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Genealogy of the extremal process of the branching random walk

Bastien Mallein

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Abstract

Let $(\mathbf{T}, V)$ be a branching random walk on the real line. The extremal process of the branching random walk is the point process of the position of particles at time n shifted by the position of the minimum. Madaule [Mad15] proved that this point process converges toward a shifted decorated Poisson point process. In this article we study the joint convergence of the extremal process with its genealogy informations. This result is then used to characterize the law of the decoration in the limiting process as well as to study the supercritical Gibbs measures of the branching random walk.

1 Introduction

A branching random walk on $\mathbb{R}$ is a discrete time particle process on the real line defined as follows. It starts with a unique particle positioned at 0 at time 0. At each new time $n \in \mathbb{N}$, each particle alive at time $(n-1)$ dies, giving birth to children that are positioned according to i.i.d. versions of a point process, shifted by the position of their parent. We denote by $\mathbf{T}$ the genealogical tree of the branching random walk. For $u \in \mathbf{T}$, we set $V(u)$ for the position of the particle u and $|u|$ for the time at which u is alive. The branching random walk is the random marked tree (T, V).

In this article, we assume that the branching random walk is in the so-called boundary case:

$$\mathbb{E}\left(\sum_{|u|=1} 1\right) > 1, \quad \mathbb{E}\left(\sum_{|u|=1} e^{-V(u)}\right) = 1 \quad \text{and} \quad \mathbb{E}\left(\sum_{|u|=1} V(u)e^{-V(u)}\right) = 0, \tag{1.1}$$

and that the reproduction law is non-lattice. Any branching random walk satisfying some mild assumption can be reduced to this case by an affine transformation (see e.g. the discussion in [BG11]). Under these assumptions, the Galton-Watson tree $\mathbf{T}$ is supercritical. Therefore, the surviving event of the branching random walk $S = \{\#\mathbf{T} = +\infty\}$ occurs with positive probability. For any $n \geq 1$, we denote by $W_n = \sum_{|u|=n} e^{-V(u)}$ and $Z_n = \sum_{|u|=n} V(u)e^{-V(u)}$. By (1.1) and the branching property of the branching random walk, (W_n) and (Z_n) are martingales.

We introduce the following integrability conditions:

$$\sigma^2 := \mathbb{E}\left(\sum_{|u|=1} V(u)^2 e^{-V(u)}\right) \in (0, +\infty) \tag{1.2}$$

$$\text{and} \quad \mathbb{E}\left(\sum_{|u|=1} e^{-V(u)} \log_+\left(\sum_{|u|=1} (1 + V(u)_+)e^{-V(u)}\right)^2\right), \tag{1.3}$$

where $x_+ = \max(x, 0)$ and $\log_+(x) = \max(\log x, 0)$. Under these assumptions, is is well-known (see [Aïd13, BK04]) that there exists Z_∞, which is a.s. positive on the survival event S, such that

$$\lim_{n\to+\infty} Z_n = Z_\infty \quad \text{and} \quad \lim_{n\to+\infty} W_n = 0 \quad \text{a.s.} \tag{1.4}$$

Assumption (1.3) is different from [Aïd13, Equation (1.4)], but we prove in Lemma A.1 that these two conditions are equivalent.

We write $M_n = \min_{|u|=n} V(u)$ for the minimal position at time n occupied by a particle and $m_n = \frac{3}{2}\log n$. The study of the asymptotic behaviour of M_n has been the subject of intense studies. Under the above integrability assumptions, Biggins [Big76] proved that $\lim_{n\to+\infty} \frac{M_n}{n} = 0$ a.s. Addario-Berry and Reed [ABR09], and Hu and Shi [HS09] independently observed that $M_n - m_n$ is tensed, while experiencing almost sure logarithmic size fluctuations. Finally, Aïdékon [Aïd13] obtained the convergence in law of $M_n - m_n$, and Chen [Che15] proved the above integrability assumptions to be optimal for this convergence in law.

We introduce some notation on point processes. In this article, a point process is understood both as a random measure on $\mathbb{R}$ that takes values in $\mathbb{Z}_+$, as well as the collection of atoms that compose this measure. Therefore, for a point process D, we authorize the notation $D = \sum_{d\in D} \delta_d$. For any $x \in \mathbb{R}$, we write $\theta_x D = \sum_{d\in D} \delta_{d+x}$ the shift of D by x.

The random measure $\sum_{|u|=n} \delta_{V(u)-m_n}$ is the extremal process of the branching random walk. Madaule [Mad15] proved the convergence in law of the extremal process toward a decorated Poisson point process with exponential intensity. More precisely, the following holds.

Fact 1.1 (Theorem 1.1 in [Mad15])**.** *Under assumptions* (1.1), (1.2) *and* (1.3)*, there exists* $c_* > 0$ *and a point process* D *satisfying* $\max D = 0$ *a.s. such that given* (ξ_n) *the atoms of a Poisson point process with intensity* $c_* Z_\infty e^x dx$*, and* $(D_n, n \geq 1)$ *i.i.d. copies of* D *we have*

$$\lim_{n\to+\infty} \left(\sum_{|u|=n} \delta_{V(u)-m_n}, Z_n \right) = \left(\sum_{n=1}^{+\infty} \sum_{d\in D_n} \delta_{\xi_n+d}, Z_\infty \right)$$

In the rest of the article, we write $\mathcal{D}$ for the law of the point process D, that we call the law of the decoration of the branching random walk. The proof of Fact 1.1 gives few information on this law, as identifies the limiting point process using its property of superposability. More general results and a characterization of decorated Poisson point processes have been studied by [SZ15].

A result similar to Fact 1.1 has been obtained for the branching Brownian motion independently by Arguin, Bovier and Kistler [ABK13], and Aïdékon, Berestycki, Brunet and Shi [ABBS13]. In this model as well, the extremal process converges toward a decorated Poisson point process. However, the decoration law is explicitly described in both these articles. In [ABBS13], the point process D corresponds to the set of positions of individuals that are close relatives to the individual realizing the minimal displacement. In [ABK13], it is described as the extremal process of the branching random walk conditioned on having an unusually small minimal displacement.

In this article, we prove that the extremal process of the branching random walk converges conjointly with additional genealogy information. Using this result, we are able to study the so-called supercritical measures on the boundary of the branching random walk, which allows to prove a conjecture of Spohn and Derrida [DS88]. In particular, we observe that these measures are purely atomic, in a "glassy" phase. Finally, we obtain the law $\mathcal{D}$ as as the limit of the positions of close relatives of the maximal displacement at time n, as in the result of [ABBS13]. We expect a result similar to [ABK13] would also holds, i.e. the law $\mathcal{D}$ could be obtained as the limit in distribution of the extremal process conditioned on having a very small minimum. Very similar result has been recently obtained by Biskup and Louidor [BL16] for the 2 dimensional Gaussian free field.

In the next section, we use Fact 1.1 as well as simple properties of the branching random walk to study the convergence in law of the extremal process with its genealogy. This result is used in Section 3 to study the supercritical measures on the boundary of the branching random walk. In Section 4, we obtain a representation of the law of the decoration of the branching random walk.

2 Convergence in law of the marked extremal process

To study the joint convergence in law of the extremal process and its genealogy, we introduce the so-called critical measure of the branching random walk. This measure has been introduced by Derrida and Spohn in [DS88]. Its existence is a consequence the precise study of the derivative martingale in [AS14]. This measure has been the subject of multiple studies [BKN$^+$14, Bur09, BDK16].

We first introduce the Ulam-Harris notation for the branching random walk. We set

$$\mathcal{U} = \cup_{n\geq 0}\mathbb{N}^n, \quad \partial\mathcal{U} = \mathbb{N}^{\mathbb{N}} \quad \text{and} \quad \overline{\mathcal{U}} = \mathcal{U} \cup \partial\mathcal{U},$$

with the convention $\mathbb{N}^0 = \{\emptyset\}$. Let $u \in \overline{\mathcal{U}}$. We denote by $|u|$ the length of u and, for $k \leq |u|$, by u_k the k first values of u. Let $u, v \in \overline{\mathcal{U}}$, we write $u < v$ if $v_{|u|} = u$, $|u \wedge v| = \max\{k \in \mathbb{N} : u_k = v_k\}$ and $u \wedge v = u_{|u\wedge v|} = v_{|u\wedge v|}$.

We observe that the application $\Psi : u \mapsto \sum_{j=1}^{|u|} 2^{-\sum_{i=1}^{j} u(i)}$ is a bijection between $\overline{\mathcal{U}}$ and $[0, 1)$. We use it to define a distance on $\overline{\mathcal{U}}$ by $d(u, v) = |\Phi(u) - \Phi(v)|$, that makes it a compact metric space.

A tree $\mathbf{T}$ is constructed as a subset of $\mathcal{U}$ such that $u \in \mathbf{T}$ with $|u| = n$ represents the $u(n)$th child of the $u(n-1)$th child of the ... of the $u(1)$th child of the root $\emptyset$. We observe that u_k is the ancestor alive at generation k of u. Similarly, we have $u < v$ if u is an ancestor of v. We denote by

$$\partial\mathbf{T} = \{u \in \partial\mathcal{U} : \forall n \in \mathbb{N}, u_n \in \mathbf{T}\}$$

the boundary of the tree, i.e. the set of all infinite rays and $\mathbf{T}(n) = \{u \in \mathbf{T} : |u| = n\}$ the nth generation of the tree. We often shorten the notation for this set into $\{|u| = n\}$.

Let $(\mathbf{T}, V)$ be a branching random walk that satisfies (1.1), (1.2) and (1.3). For $n \in \mathbb{N}$, we write $\mathcal{F}_n = \sigma(u, V(u), |u| \leq n)$. For any $u \in \mathbf{T}$, we set

$$Z_n^u = \sum_{|v|=n, v>u} (V(v) - V(u))e^{V(u)-V(v)} \quad \text{and} \quad Z_\infty^u = \liminf_{n\to+\infty} Z_n^u.$$

By the branching property of the branching random walk, $(Z_\infty^u, |u| = k)$ are i.i.d. versions of Z_∞, that are independent of $\mathcal{F}_k$. Moreover, observe that for any $k \leq n$,

$$Z_n = \sum_{|u|=k} \sum_{|v|=n, v>u} V(v)e^{-V(v)} = \sum_{|u|=k} e^{-V(u)} Z_n^u + \sum_{|u|=k} V(u)e^{-V(u)} \sum_{|v|=n, v>u} e^{-V(v)}.$$

Letting $n \to +\infty$, by (1.4), for any $k \in \mathbb{N}$, we have $Z_\infty = \sum_{|u|=k} e^{-V(u)} Z_\infty^u$ a.s.

The critical measure of the branching random walk is the random measure ν, defined on $\overline{\mathcal{U}}$ by

$$\forall u \in \mathcal{U}, \nu\left(\left\{v \in \overline{\mathcal{U}} : v > u\right\}\right) = \mathbf{1}_{\{u\in\mathbf{T}\}} e^{-V(u)} Z_\infty^u.$$

This measure is well-defined and unique by Caratheodory's extension theorem. Moreover, note that the support of ν is $\mathbf{T}$ a.s. In Lemma A.3, we give a short proof of the fact hat ν is a.s. non-atomic. To simplify notation, we set $B(u) = \left\{v \in \overline{\mathcal{U}} : v > u\right\}$ for any $u \in \mathcal{U}$. In particular, we have $\nu(B(\emptyset)) = Z_\infty$.

Conditionally on $(\mathbf{T}, V)$, we construct $(\xi_n, n \geq 1)$ the atoms of a Poisson point process with intensity $c_*\nu(B(\emptyset))e^x dx$, i.i.d. random variables $(u^{(n)}, n \geq 1)$ with distribution $\frac{\nu}{\nu(B(\emptyset))}$ and i.i.d. point processes $(D_n, n \geq 1)$ with law $\mathcal{D}$. For any $n \in \mathbb{N}$, we set

$$\mu_n = \sum_{|u|=n} \delta_{u, V(u)-m_n} \quad \text{and} \quad \mu_\infty = \sum_{n=1}^{+\infty} \sum_{d\in D_n} \delta_{u^{(n)}, \xi_n + d}.$$

Note that by classical properties of Poisson point processes $(\xi_n, u^{(n)})$ are the atoms of a Poisson point process with intensity $c_* e^x dx \otimes \nu$.

Using this notation, we observe that Madaule's convergence in law for the extremal process of the branching random walk directly implies the joint convergence in law of the extremal process and its genealogy.

Theorem 2.1. *Assuming* (1.1), (1.2) *and* (1.3), *we have* $\lim_{n\to+\infty}(\mu_n, Z_n) = (\mu_\infty, Z_\infty)$ *in law.*

Proof. For any $v \in \mathbf{T}$, we set $\mu_\infty^v(.) = \int_{B(v)} \mu_\infty(du, .) = \sum_{k=1}^{+\infty} \mathbf{1}_{\left\{u^{(k)}>v\right\}} \sum_{d\in D_k} \delta_{d+\xi_k}$. Let $k \in \mathbb{N}$, observe that conditionally on $\mathcal{F}_k$ and $(Z_\infty^v, |v| = k)$, $(\theta_{-V(v)}\mu_\infty^v, v \in \mathbf{T}(k))$ are independent decorate

Poisson point process with intensity $c_* Z_\infty^v e^x dx$ and decoration law $\mathcal{D}$. In particular, by Fact 1.1, conditionally on $\mathcal{F}_k$, we have

$$\lim_{n\to+\infty}\left(\sum_{|u|=n,u>v}\delta_{V(u)-V(v)-m_n},Z_n^v,v\in\mathbf{T}(k)\right)=\left(\theta_{-V(v)}\mu_\infty^v,Z_\infty^v,v\in\mathbf{T}(k)\right)\text{ in law.}$$

Let f be a continuous positive function with compact support, $k\in\mathbb{N}$ and $A\subset\{|v|=k\}$. By the branching property, conditionally on $\mathcal{F}_k$, the subtrees of the branching random walk rooted at points $v\in A$ are independent. Therefore

$$\begin{aligned}\lim_{n\to+\infty}\left(\mu_n(\mathbf{1}_{\{.\in A\}}f),Z_n\right)&=\lim_{n\to+\infty}\left(\sum_{|v|=k}\mathbf{1}_{\{v\in A\}}\sum_{|u|=n,u>v}f(V(u)-m_n),\sum_{|v|=k}e^{-V(v)}Z_n^v\right)\\&=\left(\sum_{v\in A}\mu_\infty^v(f),Z_\infty\right)\quad\text{ in law.}\end{aligned}$$

We conclude that $\lim_{n\to+\infty}(\mu_n,Z_n)=(\mu_\infty,Z_\infty)$ in law using [Kal02, Theorem 14.16]. □

Using this result, setting $\widehat{\mu}^x$ a point process with distribution $\theta_{-\min\mu_\infty}\mu_\infty$ conditionally on $\min\mu_\infty<-x$, we have $\lim_{x\to+\infty}\widehat{\mu}^x=D_1$ in law. This result is a straightforward consequence of [SZ15, Corollary 9]. In Section 4, we provide alternative characterizations of $\mathcal{D}$.

A straightforward consequence of Theorem 2.1, is the following estimate for the extremal process seen from the smallest position.

Corollary 2.2. *Under the same assumptions as Theorem 2.1, setting* $\mathbf{e}$ *a standard exponential random variable,* $\zeta_1=0$ *and* $(\zeta_n,n\geq 2)$ *the atoms of a Poisson point process with intensity* $\mathbf{e}e^x\mathbf{1}_{\{x>0\}}dx$*, we have*

$$\lim_{n\to+\infty}(M_n-m_n,\sum_{|u|=n}\delta_{u,V(u)-M_n})=(\log(\mathbf{e}),\sum_{n=1}^{+\infty}\sum_{d\in D_n}\delta_{\zeta_n+d}).$$

3 The supercritical Gibbs measure

Theorem 2.1 can be used to obtain a simple construction of the so-called supercritical measures on $\overline{\mathcal{U}}$, as obtained in [BRV12]. Let $(\mathbf{T},V)$ be a branching random walk. For any $n\in\mathbb{N}$, $\beta>1$ and $u\in\mathbf{T}$, we set

$$W_{n,\beta}=\sum_{|v|=n}e^{\beta(m_n-V(v))}\quad\text{and}\quad W_{n,\beta}^u=\sum_{|v|=n,v>u}e^{\beta(m_n+V(u)-V(v))}.$$

We introduce the sequence of random measures $\nu_{n,\beta}$ by writing

$$\forall u\in\mathcal{U}:|u|\leq n,\nu_{\beta,n}(B(u))=\mathbf{1}_{\{u\in\mathbf{T}\}}e^{-\beta V(u)}W_{n,\beta}^u.$$

Madaule [Mad15, Theorem 2.3] proved that $\lim_{n\to+\infty}(W_{n,\beta},Z_n)=(W_{\infty,\beta},Z_\infty)$ in law, that verifies $W_{\infty,\beta}>0$ a.s. on $\{Z_\infty>0\}$.

We recall that $(u^{(n)})$ are i.i.d. random elements of $\partial\mathcal{U}$ sampled with law $\frac{\nu}{\nu(B(\emptyset))}$. We denote by $(\xi_n^\beta,n\geq 1)$ the atoms of a Poisson point process with intensity $c_\beta\nu(B(\emptyset))e^x dx$, where we write $c_\beta=c_*\,\mathbb{E}\left(\sum_{d\in D}e^{-\beta d}\right)$. We introduce the random measure on $\partial\mathcal{U}$ defined by

$$\nu_{\beta,\infty}=\sum_{n\in\mathbb{N}}e^{-\beta\xi_n^\beta}\delta_{u^{(n)}}.$$

We note that this random measure is purely atomic, whereas ν is a.s. non-atomic on $\partial\mathcal{U}$.

Theorem 3.1. *Assuming* (1.1), (1.2) *and* (1.3), $\nu_{\beta,n}$ *converges in law to* $\nu_{\beta,\infty}$*, as* $n\to+\infty$*.*

Proof. We observe that by Theorem 2.1, for any $u \in \mathbf{T}$ we have

$$\lim_{n\to+\infty} \nu_{\beta,n}(B(u)) = \sum_{k=1}^{+\infty} \mathbf{1}_{\left\{u^{(k)}>u\right\}} \sum_{d\in D_k} e^{-\beta(\xi_k+d)} \quad \text{in law.}$$

Setting $X_k^\beta = \frac{1}{\beta}\sum_{d\in D_k} e^{-\beta d}$, we rewrite $\lim_{n\to+\infty} \nu_{\beta,n}(B(u)) = \sum_{k=1}^{+\infty} \mathbf{1}_{\left\{u^{(k)}>u\right\}} e^{-\beta(\xi_k+X_k)}$ in law. Moreover, as $(\xi_k + X_k, k \in \mathbb{N})$ are the atoms of a Poisson point process with intensity $c_\beta \nu(\emptyset) e^x dx$, we conclude that for any $j \in \mathbb{N}$, we have

$$\begin{aligned}\lim_{n\to+\infty} (\nu_{\beta,n}(B(u)), u \in \mathbf{T}(j)) &= \left(\sum_{k=1}^{+\infty} e^{-\beta\xi_k^\beta} \mathbf{1}_{\left\{u^{(n)}>u\right\}}, u \in \mathbf{T}(j)\right) \quad \text{in law,} \\ &= (\nu_{\beta,\infty}(B(u)), u \in \mathbf{T}(j)),\end{aligned}$$

which concludes the proof. □

This result indirectly implies a proof of the conjecture of Derrida and Spohn [DS88]: the rescaled distribution of the position of the most recent common ancestor of two individuals chosen independently at random according to the measure $\nu_{n,\beta}/\nu_{n,\beta}(B(\emptyset))$ converges in law toward a random measure on $[0, 1]$ with no mass on $(0, 1)$.

Theorem 3.2. *For any $n \in \mathbb{N}$ and $\beta > 1$, on the survival event $S = \{\#\mathbf{T} = +\infty\}$, we set*

$$\omega_{n,\beta} = W_{n,\beta}^{-2} \sum_{|u|=|v|=n} e^{\beta(2m_n - V(u) - V(v))} \delta_{|u\wedge v|/n}.$$

Assuming (1.1), (1.2) *and* (1.3), *conditionally on the survival of the branching random walk,*

$$\lim_{n\to+\infty} \omega_{n,\beta} = (1 - \pi_\beta)\delta_0 + \pi_\beta \delta_1 \quad \textit{in law}$$

where $\pi_\beta = \sum_{k=1}^{+\infty} p_k$ and $(p_k, k \geq 1)$ is a Poisson-Dirichlet mass partition with parameters $(\beta^{-1}, 0)$.

Proof. We first prove that for any $t \in (0, 1)$, $\omega_{n,\beta}((t, 1])$ converges in law toward π_β on S. For $k \leq n$ and $t \in [0, 1]$, we set

$$\Lambda_n^k = \frac{\sum_{|u|=|v|=n} e^{-\beta V(u)-\beta V(v)} \mathbf{1}_{\{|u\wedge v|\geq k\}}}{W_{n,\beta}^2} \quad \text{and} \quad \Delta_n^{k,t} = \frac{\sum_{|u|=|v|=n} e^{-\beta V(u)-\beta V(v)} \mathbf{1}_{\{|u\wedge v|\in[k,tn]\}}}{W_{n,\beta}^2}.$$

We observe that

$$\Lambda_n^k - \Delta_n^{k,t} \leq \omega_{n,\beta}((t, 1]) \leq \Lambda_n^k \quad \text{a.s. on S.} \tag{3.1}$$

By Theorem 2.1, as $S = \{Z_\infty > 0\}$ a.s. we have

$$\lim_{n\to+\infty} \Lambda_n^k = \Lambda_\infty^k := \frac{\sum_{|u|=k} \left(\sum_{j=1}^{+\infty} \mathbf{1}_{\left\{u^{(j)}>u\right\}} e^{-\beta\xi_j^\beta}\right)^2}{\left(\sum_{j=1}^{+\infty} e^{-\beta\xi_j^\beta}\right)^2} \quad \text{in law.}$$

Moreover, as ν is a non-atomic measure (see Lemma A.3), letting $k \to +\infty$ we obtain

$$\lim_{k\to+\infty} \Lambda_\infty^k = \frac{\sum_{j=1}^{+\infty} e^{-2\beta\xi_j^\beta}}{\left(\sum_{j=1}^{+\infty} e^{-\beta\xi_j^\beta}\right)^2} \quad \text{a.s.}$$

Using [PY97, Proposition 10], we conclude that $\lim_{k\to+\infty} \Lambda_\infty^k = \pi_\beta$ in law on S, where π_β is the sum of the square of the masses of a Poisson-Dirichlet distribution with parameters $(\beta^{-1}, 0)$.

We now study the asymptotic behaviour of $\Delta_n^{k,t}$, more precisely we prove that for any $\delta > 0$,

$$\lim_{k\to+\infty} \limsup_{n\to+\infty} \mathbb{P}\left(\Delta_n^{k,t} > \delta, \#\mathbf{T} = +\infty\right) = 0. \tag{3.2}$$

Observe, by [Mad15, Theorem 2.3], that $\lim_{\varepsilon\to 0}\lim_{n\to+\infty}\mathbb{P}\left(n^{3\beta/2}W_{n,\beta}\le\varepsilon,\#\mathbf{T}=+\infty\right)=0$, therefore it is enough to prove that for any $\varepsilon>0$,

$$\lim_{k\to+\infty}\limsup_{n\to+\infty}\mathbb{P}\left(\sum_{|u|=|v|=n}\mathbf{1}_{\{|u\wedge v|\in[k,tn]\}}e^{\beta(m_n-V(u))+\beta(m_n-V(v))}>\delta\varepsilon^2\right)=0. \tag{3.3}$$

The proof of this result, rather technical, is postponed to Lemma A.2.

Let $x\in[0,1]$ and $\delta>0$, using (3.1), we have

$$\mathbb{P}(\Lambda_n^k\le x+\delta,S)+\mathbb{P}(\Delta_n^{k,t}\ge\delta,S)\ge\mathbb{P}\left(\omega_{n,\beta}((t,1])\le x,S\right)\ge\mathbb{P}\left(\Lambda_n^k\le x,S\right).$$

Thus, letting n then k grows to $+\infty$ and using (3.2), for any $t\in(0,1)$, $\omega_{n,\beta}((t,1])$ converges in law toward π_β on S, which concludes the proof. □

Remark 3.3. With similar computations, we can obtain a "local limit" convergence for the genealogy of two individuals sampled according to the Gibbs measure. In effect, if we consider the non-rescaled measure

$$\lambda_{n,\beta}=W_{n,\beta}^{-2}\sum_{|u|=|v|=n}e^{\beta(2m_n-V(u)-V(v))}\delta_{|u\wedge v|},$$

we obtain $\lim_{n\to+\infty}\lambda_{n,\beta}=\lambda_{\infty,\beta}$ in law on S, where (p_k) is a Poisson-Dirichlet distribution with parameters $(\beta^{-1},0)$ and $\lambda_{\infty,\beta}=\sum_{k,k'=1}^{+\infty}p_kp_{k'}\delta_{|u^{(k)}\wedge u^{(k')}|}$. Note that $\lambda_{\infty,\beta}(\{+\infty\})=\pi_\beta$.

Chauvin and Rouault [CR97] studied similarly the overlap of subcritical measures, such that $\beta<1$. They proved that in this case, the measure $\omega_{n,\beta}$ converges toward δ_0, and the measure $\lambda_{n,\beta}$ converges toward a proper probability measure on $\mathbb{N}$.

4 The decoration as the close relatives of maximal displacement

In this section, we prove that the law $\mathcal{D}$ is the limiting distribution of the relative positions of the family of the individual that realizes the minimal displacement at time n. This result is similar to the one obtained in [ABBS13] for branching Brownian motion. For any $n\in\mathbb{N}$, we denote by $\widehat{u}_n$ an individual alive at time n such that $V(u)=M_n$, for example the one which is the smallest for the lexicographical order on $\mathcal{U}$.

Theorem 4.1. *For any $n\in\mathbb{N}$ and $k<n$, we set $\varrho_{n,k}=\sum_{|u|=n}\mathbf{1}_{\left\{|u\wedge\widehat{u}_n|\ge k\right\}}\delta_{V(u)-M_n}$. Under the assumptions of Theorem 2.1, we have*

$$\lim_{k\to+\infty}\lim_{n\to+\infty}\varrho_{n,k}=\lim_{k\to+\infty}\widetilde{\lim_{n\to+\infty}}\varrho_{n,n-k}=D_1\quad\text{in law,}$$

where $\widetilde{\lim_{n\to+\infty}}\varrho_{n,n-k}$ represents any accumulation point for the sequence $(\varrho_{n,n-k})$.

Observe that by Corollary 2.2, the sequence $(\varrho_{n,k},n\ge1,k\le n)$ is tensed. In effect, for any continuous positive function f, we have $\varrho_{n,k}(f)\le\varrho_{n,0}(f)=\sum_{|u|=n}f(V(u)-M_n)$. A straightforward consequence of Theorem 4.1 is the following, more intuitive convergence.

Corollary 4.2. *Let (k_n) be such that $\lim_{n\to+\infty}k_n=\lim_{n\to+\infty}n-k_n=+\infty$. Under the assumptions of Theorem 2.1, $\lim_{n\to+\infty}\varrho_{n,k_n}=D_1$ in law.*

Proof. We observe that for any $i\le j\le k$, and any continuous positive function f, we have $\varrho_{n,i}(f)\ge\varrho_{n,j}(f)\ge\varrho_{n,k}(f)$. Consequently, for any $k\in\mathbb{N}$, for all $n\ge1$ large enough, we have $\varrho_{n,k}(f)\ge\varrho_{n,k_n}(f)\ge\varrho_{n,n-k}(f)$. Applying Theorem 4.1, we have

$$\lim_{k\to+\infty}\limsup_{n\to+\infty}\mathbb{P}(\varrho_{n,k_n}(f)-\varrho_{n,k}(f)>\varepsilon)=0\quad\text{for any }\varepsilon>0,$$

which concludes the proof. □

The first limit in distribution for Theorem 4.1 is a straightforward consequence of Fact 1.1.

Lemma 4.3. *We have* $\lim_{k\to+\infty}\lim_{n\to+\infty}\varrho_{n,k}=D$ *in law.*

Proof. Using Fact 1.1, we observe that for any $k\in\mathbb{N}$, conditionally on $\mathcal{F}_k$,

$$\lim_{n\to+\infty}\left(\sum_{|v|=n,v>u}\delta_{V(v)-m_n},Z_n^u,u\in\mathbf{T}(k)\right)=(\mu_\infty^u,Z_\infty^u,u\in\mathbf{T}(k))\ \text{ in law.}$$

Therefore, setting $M_n^u=\min_{|v|=n,v>u}V(v)$, we have in particular

$$\lim_{n\to+\infty}\sum_{|u|=k}\mathbf{1}_{\{M_n^u=M_n\}}\sum_{|v|=n,v>u}\delta_{V(v)-m_n}=\sum_{|u|=k}\mathbf{1}_{\left\{u^{(1)}>u\right\}}\mu_\infty^u\quad\text{in law.}$$

Observe that $\sum_{|u|=k}\mathbf{1}_{\left\{u^{(1)}>u\right\}}\mu_\infty^u=\sum_{n=1}^{+\infty}\sum_{d\in D_n}\mathbf{1}_{\left\{u_k^{(n)}=u_k^{(1)}\right\}}\delta_{\xi_n+d}$.

Let f be a continuous positive function with compact support, we prove that

$$\lim_{k\to+\infty}\sum_{n=2}^{+\infty}\sum_{d\in D_n}\mathbf{1}_{\left\{u_k^{(n)}=u_k^{(1)}\right\}}f(\xi_n+d)=0\quad\text{in probability.}$$

In effect, for any $k\in\mathbb{N}$, we have

$$\mathbb{E}\left(\sum_{n=2}^{+\infty}\sum_{d\in D_n}f(\xi_n+d)\mathbf{1}_{\left\{u_k^{(n)}=u_k^{(1)}\right\}}\middle|\mathcal{F}_k\right)=\sum_{|u|=k}\frac{\nu(u)}{\nu(\emptyset)}\sum_{n=2}^{+\infty}\mathbb{P}(u_k^{(n)}=u|\mathcal{F}_k)\,\mathbb{E}(g(\xi_n)),$$

where $g:x\mapsto\mathbb{E}\left(\sum_{d\in D}f(x+d)\right)$. Therefore,

$$\mathbb{E}\left(\sum_{n=2}^{+\infty}\sum_{d\in D_n}f(\xi_n+d)\mathbf{1}_{\left\{u_k^{(n)}=u_k^{(1)}\right\}}\middle|\mathcal{F}_k\right)=\left(\frac{\sum_{|u|=k}\nu(u)^2}{\nu(\emptyset)^2}\right)\sum_{n=2}^{+\infty}\mathbb{E}(g(\xi_n)).$$

As $\lim_{k\to+\infty}\max_{|u|=k}\nu(u)=0$ a.s. (see Lemma A.3), we conclude that

$$\lim_{k\to+\infty}\sum_{n=2}^{+\infty}\sum_{d\in D_n}f(\xi_n+d)\mathbf{1}_{\left\{u_k^{(n)}=u_k^{(1)}\right\}}=0\quad\text{in probability.}$$

This result yields that $\lim_{k\to+\infty}\sum_{|u|=k}\mathbf{1}_{\left\{u^{(1)}>u\right\}}\mu_\infty^u(f)=\tau_{\xi_1}D(f)$ in law. We conclude the proof observing that we chose the law of the decoration such that $\max D=0$ a.s. □

To complete the proof of Theorem 4.1, we first observe that the genealogy of individuals close to the maximal displacement at time n in the branching random walk are either close relatives, or their most recent common ancestor is a close relative to the root. This well-known estimate can be found for example in [Mal16, Theorem 4.5]. For any $z\geq 1$, we have

$$\lim_{k\to+\infty}\limsup_{n\to+\infty}\mathbb{P}\left(\exists u,v\in\mathbf{T}(n):V(u),V(v)\leq m_n+z,|u\wedge v|\in[k,n-k]\right)=0.\tag{4.1}$$

Lemma 4.4. *For any $k\in\mathbb{N}$, we set $(n_p^k,p\geq 1)$ an increasing sequence such that $(\varrho_{n_p^k,n_p^k-k})$ converges. We have*

$$\lim_{k\to+\infty}\lim_{p\to+\infty}\varrho_{n_p^k,n_p^k-k}=D\quad\textit{in law.}$$

Proof. For any positive continuous function f with compact support and $k\in\mathbb{N}$, we have

$$\varrho_{n,k}(f)-\varrho_{n,n-k}(f)=\sum_{|u|=n}\mathbf{1}_{\left\{|u\wedge\widehat{u}_n|\in[k,n-k]\right\}}f(V(u)-M_n).$$

We write $z = \sup\{x \geq 0 : f(x) > 0\}$, for any $y \geq 0$, we have

$$\begin{aligned}\mathbb{P}\left(\varrho_{n,k}(f) - \varrho_{n,n-k}(f) > 0\right) &\leq \mathbb{P}\left(\exists u \in \mathbf{T}(n) : |u \wedge \widehat{u}_n| \in [k, n-k], V(u) - M_n \leq z\right) \\ &\leq \mathbb{P}(M_n - m_n \geq y) + \mathbb{P}\left(\exists u, v \in \mathbf{T}(n) : |u \wedge v| \in [k, n-k], V(u), V(v) \leq m_n + y + z\right).\end{aligned}$$

Letting n then $k \to +\infty$, we have by (4.1),

$$\limsup_{k\to+\infty} \limsup_{n\to+\infty} \mathbb{P}\left(\varrho_{n,k}(f) - \varrho_{n,n-k}(f) > 0\right) \leq \sup_{n\in\mathbb{N}} \mathbb{P}(M_n \geq m_n + y).$$

Moreover, $(M_n - m_n)$ is tensed, by [Aïd13], thus letting $y \to +\infty$, we conclude that

$$\lim_{k\to+\infty} \limsup_{n\to+\infty} \mathbb{P}\left(\varrho_{n,k}(f) - \varrho_{n,n-k}(f) > 0\right) = 0.$$

Using Lemma 4.3, we conclude the proof. □

We were not able to study the limiting distribution of $\varrho_{n,n-k}$, but this law probably exists.

Conjecture 4.5. *For any $k \in \mathbb{N}$, there exists a point process ϱ_k such that $\lim_{n\to+\infty} \varrho_{n,n-k} = \varrho_k$.*

A Some technical results

In this section, we provide some technical estimates on the branching random walks. We first prove that (1.3) is equivalent to the usual integrability conditions for the branching random walk.

Lemma A.1. *Under assumptions* (1.1) *and* (1.2)*, the condition* (1.3) *is equivalent to*

$$\mathbb{E}\left(\sum_{|u|=n} e^{-V(u)} \log_+\left(\sum_{|u|=n} e^{-V(u)}\right)^2\right) < +\infty$$

$$\mathbb{E}\left(\sum_{|u|=n} V(u)_+ e^{-V(u)} \log_+\left(\sum_{|u|=n} V(u)_+ e^{-V(u)}\right)\right) < +\infty$$

Proof. The reciprocal part is a direct consequence of [Aïd13, Lemma B.1]. To prove the direct part, we first observe that by (1.3),

$$\begin{aligned}&\mathbb{E}\left(\sum_{|u|=n} e^{-V(u)} \log_+\left(\sum_{|u|=n} e^{-V(u)}\right)^2\right) \\ &\qquad \leq \mathbb{E}\left(\sum_{|u|=n} e^{-V(u)} \log_+\left(\sum_{|u|=n} (1 + V(u)_+) e^{-V(u)}\right)^2\right) < +\infty.\end{aligned}$$

We now use the celebrated spinal decomposition of the branching random walk, introduced by Lyons [Lyo97]. Loosely speaking, it is an alternative description of the law of the branching random walk biased by the martingale (W_n), as the law of a branching random walk $(\mathbf{T}, V)$ with a distinguished spine $w \in \partial\mathbf{T}$ that makes more children than usual. For any $u \in \mathbf{T}$, we write $\xi(u) = \log_+\left(\sum_{v\in\Omega(\pi u)} \sum_{|u|=1} V(u)_+ e^{-V(u)}\right)$. We denote by $\widehat{\mathbb{P}} = W_n.\mathbb{P}$ the size-biased distribution, and refer to [Lyo97] for more details on the spinal decomposition. We have

$$\begin{aligned}\mathbb{E}\left(\sum_{|u|=n} V(u)_+ e^{-V(u)} \log_+\left(\sum_{|u|=n} V(u)_+ e^{-V(u)}\right)\right) &= \widehat{\mathbb{E}}\left(\xi(w_1) V(w_1)_+\right) \\ &\leq \widehat{\mathbb{E}}\left(V(w_1)^2\right)^{1/2} \widehat{\mathbb{E}}\left(\xi(w_1)^2\right)^{1/2} < +\infty,\end{aligned}$$

by Cauchy-Schwarz inequality, using (1.2) and (1.3) to conclude. □

We now prove that (3.3) holds.

Lemma A.2. *For any $\beta > 1$ and $k \leq n$, we set*

$$R^{\beta}_{n,k} = \sum_{|u|=|v|=n} \mathbf{1}_{\{|u\wedge v|\in[k,n-k]\}} e^{\beta(m_n - V(u))+\beta(m_n-V(v))}.$$

For any $\varepsilon > 0$, we have $\lim_{k\to+\infty} \limsup_{n\to+\infty} \mathbb{P}\left(R^{\beta}_{n,k} \geq \varepsilon\right) = 0$.

Proof. To prove this result, we first introduce some notation. For any $u \in \mathbf{T}$, we set

$$\xi(u) = \log \sum_{|v|=|u|+1, v>u} (1 + (V(v) - V(u))_+) e^{V(u)-V(v)}.$$

For any $n \in \mathbb{N}$ and $k \leq n$, we write $f_n(k) = \frac{3}{2}\log\frac{n+1}{n-k+1}$ and, for $y, z, h \geq 0$,

$$\begin{aligned}
\mathcal{A}_n(y) &= \{|u| \leq n : V(u_j) \geq f_n(j) - y, j \leq |u|\}, \\
\overline{\mathcal{A}}_n(y,h) &= \{|u| = n : u \in \mathcal{A}_n(y), V(u) - f_n(n) + y \in [h-1, h]\} \\
\mathcal{B}_n(y,z) &= \{|u| \leq n : \xi(u_j) \leq z + (V(u_j) - f_n(j) + y)/2, j \leq |u|\}.
\end{aligned}$$

We introduce branching random walk estimates obtained in [Mal16]. There exists $C > 0$ and a function χ such that $\lim_{z\to+\infty}\chi(z) = 0$ such that for any $k \leq n$ and $y, z, h \geq 1$ we have

$$\mathbb{P}\left(\mathcal{A}_n(y) \neq \emptyset\right) \leq Cye^{-y},\ \mathbb{P}\left(\overline{\mathcal{A}}_n(y,h) \cap \mathcal{B}^c_n(y,z) \neq \emptyset\right) \leq \chi(z)yhe^{h-y},\ \mathbb{E}\left(\#\overline{\mathcal{A}}_n(y,h)\right) \leq Cyhe^{h-y}$$
$$\text{and } \mathbb{P}\left(\exists u, v \in \overline{\mathcal{A}}_n(y,h) \cap \mathcal{B}_n(y,z) : |u\wedge v| \in [k, n-k]\right) \leq C\frac{zyh^2e^{2h-y}}{k^{1/2}}. \quad \text{(A.1)}$$

In the rest of this proof, C is a large positive constant, that depends only on the law of the branching random walk, and may change from line to line.

We decompose $R^{\beta}_{n,k}$ into three parts, that we bound separately. For any $h \geq 0$, we have

$$R^{\beta}_{n,k} \leq \widetilde{R}^{\beta}_{n,k}(h) + 2W_{n,\beta}(h)W_{n,\beta},$$

where we denote by $\widetilde{R}^{\beta}_{n,k}(y,h) = \sum_{|u|=|v|=n} \mathbf{1}_{\{|u\wedge v|\in[k,n-k]\}}\mathbf{1}_{\{V(u)-m_n\leq h\}} e^{\beta(m_n-V(u))+\beta(m_n-V(v))}$, and by $W_{n,\beta}(h) = \sum_{|u|=n}\mathbf{1}_{\{V(u)-m_n\geq h\}}e^{\beta(m_n-V(u))}$.

By (A.1), for any $y, h \geq 0$, we have

$$\begin{aligned}
\mathbb{E}\left(\sum_{|u|=n}\mathbf{1}_{\{u\in\mathcal{A}_n(y), V(u)\geq m_n+h\}}e^{\beta(m_n-V(u))}\right) &= \sum_{j=h+1}^{+\infty} e^{-\beta(j-1)}\,\mathbb{E}\left(\#\overline{\mathcal{A}}_n(y, j+y)\right) \\
&\leq Cye^{-y}\sum_{j=h+1}^{+\infty}(j+y)e^{(1-\beta)j} \leq Cy(h+y)e^{(1-\beta)h}.
\end{aligned}$$

With similar computations, we have $E\left(\sum_{|u|=n}\mathbf{1}_{\{u\in\mathcal{A}_n(y)\}}e^{\beta(m_n-V(u))}\right) \leq Cye^{(\beta-1)y}$. Using the Markov inequality, there exists $C > 0$ such that for any $\varepsilon \geq 0$ and $y, h \geq 0$, we have

$$\mathbb{P}\left(W_{n,\beta}(h) \geq \varepsilon\right) \leq \mathbb{P}\left(\mathcal{A}_n(y) \neq \emptyset\right) + Cy(h+y)e^{(1-\beta)h}/\varepsilon \leq Cye^{-y} + Cy(h+y)e^{(1-\beta)h}/\varepsilon,$$

and similarly for any $A > 0$, $\mathbb{P}(W_{n,\beta} \geq A) \leq Cye^{-y} + Cy^2e^{(\beta-1)y}/A$. Thus, for any $\delta > 0$, we have

$$\begin{aligned}
\mathbb{P}\left(W_{n,\beta}(h)W_{n,\beta} \geq \delta\right) &\leq \mathbb{P}(W_{n,\beta}(h) \geq \delta\varepsilon) + \mathbb{P}(W_{n,\beta} \geq 1/\varepsilon) \\
&\leq Cye^{-y} + Cy(h+y)e^{(1-\beta)h}/(\delta\varepsilon) + C\varepsilon ye^{(\beta-1)y}.
\end{aligned}$$

Choosing $y \geq 1$ large enough, then $\varepsilon > 0$ small enough and h large enough, we obtain

$$\sup_{n\in\mathbb{N}} \mathbb{P}\left(W_{n,\beta}(h)W_{n,\beta} \geq 2\delta\right) \leq \delta.$$

In a second time,we bound $\widetilde{R}^{\beta}_{n,k}$, by observing that for any $y, z \geq 0$,

$$\begin{aligned}\mathbb{P}(\widetilde{R}^{\beta}_{n,k}(h) \neq 0) \leq \mathbb{P}(\mathcal{A}_n(y) \neq \emptyset) + \sum_{j=0}^{h+y} P\left(\overline{\mathcal{A}}_n(y,j) \cap \mathcal{B}^c_n(y,z) \neq \emptyset\right) \\ + \sum_{j=0}^{h+y} \mathbb{P}\left(\exists u, v \in \overline{\mathcal{A}}_n(y,j) \cap \mathcal{B}_n(y,z) : |u \wedge v| \in [k, n-k]\right) \\ \leq Cye^{-y} + \chi(z)y(h+y)e^h + C\frac{zy(y+h)^2e^{2h+y}}{k^{1/2}},\end{aligned}$$

using again (A.1).

As a consequence, for any $\delta > 0$, we can choose $y \geq 1$, $\varepsilon > 0$, and $h \geq 0$ large enough such that for any $k, z \geq 0$ and $n \geq k$, we have

$$\mathbb{P}\left(R^{\beta}_{n,k} \geq \delta\right) \leq \delta + \chi(z)y(h+y)e^h + C\frac{zy(y+h)^2e^{2h+y}}{k^{1/2}}.$$

Setting $z = k^{1/4}$, we conclude that $\limsup_{k\to+\infty} \limsup_{n\to+\infty} \mathbb{P}\left(R^{\beta}_{n,k} \geq \delta\right) \leq \delta$, which concludes the proof. □

Finally, we prove that the measure ν is a.s. non-atomic.

Lemma A.3. *Under assumptions* (1.1), (1.2) *and* (1.3), *we have*

$$\lim_{n\to+\infty} \max_{|u|=n} \nu(B(u)) = 0 \quad a.s.$$

Proof. We first prove that Z_∞ has finite mean. We introduce the renewal function of the spine of the branching random walk, defined by

$$R : x \mapsto \mathbf{1}_{\{x\geq 0\}}\, \mathbb{E}\left(\sum_{u\in\mathbf{T}} e^{-V(u)}\mathbf{1}_{\{V(u)\geq -x\}}\mathbf{1}_{\{V(u_j)<0, 1\leq j<|u|\}}\right).$$

It is well-known (see e.g. [Fel71, Chapter XII]) that there exists $c_1 > 0$ such that $c_1(1+x) \leq R(x)$ for all $x \geq 0$. In particular, we have

$$\forall n \geq 0, c_1 Z_n \leq \sum_{|u|=n} R(V(u))e^{-V(u)} =: D_n \quad \text{a.s.}$$

Moreover, we observe that for a branching random walk satisfying (1.1), the process (D_n) is a non-negative martingale. By [Aïd13, BK04], we have $\lim_{n\to+\infty} D_n = D_\infty$ a.s. and in L^1 under assumptions (1.2) and (1.3). As $0 \leq Z_\infty \leq \frac{D_\infty}{c_1}$ a.s., this yields $\mathbb{E}(Z_\infty) < +\infty$.

We now observe that for any $\varepsilon > 0$ and $n, p \in \mathbb{N}$, using the Markov inequality we have

$$\mathbb{P}\left(\max_{|u|=n} \nu(B(u)) \geq \varepsilon \middle| \mathcal{F}_n\right) \leq \sum_{|u|=n} \mathbb{P}\left(e^{-V(u)}Z^u_\infty \geq \varepsilon \middle| \mathcal{F}_n\right) \leq \frac{\mathbb{E}(Z_\infty)}{\varepsilon} \sum_{|u|=n} e^{-V(u)} \quad \text{a.s.}$$

This yields $\lim_{n\to+\infty} \mathbb{P}\left(\max_{|u|=n} \nu(B(u)) \geq \varepsilon \middle| \mathcal{F}_n\right) = 0$ a.s. As $\max_{|u|=n} \nu(B(u))$ is decreasing in n, we conclude that $\lim_{n\to+\infty} \max_{|u|=n} B(u) = 0$ a.s. □

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