_2} \right) - \frac{\partial}{\partial x_2} \left(u \wedge \frac{\partial u}{\partial x_1} \right) \right] = \frac{1}{2} \operatorname{curl} (u \wedge \nabla u)\,. \tag{2.10}$$

From (2.6)–(2.8) we deduce that

$$J(uv) = Ju + Jv, \qquad \forall\, u, v \in W^{1,1}(\Omega; \mathbb{S}^1), \tag{2.11}$$

$$J(\overline{u}) = -Ju, \qquad \forall\, u \in W^{1,1}(\Omega; \mathbb{S}^1), \tag{2.12}$$

$$J(e^{\imath\varphi}) = 0, \qquad \forall\, \varphi \in W^{1,1}(\Omega; \mathbb{R}), \tag{2.13}$$

i.e.,

$$J(u) = 0, \; \forall\, u \in \mathcal{E}, \tag{2.14}$$

and thus

$$u \sim v \Longrightarrow Ju = Jv. \tag{2.15}$$

When Ω is *simply connected* the converse is also true, so that

$$u \sim v \Longleftrightarrow Ju = Jv; \tag{2.16}$$

in other words,

$$\mathcal{E}(u) = \{v \in W^{1,1}(\Omega; \mathbb{S}^1);\ Ju = Jv\}. \tag{2.17}$$

This fact is originally due to Demengel [20], with roots in [3]; simpler proofs can be found in [12, 9, 18].

In order to have a more concrete perception of the equivalence relation $u \sim v$ it is instructive to understand what it means when $N = 2$ and Ω is simply connected, for $u, v \in \mathcal{R}$ where

$$\mathcal{R} = \{u \in W^{1,1}(\Omega; \mathbb{S}^1);\ u \text{ is smooth in } \Omega \text{ except at a finite number of points}\}. \tag{2.18}$$

The class $\mathcal{R}$ plays an important role since it is dense in $W^{1,1}(\Omega; \mathbb{S}^1)$ (see [4, 12]).

If $u \in \mathcal{R}$ then

$$Ju = \pi \sum_j d_j \delta_{a_j}, \tag{2.19}$$

where the a_j's are the singular points of u and $d_j := \deg(u, a_j)$, i.e., the topological degree of u restricted to any small circle centered at a_j; see [8, 13, 12] and also [2, end of Section 6] for the special case where $u(x) = x/|x|$. In particular, when $u, v \in \mathcal{R}$,

$$\begin{aligned} u \sim v \Longleftrightarrow [&u \text{ and } v \text{ have the same singularities} \\ &\text{and the same degree at each singularity}]. \end{aligned} \tag{2.20}$$

2.2 $\Sigma(u)$ computed by duality

An equivalent formula to (1.4) is

$$\Sigma(u) = \inf_{\varphi \in W^{1,1}(\Omega; \mathbb{R})} \int_\Omega |u \wedge \nabla u - \nabla \varphi|. \tag{2.21}$$

Indeed, from (2.6)–(2.8) we have $ue^{-\imath\varphi} \wedge \nabla(ue^{-\imath\varphi}) = u \wedge \nabla u - \nabla\varphi$, and by (2.5), $|\nabla(ue^{-\imath\varphi})| = |u \wedge \nabla u - \nabla\varphi|$, which yields (2.21).

Next we apply the following standard consequence of the Hahn-Banach theorem:

$$\operatorname{dist}(p, M) = \inf_{m \in M} \|p - m\| = \max\{< \xi, p >;\ \xi \in M^\perp, \|\xi\| \le 1\}, \tag{2.22}$$

where E is a Banach space, $p \in E$, and M is a linear subspace of E (see e.g., [6, Section 1.4, Example 3]). If we take $E = L^1(\Omega; \mathbb{R}^N)$, $p = u \wedge \nabla u$, $M = \{\nabla\varphi;\ \varphi \in W^{1,1}(\Omega; \mathbb{R})\}$, then we have

$$M^\perp = \{\xi \in L^\infty(\Omega; \mathbb{R}^N);\ \operatorname{div} \xi = 0 \text{ in } \Omega \text{ and } \xi \cdot \nu = 0 \text{ on } \partial\Omega\}, \tag{2.23}$$

where ν is the outward normal to $\partial\Omega$. Here the condition [$\operatorname{div}\xi = 0$ in Ω and $\xi\cdot\nu = 0$ on $\partial\Omega$] is understood in the weak sense [$\int_\Omega \xi\cdot\nabla\varphi = 0$, $\forall\varphi\in W^{1,1}(\Omega;\mathbb{R})$], or equivalently, [$\int_\Omega \xi\cdot\nabla\varphi = 0$, $\forall\varphi\in C^\infty(\overline{\Omega};\mathbb{R})$]. Inserting (2.23) in (2.22) yields

$$\Sigma(u) = \max\{\int_\Omega (u\wedge\nabla u)\cdot\xi;\ \xi\in M^\perp, \|\xi\|_{L^\infty}\le 1\}. \tag{2.24}$$

Next we assume that $N = 2$ and Ω is *simply connected.* We claim that for every $u\in W^{1,1}(\Omega;\mathbb{S}^1)$,

$$\Sigma(u) = \max\{\int_\Omega (u\wedge\nabla u)\cdot\nabla^\perp\zeta;\ \zeta\in W_0^{1,\infty}(\Omega;\mathbb{R}) \text{ and } \|\nabla\zeta\|_{L^\infty}\le 1\}, \tag{2.25}$$

where $\nabla^\perp\zeta = (-\partial\zeta/\partial x_2, \partial\zeta/\partial x_1)$.

Proof of (2.25). In view of (2.23)–(2.24) it suffices to show that

$$\{\xi\in L^\infty(\Omega;\mathbb{R}^2); \operatorname{div}\xi = 0 \text{ in } \Omega \text{ and } \xi\cdot\nu = 0 \text{ on } \partial\Omega\} = \{\nabla^\perp\zeta;\ \zeta\in W_0^{1,\infty}(\Omega;\mathbb{R})\}. \tag{2.26}$$

For the inclusion "$\supset$", we verify that

$$\int_\Omega \nabla^\perp\zeta\cdot\nabla\varphi = 0,\ \forall\varphi\in C^\infty(\overline{\Omega};\mathbb{R});$$

this is clear since $\operatorname{curl}(\nabla\varphi) = 0$ and $\zeta = 0$ on $\partial\Omega$.

For the inclusion "$\subset$", we start with some $\xi\in L^\infty(\Omega;\mathbb{R}^2)$ such that

$$\int_\Omega \xi\cdot\nabla\varphi = 0,\ \forall\varphi\in W^{1,1}(\Omega;\mathbb{R}). \tag{2.27}$$

Set $\bar\xi := \begin{cases}\xi, & \text{in } \Omega\\ 0, & \text{in } \mathbb{R}^2\setminus\Omega\end{cases}$. Then, by (2.27),

$$\int_{\mathbb{R}^2} \bar\xi\cdot\nabla\Phi = \int_\Omega \xi\cdot\nabla(\Phi|_\Omega) = 0,\ \forall\Phi\in C_c^1(\mathbb{R}^2;\mathbb{R}). \tag{2.28}$$

Thus we may invoke the generalized Poincaré lemma in $\mathbb{R}^2$ and conclude that $\bar\xi = \nabla^\perp\bar\zeta$ for some $\bar\zeta\in W^{1,\infty}(\mathbb{R}^2;\mathbb{R})$. Clearly, $\zeta = \bar\zeta|_\Omega \in W^{1,\infty}(\Omega;\mathbb{R})$, $\nabla^\perp\zeta = \xi$ and ζ is constant on $\partial\Omega$ (since $\partial\Omega$ is connected because Ω is simply connected). □

Remark 2.1. Equality (2.25) is originally due to [13, Thm 2] (with a much more complicated proof).

Finally we give a geometric interpretation for $\Sigma(u)$ when $\Omega\subset\mathbb{R}^2$ is simply connected and $u\in\mathcal{R}$. We first need some notation. Given $a, b\in\overline{\Omega}$, set

$$d_\Omega(a,b) = \min\{|a-b|, d(a,\partial\Omega) + d(b,\partial\Omega)\} = \inf_\Gamma \operatorname{length}(\Gamma\cap\Omega), \tag{2.29}$$

where the $\inf_\Gamma$ is taken over all curves $\Gamma\subset\mathbb{R}^2$ joining a to b. Clearly d_Ω is a semi-metric on $\overline{\Omega}$; moreover

$$d_\Omega(a,b) = 0 \iff [\text{either } a = b \text{ or } a, b\in\partial\Omega].$$

Thus we may identify $\partial\Omega$ as a single point in $\overline{\Omega}$, still denoted $\partial\Omega$.

Given $(\boldsymbol{a},\boldsymbol{d})=(a_1,a_2,\ldots,a_l,d_1,d_2,\ldots,d_l)$ with $a_j\in\Omega$ and $d_j\in\mathbb{Z}$, $\forall j$, we set

$$D=-\sum_{j=1}^{l}d_j, \tag{2.30}$$

and we consider the collection $(a_1,a_2,\ldots,a_l,\partial\Omega)$ in $\overline{\Omega}$ affected with the integer coefficients $(d_1,d_2,\ldots,d_l,D)$. We then repeat the points a_j's and $\partial\Omega$ according to their multiplicities, i.e., $d_1,d_2,\ldots,d_l$ and D, and we rewrite them as a collection of m positive points (P_j) and m negative points (N_j), $1\le j\le m$ (this is possible by (2.30)). Finally we define

$$L(\boldsymbol{a},\boldsymbol{d})=\min_{\sigma\in\mathcal{S}_m}\sum_{j=1}^{m}d_\Omega(P_j,N_{\sigma(j)}), \tag{2.31}$$

where $\mathcal{S}_m$ denotes the set of permutations of $\{1,2,\ldots,m\}$.

We are now ready to state our main claim:

$$\Sigma(u)=2\pi L(\boldsymbol{a},\boldsymbol{d}),\ \forall\, u\in\mathcal{R}, \tag{2.32}$$

where the a_j's are the singular points of u and $d_j=\deg(u,a_j)$.

Remark 2.2. A variant of formula (2.32) where $\Omega=\mathbb{S}^2$ (and thus $\partial\Omega=\emptyset$) appears originally in [13], but the core of the proof goes back to [8].

Here is a sketch of the proof of (2.32). From (2.10) and (2.19) we have

$$-\int_\Omega (u\wedge\nabla u)\cdot\nabla^\perp\zeta=2\pi\sum_{j=1}^{l}d_j\zeta(a_j),\ \forall\,\zeta\in W_0^{1,\infty}(\Omega;\mathbb{R}). \tag{2.33}$$

Set $W^{1,\infty}_{\text{const}}(\Omega;\mathbb{R})=\{\zeta\in W^{1,\infty}(\Omega;\mathbb{R});\ \zeta=\text{const on }\partial\Omega\}$ and let $\zeta\in W^{1,\infty}_{\text{const}}(\Omega;\mathbb{R})$. From (2.33) applied to $\zeta-\zeta(\partial\Omega)$ we obtain

$$-\int_\Omega (u\wedge\nabla u)\cdot\nabla^\perp\zeta=2\pi\left(\sum_{j=1}^{l}d_j\zeta(a_j)+D\zeta(\partial\Omega)\right). \tag{2.34}$$

Combining (2.25) and (2.34) we see that

$$\Sigma(u)=2\pi\max\left\{\sum_{j=1}^{m}(\zeta(P_j)-\zeta(N_j))\,;\,\zeta\in W^{1,\infty}_{\text{const}}(\Omega;\mathbb{R})\text{ and }\|\nabla\zeta\|_{L^\infty}\le 1\right\}. \tag{2.35}$$

Next we observe that for every $\zeta:\overline{\Omega}\to\mathbb{R}$ the following conditions are equivalent:

$$\zeta\in W^{1,\infty}_{\text{const}}(\Omega;\mathbb{R})\text{ and }\|\nabla\zeta\|_{L^\infty}\le 1 \tag{2.36}$$

and

$$|\zeta(x)-\zeta(y)|\le d_\Omega(x,y),\ \forall\, x,y\in\overline{\Omega}. \tag{2.37}$$

Thus (2.35) becomes

$$\Sigma(u)=2\pi\max\left\{\sum_{j=1}^{m}(\zeta(P_j)-\zeta(N_j))\,;\,\zeta\text{ satisfying (2.37)}\right\}. \tag{2.38}$$

Finally we invoke the formula

$$\max \left\{ \sum_{j=1}^{m} (\zeta(P_j) - \zeta(N_j))\,;\, \zeta \text{ satisfying (2.37)} \right\} = L(\boldsymbol{a}, \boldsymbol{d}) \tag{2.39}$$

to conclude that $\Sigma(u) = 2\pi L(\boldsymbol{a}, \boldsymbol{d})$.

Relation (2.39) appears originally in [8, Lemma 4.2]. The proof in [8] combines a theorem of Kantorovich with Birkhoff's theorem on doubly stochastic matrices. An elementary proof of (2.39), totally self-contained, is presented in [5] (see also [7]); it is related in spirit to the proof of the celebrated result of Rockafellar concerning cyclically monotone operators.

2.3 Optimal lifting

It is known (see [21, Section 6.2] and [19, 24, 12]) that every $u \in W^{1,1}(\Omega; \mathbb{S}^1)$ can be written as $u = e^{\imath\varphi}$ with $\varphi \in BV(\Omega; \mathbb{R})$. In fact, there are many such φ's in BV and it is natural to introduce the quantity

$$E(u) = \inf \left\{ \int_\Omega |D\varphi|;\ \varphi \in BV(\Omega; \mathbb{R}) \text{ such that } u = e^{\imath\varphi} \right\}. \tag{2.40}$$

Then,

$$E(u) = \int_\Omega |\nabla u| + \Sigma(u). \tag{2.41}$$

Formula (2.41) was originally established in [13] when $N = 2$ (and $\Omega = \mathbb{S}^2$). The nontrivial extension to $N \geq 2$ can be deduced from results of Poliakovsky [25], see also [12] for a direct approach.

2.4 Relaxed energy

The *relaxed energy* is defined for every $u \in W^{1,1}(\Omega; \mathbb{S}^1)$ by

$$R(u) = \inf \left\{ \liminf_{n\to\infty} \int_\Omega |\nabla u_n|;\ u_n \in C^\infty(\overline{\Omega}; \mathbb{S}^1),\ u_n \to u \text{ a.e. on } \Omega \right\},$$

where the first inf means that the infimum is taken over all sequences (u_n) in $C^\infty(\overline{\Omega}; \mathbb{S}^1)$ such that $u_n \to u$ a.e. on Ω. [In general there is no sequence (u_n) in $C^\infty(\overline{\Omega}; \mathbb{S}^1)$ such that $u_n \to u$ in $W^{1,1}$, unless $Ju = 0$. However, it is always possible to find a sequence (u_n) in $C^\infty(\overline{\Omega}; \mathbb{S}^1)$ such that $u_n \to u$ a.e. on Ω.] Assume that Ω is simply connected, then

$$R(u) = \int_\Omega |\nabla u| + \Sigma(u),$$

see [13] for $N = 2$ and [12] for $N \geq 3$.

Remark 2.3. We did not investigate the natural question concerning a generalization of Theorem 1.1 to $BV(\Omega; \mathbb{S}^1)$ when the classes $\{\mathcal{E}(u)\}$ and the quantity $\Sigma(u)$ are appropriately adapted.

3 Motivation

In order to illustrate the significance of the results of Theorem 1.1 it is instructive to explain it in a special case involving maps with a finite number of singularities. Moreover, this allows us to compare the problem to an analogous one involving the Dirichlet energy of $\mathbb{S}^2$-valued maps on three dimensional domains, whose study was initiated in [8]. Since in both cases the energy scales like length, one may expect similar results; as we shall see below the analogy is not complete. We start with the problem in $\mathbb{R}^3$. Consider for simplicity $\Omega = B_R(0) \subset \mathbb{R}^3$. Analogously to (2.18) we consider the set $\mathcal{R}$ of maps in $H^1(\Omega; \mathbb{S}^2)$ which are smooth on $\overline{\Omega}$, except at (at most) a finite number of singularities. With each k-tuple of distinct points $\boldsymbol{a} = (a_1, \dots, a_k) \in \Omega^k$ and corresponding degrees $\boldsymbol{d} = (d_1, \dots, d_k) \in \mathbb{Z}^k$ we associate the following class of maps in $\mathcal{R}$:

$$\mathcal{E}_{\boldsymbol{a},\boldsymbol{d}} := \left\{ u \in C^\infty \left(\overline{\Omega} \setminus \bigcup_{j=1}^{k} \{a_j\}; \mathbb{S}^2 \right); \nabla u \in L^2(\Omega) \text{ and } \deg(u, a_j) = d_j, \forall\, j \right\}. \tag{3.1}$$

[Here, $\deg(u, a_j) = d_j$ means that the restriction of u to any small sphere around a_j has topological degree d_j.] In the case where $k = 0$ the resulting class is $C^\infty(\overline{\Omega}; \mathbb{S}^2)$. There are three natural questions that we want to discuss:

(i) What is the least energy of a map in $\mathcal{E}_{\boldsymbol{a},\boldsymbol{d}}$ i.e., the value of

$$\Sigma^{(2)}_{\boldsymbol{a},\boldsymbol{d}} := \inf_{u \in \mathcal{E}_{\boldsymbol{a},\boldsymbol{d}}} \int_\Omega |\nabla u|^2 \ ? \tag{3.2}$$

(ii) Consider two sets of distinct points in Ω, $\boldsymbol{a} = (a_1, a_2, \dots, a_k)$ and $\boldsymbol{b} = (b_1, b_2, \dots, b_l)$, each with associated vectors of degrees, $\boldsymbol{d} \in \mathbb{Z}^k$ and $\boldsymbol{e} \in \mathbb{Z}^l$, respectively. What is the H^1-distance between $\mathcal{E}_{\boldsymbol{a},\boldsymbol{d}}$ and $\mathcal{E}_{\boldsymbol{b},\boldsymbol{e}}$ i.e., analogously to (1.10),

$$\operatorname{dist}^2_{H^1}(\mathcal{E}_{\boldsymbol{a},\boldsymbol{d}}, \mathcal{E}_{\boldsymbol{b},\boldsymbol{e}}) := \inf_{u \in \mathcal{E}_{\boldsymbol{a},\boldsymbol{d}}} \inf_{v \in \mathcal{E}_{\boldsymbol{b},\boldsymbol{e}}} \int_\Omega |\nabla(u - v)|^2 \ ? \tag{3.3}$$

i.e., what is the least energy required to pass from singularities located at $\{a_j\}_{j=1}^k$, with degrees $\{d_j\}_{j=1}^k$, to singularities located at $\{b_j\}_{j=1}^l$, with degrees $\{e_j\}_{j=1}^l$?

(iii) Similarly, by analogy with (1.11), what is the value of

$$\operatorname{Dist}^2_{H^1}(\mathcal{E}_{\boldsymbol{a},\boldsymbol{d}}, \mathcal{E}_{\boldsymbol{b},\boldsymbol{e}}) := \sup_{u \in \mathcal{E}_{\boldsymbol{a},\boldsymbol{d}}} \inf_{v \in \mathcal{E}_{\boldsymbol{b},\boldsymbol{e}}} \int_\Omega |\nabla(u - v)|^2 \ ? \tag{3.4}$$

Question (i) was originally tackled by [8]; their motivation came from a question of J. Ericksen concerning the least energy required to produce a liquid crystal configuration with prescribed singularities. Quite surprisingly it turns out that the value of this least energy can be computed explicitly in terms of *geometric* quantities. In the special case (3.2) their formula becomes

$$\Sigma^{(2)}_{\boldsymbol{a},\boldsymbol{d}} = 8\pi L(\boldsymbol{a}, \boldsymbol{d}), \tag{3.5}$$

where $L(\boldsymbol{a}, \boldsymbol{d})$ is defined as in (2.31).

On the other hand, it seems that Question (ii) was never treated in the literature. Using the results of [8] one can show that if $(\boldsymbol{a}, \boldsymbol{d}) \neq (\boldsymbol{b}, \boldsymbol{e})$ then for every *fixed* $u \in \mathcal{E}_{\boldsymbol{a},\boldsymbol{d}}$ we have

$$\operatorname{dist}_{H^1}(u, \mathcal{E}_{\boldsymbol{b},\boldsymbol{e}}) > 0. \tag{3.6}$$

What is quite surprising is that for *all* pairs of classes we have

$$\operatorname{dist}_{H^1}(\mathcal{E}_{\boldsymbol{a},\boldsymbol{d}},\mathcal{E}_{\boldsymbol{b},\boldsymbol{e}})=0. \tag{3.7}$$

The basic ingredient behind (3.7) is the following fact: for every pair of integers $d_1\neq d_2$ we have

$$\inf\left\{\int_{\mathbb{S}^2}|\nabla(F_1-F_2)|^2;\ F_j\in H^1(\mathbb{S}^2;\mathbb{S}^2),\deg(F_j)=d_j\ \text{ for } j=1,2\right\}=0. \tag{3.8}$$

Formula (3.8) was established in [23] (see also [14] for generalizations) following the same idea used by Brezis and Nirenberg [17] in the setting of degree theory in $H^{1/2}(\mathbb{S}^1;\mathbb{S}^1)$.

As for Question (iii), the "dipole removing" technique of Bethuel [3] (with roots in [8]) can be applied to derive the upper bound

$$\operatorname{Dist}^2_{H^1}(\mathcal{E}_{\boldsymbol{a},\boldsymbol{d}},\mathcal{E}_{\boldsymbol{b},\boldsymbol{e}})\le 8\pi L(\boldsymbol{c},\boldsymbol{f}), \tag{3.9}$$

where

$$\boldsymbol{c}=(a_1,\dots,a_k,b_1,\dots,b_l)\in\Omega^{k+l}\ \text{and}\ \boldsymbol{f}=(d_1,\dots,d_k,-e_1,\dots,-e_l)\in\mathbb{Z}^{k+l}. \tag{3.10}$$

We suspect that equality holds in (3.9).

It is possible to associate with every $u\in H^1(\Omega;\mathbb{S}^2)$ a "natural" class $\mathcal{E}(u)$, in the spirit of (2.17). Formulas (3.7) and (3.9), as well as their extensions to arbitrary classes $\mathcal{E}(u)$, $\mathcal{E}(v)$, are established in [16]. We also present in [16] evidence that equality holds in (3.9) by establishing the following analogue of (1.20):

$$\sup_{\substack{\boldsymbol{a},\boldsymbol{d}\\ \boldsymbol{d}\neq\boldsymbol{0}}}\ \sup_{u\in\mathcal{E}_{\boldsymbol{a},\boldsymbol{d}}}\frac{d^2_{H^1}(u,C^\infty(\overline{\Omega};\mathbb{S}^2))}{8\pi L(\boldsymbol{a},\boldsymbol{d})}=1. \tag{3.11}$$

Next we consider similar questions for $W^{1,1}(\Omega;\mathbb{S}^1)$. For simplicity let $\Omega=B_R(0)\subset\mathbb{R}^2$. By analogy with (3.1), for $\boldsymbol{a}=(a_1,\dots,a_k)\in\Omega^k$ and $\boldsymbol{d}=(d_1,\dots,d_k)\in\mathbb{Z}^k$ we consider the following class of maps in $\mathcal{R}$:

$$\mathcal{E}_{\boldsymbol{a},\boldsymbol{d}}:=\left\{u\in C^\infty\left(\overline{\Omega}\setminus\bigcup_{j=1}^k\{a_j\};\mathbb{S}^1\right);\ \nabla u\in L^1(\Omega)\ \text{and}\ \deg(u,a_j)=d_j,\ \forall\, j\right\}. \tag{3.12}$$

The analogous questions to (i)–(iii) are then:

(i') What is the value of

$$\Sigma^{(1)}_{\boldsymbol{a},\boldsymbol{d}}:=\inf_{u\in\mathcal{E}_{\boldsymbol{a},\boldsymbol{d}}}\int_\Omega|\nabla u|\ ? \tag{3.13}$$

(ii') For any pair $\boldsymbol{a}\in\Omega^k$, $\boldsymbol{b}\in\Omega^l$ and associated vectors of degrees, $\boldsymbol{d}\in\mathbb{Z}^k$ and $\boldsymbol{e}\in\mathbb{Z}^l$,what is the $W^{1,1}$-distance between $\mathcal{E}_{\boldsymbol{a},\boldsymbol{d}}$ and $\mathcal{E}_{\boldsymbol{b},\boldsymbol{e}}$,

$$\operatorname{dist}_{W^{1,1}}(\mathcal{E}_{\boldsymbol{a},\boldsymbol{d}},\mathcal{E}_{\boldsymbol{b},\boldsymbol{e}}):=\inf_{u\in\mathcal{E}_{\boldsymbol{a},\boldsymbol{d}}}\ \inf_{v\in\mathcal{E}_{\boldsymbol{b},\boldsymbol{e}}}\int_\Omega|\nabla(u-v)|\ ? \tag{3.14}$$

(iii') What is the value of

$$\operatorname{Dist}_{W^{1,1}}(\mathcal{E}_{\boldsymbol{a},\boldsymbol{d}},\mathcal{E}_{\boldsymbol{b},\boldsymbol{e}}):=\sup_{u\in\mathcal{E}_{\boldsymbol{a},\boldsymbol{d}}}\ \inf_{v\in\mathcal{E}_{\boldsymbol{b},\boldsymbol{e}}}\int_\Omega|\nabla(u-v)|\ ? \tag{3.15}$$

The answer to Question (i') is given by the results in §2.2. Indeed, setting $u_{\boldsymbol{a},\boldsymbol{d}}(\zeta) := \prod_{j=1}^{k} \left(\frac{\zeta - a_j}{|\zeta - a_j|}\right)^{d_j}$, we get from (2.32) that

$$\Sigma^{(1)}_{\boldsymbol{a},\boldsymbol{d}} = \Sigma(u_{\boldsymbol{a},\boldsymbol{d}}) = 2\pi L(\boldsymbol{a}, \boldsymbol{d}), \tag{3.16}$$

which is completely analogous to (3.5). [Here we used the density of $\mathcal{E}_{\boldsymbol{a},\boldsymbol{d}}$ (for the $W^{1,1}$-topology) in $\mathcal{E}(u_{\boldsymbol{a},\boldsymbol{d}})$ (see [4, 12]).]

On the other hand, the situation with Question (ii') is completely different. In contrast with (3.7), here $\mathrm{dist}_{W^{1,1}}(\mathcal{E}_{\boldsymbol{a},\boldsymbol{d}}, \mathcal{E}_{\boldsymbol{b},\boldsymbol{e}})$ is strictly positive when $(\boldsymbol{a}, \boldsymbol{d}) \neq (\boldsymbol{b}, \boldsymbol{e})$. The explicit value of this infimum can be computed in terms of geometric quantities. Actually, (1.12) of Theorem 1.1 asserts that

$$\mathrm{dist}_{W^{1,1}}(\mathcal{E}_{\boldsymbol{a},\boldsymbol{d}}, \mathcal{E}_{\boldsymbol{b},\boldsymbol{e}}) = \frac{2}{\pi}\,\Sigma(u_{\boldsymbol{a},\boldsymbol{d}}\,\overline{u}_{\boldsymbol{b},\boldsymbol{e}}) = 4L(\boldsymbol{c}, \boldsymbol{f}), \tag{3.17}$$

where $\boldsymbol{c}$ and $\boldsymbol{f}$ are given by (3.10). Indeed, the last equality in (3.17) follows from (3.16) when applied to the map $u_{\boldsymbol{a},\boldsymbol{d}}\,\overline{u}_{\boldsymbol{b},\boldsymbol{e}}$ which has singularities precisely at the points $\{c_j\}_{j=1}^{k+l}$, with associated singularities $\{f_j\}_{j=1}^{k+l}$. Similarly, the second part of Theorem 1.1, (1.13), asserts that

$$\sup_{u\in\mathcal{E}_{\boldsymbol{a},\boldsymbol{d}}} \inf_{v\in\mathcal{E}_{\boldsymbol{b},\boldsymbol{e}}} \int_\Omega |\nabla(u - v)| = \Sigma(u_{\boldsymbol{a},\boldsymbol{d}}\,\overline{u}_{\boldsymbol{b},\boldsymbol{e}}) = 2\pi L(\boldsymbol{c}, \boldsymbol{f}).$$

We also present an interpretation of Theorem 1.1 when $\Omega \subset \mathbb{R}^3$. Fix two disjoint smooth closed oriented curves $\Gamma_1, \Gamma_2 \subset \Omega$ and consider for $j = 1, 2$

$$\mathcal{E}_{\Gamma_j} = \left\{u \in C^\infty(\overline{\Omega} \setminus \Gamma_j; \mathbb{S}^1);\ \nabla u \in L^1(\Omega) \text{ and } \deg(u, \Gamma_j) = +1\right\}$$

($\deg(u, \Gamma_j) = +1$ means that $\deg(u, C_j) = +1$ for every small circle $C_j \subset \Omega \setminus \Gamma_j$ linking Γ_j). In this case Theorem 1.1 asserts that

$$\inf_{u\in\mathcal{E}_{\Gamma_1}} \inf_{v\in\mathcal{E}_{\Gamma_2}} \int_\Omega |\nabla(u - v)| = 4 \inf_S \mathrm{area}(S \cap \Omega),$$

where $\inf_S$ is taken over all surfaces $S \subset \mathbb{R}^3$ such that $\partial S = \Gamma_1 \cup \Gamma_2$, and

$$\sup_{u\in\mathcal{E}_{\Gamma_1}} \inf_{v\in\mathcal{E}_{\Gamma_2}} \int_\Omega |\nabla(u - v)| = 2\pi \inf_S \mathrm{area}(S \cap \Omega).$$

For more details on this case, see [1, 8, 12].

4 Proof of (1.12) in Theorem 1.1

4.1 A basic lower bound inequality

We begin with a simple lemma about composition with Lipschitz maps; it provides a very useful device for constructing maps in the same equivalence class, or in the class $\mathcal{E}(1)$.

Lemma 4.1. *Let $T \in Lip(\mathbb{S}^1; \mathbb{S}^1)$ be a map of degree D. Then*

$$T \circ u \sim u^D,\ \forall\, u \in W^{1,1}(\Omega; \mathbb{S}^1). \tag{4.1}$$

Proof. Since $T(z)\bar{z}^D$ is a Lipschitz self-map of $\mathbb{S}^1$ of zero degree, there exists $g \in \mathrm{Lip}(\mathbb{S}^1;\mathbb{R})$ such that $T(z)\bar{z}^D = e^{\imath g(z)}$. The function $\varphi(x) = g(u(x))$ belongs to $W^{1,1}(\Omega;\mathbb{R})$ and satisfies $T(u(x)) = (u(x))^D e^{\imath\varphi(x)}$, and (4.1) follows by the definition of the equivalence relation. □

The next simple lemma is essential for the proof of the lower bound in (1.19).

Lemma 4.2. *For any $w \in W^{1,1}(\Omega;\mathbb{S}^1)$ we have*

$$\int_\Omega |\nabla(|w-1|)| \geq \frac{2}{\pi}\Sigma(w). \tag{4.2}$$

Proof. As in [27], we define $T : \mathbb{S}^1 \to \mathbb{S}^1$ by

$$T(e^{\imath\varphi}) := e^{\imath\theta} \text{ with } \theta = \theta(\varphi) = \pi\sin(\varphi/2),\ \forall\,\varphi \in (-\pi,\pi], \tag{4.3}$$

so that

$$|e^{\imath\varphi} - 1| = 2|\sin(\varphi/2)| = \frac{2}{\pi}|\theta|. \tag{4.4}$$

Clearly T is of class C^1 and its degree equals one. We claim that

$$|\nabla(|w-1|)| = \frac{2}{\pi}|\nabla(T\circ w)| \text{ a.e.} \tag{4.5}$$

This is a consequence of the standard fact that, if $F \in \mathrm{Lip}(\mathbb{S}^1;\mathbb{R}^2) \cap C^1(\mathbb{S}^1 \setminus \{1\})$ and $w \in W^{1,1}(\Omega;\mathbb{S}^1)$, then $F\circ w \in W^{1,1}(\Omega;\mathbb{S}^1)$ and, moreover,

$$\nabla(F\circ w) = \begin{cases} \dot{F}(w)\nabla w & \text{a.e. in } [w \neq 1] \\ 0 & \text{a.e. in } [w = 1] \end{cases}.$$

Integration of (4.5) leads to

$$\int_\Omega |\nabla|w-1|| = \frac{2}{\pi}\int_\Omega |\nabla(T\circ w)|. \tag{4.6}$$

By Lemma 4.1, we have $\Sigma(T\circ w) = \Sigma(w)$, and therefore (4.2) follows from (4.6). □

Corollary 4.3. *For every $u, v \in W^{1,1}(\Omega;\mathbb{S}^1)$ we have*

$$\int_\Omega |\nabla(u-v)| \geq \frac{2}{\pi}\Sigma(u\bar{v}). \tag{4.7}$$

Proof. Setting $w = u\bar{v}$ and applying (4.2) yields

$$\int_\Omega |\nabla(u-v)| \geq \int_\Omega |\nabla(|u-v|)| = \int_\Omega |\nabla(|w-1|)| \geq \frac{2}{\pi}\Sigma(u\bar{v}).$$ □

4.2 Proof of (1.12)

We begin by introducing some notation. For an open arc in $\mathbb{S}^1$ we use the notation

$$\mathcal{A}(\alpha,\beta) = \{\, e^{\imath\theta};\ \theta \in (\alpha,\beta)\} \tag{4.8}$$

for any $\alpha < \beta$. We shall also use a specific notation for half-circles; for every $\zeta \in \mathbb{S}^1$ write $\zeta = e^{\imath\varphi}$ with $\varphi \in (-\pi,\pi]$ and denote $I(\zeta,-\zeta) = \mathcal{A}(\varphi,\varphi+\pi)$. Note that

$$z \in I(\zeta,-\zeta) \Longleftrightarrow \zeta \in I(-z,z). \tag{4.9}$$

For each $\zeta = e^{\imath\varphi} \in \mathbb{S}^1$ define a map $P_\zeta : \mathbb{S}^1 \to \overline{I(\zeta, -\zeta)}$ by

$$P_\zeta(z) = \begin{cases} z, & \text{if } z = e^{\imath\theta} \in I(\zeta, -\zeta) \\ e^{\imath(2\varphi - \theta)} = \zeta^2 \overline{z}, & \text{if } z \notin I(\zeta, -\zeta) \end{cases}, \tag{4.10}$$

so that for $z \notin I(\zeta, -\zeta)$, $P_\zeta(z)$ is the reflection of z with respect to the line $\ell_\zeta = \{t\zeta;\, t \in \mathbb{R}\}$. Next we state

Proposition 4.4. *For every $u \in W^{1,1}(\Omega; \mathbb{S}^1)$ we have*

$$\int_{\mathbb{S}^1} \left(\int_\Omega |\nabla(u - P_\zeta \circ u)|\, dx \right) d\zeta = 4 \int_\Omega |\nabla u|\, dx. \tag{4.11}$$

Proof. For each $\zeta \in \mathbb{S}^1$ set $v_\zeta := P_\zeta \circ u$. By Lemma 4.1, $v_\zeta \in \mathcal{E}(1)$, since $\deg P_\zeta = 0$. We note that

$$z - P_\zeta(z) = \begin{cases} 0, & \text{if } z \in I(\zeta, -\zeta) \\ z - \zeta^2\, \overline{z}, & \text{if } z \notin I(\zeta, -\zeta) \end{cases}. \tag{4.12}$$

Set $w_\zeta := u - v_\zeta$. Using (4.12), we find that for every $\zeta = e^{\imath\varphi}$ and a.e. $x \in \Omega$ we have

$$\nabla w_\zeta(x) = \begin{cases} 0, & \text{if } u(x) \in I(\zeta, -\zeta) \\ \nabla u(x) - \zeta^2\, \nabla \overline{u}(x), & \text{if } u(x) \notin I(\zeta, -\zeta) \end{cases}. \tag{4.13}$$

Therefore, for a.e. $x \in \Omega$ we have

$$|\nabla w_\zeta(x)| = \begin{cases} 0, & \text{if } u(x) \in I(\zeta, -\zeta) \\ 2|\cos(\theta - \varphi)||\nabla u(x)|, & \text{if } u(x) = e^{\imath\theta} \notin I(\zeta, -\zeta) \end{cases}. \tag{4.14}$$

Indeed, we justify (4.14) e.g. when $\zeta = 1$. In view of (4.13), we have to prove that

$$|\nabla \mathrm{Im}\, u(x)| = |\mathrm{Re}\, u(x)|\, |\nabla u(x)| \quad \text{for a.e. } x. \tag{4.15}$$

If we differentiate the identity $|u|^2 \equiv 1$, we obtain

$$\mathrm{Re}\, u\, \nabla(\mathrm{Re}\, u) + \mathrm{Im}\, u\, \nabla(\mathrm{Im}\, u) = 0 \text{ a.e.};$$

this easily implies (4.15).

Using (4.9) we find that, with $u(x) = e^{\imath\theta}$ and

$$A(x) = \{\varphi \in (-\pi, \pi];\, u(x) \notin I(e^{\imath\varphi}, -e^{\imath\varphi})\},$$

we have

$$\begin{aligned} \int_{\mathbb{S}^1} \int_\Omega |\nabla w_\zeta(x)|\, dx\, d\zeta &= \int_{-\pi}^{\pi} \int_\Omega \chi_{A(x)}(\varphi)\, 2|\cos(\theta - \varphi)||\nabla u(x)|\, dx\, d\varphi \\ &= \int_\Omega |\nabla u(x)| \Big(\int_\theta^{\theta+\pi} 2|\cos(\theta - \varphi)|\, d\varphi \Big)\, dx = 4 \int_\Omega |\nabla u(x)|\, dx, \end{aligned}$$

which is (4.11). Here we have used $\int_\theta^{\theta+\pi} 2|\cos(\theta - \varphi)|\, d\varphi = \int_0^\pi 2|\cos t|\, dt = 4$. $\square$

The identity (4.11) is a key tool in the proof of "$\le$" in (1.12). For the convenience of the reader we shall present first the slightly simpler proof when $v_0 = 1$.

Proof of "$\le$" in (1.12) for $v_0 = 1$. By Corollary 4.3 we have

$$\inf_{u\sim u_0} d_{W^{1,1}}(u,\mathcal{E}(1)) \ge \frac{2}{\pi}\Sigma(u_0).$$

Use (1.4) to choose a sequence $\{u_n\} \subset \mathcal{E}(u_0)$ with $\lim_{n\to\infty}\int_\Omega |\nabla u_n| = \Sigma(u_0)$. Use Proposition 4.4 to choose $\zeta_n \in \mathbb{S}^1$ such that

$$\int_\Omega |\nabla(u_n - P_{\zeta_n}\circ u_n)| \le \frac{2}{\pi}\int_\Omega |\nabla u_n|,$$

implying that $\lim_{n\to\infty} d_{W^{1,1}}(u_n,\mathcal{E}(1)) = \dfrac{2}{\pi}\Sigma(u_0)$. □

Next we turn to the general case.

Proof of "$\le$" in (1.12) for general v_0. By (4.7) we have

$$\mathrm{dist}_{W^{1,1}}(\mathcal{E}(u_0),\mathcal{E}(v_0)) \ge (2/\pi)\,\Sigma(u_0\overline{v}_0),$$

so we need to prove that this is actually an equality. By Proposition 1.2 there exists a sequence $\{w_n\}$ satisfying $w_n \sim u_0\overline{v}_0$ for all n, $\lim_{n\to\infty} w_n = 1$ a.e., and

$$\int_\Omega |\nabla w_n| = \Sigma(u_0\overline{v}_0) + \varepsilon_n, \tag{4.16}$$

with $\varepsilon_n \searrow 0$. By Proposition 4.4 we get

$$\int_{\mathbb{S}^1}\int_\Omega |\nabla(w_n - P_\zeta\circ w_n)|\,dx\,d\zeta = 4\int_\Omega |\nabla w_n|\,dx = 4(\Sigma(u_0\overline{v}_0)+\varepsilon_n). \tag{4.17}$$

Hence, there exists $\zeta_n \in \mathbb{S}^1_- := \{z = e^{\imath\theta};\ \theta\in[-\pi,0]\}$ such that

$$\int_\Omega |\nabla(w_n - P_{\zeta_n}\circ w_n)| + \int_\Omega |\nabla(w_n - P_{-\zeta_n}\circ w_n)| \le \frac{4}{\pi}(\Sigma(u_0\overline{v}_0)+\varepsilon_n).$$

By (4.7) we have

$$\frac{2}{\pi}\Sigma(u_0\overline{v}_0) \le \min\left(\int_\Omega |\nabla(w_n - P_{\zeta_n}\circ w_n)|, \int_\Omega |\nabla(w_n - P_{-\zeta_n}\circ w_n)|\right),$$

and thus

$$\lim_{n\to\infty}\int_\Omega |\nabla(w_n - P_{\zeta_n}\circ w_n)| = \frac{2}{\pi}\Sigma(u_0\overline{v}_0). \tag{4.18}$$

Passing to a subsequence, we may assume $\zeta_n \to \zeta \in \mathbb{S}^1_-$. Therefore, $P_\zeta(1) = 1$. Denote $F_n := P_{\zeta_n}\circ w_n$. Since $w_n \to 1$ a.e., we have $\lim_{n\to\infty} F_n = \lim_{n\to\infty} P_\zeta\circ w_n = 1$ a.e., and it follows that

$$F_n - w_n \to 0 \text{ a.e.} \tag{4.19}$$

For any v such that $v\sim v_0$ we have $vF_n \sim v_0$, $vw_n \sim u_0$ and

$$\frac{2}{\pi}\Sigma(u_0\overline{v}_0) \le \int_\Omega |\nabla(vF_n - vw_n)| \le \int_\Omega |\nabla(F_n - w_n)| + \int_\Omega |\nabla v||F_n - w_n|. \tag{4.20}$$

From (4.18)-(4.20) we deduce that

$$\lim_{n\to\infty}\int_\Omega |\nabla(vF_n - vw_n)| = \frac{2}{\pi}\Sigma(u_0\overline{v}_0),$$

and the result follows. □

5 Proof of (1.13) in Theorem 1.1

5.1 An upper bound for $\mathrm{Dist}_{W^{1,1}}$

This short subsection is devoted to the proof of the following

Proposition 5.1. *For every u_0, v_0 in $W^{1,1}(\Omega, \mathbb{S}^1)$ we have,*

$$\mathrm{Dist}_{W^{1,1}}(\mathcal{E}(u_0), \mathcal{E}(v_0)) = \sup_{u \sim u_0} d_{W^{1,1}}(u, \mathcal{E}(v_0)) \leq \Sigma(u_0 \overline{v}_0). \tag{5.1}$$

Proof. We adapt an argument from [12]. By Proposition 1.2 there exists a sequence $\{w_n\} \subset W^{1,1}(\Omega; \mathbb{S}^1)$ satisfying $w_n \sim u_0\overline{v}_0$, $w_n \to 1$ a.e., and $\lim_{n\to\infty} \int_\Omega |\nabla w_n| = \Sigma(u_0\overline{v}_0)$. For a given $u \in \mathcal{E}(u_0)$ define $v_n = u\overline{w}_n$ for all n. Then, $v_n \sim v_0$ and

$$\begin{aligned} d_{W^{1,1}}(u, \mathcal{E}(v_0)) \leq \int_\Omega |\nabla(u - v_n)| &= \int_\Omega |\nabla(u(1 - \overline{w}_n))| \\ &\leq \int_\Omega |1 - w_n||\nabla u| + \int_\Omega |\nabla w_n| \to \Sigma(u_0\overline{v}_0). \end{aligned}$$ □

5.2 A lower bound for $\mathrm{Dist}_{W^{1,1}}$

We begin with the following elementary geometric lemma.

Lemma 5.2. *Let z_1 and z_2 be two points in $\mathbb{S}^1$ satisfying, for some $\varepsilon \in (0, \pi/2)$,*

$$d_{\mathbb{S}^1}(z_1, z_2) \in (\varepsilon, \pi - \varepsilon). \tag{5.2}$$

If the vectors $v_1, v_2 \in \mathbb{R}^2$ satisfy

$$v_j \perp z_j,\ j = 1, 2, \tag{5.3}$$

then

$$|v_1 - v_2| \geq (\sin\varepsilon)|v_j|, j = 1, 2, \tag{5.4}$$

and in particular

$$|v_1 - v_2|^2 \geq \left(\frac{\sin^2\varepsilon}{2}\right)(|v_1|^2 + |v_2|^2). \tag{5.5}$$

Note that the inequality (5.5) can be viewed as a "reverse triangle inequality".

Proof. From the assumptions (5.2)–(5.3) it follows that

$$< v_1, v_2 >\leq (\cos\varepsilon)|v_1||v_2|,$$

and then

$$|v_1 - v_2|^2 \geq |v_1|^2 + |v_2|^2 - 2(\cos\varepsilon)|v_1||v_2| \geq (\sin\varepsilon)^2|v_j|^2,\ j = 1, 2.$$ □

An immediate consequence of Lemma 5.2 is

Lemma 5.3. *Let $v, \widetilde{u} \in W^{1,1}(\Omega;\mathbb{S}^1)$ and denote, for $\varepsilon \in (0,\pi/2)$,*

$$\begin{aligned} A_\varepsilon :&= \{x \in \Omega;\, d_{\mathbb{S}^1}(\widetilde{u}(x), v(x)) \in (\varepsilon, \pi-\varepsilon)\} \\ &= \{x \in \Omega;\, 2\sin(\varepsilon/2) < |\widetilde{u}(x) - v(x)| < 2\cos(\varepsilon/2)\}. \end{aligned} \tag{5.6}$$

Then

$$|\nabla(\widetilde{u} - v)| \ge (\sin\varepsilon)|\nabla\widetilde{u}| \text{ a.e. in } A_\varepsilon. \tag{5.7}$$

Proof. Since $v \perp v_{x_i}$ and $\widetilde{u} \perp \widetilde{u}_{x_i}$ a.e. on Ω for $i = 1,\dots,N$, we may apply Lemma 5.2 with $z_1 = \widetilde{u}(x), z_2 = v(x), v_1 = \widetilde{u}_{x_i}(x)$ and $v_2 = v_{x_i}(x)$ to obtain

$$|\widetilde{u}_{x_i} - v_{x_i}|^2 \ge (\sin\varepsilon)^2|\widetilde{u}_{x_i}|^2, \quad \text{a.e. in } A_\varepsilon,\ i = 1,\dots,N.$$

Summing over i yields (5.7). □

The next lemma is the main ingredient in the proof of Proposition 1.3.

Lemma 5.4. *Let $u, \widetilde{u}, v \in W^{1,1}(\Omega;\mathbb{S}^1)$, $\varepsilon \in (0,\pi/20)$ and A_ε as in (5.6). Assume that*

$$|u(x) - \widetilde{u}(x)| \le \varepsilon,\ \forall\, x \in \Omega. \tag{5.8}$$

Then,

$$\int_{A_\varepsilon} |\nabla(v - \widetilde{u})| \ge (1-6\varepsilon)\Sigma(v\overline{u}) - 2\int_{A_\varepsilon} |\nabla u|. \tag{5.9}$$

Proof. Note first that (5.8) implies that $\widetilde{u} \sim u$. Indeed, the image of the map $\widetilde{u}\,\overline{u}$ is contained in an arc of $\mathbb{S}^1$ of length$\le 2\arcsin(\varepsilon/2)$, so there exists $\varphi \in W^{1,1}(\Omega;\mathbb{R})$ such that $\widetilde{u} = e^{\imath\varphi}u$. Hence, setting $w := v/u = v\,\overline{u}$ and $\widetilde{w} := v/\widetilde{u}$, we have also $\widetilde{w} \sim w$. Consider the map

$$W := \overline{u}(v - \widetilde{u}) + 1 = w + (1 - \widetilde{u}/u). \tag{5.10}$$

By the triangle inequality,

$$|\nabla W| = |\nabla\left(\overline{u}(v - \widetilde{u})\right)| \le 2|\nabla\overline{u}| + |\nabla(v - \widetilde{u})|,$$

whence

$$\int_{A_\varepsilon} |\nabla(v - \widetilde{u})| \ge \int_{A_\varepsilon} |\nabla W| - 2\int_{A_\varepsilon} |\nabla u|. \tag{5.11}$$

By (5.8), $|W - w| = |1 - \widetilde{u}/u| = |u - \widetilde{u}| \le \varepsilon$ in Ω. Hence

$$||W| - 1| \le |W - w| \le \varepsilon \quad \text{in } \Omega, \tag{5.12}$$

and also

$$|\widetilde{w} - w| = |\widetilde{u} - u| \le \varepsilon \quad \text{in } \Omega. \tag{5.13}$$

Consider the map $\widetilde{W} := W/|W|$, which thanks to (5.12) belongs to $W^{1,1}(\Omega;\mathbb{S}^1)$. Furthermore, again by (5.12),

$$|\widetilde{W} - w| \le |\widetilde{W} - W| + |W - w| \le 2\varepsilon \text{ in } \Omega, \tag{5.14}$$

implying in particular that

$$\widetilde{W} \in \mathcal{E}(w). \tag{5.15}$$

Combining (5.14) with (5.13) yields

$$|\widetilde{W} - \widetilde{w}| \leq 3\varepsilon \text{ and } d_{\mathbb{S}^1}(\widetilde{W}, \widetilde{w}) \leq 6\varepsilon \text{ in } \Omega. \tag{5.16}$$

A direct consequence of (5.12) is the pointwise inequality in Ω

$$|\nabla W| \geq (1-\varepsilon)|\nabla \widetilde{W}|,$$

which together with (5.11) yields

$$\int_{A_\varepsilon} |\nabla(v - \widetilde{u})| \geq (1-\varepsilon)\int_{A_\varepsilon} |\nabla \widetilde{W}| - 2\int_{A_\varepsilon} |\nabla u|. \tag{5.17}$$

Since

$$A_\varepsilon = \{x \in \Omega;\, \widetilde{w}(x) \in \mathcal{A}(\varepsilon, \pi - \varepsilon) \cup \mathcal{A}(\pi + \varepsilon, 2\pi - \varepsilon)\} \text{ (see (5.6) and (4.8))},$$

we deduce from (5.16) that

$$B_\varepsilon := \{x \in \Omega;\, \widetilde{W}(x) \in \mathcal{A}(7\varepsilon, \pi - 7\varepsilon) \cup \mathcal{A}(\pi + 7\varepsilon, 2\pi - 7\varepsilon)\} \subseteq A_\varepsilon. \tag{5.18}$$

For each $\delta \in (0, \pi/2)$ consider the map $K_\delta : \mathbb{S}^1 \to \mathbb{S}^1$ defined by

$$K_\delta(e^{\imath\theta}) := \begin{cases} 1, & \text{if } -\delta \leq \theta < \delta \\ e^{\imath\pi(\theta-\delta)/(\pi-2\delta)}, & \text{if } \delta \leq \theta < \pi - \delta \\ -1, & \text{if } \pi - \delta \leq \theta < \pi + \delta \\ -e^{\imath\pi(\theta-\pi-\delta)/(\pi-2\delta)}, & \text{if } \pi + \delta \leq \theta < 2\pi - \delta \end{cases}. \tag{5.19}$$

Clearly $K_\delta \in \mathrm{Lip}(\mathbb{S}^1; \mathbb{S}^1)$ with $\|\dot{K}_\delta\|_\infty = \pi/(\pi - 2\delta)$ and $\deg(K_\delta) = 1$. Therefore, by (5.15) and Lemma 4.1

$$w_1 := K_{7\varepsilon} \circ \widetilde{W} \in \mathcal{E}(w). \tag{5.20}$$

Note that by definition, $\nabla w_1 = 0$ a.e. on $\Omega \setminus B_\varepsilon$, so by (5.18) and (5.20) we have

$$\int_{A_\varepsilon} |\nabla \widetilde{W}| \geq \int_{B_\varepsilon} |\nabla \widetilde{W}| \geq (1-5\varepsilon)\int_{B_\varepsilon} |\nabla w_1| = (1-5\varepsilon)\int_\Omega |\nabla w_1| \geq (1-5\varepsilon)\Sigma(w). \tag{5.21}$$

Plugging (5.21) in (5.17) yields

$$\begin{aligned} \int_{A_\varepsilon} |\nabla(v - \widetilde{u})| &\geq (1-\varepsilon)\int_{B_\varepsilon} |\nabla \widetilde{W}| - 2\int_{A_\varepsilon} |\nabla u| \\ &\geq (1-\varepsilon)(1-5\varepsilon)\Sigma(w) - 2\int_{A_\varepsilon} |\nabla u| \geq (1-6\varepsilon)\Sigma(w) - 2\int_{A_\varepsilon} |\nabla u|, \end{aligned} \tag{5.22}$$

and (5.9) follows. □

The next result is a direct consequence of Lemma 5.4.

Corollary 5.5. *There exists a universal constant C such that for every $\varepsilon > 0$ we have*

$$n \geq 1/\varepsilon^2 \implies \int_\Omega |\nabla(T_n \circ u) - v| \geq (1 - C\varepsilon)\,\Sigma(u\overline{v}),\ \forall\, u, v \in W^{1,1}(\Omega;\mathbb{S}^1). \tag{5.23}$$

Proof. We shall use two basic properties of T_n:

$$d_{\mathbb{S}^1}(x, T_n(x)) \leq \frac{\pi(n-1)}{n^2},\ \forall\, x \in \mathbb{S}^1, \tag{5.24}$$

$$|\dot{T}_n| \geq n - 2 \text{ a.e. in } \mathbb{S}^1. \tag{5.25}$$

Clearly it suffices to consider $\varepsilon < \pi/20$. Hence for $n \geq 1/\varepsilon^2$ we can apply Lemma 5.4 with $\widetilde{u} := T_n \circ u$ (thanks to (5.24)). By (5.25) we have

$$|\nabla(T_n \circ u)| \geq (n-2)|\nabla u| \text{ a.e. on } \Omega, \tag{5.26}$$

so combining (5.7) and (5.9) gives (recall that A_ε is defined in (5.6)):

$$\begin{aligned}\int_{A_\varepsilon} |\nabla(T_n \circ u - v)| &\geq (1 - 6\varepsilon)\Sigma(u\overline{v}) - \frac{2}{(n-2)\,\sin\varepsilon}\int_{A_\varepsilon} |\nabla(T_n \circ u - v)| \\ &\geq (1 - 6\varepsilon)\Sigma(u\overline{v}) - \frac{3}{n\varepsilon}\int_{A_\varepsilon} |\nabla(T_n \circ u - v)|;\end{aligned}$$

this leads easily to (5.23). □

Proof of Proposition 1.3. Recall (see (5.1)) that

$$\mathrm{Dist}_{W^{1,1}}(\mathcal{E}(u_0), \mathcal{E}(v_0)) = \sup_{u \sim u_0} d_{W^{1,1}}(u, \mathcal{E}(v_0)) \leq \Sigma(u_0\overline{v}_0), \tag{5.27}$$

and in particular, $\forall\, n \geq 3$,

$$d_{W^{1,1}}(T_n \circ u_0, \mathcal{E}(v_0)) \leq \Sigma(u_0\overline{v}_0). \tag{5.28}$$

On the other hand, from Corollary 5.5 we know that, $\forall\, \varepsilon > 0$, $\forall\, n \geq 1/\varepsilon^2$,

$$d_{W^{1,1}}(T_n \circ u_0, \mathcal{E}(v_0)) \geq (1 - C\varepsilon)\Sigma(u_0\overline{v}_0). \tag{5.29}$$

We conclude combining (5.28) and (5.29). □

Proof of (1.13). Use (1.18) and (5.27). □

5.3 About equality cases in (1.19)

It is interesting to decide whether there exist maps $u \in W^{1,1}(\Omega;\mathbb{S}^1)$ for which equality holds in any of the two inequalities in (1.19). Consider the following properties of a smooth bounded domain Ω in $\mathbb{R}^N$, $N \geq 2$:

(P_1) There exists $u \in W^{1,1}(\Omega;\mathbb{S}^1)$ such that

$$\int_\Omega |\nabla u| = \Sigma(u) > 0. \tag{5.30}$$

(P_2) There exists $u \in W^{1,1}(\Omega;\mathbb{S}^1)$ such that

$$d_{W^{1,1}}(u, \mathcal{E}(1)) = \Sigma(u) > 0. \tag{5.31}$$

(P_2^*) There exist $u \in W^{1,1}(\Omega;\mathbb{S}^1)$ with $\Sigma(u) > 0$ and $v \in \mathcal{E}(1)$ for which

$$\int_\Omega |\nabla(u-v)| = d_{W^{1,1}}(u,\mathcal{E}(1)) = \Sigma(u). \tag{5.32}$$

(P_3) There exists $u \in W^{1,1}(\Omega;\mathbb{S}^1)$ such that

$$d_{W^{1,1}}(u,\mathcal{E}(1)) = \frac{2}{\pi}\Sigma(u) > 0. \tag{5.33}$$

(P_3^*) There exist $u \in W^{1,1}(\Omega;\mathbb{S}^1)$ with $\Sigma(u) > 0$ and $v \in \mathcal{E}(1)$ for which

$$\int_\Omega |\nabla(u-v)| = d_{W^{1,1}}(u,\mathcal{E}(1)) = \frac{2}{\pi}\Sigma(u). \tag{5.34}$$

Very little is known about domains satisfying any of the above properties. The unit disc $\Omega = B(0,1)$ in $\mathbb{R}^2$ is an example of a domain for which (P_1) is satisfied. Indeed, for $u = x/|x|$ it is straightforward that

$$\Sigma\left(\frac{x}{|x|}\right) = 2\pi = \int_\Omega \left|\nabla\left(\frac{x}{|x|}\right)\right| \quad \text{(see also [13, 12])}$$

whence (P_1) holds. In view of the following proposition we know that (P_3^*) is also satisfied for $\Omega = B(0,1)$ in $\mathbb{R}^2$.

Proposition 5.6. *Properties (P_1) and (P_3^*) are equivalent. More precisely, let $u \in W^{1,1}(\Omega;\mathbb{S}^1)$ with $\Sigma(u) > 0$. Then, the following are equivalent:*

(a) u satisfies (5.30).

(b) There exist $u_0 \in \mathcal{E}(u)$ and $v \in \mathcal{E}(1)$ such that $\displaystyle\int_\Omega |\nabla(u_0 - v)| = \frac{2}{\pi}\Sigma(u)$.

Proof of "(a) $\Longrightarrow$ (b)". Use Proposition 4.4 to find $\zeta_0 \in \mathbb{S}^1$ such that $v = P_{\zeta_0} \circ u \in \mathcal{E}(1)$ satisfies

$$d_{W^{1,1}}(u,\mathcal{E}(1)) \le \int_\Omega |\nabla(u-v)| \le \frac{2}{\pi}\int_\Omega |\nabla u| = \frac{2}{\pi}\Sigma(u), \tag{5.35}$$

and the result follows, with $u_0 = u$, since by (1.21) we have

$$d_{W^{1,1}}(u,\mathcal{E}(1)) \ge \frac{2}{\pi}\Sigma(u). \tag{5.36}$$

Proof of "(b) $\Longrightarrow$ (a)". Let u_0 and v be as in statement *(b)*. Set $w_0 := u_0\overline{v}$, so that $w_0 \sim u_0$. By assumption and (4.2) we have:

$$\frac{2}{\pi}\Sigma(u_0) \le \int_\Omega |\nabla(|w_0 - 1|)| = \int_\Omega |\nabla(|u_0 - v|)| \le \int_\Omega |\nabla(u_0 - v)| = \frac{2}{\pi}\Sigma(u_0). \tag{5.37}$$

Set $w_1 := T \circ w_0$, where $T : \mathbb{S}^1 \to \mathbb{S}^1$ is given by (4.3). By Lemma 4.1, $w_1 \sim w_0 \sim u_0$, and by (4.6) and (5.37) we obtain that

$$\int_\Omega |\nabla w_1| = \frac{\pi}{2}\int_\Omega |\nabla(|w_0 - 1|)| = \Sigma(u).$$

$\square$

We do not know any domain Ω in $\mathbb{R}^2$ for which (5.30) fails and we ask:

Open Problem 1. Is there a domain Ω in $\mathbb{R}^N$, $N \geq 2$, for which property (P_1) (respectively, (P_3)) does not hold?

It seems plausible that if Ω is the interior of a non circular ellipse, then (P_1) and (P_3) fail. We also do not know whether properties (P_3) and (P_3^*) are equivalent.

Concerning properties (P_2) and (P_2^*) we know even less:

Open Problem 2. Is there a domain Ω for which (P_2) holds (respectively, fails)?

We suspect that (P_2) is satisfied in *every* domain, but we do not know *any* such domain. In particular, we do not know what happens when Ω is a disc in $\mathbb{R}^2$.

6 Distances in $W^{1,p}(\Omega;\mathbb{S}^1)$, $1 < p < \infty$

Throughout this section we study classes in $W^{1,p}(\Omega;\mathbb{S}^1)$, where $1 < p < \infty$ and Ω is a smooth bounded domain in $\mathbb{R}^N$, $N \geq 2$. We give below the proofs of the results stated in the Introduction.

6.1 Proof of Theorem 1.5

Proof of Theorem 1.5. The result is a direct consequence of the following analog of Corollary 4.3: for every $u, v \in W^{1,p}(\Omega;\mathbb{S}^1)$ we have

$$\|\nabla(u-v)\|_{L^p(\Omega)} \geq \left(\frac{2}{\pi}\right) \inf_{w\sim u\overline{v}} \|\nabla w\|_{L^p(\Omega)}. \tag{6.1}$$

The proof of (6.1) uses an argument identical to the one used in the proofs of Lemma 4.2 and Corollary 4.3. Indeed, we first note that

$$\int_\Omega |\nabla(u-v)|^p \geq \int_\Omega |\nabla(|u-v|)|^p = \int_\Omega |\nabla(|u\overline{v}-1|)|^p. \tag{6.2}$$

Next, by (4.5) we have

$$\int_\Omega |\nabla(|u\overline{v}-1|)|^p = \left(\frac{2}{\pi}\right)^p \int_\Omega |\nabla(T\circ(u\overline{v}))|^p \geq \left(\frac{2}{\pi}\right)^p \inf_{w\sim u\overline{v}} \int_\Omega |\nabla w|^p. \tag{6.3}$$

The result clearly follows by combining (6.2) with (6.3). $\square$

6.2 Proof of Theorem 1.7

We shall need the following technical lemma.

Lemma 6.1. *For every $w_0 \in W^{1,p}(\Omega;\mathbb{S}^1)$ we have*

$$\inf_{w\sim w_0} \|\nabla(|w-1|)\|_{L^p(\Omega)} = \left(\frac{2}{\pi}\right) \inf_{w\sim w_0} \|\nabla w\|_{L^p(\Omega)}. \tag{6.4}$$

Proof. The inequality "$\geq$" follows from (6.3) (taking $v = 1$) so it remains to prove the reverse inequality. The argument is almost identical to the one used in the proof of [27, Prop 3.1]; we reproduce the argument for the convenience of the reader. We shall need the inverse $S := T^{-1}$ of $T : \mathbb{S}^1 \to \mathbb{S}^1$ that was defined in (4.3). It is given by:

$$S(e^{\imath\theta}) = e^{\imath\phi}, \quad \text{with } \phi = 2\sin^{-1}(\theta/\pi),\ \forall\, \theta \in (-\pi, \pi].$$

This map belongs to $C(\mathbb{S}^1;\mathbb{S}^1) \cap C^1(\mathbb{S}^1 \setminus \{-1\};\mathbb{S}^1)$ but it is not Lipschitz. We therefore define, for each small $\varepsilon > 0$, an approximation S_ε by:

$$S_\varepsilon(e^{\imath\theta}) = e^{\imath\phi} \quad \text{with } \phi = 2\sin^{-1}\left(J_\varepsilon(\theta/\pi)\right),\ \forall\, \theta \in (-\pi, \pi], \tag{6.5}$$

where J_ε satisfies:

$$\begin{aligned}
&J_\varepsilon(\pm 1) = \pm 1,\ J_\varepsilon'(\pm 1) = 0,\\
&J_\varepsilon(t) = t,\ \text{for } |t| \leq 1 - \varepsilon,\\
&0 < J_\varepsilon'(t) < c_0,\ \text{for } |t| < 1,\\
&\frac{c_1}{\varepsilon} \leq |J_\varepsilon''(t)| \leq \frac{c_2}{\varepsilon},\ \text{for } 1 - \frac{\varepsilon}{2} \leq |t| \leq 1,
\end{aligned} \tag{6.6}$$

for some positive constants c_0, c_1, c_2 (independent of ε). Clearly $S_\varepsilon \in C^1(\mathbb{S}^1;\mathbb{S}^1)$ with $\deg(S_\varepsilon) = 1$, so by Lemma 4.1, for any $w \in \mathcal{E}(w_0)$ we have $w_\varepsilon := S_\varepsilon \circ w \in \mathcal{E}(w_0)$. Since $|S_\varepsilon(e^{\imath\theta}) - 1| = 2|J_\varepsilon(\theta/\pi)|$ it follows from (6.6) that

$$\left|\frac{d}{d\theta}\left(|S_\varepsilon(e^{\imath\theta}) - 1|\right)\right| \leq C,\ \forall\, \theta,\ \forall\, \varepsilon. \tag{6.7}$$

Put $A_\varepsilon := \{x \in \Omega : \ w(x) \in \mathcal{A}(-\pi(1-\varepsilon), \pi(1-\varepsilon))\}$. By (4.5),

$$|\nabla|w_\varepsilon - 1|| = \frac{2}{\pi}|\nabla w| \text{ a.e. on } A_\varepsilon,$$

while, by (6.7),

$$|\nabla|w_\varepsilon - 1|| \leq C|\nabla w| \text{ a.e. on } \Omega.$$

Therefore, by dominated convergence,

$$\lim_{\varepsilon \to 0} \int_\Omega |\nabla|w_\varepsilon - 1||^p = \left(\frac{2}{\pi}\right)^p \int_\Omega |\nabla w|^p,$$

and since the above is valid for any $w \in \mathcal{E}(w_0)$, the inequality "$\leq$" in (6.4) follows. $\square$

The next lemma is the main ingredient of the proof of Theorem 1.7.

Lemma 6.2. *For every $w \in W^{1,p}(\Omega;\mathbb{S}^1)$ and $0 < \delta < 1$ there exist a set $A = A(w, \delta) \subset \Omega$ and two functions $w_0, w_1 \in W^{1,p}(\Omega;\mathbb{S}^1)$ such that:*
(i) $w_1 = \overline{w}_0$ in $\Omega \setminus A$;
(ii) $w_0 = w_1$ in A;
(iii) $w_1 \in \mathcal{E}(w)$ and $w_0 \in \mathcal{E}(1)$;
(iv) $\int_\Omega |\nabla(w_1 - w_0)|^p \leq (1 + C_p\delta) \int_\Omega |\nabla|w - 1||^p$;
(v) $\int_\Omega |\nabla w_1|^p = \int_\Omega |\nabla w_0|^p \leq C(\delta, p) \int_\Omega |\nabla w|^p$.

Proof. Let I denote the open arc of $\mathbb{S}^1$, $I := \mathcal{A}(2\pi-\delta, 2\pi) = \{e^{\imath\theta} : \theta \in (2\pi-\delta, 2\pi)\}$, and let $A := w^{-1}(I)$. Define $T_1 : \mathbb{S}^1 \to \mathbb{S}^1$ by

$$T_1(e^{\imath\theta}) = \begin{cases} e^{\imath\pi\theta/(2\pi-\delta)}, & \text{if } 0 \le \theta \le 2\pi-\delta \\ (-1)e^{\imath\pi(\theta-(2\pi-\delta))/\delta}, & \text{if } 2\pi-\delta < \theta < 2\pi \end{cases}.$$

Note that the image of T_1, restricted to the arc $\mathbb{S}^1 \setminus I$ is $\mathbb{S}^1_+$, and that $\mathbb{S}^1_+$ is covered counterclockwise. Similarly, on the arc I, the image of T_1 is $\mathbb{S}^1_-$, covered again counterclockwise. It follows that $\deg(T_1) = 1$. Next we define $T_0 : \mathbb{S}^1 \to \mathbb{S}^1$ by $P_{-1} \circ T_1$ (see (4.10)), or explicitly by

$$T_0(e^{\imath\theta}) := \begin{cases} \overline{T_1(e^{\imath\theta})}, & \text{if } 0 \le \theta \le 2\pi-\delta \\ T_1(e^{\imath\theta}), & \text{if } 2\pi-\delta < \theta < 2\pi \end{cases}.$$

Clearly $\deg(T_0) = 0$. Define $w_0 := T_0 \circ w$ and $w_1 := T_1 \circ w$.

Properties *(i)–(ii)* are direct consequences of the definition of w_0, w_1. The fact that $w_0 \in \mathcal{E}(1)$ and $w_1 \in \mathcal{E}(w)$ (i.e., property *(iii)*) follows from Lemma 4.1. Since T_0 and T_1 are Lipschitz maps (actually, piecewise smooth, with a single corner at $z = e^{\imath(2\pi-\delta)}$), the chain rule implies that

$$|\nabla w_0| = |\nabla w_1| \le \begin{cases} (\pi/\delta)\,|\nabla w|, & \text{a.e. in } A \\ (\pi/(2\pi-1))\,|\nabla w|, & \text{a.e. in } \Omega \setminus A \end{cases},$$

whence property *(v)*. Finally, in order to verify property *(iv)* we first notice that on $\Omega \setminus A$ we have

$$\widetilde{w} := w_1\,\overline{w}_0 = w_1^2 = Q \circ w,$$

where $Q(e^{\imath\theta}) := e^{2\imath\theta\pi/(2\pi-\delta)}$ for $\theta \in (0, 2\pi-\delta)$. Therefore,

$$\begin{aligned} \int_\Omega |\nabla(w_1-w_0)|^p &= \int_{\Omega\setminus A} |\nabla(w_1-w_0)|^p = \int_{\Omega\setminus A} |\nabla|w_1-w_0||^p = \int_{\Omega\setminus A} \big|\nabla|\widetilde{w}-1|\big|^p \\ &\le (1+C_p\delta)\int_{\Omega\setminus A} \big|\nabla|w-1|\big|^p \le (1+C_p\delta)\int_\Omega |\nabla|w-1||^p. \end{aligned}$$

□

We are now in a position to present the

Proof of Theorem 1.7. In view of Theorem 1.5 we only need to prove the inequality "≤" in (1.28). For any $w \in \mathcal{E}(u_0)$ we may apply Lemma 6.2 with a sequence $\delta_n \to 0$ to obtain that

$$d_{W^{1,p}}(\mathcal{E}(u_0), \mathcal{E}(1)) \le \inf_{w\sim u_0} \|\nabla|w-1|\|_{L^p(\Omega)},$$

and the result follows from Lemma 6.1. □

6.3 Proof of Theorem 1.9

We next turn to the unboundedness of the $\mathrm{Dist}_{W^{1,p}}$-distance between distinct classes.

Proof of Theorem 1.9 when $u_0 = 1$. For every $n \geq 1$ let

$$u_n := e^{\imath n x_1} \text{ (we write } x = (x_1, \dots, x_N)),$$

so clearly $u_n \in C^\infty(\Omega; \mathbb{S}^1) \subset \mathcal{E}(1)$. We claim that

$$\lim_{n\to\infty} d_{W^{1,p}}(u_n, \mathcal{E}(v_0)) = \infty, \tag{6.8}$$

which implies of course (1.29) in this case. Fix a small $\varepsilon > 0$, e.g., $\varepsilon = \pi/8$. For each $v \in \mathcal{E}(v_0)$ let $w_n := \overline{u}_n\, v$ and define the set A_ε as in (5.6), with $\widetilde{u} = u_n$. Note that $|\nabla u_n(x)| = n$, $x \in \Omega$, so by Lemma 5.3 we have

$$|\nabla(u_n - v)| \geq n \sin\varepsilon \text{ a.e. in } A_\varepsilon. \tag{6.9}$$

Therefore,

$$\int_{A_\varepsilon} |\nabla(u_n - v)|^p \geq |A_\varepsilon| (\sin\varepsilon)^p n^p = c_1 |A_\varepsilon| n^p. \tag{6.10}$$

Using (5.5) instead of (5.4) in the computation leading to (6.9) yields

$$|\nabla(u_n - v)| \geq \left(\frac{\sin\varepsilon}{2}\right) (|\nabla u_n| + |\nabla v|) \geq \left(\frac{\sin\varepsilon}{2}\right) |\nabla w_n|, \quad \text{a.e. in } A_\varepsilon. \tag{6.11}$$

We set $\widetilde{w}_n := K_\varepsilon \circ w_n$ (see (5.19)) and recall that $K_\varepsilon \in \mathrm{Lip}(\mathbb{S}^1; \mathbb{S}^1)$, $\|\dot{K}_\varepsilon\|_\infty = \pi/(\pi - 2\varepsilon)$ and $\deg(K_\varepsilon) = 1$. We have $\widetilde{w}_n \in \mathcal{E}(v_0)$ and $\nabla \widetilde{w}_n = 0$ a.e. in $\Omega \setminus A_\varepsilon$. By (1.4),

$$\int_{A_\varepsilon} |\nabla \widetilde{w}_n| = \int_\Omega |\nabla \widetilde{w}_n| \geq \Sigma(v_0). \tag{6.12}$$

Using Hölder inequality and (6.12) gives

$$\int_{A_\varepsilon} |\nabla \widetilde{w}_n|^p \geq \frac{\left(\int_{A_\varepsilon} |\nabla \widetilde{w}_n|\right)^p}{|A_\varepsilon|^{p-1}} \geq \frac{(\Sigma(v_0))^p}{|A_\varepsilon|^{p-1}}. \tag{6.13}$$

Since $|\nabla w_n| \geq (1 - 2\varepsilon/\pi) |\nabla \widetilde{w}_n|$ on Ω we obtain by combining (6.11) and (6.13) that

$$\int_{A_\varepsilon} |\nabla(u_n - v)|^p \geq \frac{c_2}{|A_\varepsilon|^{p-1}}, \tag{6.14}$$

whence,

$$|A_\varepsilon| \geq c_2^{1/(p-1)} \left(\int_{A_\varepsilon} |\nabla(u_n - v)|^p\right)^{-1/(p-1)}. \tag{6.15}$$

Plugging (6.15) in (6.10) finally yields

$$\int_{A_\varepsilon} |\nabla(u_n - v)|^p \geq c_3 n^{p-1},$$

and (6.8) follows.

Proof of Theorem Theorem 1.9 in the general case. Consider an arbitrary $u_0 \in W^{1,p}(\Omega; \mathbb{S}^1)$. We set $u_n := e^{\imath n x_1} u_0 \in \mathcal{E}(u_0)$. By the triangle inequality,

$$|\nabla(e^{\imath n x_1} - \overline{u}_0\, v)| = |\nabla\left(\overline{u}_0 (u_n - v)\right)| \leq |\nabla(u_n - v)| + 2|\nabla \overline{u}_0|.$$

Therefore,

$$\|\nabla(u_n - v)\|_{L^p(\Omega)} \geq \|\nabla(e^{\imath n x_1} - \overline{u}_0 v)\|_{L^p(\Omega)} - 2\|\nabla u_0\|_{L^p(\Omega)},$$

and the result follows from the first part of the proof. □

6.4 An example of strict inequality in (1.27)

Proposition 6.3. *There exist a smooth bounded simply connected domain* Ω *in* $\mathbb{R}^2$ *and* $u_0, v_0 \in \bigcap_{1\le p<2} W^{1,p}(\Omega;\mathbb{S}^1)$ *such that*

$$\mathrm{dist}_{W^{1,p}}(\mathcal{E}(u_0),\mathcal{E}(v_0)) > \left(\frac{2}{\pi}\right) \inf_{w\sim u_0\overline{v}_0} \|\nabla w\|_{L^p(\Omega)},\ \forall\, 1<p<2. \tag{6.16}$$

Proof. The construction resembles the one used in the proof of [27, Proposition 4.1] (for a multiply connected domain and $p=2$), but the details of the proof are quite different.

Step 1. Definition of Ω_ε and u_0, v_0
Consider the three unit discs with centers at the points $a_- := (-3,0), a := (0,0)$ and $a_+ := (3,0)$, respectively:

$$B_- := B(a_-,1),\ B := B(a,1)\ \text{ and } B_+ := B(a_+,1).$$

For a small $\varepsilon \in (0,1/4)$, to be determined later, define the domain Ω_ε by

$$\Omega_\varepsilon := B_- \cup B \cup B_+ \cup \{(x_1,x_2);\ x_1 \in (-3,3), x_2 \in (-\varepsilon,\varepsilon)\}. \tag{6.17}$$

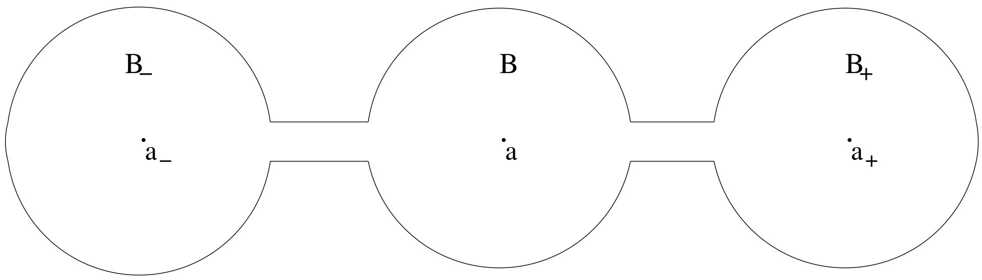

Figure 1: The domain Ω_ε (before smoothing)

Hence, B is connected to B_- and B_+ by two narrow tubes (see Figure 1). We can enlarge Ω_ε slightly near the "corners"=the contact points of the tubes with the circles, in order to have a smooth Ω_ε. But we do it keeping the following property:

$$\Omega_\varepsilon \text{ is symmetric with respect to reflections in both the } x \text{ and } y\text{-axis.} \tag{6.18}$$

For later use we denote by Ω_+ and Ω_- the two components of $\Omega_\varepsilon \setminus \overline{B}$ (with $\Omega_+ \subset \{z;\ \mathrm{Re}\, z > 0\}$). We define the maps $u_0, v_0 \in \bigcap_{1\le p<2} W^{1,p}(\Omega_\varepsilon;\mathbb{S}^1)$ by

$$u_0 := \left(\frac{x-a_-}{|x-a_-|}\right)\left(\frac{x}{|x|}\right)^2\left(\frac{x-a_+}{|x-a_+|}\right),\ v_0 := \left(\frac{x-a_-}{|x-a_-|}\right)\left(\frac{x}{|x|}\right)\left(\frac{x-a_+}{|x-a_+|}\right), \tag{6.19}$$

and then

$$w_0 := u_0\,\overline{v}_0 = \frac{x}{|x|}. \tag{6.20}$$

Step 2. Properties of energy minimizers in $\mathcal{E}(w_0)$
We denote by $W_\varepsilon \in W^{1,p}(\Omega_\varepsilon;\mathbb{S}^1)$ a map realizing the minimum in

$$S_\varepsilon = \inf_{w\sim w_0} \|\nabla w\|_{L^p(\Omega_\varepsilon)}. \tag{6.21}$$

Note that the minimizer W_ε is unique, up to multiplication by a complex constant of modulo one. This follows from the strict convexity of the functional:

$$F(\varphi) = \int_{\Omega_\varepsilon} \left| \nabla \left(e^{\imath\varphi} \frac{x}{|x|} \right) \right|^p \text{ over } \varphi \in W^{1,p}(\Omega_\varepsilon; \mathbb{R}). \tag{6.22}$$

We next claim that

$$\int_B \left| \nabla \left(\frac{x}{|x|} \right) \right|^p \le \int_B |\nabla W_\varepsilon|^p \le \int_{\Omega_\varepsilon} |\nabla W_\varepsilon|^p \le \int_B \left| \nabla \left(\frac{x}{|x|} \right) \right|^p + C\varepsilon^2, \tag{6.23}$$

for some constant C. Here and in the sequel we denote by C different constants that are independent of ε and p. Indeed, the first inequality in (6.23) is clear since the restriction of W_ε to B belongs to the class of $x/|x|$ in B, and the latter map is a minimizer of the energy in its class (see Remark 6.5 below). For the proof of the last inequality in (6.23) it suffices to construct a comparison map $\widetilde{w} \in \mathcal{E}(w_0)$ as follows. We first set $\widetilde{w} = x/|x|$ in B. Then extend it to $\Omega_+ \cap \{x_1 \le 1+\varepsilon\}$ in such a way that $\widetilde{w} \equiv \zeta$ (for some constant $\zeta \in \mathbb{S}^1$) on $\Omega_+ \cap \{x_1 = 1+\varepsilon\}$. Such an extension can be constructed with $\|\nabla\widetilde{w}\|_{L^\infty} \le C$, whence

$$\int_{\Omega_+ \cap \{x_1 < 1+\varepsilon\}} |\nabla\widetilde{w}|^p \le C\varepsilon^2.$$

In the remaining part of Ω_+, namely $\Omega_+ \cap \{x_1 > 1+\varepsilon\}$ we simply set $\widetilde{w} \equiv \zeta$. We use a similar construction for $\widetilde{w}$ on Ω_-, and this completes the proof of (6.23).

We shall also use a certain symmetry property of W_ε. We claim that:

$$W_\varepsilon(x) = -W_\varepsilon(-x) \text{ in } \Omega_\varepsilon. \tag{6.24}$$

Indeed, since $W_\varepsilon(-x)$ is also a minimizer in (6.21), we must have

$$W_\varepsilon(-x) = e^{\imath\alpha} W_\varepsilon(x) \text{ for some constant } \alpha \in \mathbb{R}. \tag{6.25}$$

Write

$$W_\varepsilon = e^{\imath\Psi_\varepsilon} \left(\frac{x}{|x|} \right), \text{ with } \Psi_\varepsilon \in W^{1,p}(\Omega_\varepsilon; \mathbb{R}). \tag{6.26}$$

Plugging (6.26) in (6.25) gives

$$-e^{\imath\Psi_\varepsilon(-x)} \left(\frac{x}{|x|} \right) = e^{\imath\alpha} e^{\imath\Psi_\varepsilon(x)} \left(\frac{x}{|x|} \right),$$

whence $e^{\imath(\Psi_\varepsilon(-x)-\Psi_\varepsilon(x))} = -e^{\imath\alpha}$. It follows that $\Psi_\varepsilon(-x)-\Psi_\varepsilon(x) \equiv \mathit{const}$ in Ω_ε. Since $\Psi_\varepsilon(-x)-\Psi_\varepsilon(x)$ is odd, it follows that the constant must be zero. Hence $\Psi_\varepsilon(-x) = \Psi_\varepsilon(x)$ a.e. in Ω_ε, $e^{\imath\alpha} = -1$ and (6.24) follows from (6.25).

The main property of W_ε that we need is the following: there exists $\zeta_\varepsilon \in \mathbb{S}^1$ such that

$$|W_\varepsilon - \zeta_\varepsilon| \le c_0\, \varepsilon^{2/p} \text{ on } B_+, \tag{6.27}$$

$$|W_\varepsilon + \zeta_\varepsilon| \le c_0\, \varepsilon^{2/p} \text{ on } B_-. \tag{6.28}$$

In order to verify (6.27)–(6.28) we first notice that we may write $W_\varepsilon = e^{\imath\Phi_\varepsilon}$ in $\Omega_\varepsilon \cap \{x_1 > 1\}$. Using (6.23) and Fubini Theorem we can find $t_\varepsilon \in (1, 3/2)$ such that the segment $I_\varepsilon = \{(t_\varepsilon, x_2);\, x_2 \in (-\varepsilon, \varepsilon)\}$ satisfies

$$\int_{I_\varepsilon} |\nabla\Phi_\varepsilon|^p = \int_{I_\varepsilon} |\nabla W_\varepsilon|^p \le C\varepsilon^2.$$

By Hölder inequality it follows that $|\Phi_\varepsilon(z_1)-\Phi_\varepsilon(z_2)| \le C\varepsilon^{2/p}$ for all $z_1, z_2 \in I_\varepsilon$. Hence, there exists $\alpha_\varepsilon \in \mathbb{R}$ satisfying

$$|\Phi_\varepsilon(z)-\alpha_\varepsilon| \le C\varepsilon^{2/p},\ \forall\, z \in I_\varepsilon. \tag{6.29}$$

We claim that (6.29) continues to hold in $G_\varepsilon := \Omega_\varepsilon \cap \{x_1 > t_\varepsilon\}$, i.e.,

$$|\Phi_\varepsilon(x)-\alpha_\varepsilon| \le C\varepsilon^{2/p},\ \forall\, x \in G_\varepsilon. \tag{6.30}$$

Indeed, defining

$$\widetilde{\Phi}_\varepsilon(x) := \max\left(\alpha_\varepsilon - C\varepsilon^{2/p}, \min(\Phi_\varepsilon(x), \alpha_\varepsilon + C\varepsilon^{2/p})\right),$$

and then $\widetilde{W}_\varepsilon := e^{\imath\widetilde{\Phi}_\varepsilon}$, we clearly have $\int_{G_\varepsilon} |\nabla \widetilde{W}_\varepsilon|^p \le \int_{G_\varepsilon} |\nabla W_\varepsilon|^p$, with strict inequality, unless (6.30) holds. Setting $\zeta_\varepsilon := e^{\imath\alpha_\varepsilon}$, we deduce (6.27) from (6.30). Finally, using the symmetry properties, (6.18) of Ω_ε and (6.24) of Ψ_ε, we easily deduce (6.28) from (6.27).

Step 3. A basic estimate for maps in $W^{1,p}(\mathbb{S}^1;\mathbb{S}^1)$

The following claim provides a simple estimate which is essential for the proof. The case $p=2$ was proved in [27, Lemma 4.1] and the generalization to any $p \ge 1$ is straightforward. We include the proof for the convenience of the reader.

Claim. *For any $p \ge 1$, let $f, g \in W^{1,p}(\mathbb{S}^1;\mathbb{S}^1)$ satisfy:*

$$\deg f = \deg g = k \ne 0 \ \textit{and}\ |(f-g)(\zeta)| = \eta > 0,$$

for some point $\zeta \in \mathbb{S}^1$. Then,

$$\int_{\mathbb{S}^1} |\dot f - \dot g|^p \ge \frac{2\eta^p}{\pi^{p-1}}. \tag{6.31}$$

Proof of Claim. Set $w := f - g = w_1 + \imath w_2$. We may assume without loss of generality that $w(1) = (f-g)(1) = \eta\,\imath$. Since $\deg(g) \ne 0$, there exists a point $\theta_1 \in (0, 2\pi)$ such that $g(e^{\imath\theta_1}) = \imath$, whence $w_2(e^{\imath\theta_1}) = -t\imath$ for some $t \ge 0$. Hölder's inequality, and a straightforward computation yield

$$\int_{\mathbb{S}^1} |w'|^p \ge \int_{\mathbb{S}^1} |w_2'|^p \ge \frac{(\eta+t)^p}{\theta_1^{p-1}} + \frac{(\eta+t)^p}{(2\pi-\theta_1)^{p-1}} \ge 2\frac{(\eta+t)^p}{\pi^{p-1}} > \frac{2\eta^p}{\pi^{p-1}},$$

and (6.31) follows. □

Remark 6.4. We thank an anonymous referee for suggesting a simplification of our original argument for the proof of the Claim, and for pointing out that it holds under the weaker assumption: either f or g has a nontrivial degree.

Step 4. Conclusion

Consider two sequences $\{u_n\} \subset \mathcal{E}(u_0)$ and $\{v_n\} \subset \mathcal{E}(v_0)$ such that

$$\lim_{n\to\infty} \|\nabla(u_n - v_n)\|_{L^p(\Omega_\varepsilon)} = \mathrm{dist}_{W^{1,p}}(\mathcal{E}(u_0), \mathcal{E}(v_0)). \tag{6.32}$$

By a standard density argument we may assume that $u_n, v_n \in C^\infty(\overline{\Omega}_\varepsilon \setminus \{a_-, a, a_+\})$ for all n. Assume by contradiction that

$$\mathrm{dist}_{W^{1,p}}(\mathcal{E}(u_0), \mathcal{E}(v_0)) = \left(\frac{2}{\pi}\right) S_\varepsilon = \left(\frac{2}{\pi}\right) \min_{w\sim w_0} \|\nabla w\|_{L^p(\Omega_\varepsilon)} \text{ (see (6.21))}. \tag{6.33}$$

By (6.33) and (6.23) there exists a constant C_0 such that

$$\int_{\Omega_\varepsilon} |\nabla(u_n - v_n)|^p \leq \left(\frac{2}{\pi}\right)^p \int_B \left|\nabla\left(\frac{x}{|x|}\right)\right|^p + C_0\varepsilon^2,\ \forall\, n \geq n_0(\varepsilon). \tag{6.34}$$

Set $w_n := u_n\,\overline{v}_n$ and note that by the same computation as in (6.2)–(6.3) we have

$$\int_{\Omega_\varepsilon} |\nabla(u_n - v_n)|^p \geq \int_{\Omega_\varepsilon} |\nabla(|u_n - v_n|)|^p = \left(\frac{2}{\pi}\right)^p \int_{\Omega_\varepsilon} |\nabla(T\circ w_n)|^p \geq \left(\frac{2}{\pi}\right)^p S_\varepsilon^p \tag{6.35}$$

(recall that T is defined in (4.3)). Combining (6.32),(6.33) and (6.35) yields that $\widetilde{w}_n := T\circ w_n$ satisfies

$$\lim_{n\to\infty} \|\nabla\widetilde{w}_n\|_{L^p(\Omega_\varepsilon)} = S_\varepsilon,$$

and up to passing to a subsequence we have

$$W_\varepsilon = \lim_{n\to\infty} \widetilde{w}_n \ \text{ in } W^{1,p}(\Omega;\mathbb{S}^1), \tag{6.36}$$

where W_ε is a minimizer in (6.21). Recall that W_ε is unique up to rotations; the particular W_ε in (6.36) is chosen by the subsequence. For any $\zeta \in \mathbb{S}^1$ we have $\max(|\zeta-1|,|\zeta+1|) \geq \sqrt{2}$. In particular, for ζ_ε associated with W_ε (see (6.27)–(6.28)) we may assume without loss of generality that

$$|\zeta_\varepsilon - 1| \geq \sqrt{2}. \tag{6.37}$$

By (6.36) and Egorov Theorem there exists $A_\varepsilon \subset \Omega_\varepsilon$ satisfying

$$|A_\varepsilon| \leq \varepsilon \ \text{ and } \widetilde{w}_n \to W_\varepsilon \text{ uniformly on } \Omega_\varepsilon \setminus A_\varepsilon, \tag{6.38}$$

again, after passing to a subsequence. Combining (6.38) with (6.37) and (6.27) yields

$$|\widetilde{w}_n - 1| \geq \sqrt{2} - 2c_0\,\varepsilon \ \text{ on } B_+ \setminus A_\varepsilon,\ \forall\, n \geq n_1(\varepsilon),$$

and choosing $\varepsilon < (\sqrt{2}-1)/(2c_0)$ guarantees that

$$|\widetilde{w}_n - 1| \geq 1 \ \text{ on } B_+ \setminus A_\varepsilon,\ \forall\, n \geq n_1(\varepsilon). \tag{6.39}$$

Going back to the definition of T in (4.3), we find by a simple computation the following equivalences for $e^{\imath\theta} = T(e^{\imath\varphi})$ (with $\theta \in (-\pi,\pi)$):

$$|T(e^{\imath\varphi})-1| \geq 1 \iff |\theta| = \pi|\sin(\varphi/2)| \geq \pi/3 \iff |e^{\imath\varphi}-1| = 2|\sin(\varphi/2)| \geq 2/3. \tag{6.40}$$

Using (6.40) we may rewrite (6.39) in terms of the original sequence $\{w_n\}$:

$$|u_n - v_n| = |w_n - 1| \geq 2/3 \ \text{ on } B_+ \setminus A_\varepsilon,\ \forall\, n \geq n_1(\varepsilon). \tag{6.41}$$

Consider the set

$$\Lambda_\varepsilon = \{r \in (1/2,1);\, \partial B(a_+,r) \subset A_\varepsilon\}. \tag{6.42}$$

By (6.38) we clearly have $\varepsilon \geq |A_\varepsilon| \geq (1/2)|\Lambda_\varepsilon|\cdot 2\pi$, whence

$$|\Lambda_\varepsilon| \leq \frac{\varepsilon}{\pi}. \tag{6.43}$$

For $n \geq n_1(\varepsilon)$ we have: on each circle $\partial B(a_+, r)$ with $r \in (1/2, 1) \setminus \Lambda_\varepsilon$ there exists at least one point where $|u_n - v_n| \geq 2/3$. Thus we may apply the Claim from Step 3 with $\eta := 2/3$, $f := u_n\big|_{\partial B(a_+,r)}$ and $g := v_n\big|_{\partial B(a_+,r)}$ to obtain by (6.31) (after a suitable rescaling):

$$\int_{\partial B(a_+,r)} |\nabla(u_n - v_n)|^p \geq 2(r\pi)^{1-p} \left(\frac{2}{3}\right)^p \geq (2\pi)\left(\frac{2}{3\pi}\right)^p := \gamma_p. \tag{6.44}$$

Integrating (6.44), taking into account (6.43), yields

$$\int_{B_+} |\nabla(u_n - v_n)|^p \geq \int_{(1/2,1)\setminus\Lambda_\varepsilon} \int_{\partial B(a_+,r)} |\nabla(u_n - v_n)|^p \geq (1/2 - \varepsilon/\pi)\gamma_p. \tag{6.45}$$

In addition, by (1.27), applied to $u_n\big|_B, v_n\big|_B$, we clearly have

$$\int_B |\nabla(u_n - v_n)|^p \geq \left(\frac{2}{\pi}\right)^p \int_B \left|\nabla\left(\frac{x}{|x|}\right)\right|^p,$$

which together with (6.45) gives

$$\int_{\Omega_\varepsilon} |\nabla(u_n - v_n)|^p \geq \left(\frac{2}{\pi}\right)^p \int_B \left|\nabla\left(\frac{x}{|x|}\right)\right|^p + (1/2 - \varepsilon/\pi)\gamma_p,\ \forall\, n \geq n_1(\varepsilon). \tag{6.46}$$

The inequality (6.46) clearly contradicts (6.34) for n large enough if ε is chosen sufficiently small. □

Remark 6.5. In the course of the proof of Proposition 6.3 we used the following fact:

Let $1 \leq p < 2$ and let Ω be the unit disc. Set $u_0(x) := x/|x|$, $\forall\, x \in \Omega$. Then

$$\int_\Omega |\nabla u|^p \geq \int_\Omega |\nabla u_0|^p,\ \forall\, u \in \mathcal{E}(u_0). \tag{6.47}$$

We sketch the proof for the convenience of the reader.

Let $u \in \mathcal{E}(u_0)$. Let $C_r := \{z;\, |z| = r\}$. Since we may write $u = e^{\imath\varphi}\, u_0$, with $\varphi \in W^{1,p}$, for a.e. $r \in (0,1)$ we have $u\big|_{C_r} \in W^{1,p}(C_r; \mathbb{S}^1)$ and $\deg\left(u\big|_{C_r}\right) = 1$. This implies that for a.e. $r \in (0,1)$ we have

$$\int_{C_r} |\nabla u| \geq \int_{C_r} |\dot{u}| = \int_{C_r} |u \wedge \dot{u}| \geq \int_{C_r} u \wedge \dot{u} = 2\pi = \int_{C_r} u_0 \wedge \dot{u}_0 = \int_{C_r} |\nabla u_0|. \tag{6.48}$$

In case $p = 1$ integration over $r \in (0,1)$ of (6.48) yields (6.47). In case $1 < p < 2$ we use (6.48) and Hölder inequality, and then integration over r yields

$$\int_\Omega |\nabla u|^p \geq 2\pi \int_0^1 \frac{dr}{r^{p-1}} = \frac{2\pi}{2-p} = \int_\Omega |\nabla u_0|^p, \text{ and (6.47) follows.}$$

Examining the equality cases for the inequalities in (6.48) (and in Hölder inequality when $1 < p < 2$) we obtain in addition the following conclusion: equality holds in (6.47) if and only if

(i) for $1 < p < 2$, $u = e^{\imath\alpha} u_0$ for some constant α;
(ii) for $p = 1$, $u(re^{\imath\theta}) = e^{i\varphi(\theta)}$ where $\varphi \in W^{1,1}([0, 2\pi]; \mathbb{R})$ satisfies $\varphi(2\pi) - \varphi(0) = 2\pi$ and $\varphi' \geq 0$ a.e. on $[0, 2\pi]$.
The difference between (i) and (ii) is the main reason why for the same u_0 and v_0 as in the proof of Proposition 6.3, we have the strict inequality (6.16) for $1 < p < 2$, while for $p = 1$ the equality (1.12) holds.

Remark 6.6. Consider the maps $u_1 := x/|x|$ and $v_1 := 1$ in Ω_ε (as in the proof of Proposition 6.3). By Theorem 1.7 we have $\mathrm{dist}_{W^{1,p}}(\mathcal{E}(u_1), \mathcal{E}(v_1)) = (2/\pi) \inf_{w\sim u_1\overline{v}_1} \|\nabla w\|_{L^p(\Omega_\varepsilon)}$. Therefore, we have

$$\mathrm{dist}_{W^{1,p}}(\mathcal{E}(u_1), \mathcal{E}(v_1)) \neq \mathrm{dist}_{W^{1,p}}(\mathcal{E}(u_0), \mathcal{E}(v_0))$$

although $u_1\overline{v}_1 = u_0\overline{v}_0$. This shows that in general it is not even true that $\mathrm{dist}_{W^{1,p}}(\mathcal{E}(u), \mathcal{E}(v))$ depends only on $\mathcal{E}(u\overline{v})$ when $1 < p < 2$. A similar phenomenon occurs when Ω is multiply connected and $p = 2$ (see [27, Remark 4.1]); a comparable argument works for $p > 2$.

Appendix. Proof of Proposition 1.2

Proof of Proposition 1.2. We fix a sequence $\varepsilon_n \searrow 0$ and use (1.4) to find a sequence $\{v_n\} \subset \mathcal{E}(u)$ such that

$$\int_\Omega |\nabla v_n| \leq \Sigma(u) + \varepsilon_n, \ \forall\, n. \tag{A.1}$$

For $\theta \in [0, 2\pi)$ define $\Psi_{n,\theta} \in \mathrm{Lip}(\mathbb{S}^1; \mathbb{S}^1)$ by

$$\Psi_{n,\theta}(z) := \begin{cases} e^{\imath\pi(1+2(\varphi-\theta)/\varepsilon_n)}, & \text{if } z = e^{\imath\varphi} \in \mathcal{A}(\theta - \varepsilon_n/2, \theta + \varepsilon_n/2) \\ 1, & \text{if } z \notin \mathcal{A}(\theta - \varepsilon_n/2, \theta + \varepsilon_n/2) \end{cases}. \tag{A.2}$$

Clearly $\deg \Psi_{n,\theta} = 1$, so setting $w_{n,\theta} := \Psi_{n,\theta} \circ v_n$, we have, by Lemma 4.1, $w_{n,\theta} \sim v_n \sim u$. Moreover,

$$|\nabla w_{n,\theta}(x)| = \begin{cases} (2\pi/\varepsilon_n)\, |\nabla v_n(x)|, & \text{if } v_n(x) \in \mathcal{A}(\theta - \varepsilon_n/2, \theta + \varepsilon_n/2) \\ 0, & \text{if } v_n(x) \notin \mathcal{A}(\theta - \varepsilon_n/2, \theta + \varepsilon_n/2) \end{cases}. \tag{A.3}$$

Set $A_n(x) := \{\theta \in [0, 2\pi); v_n(x) \in \mathcal{A}(\theta - \varepsilon_n/2, \theta + \varepsilon_n/2)\}$. We have

$$\int_0^{2\pi} \int_\Omega |\nabla w_{n,\theta}|\, dx\, d\theta = \frac{2\pi}{\varepsilon_n} \int_\Omega |\nabla v_n(x)|\, |A_n(x)|\, dx = 2\pi \int_\Omega |\nabla v_n|, \tag{A.4}$$

and

$$\int_0^{2\pi} |\{w_{n,\theta} \neq 1\}|\, d\theta = \int_\Omega |A_n(x)|\, dx = \varepsilon_n\, |\Omega|. \tag{A.5}$$

Combining (A.1) with (A.4)–(A.5) yields

$$\int_0^{2\pi} \left(|\{w_{n,\theta} \neq 1\}|/\varepsilon_n^{1/2} + \int_\Omega |\nabla w_{n,\theta}| \right) d\theta \leq |\Omega|\, \varepsilon_n^{1/2} + 2\pi(\Sigma(u) + \varepsilon_n). \tag{A.6}$$

From (A.6) it follows that there exists $\theta_n \in [0, 2\pi)$ such that

$$|\{w_{n,\theta_n} \neq 1\}|/\varepsilon_n^{1/2} + \int_\Omega |\nabla w_{n,\theta_n}| \leq |\Omega|\, \varepsilon_n^{1/2}/(2\pi) + \Sigma(u) + \varepsilon_n;$$

so clearly a subsequence of $u_n := w_{n,\theta_n}$ satisfies (1.15).

□

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