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Centers and partial volumes of convex cones I. Basic theory

Alberto Seeger · Mounir Torki

Abstract We study the concept of least partial volume of a proper cone in $\mathbb{R}^n$. This notion is a reasonable alternative to the classical concept of solid angle. In tandem, we study the concept of volumetric center of a proper cone. We compare this kind of center with the old notion of incenter.

Keywords Partial volume of a convex cone · Solid angle · Volumetric center · Incenter

1 Introduction

Let Π_n denote the set of proper cones in $\mathbb{R}^n$. A closed convex cone is said to be proper if it is pointed and has nonempty interior. Perhaps the most natural way to define the "volume" of a proper cone K in $\mathbb{R}^n$ is by setting

$$\text{btv}(K) := \text{vol}_n(K \cap \mathbb{B}_n), \tag{1}$$

where $\mathbb{B}_n$ is the n-dimensional closed unit ball and $\text{vol}_n(\cdot)$ stands for the n-dimensional Lebesgue measure. By an obvious reason, one refers to the positive number (1) as the

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ball-truncated volume of K. The expression (1) is, up to a multiplicative constant, equal to the solid angle of K. Indeed,

$$\mathrm{btv}(K) = \frac{1}{n}\,\mathrm{vol}_{n-1}(K \cap \mathbb{S}_n),$$

where $\mathbb{S}_n$ stands for the unit sphere of $\mathbb{R}^n$. The computation of ball-truncated volumes in spaces of dimension higher than three has been the object of several publications in the last decade, cf. (Gourion and Seeger 2010; Ribando 2006). For a vast majority of proper cones arising in practice, it is hopeless to derive an easily computable formula for evaluating the ball-truncated volume, so one has to resort to all kinds of numerical schemes: use of multivariate power series as in (Ribando 2006), numerical integration methods as in (Gourion and Seeger 2010, Section 3), probabilistic methods as in (Gourion and Seeger 2010, Section 5), etc.

Our work has essentially two goals. The first one is to study in depth the concept of least partial volume, which is a fruitful alternative to the concept of ball-truncated volume.

Definition 1.1 The partial volume of $K \in \Pi_n$ relative to $x \in \mathbb{R}^n$ is the possibly infinite number

$$v_K(x) := \mathrm{vol}_n(K \cap H_x),$$

where H_x is the closed half-space given by $H_x := \{u \in \mathbb{R}^n : \langle x, u\rangle \leq 1\}$. The finite number

$$\mathrm{lpv}(K) := \min_{x \in \mathbb{S}_n} \mathrm{vol}_n(K \cap H_x) \tag{2}$$

is called the least partial volume of K.

The functions $\mathrm{lpv} : \Pi_n \to \mathbb{R}$ and $\mathrm{btv} : \Pi_n \to \mathbb{R}$ share a number of properties: monotonicity with respect to set inclusion, invariance under orthogonal transformations, etc. These functions differ however in a substantial way.

As we shall see in Sect. 3, the minimization problem (2) has exactly one solution. Geometrically speaking, such solution can be viewed as a "center" of the dual cone

$$K^* := \{y \in \mathbb{R}^n : \langle y, x\rangle \geq 0 \text{ for all } x \in K\}. \tag{3}$$

Since (3) is proper as well and has K as dual cone, we suggest to consider the following definition.

Definition 1.2 Let $K \in \Pi_n$. The volumetric center of K, denoted by $\varrho(K)$, is the unique solution to the minimization problem

$$\mathrm{lpv}(K^*) = \min_{x \in \mathbb{S}_n} \mathrm{vol}_n(K^* \cap H_x). \tag{4}$$

The analysis and computation of volumetric centers is the second goal of our work. In particular, we discuss the link between the volumetric center and the incenter of

a proper cone. The *incenter* of $K \in \Pi_n$, denoted by $\xi(K)$, is defined as the unique solution to the maximization problem

$$r(K) := \max_{x \in \mathbb{S}_n \cap K} \operatorname{dist}(x, \partial K),$$

where $\operatorname{dist}(x, \partial K)$ stands for the distance from x to the boundary of K. The vector $\xi(K)$ belongs clearly to the interior of K. The coefficient $r(K)$ is called the *inradius* of K. The reader is conveyed to Henrion and Seeger (2010a, b, 2011), and Seeger and Torki (2014) for general material concerning the theory of incenters.

2 Basic facts on partial volumes

We start the discussion by recalling some basic properties of the extended real valued function $v_K : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$. Such a function is known in the optimization community, specially among practitioners of interior point methods [cf. (Güler 1996; Nesterov and Nemirovskii 1994; Truong and Tuncel 2004)]. For instance, Güler (1996, Theorem 4.1) has derived the integral representation formula

$$v_K(x) = \frac{1}{n!} \int_K e^{-\langle x, y \rangle} dy \tag{5}$$

for all $x \in \operatorname{int}(K^*)$. As a function of the vector x, the integral on the right-hand side of (5) corresponds to the Koszul-Vinberg characteristic function of K^*. The next proposition sets straight a number of technical details.

Proposition 2.1 *Let $K \in \Pi_n$. One has:*

(a) $v_K(x) \geq \|x\|^{-n} \operatorname{btv}(K)$ for all $x \neq 0$. In particular, v_K is positive on $\mathbb{R}^n$.
(b) $v_K(x) < \infty$ if and only if $x \in \operatorname{int}(K^*)$.
(c) $v_K(tx) = t^{-n} v_K(x)$ for all $t > 0$ and $x \in \mathbb{R}^n$.
(d) v_K is convex. In fact, v_K is strictly log-convex on $\operatorname{int}(K^*)$.
(e) v_K is infinitely often differentiable on $\operatorname{int}(K^*)$.

Proof (a) For all $x \neq 0$, one has $H_x \supseteq \|x\|^{-1} \mathbb{B}_n$. Hence,

$$v_K(x) \geq \operatorname{vol}_n \left[K \cap (\|x\|^{-1} \mathbb{B}_n) \right] = \operatorname{vol}_n \left[\|x\|^{-1} (K \cap \mathbb{B}_n) \right] = \|x\|^{-n} \operatorname{btv}(K).$$

(b) Let $x \in \mathbb{R}^n$. The set $K \cap H_x$ is closed, convex, and has nonempty interior. Hence, $v_K(x)$ is finite if and only if $K \cap H_x$ is bounded. Note that $K \cap H_x$ is bounded exactly when $x \in \operatorname{int}(K^*)$. (c) Take the n-dimensional Lebesgue measure on each side of the equality

$$K \cap H_{tx} = t^{-1}(K \cap H_x).$$

(d) and (e) See (Faraut and Korányi 1994, Proposition I.3.3) and (Faraut and Korányi 1994, Proposition I.3.1), respectively. □

The function v_K has many other interesting properties. For instance, it is known that v_K behaves as a barrier function for K^*. Such a property plays an important role in the analysis of interior point methods. The following result is a refinement of (Faraut and Korányi 1994, Proposition I.3.2).

Proposition 2.2 *Let $K \in \Pi_n$ and $\{x_\nu\}_{\nu\in\mathbb{N}}$ be a bounded sequence in the interior of K^*. Then*

$$\lim_{\nu\to\infty} v_K(x_\nu) = \infty \quad \textit{if and only if} \quad \lim_{\nu\to\infty} \operatorname{dist}(x_\nu, \partial K^*) = 0. \tag{6}$$

Proof As pointed out in (Seeger 2012, Lemma 5.1), one has

$$\int_K e^{-\langle x,y\rangle} dy \;\le\; \left(\frac{1}{\operatorname{dist}(x, \partial K^*)}\right)^n \int_K e^{-\|y\|} dy \tag{7}$$

for all $x \in \operatorname{int}(K^*)$. The above inequality proves the "only if" part of (6). The proof of the "if" part runs as follows. Suppose that $\operatorname{dist}(x_\nu, \partial K^*)$ goes to 0, but that

$$\kappa := \liminf_{\nu\to\infty} v_K(x_\nu)$$

is finite. One must arrive to a contradiction. Take a subsequence $\{x_{\varphi(\nu)}\}_{\nu\in\mathbb{N}}$ such that

$$\lim_{\nu\to\infty} v_K(x_{\varphi(\nu)}) = \kappa.$$

Without loss of generality, one may suppose that $\bar{x} := \lim_{\nu\to\infty} x_{\varphi(\nu)}$ exists. By passing to the limit on the right-hand side of

$$\operatorname{dist}(\bar{x}, \partial K^*) \le \|\bar{x} - x_{\varphi(\nu)}\| + \operatorname{dist}(x_{\varphi(\nu)}, \partial K^*),$$

one sees that $\bar{x} \in \partial K^*$. In such a case (Faraut and Korányi 1994, Proposition I.3.2) implies that $v_K(x_{\varphi(\nu)})$ goes to infinity, a clear contradiction. □

The "if" part of (6) is no longer true if the sequence $\{x_\nu\}_{\nu\in\mathbb{N}}$ is unbounded. The combination of Proposition 2.1(a) and the inequality (7) leads to the following sandwich for $\operatorname{lpv}(K)$.

Proposition 2.3 *For all $K \in \Pi_n$, one has*

$$\operatorname{btv}(K) \;\le\; \operatorname{lpv}(K) \;\le\; \big[r(K^*)\big]^{-n} \operatorname{btv}(K). \tag{8}$$

Proof The first inequality in (8) follows from Proposition 2.1(a). Note that

$$\int_K e^{-\|y\|} dy \;=\; \int_0^\infty \left(\int_{K\cap t\mathbb{S}_n} e^{-t} dy\right) dt \;=\; (n-1)!\,\operatorname{vol}_{n-1}(K \cap \mathbb{S}_n).$$

Hence, (7) can be rewritten as

$$v_K(x) \;\leq\; \left(\frac{1}{\operatorname{dist}(x,\,\partial K^*)}\right)^n \operatorname{btv}(K).$$

It suffices now to take the minimum with respect to $x \in \mathbb{S}_n \cap \operatorname{int}(K^*)$. □

As seen in the proof of Proposition 2.1(b), if $K \in \Pi_n$ and $x \in \operatorname{int}(K^*)$, then $K \cap H_x$ is an n-dimensional convex body. Sometimes it is easier to compute the volume of a convex body by evaluating the volume of an associated convex body that has one dimension less. This is the basic idea behind the following proposition.

Proposition 2.4 *Let $K \in \Pi_n$ and $x \in \operatorname{int}(K^*)$. Then*

$$v_K(x) = \frac{1}{n\|x\|}\operatorname{vol}_{n-1}(K \cap \partial H_x), \tag{9}$$

with ∂H_x denoting the boundary of H_x.

Proof Formula (9) appears already in the proof of (Güler 1996, Theorem 4.1), at least for the particular case in which $x \in \operatorname{int}(K^*)$ is a unit vector. If $x \in \operatorname{int}(K^*)$ is not a unit vector, then we write

$$\begin{aligned} v_K(x) = v_K\left(\|x\|\frac{x}{\|x\|}\right) \;&=\; \frac{1}{\|x\|^n}\, v_K\left(\frac{x}{\|x\|}\right) \\ &= \frac{1}{n\|x\|^n}\operatorname{vol}_{n-1}\left(K \cap \partial H_{\|x\|^{-1}x}\right) \end{aligned}$$

and observe that $K \cap \partial H_{\|x\|^{-1}x} = \|x\|\,(K \cap \partial H_x)$. Hence,

$$v_K(x) \;=\; \frac{1}{n\|x\|^n}\operatorname{vol}_{n-1}\left(\|x\|\,(K \cap \partial H_x)\right) \;=\; \frac{1}{n\|x\|}\operatorname{vol}_{n-1}\left(K \cap \partial H_x\right),$$

as desired. □

In the sequel, $\mathbb{GL}(n)$ denotes the general linear group of nonsingular matrices of order n and the superscript "T" stands for transposition.

Proposition 2.5 *Let $K = G(Q)$ with $G \in \mathbb{GL}(n)$. Then, for all $x \in \operatorname{int}(K^*)$, one has*

$$v_K(x) = |\det G|\, v_Q(G^T x).$$

Proof One can easily check that

$$\begin{aligned} &\operatorname{int}(K^*) = \{x \in \mathbb{R}^n : G^T x \in \operatorname{int}(Q^*)\}, \\ &K \cap H_x = G\left(Q \cap H_{G^T x}\right). \end{aligned}$$

So, it suffices to take the n-dimensional Lebesgue measure on each side of the last equality. □

Example 2.6 As pointed out in Güler (1996), one has $v_{\mathbb{R}^n_+}(x) = \left[n!\,\Pi_{i=1}^n x_i\right]^{-1}$ for all $x \in \mathrm{int}(\mathbb{R}^n_+)$. Consider now an arbitrary simplicial cone, i.e., a proper cone of the form $K = G(\mathbb{R}^n_+)$ with $G \in \mathbb{GL}(n)$. One gets

$$v_K(x) = \frac{1}{n!}\,\frac{|\det G|}{\Pi_{i=1}^n \langle g_i, x\rangle}$$

for all $x \in \mathrm{int}(K^*)$.

3 Basic facts on volumetric centers

We now introduce the concept of volumetric center into the discussion. The idea is to use the following simple but fundamental fact.

Proposition 3.1 *Let $K \in \Pi_n$. Then v_K admits a unique minimizer on $\mathbb{S}_n$. Furthermore, such a minimizer lies in the interior of K^*.*

Proof In view of the statements (a) and (c) of Proposition 2.1, the minimization problem (2) has the same optimal value and the same solution set as

$$\begin{cases} \text{minimize } \; v_K(x) \\ x \in \mathbb{B}_n. \end{cases} \tag{10}$$

The solution set to (10) has at most one element, because it is a convex set contained in $\mathbb{S}_n$. It remains to prove that (10) has at least one solution. Let $\{x_\nu\}_{\nu\in\mathbb{N}}$ be a sequence in $\mathbb{B}_n \cap \mathrm{int}(K^*)$ such that

$$\lim_{\nu\to\infty} v_K(x_\nu) = \mathrm{lpv}(K).$$

Without loss of generality, one may suppose that $\bar{x} := \lim_{\nu\to\infty} x_\nu$ exists. By using Proposition 2.2, one deduces that $\bar{x}$ is a solution to (10) and that $\bar{x} \in \mathrm{int}(K^*)$. □

A result similar to Proposition 3.1 is obtained by exchanging the roles of K and K^*. As mentioned in Sect. 1, the unique solution to (4) is denoted by $\varrho(K)$ and it is called the volumetric center of K.

The following result is easy and consistent with geometric intuition. The first formula in Proposition 3.2 says that $\mathrm{lpv} : \Pi_n \to \mathbb{R}$ is invariant under orthogonal transformations. The second formula says that an orthogonal transformation acting on a proper cone modifies its volumetric center as expected. The notation $\mathbb{O}(n)$ refers to the set of orthogonal matrices of order n.

Proposition 3.2 *Let $K \in \Pi_n$. Then, for all $U \in \mathbb{O}(n)$, one has*

$$\begin{aligned} \mathrm{lpv}(U(K)) &= \mathrm{lpv}(K), \\ \varrho(U(K)) &= U(\varrho(K)). \end{aligned}$$

More generally, for any pair A, B of invertible matrices such that $A^TA = B^TB$, one can write

$$\mathrm{lpv}(A(K)) = \mathrm{lpv}(B(K)), \tag{11}$$
$$A^{-1}\varrho(A(K)) = B^{-1}\varrho(B(K)). \tag{12}$$

Proof Note that

$$\begin{aligned}\mathrm{lpv}(U(K)) &= \min\{\mathrm{vol}_n[U(K)\cap H_x] : x\in\mathbb{S}_n\}\\ &= \min\{\mathrm{vol}_n\left[U(K\cap H_{U^Tx})\right] : x\in\mathbb{S}_n\}\\ &= \min\{\mathrm{vol}_n\left(K\cap H_{U^Tx}\right) : x\in\mathbb{S}_n\}.\end{aligned}$$

The change of variables $y = U^Tx$ leads to

$$\mathrm{lpv}(U(K)) \;=\; \min\{\mathrm{vol}_n(K\cap H_y) : y\in\mathbb{S}_n\} \;=\; \mathrm{lpv}(K),$$

and shows that $\varrho(K) = U^T\varrho(U(K))$. For proving (11) and (12), one just needs to observe that $A(K)$ is the image of $B(K)$ under the orthogonal matrix AB^{-1}. □

The next theorem characterizes the volumetric center of $K\in\Pi_n$ as the unique fixed point of the gradient map $\nabla\Gamma_K : \mathrm{int}(K)\to\mathbb{R}^n$, where $\Gamma_K : \mathrm{int}(K)\to\mathbb{R}$ is the strictly concave function given by

$$\Gamma_K(x) := -\frac{1}{n}\log[v_{K^*}(x)].$$

In view of (5), one has

$$\nabla\Gamma_K(x) = -\frac{\nabla v_{K^*}(x)}{nv_{K^*}(x)} = \frac{1}{n}\,\frac{\int_{K^*} ye^{-\langle x,y\rangle}dy}{\int_{K^*} e^{-\langle x,y\rangle}dy}\,.$$

For subsequence use, we recall that

$$\begin{cases}\nabla\Gamma_K \text{ is a bijection between } \mathrm{int}(K) \text{ and } \mathrm{int}(K^*),\\ \nabla\Gamma_K(tx) = t^{-1}\nabla\Gamma_K(x) \text{ for all } t>0 \text{ and } x\in\mathrm{int}(K),\\ \text{the Hessian matrix } \nabla^2\Gamma_K(x) \text{ is negative definite for all } x\in\mathrm{int}(K).\end{cases} \tag{13}$$

The properties listed in (13) can be found for instance in (Faraut and Korányi, 1994, Section I.3).

Theorem 3.3 *Let $K\in\Pi_n$. Then $\varrho(K)$ is the unique solution to the fixed point problem*

$$x\in\mathrm{int}(K),\quad x = \nabla\Gamma_K(x). \tag{14}$$

In particular, $\varrho(K)$ belongs not only to $\mathrm{int}(K)$*, but also to* $\mathrm{int}(K^*)$.

Proof By writing down the optimality conditions for the minimization of the convex function v_{K^*} on the convex set $\mathbb{B}_n$, one sees that $\varrho(K)$ is equal to the unique solution to the system

$$x \in \mathbb{S}_n \cap \operatorname{int}(K), \quad -\nabla v_{K^*}(x) \in N_{\mathbb{B}_n}(x),$$

where $N_{\mathbb{B}_n}(x)$ is the normal cone to $\mathbb{B}_n$ at x. This system can be written in the equivalent form

$$x \in \operatorname{int}(K), \quad \nabla v_{K^*}(x) = \langle \nabla v_{K^*}(x), x \rangle\, x. \tag{15}$$

The equality stated in (15) contains implicitly the information that x is a unit vector. Since v_{K^*} is positively homogeneous of degree $-n$, the Euler equation

$$\langle \nabla v_{K^*}(x), x \rangle = -n v_{K^*}(x)$$

holds for all $x \in \operatorname{int}(K)$. Hence, the system (15) can be rewritten as in (14). □

Remark 3.4 Vinberg (1963) suggested to use the unique solution to the fixed point problem (14) as definition of "center" for a homogeneous cone $K \in \Pi_n$, but in fact one can dispense from homogeneity. One says that $K \in \Pi_n$ is *homogeneous* if, for all $x, y \in \operatorname{int}(K)$, there exists $A \in \mathbb{GL}(n)$ such that $A(K) = K$ and $Ax = y$.

The fixed point characterization of $\varrho(K)$ can be exploited in manifold ways. For instance, one can use it to derive a formula for computing the volumetric center of a Cartesian product of finitely many proper cones.

Corollary 3.5 *Let $n = n_1 + \cdots + n_q$ with $n_1, \ldots, n_q \geq 1$. Let K be the Cartesian product of the cones $K_1 \in \Pi_{n_1}, \ldots, K_q \in \Pi_{n_q}$. Then $K \in \Pi_n$ and*

$$n^{1/2} \varrho\,(K) = \left(n_1^{1/2} \varrho(K_1), \ldots, n_q^{1/2} \varrho(K_q) \right), \tag{16}$$

$$\frac{\operatorname{lpv}\,(K)}{\kappa_n} = \frac{\operatorname{lpv}(K_1)}{\kappa_{n_1}} \cdots \frac{\operatorname{lpv}(K_q)}{\kappa_{n_q}}, \tag{17}$$

where $\kappa_n = n^{n/2}/n!$.

Proof Clearly, K^* is the Cartesian product of the K_i^*'s. A matter of computation shows that, for all $x = (x_1, \ldots, x_q) \in \operatorname{int}(K)$, one has

$$\begin{aligned} v_{K^*}(x) &= \frac{n_1! \ldots n_q!}{n!}\, v_{K_1^*}(x_1) \ldots v_{K_q^*}(x_q), \\ \nabla \Gamma_K(x) &= \left(\frac{n_1}{n} \nabla \Gamma_{K_1}(x_1), \ldots, \frac{n_q}{n} \nabla \Gamma_{K_q}(x_q) \right). \end{aligned} \tag{18}$$

The combination of (13), (18), and Theorem 3.3, leads to (16). On the other hand,

$$\begin{aligned} v_{K^*}(\varrho(K)) &= \frac{n_1! \ldots n_q!}{n!}\, v_{K_1^*}\left(\sqrt{\frac{n_1}{n}} \varrho(K_1) \right) \ldots v_{K_q^*}\left(\sqrt{\frac{n_q}{n}} \varrho(K_q) \right) \\ &= \frac{\kappa_n}{\kappa_{n_1} \ldots \kappa_{n_q}}\, v_{K_1^*}\left(\varrho(K_1) \right) \ldots v_{K_q^*}\left(\varrho(K_q) \right). \end{aligned}$$

In other words,

$$\frac{\mathrm{lpv}\,(K^*)}{\kappa_n} = \frac{\mathrm{lpv}(K_1^*)}{\kappa_{n_1}} \cdots \frac{\mathrm{lpv}(K_q^*)}{\kappa_{n_q}}\,. \tag{19}$$

The equality (17) is obtained from (19) by exchanging the roles of K_i and K_i^*. □

4 Analysis of a few particular cones

Computing a volumetric center may be difficult or not depending on the geometric nature of the cone. The computation of $\varrho(K)$ is greatly simplified if $K \in \Pi_n$ is symmetric with respect to one or several linear subspaces. Symmetry with respect to a linear subspace is understood in the classical sense, i.e., invariance with respect to reflection through that subspace [(cf. Barker and Carlson 1979, Definition 1)].

Lemma 4.1 *Let L be a linear subspace with respect to which $K \in \Pi_n$ is symmetric. Then $\varrho(K) \in L$.*

Proof Combine Proposition 3.2 and the symmetry principle established in (Seeger and Torki 2014, Theorem 2.4). □

Let $\mathcal{S}(K)$ denote the intersection of all linear subspaces with respect to which $K \in \Pi_n$ is symmetric. The dimension of the subspace $\mathcal{S}(K)$ is called the *symmetry rank* of K. We mention in passing that $\mathcal{S}(K^*) = \mathcal{S}(K)$ for all $K \in \Pi_n$. In particular, K^* and K have the same symmetry rank.

Theorem 4.2 *Suppose that $K \in \Pi_n$ has symmetry rank equal to 1, i.e., there exists a vector $c \in \mathbb{S}_n$ such that $\mathcal{S}(K) = \mathbb{R}c$. Then*

$$\varrho(K) = \varrho(K^*) = \begin{cases} c & \text{if } c \in K, \\ -c & \text{if } c \notin K. \end{cases}$$

Proof By applying Lemma 4.1 first to K and then to K^*, one gets $\varrho(K) = \pm c$ and $\varrho(K^*) = \pm c$, respectively. Since the unit vectors $\varrho(K) \in \mathrm{int}(K)$ and $\varrho(K^*) \in \mathrm{int}(K^*)$ cannot be opposite, one necessarily has $\varrho(K) = \varrho(K^*)$. □

By the way, saying that $K \in \Pi_n$ has symmetry rank equal to 1 is weaker than saying that K is symmetric with respect to a one-dimensional subspace. The next two corollaries are obtained straightforwardly from Theorem 4.2. We just give short sketches of theirs proofs.

Corollary 4.3 *Let $K \in \Pi_n$ be permutation invariant, i.e., $P(K) = K$ for all permutation matrix P. Then*

$$\varrho(K) = \varrho(K^*) = \begin{cases} \mathbf{1_n}/\sqrt{n} & \text{if } \mathbf{1_n} \in K, \\ -\mathbf{1_n}/\sqrt{n} & \text{if } \mathbf{1_n} \notin K, \end{cases}$$

where $\mathbf{1_n}$ stands for the n-dimensional vector of ones.

Proof As shown in (Seeger and Torki 2014, Example 2.5), if $K \in \Pi_n$ is permutation invariant, then $\mathcal{S}(K)$ is equal to the line generated by $\mathbf{1_n}$. □

Example 4.4 Since $\mathbb{R}^n_+$ is permutation invariant and contains $\mathbf{1_n}$, one has $\varrho(\mathbb{R}^n_+) = \mathbf{1_n}/\sqrt{n}$. Thus,

$$\operatorname{lpv}(\mathbb{R}^n_+) = v_{\mathbb{R}^n_+}\left(\frac{\mathbf{1_n}}{\sqrt{n}}\right) = \frac{n^{n/2}}{n!}\,.$$

A *top-heavy cone* in $\mathbb{R}^n$ is a proper cone that can be expressed as epigraph of a norm on $\mathbb{R}^{n-1}$. Such class of cones has been studied in depth by Fiedler and Haynsworth (1973), see also Barker and Carlson (1979).

Corollary 4.5 *Let $K \in \Pi_n$ be top-heavy up to orthogonal transformation, i.e., there exist an orthogonal matrix $U = [u_1, \ldots, u_n]$ of order n and a norm Φ on $\mathbb{R}^{n-1}$ such that $K = U$ (epi Φ), where*

$$\operatorname{epi}\Phi := \left\{x \in \mathbb{R}^n : \Phi(x_1, \ldots, x_{n-1}) \le x_n\right\}.$$

Then $\varrho(K) = \varrho(K^*) = u_n$. Furthermore,

$$\operatorname{lpv}(K) = \frac{1}{n}\operatorname{vol}_{n-1}(B_\Phi),$$

where B_Φ denotes the closed unit ball associated to Φ.

Proof In view of Proposition 3.2, it suffices to consider the case in which U is equal to I_n, the identity matrix of order n. In such a case, K is symmetric with respect to the line generated by n-th canonical vector $e_n := (0, \ldots, 0, 1)^T$. Thus, $\varrho(K) = \varrho(K^*) = e_n$ and

$$\operatorname{lpv}(K) = v_K(e_n) = \frac{1}{n}\operatorname{vol}_{n-1}\left(K \cap \partial H_{e_n}\right).$$

For completing the proof we observe that $K \cap \partial H_{e_n}$ is equal to $B_\Phi \times \{1\}$. □

There are plenty of interesting proper cones that fit into the setting of Corollary 4.5. By way of example we mention

$$\mathbb{L}_n := \left\{x \in \mathbb{R}^n : \left(\sum_{i=1}^{n-1} x_i^2\right)^{1/2} \le x_n\right\} \qquad \text{(Lorentz or circular cone)},$$

$$\mathbb{K}_{n,p} := \left\{x \in \mathbb{R}^n : \left(\sum_{i=1}^{n-1} |x_i|^p\right)^{1/p} \le x_n\right\} \qquad (\ell^p\text{- cone with } 1 \le p < \infty),$$

$$\mathbb{K}_{n,\infty} := \left\{x \in \mathbb{R}^n : \max_{1\le i\le n-1} |x_i| \le x_n\right\} \qquad (\ell^\infty\text{- cone}),$$

$$\mathbb{E}_M := \left\{(z,t) \in \mathbb{R}^n : \sqrt{\langle z, Mz\rangle} \le t\right\} \qquad \text{(upward ellipsoidal cone)}.$$

The upward ellipsoidal cone $\mathbb{E}_M$ is defined in terms of a symmetric positive definite matrix M of order $n-1$. All the above cones have the vector e_n as volumetric center. Their least partial volumes are given by the following explicit formulas:

$$\begin{aligned}
\operatorname{lpv}(\mathbb{L}_n) &= \frac{\pi^{(n-1)/2}}{n\,\Gamma((n+1)/2)}\,,\\
\operatorname{lpv}\left(\mathbb{K}_{n,p}\right) &= \frac{[2\,\Gamma(1+1/p)]^{n-1}}{n\,\Gamma(1+(n-1)/p)}\,,\\
\operatorname{lpv}\left(\mathbb{K}_{n,\infty}\right) &= 2^{n-1}/n,\\
\operatorname{lpv}(\mathbb{E}_M) &= \frac{\operatorname{lpv}(\mathbb{L}_n)}{\sqrt{\det M}}\,,
\end{aligned}$$

where Γ stands for the usual Euler gamma function.

As shown above, the least partial volume of the upward ellipsoidal cone $\mathbb{E}_M$ can be computed in a straightforward manner. To the best of our knowledge, there is no simple formula for evaluating the ball-truncated volume of $\mathbb{E}_M$. Another useful proper cone that fits into the setting of Corollary 4.5 is the revolution cone

$$\operatorname{rev}(c,s) := \{x \in \mathbb{R}^n : s\,\|x\| \le \langle c, x\rangle\},$$

where $c \in \mathbb{S}_n$ determines the revolution axis and $s \in]0,1[$ is a parameter reflecting the degree of aperture of the cone. One gets

$$\begin{aligned}
\varrho\,(\operatorname{rev}(c,s)) &= c,\\
\operatorname{lpv}(\operatorname{rev}(c,s)) &= \left(\frac{1}{s^2}-1\right)^{\frac{n-1}{2}} \operatorname{lpv}(\mathbb{L}_n)\,.
\end{aligned}$$

The next proposition concerns the case of a general ellipsoidal cone, i.e., not necessarily upward. By definition, an *ellipsoidal cone* in $\mathbb{R}^n$ is an image of the Lorentz cone $\mathbb{L}_n$ under a nonsingular matrix, cf. Stern and Wolkowicz (1991). Ellipsoidal cones have been studied under different angles by a number of authors, see for instance Bhattacharya et al. (2004) for an application of ellipsoidal cones in control theory.

Proposition 4.6 *Let $K = G(\mathbb{L}_n)$ with $G \in \mathbb{GL}(n)$. Then*

$$\operatorname{lpv}(K) = (-\mu)^{n/2}\,|\det G|\,\operatorname{lpv}(\mathbb{L}_n)\,,$$

where μ is the smallest eigenvalue of the symmetric matrix

$$G^{\diamond} := G^{-T}(I_n - 2e_n e_n^T)G^{-1}.$$

Furthermore, $\varrho\,(K)$ is an eigenvector of $G^{\diamond}$ associated to the eigenvalue μ. More precisely, $\varrho\,(K)$ is the unique solution to

$$G^{\diamond}x = \mu\,x, \quad \|x\| = 1, \quad \langle G^{-T}e_n, x\rangle > 0. \tag{20}$$

Proof Following similar steps as in (Seeger 2013, Lemma 5), one can check that

$$K = U(\mathbb{E}_M) \tag{21}$$

for some $U \in \mathbb{O}(n)$ and some positive definite symmetric matrix M of order $n-1$. We briefly explain the construction of U and M. The matrix $G^\diamond$ is clearly symmetric. Let

$$\lambda_1(G^\diamond) \geq \ldots \geq \lambda_{n-1}(G^\diamond) \geq \lambda_n(G^\diamond)$$

be the eigenvalues of $G^\diamond$ arranged in nonincreasing order. By applying Silvester's inertia theorem, one sees that $\mu = \lambda_n(G^\diamond)$ is negative and all the other eigenvalues of $G^\diamond$ are positive. Hence, the diagonal matrix

$$M := -\mu^{-1}\mathrm{Diag}\left(\lambda_1(G^\diamond), \ldots, \lambda_{n-1}(G^\diamond)\right)$$

is positive definite. Let the columns of U be formed with an orthonormal basis of eigenvectors of $G^\diamond$. As last column of U we take the vector $\bar{x} \in \mathbb{R}^n$ which solves (20). Such choices of M and U lead to the representation formula (21), from where one gets $\varrho(K) = Ue_n = \bar{x}$ and

$$\mathrm{lpv}(K) \;=\; \mathrm{lpv}(\mathbb{E}_M) \;=\; \frac{\mathrm{lpv}\,(\mathbb{L}_n)}{\sqrt{\det M}}\,. \tag{22}$$

It remains now to substitute $\det M = (\det G)^{-2}(-\mu)^{-n}$ into (22). □

The dual of an ellipsoidal cone is an ellipsoidal cone. The next corollary is a consequence of the representation formula (21) and the fact that

$$(\mathbb{E}_M)^* = \mathbb{E}_{M^{-1}}.$$

Corollary 4.7 *Let K be an ellipsoidal cone in $\mathbb{R}^n$. Then*

$$\varrho(K^*) = \varrho(K), \tag{23}$$

$$\mathrm{lpv}(K^*)\,\mathrm{lpv}(K) = \left[\mathrm{lpv}(\mathbb{L}_n)\right]^2. \tag{24}$$

Proof One can represent K as in (21). In such a case, $K^* = U(\mathbb{E}_{M^{-1}})$ and

$$U^T\varrho(K^*) \;=\; \varrho(\mathbb{E}_{M^{-1}}) \;=\; \varrho(\mathbb{E}_M) \;=\; U^T\varrho(K).$$

This proves (23). On the other hand,

$$\begin{aligned}\frac{\mathrm{lpv}(K^*)}{\mathrm{lpv}(\mathbb{L}_n)} = \frac{\mathrm{lpv}(\mathbb{E}_{M^{-1}})}{\mathrm{lpv}(\mathbb{L}_n)} \;&=\; \frac{1}{\sqrt{\det(M^{-1})}} \;=\; \left(\frac{1}{\sqrt{\det M}}\right)^{-1} \\ &= \left(\frac{\mathrm{lpv}(\mathbb{E}_M)}{\mathrm{lpv}(\mathbb{L}_n)}\right)^{-1} \;=\; \left(\frac{\mathrm{lpv}(K)}{\mathrm{lpv}(\mathbb{L}_n)}\right)^{-1}.\end{aligned}$$

This proves (24). □

5 Continuity issues

Let Ξ_n denote the set of nontrivial closed convex cones in $\mathbb{R}^n$. Recall that a closed convex cone is nontrivial if it is different from the zero cone and the whole space. Topological and continuity issues on Ξ_n are relative to the spherical metric δ. By definition, the spherical distance between a pair P, Q of elements in Ξ_n is the number

$$\delta(P, Q) := \text{haus}(P \cap \mathbb{S}_n, Q \cap \mathbb{S}_n),$$

where

$$\text{haus}(C, D) := \max\left\{\max_{x\in C} \text{dist}(x, D), \max_{x\in D} \text{dist}(x, C)\right\}$$

stands for the classical Pompeiu–Hausdorff distance between a pair C, D of nonempty compact subsets of $\mathbb{R}^n$. Convergence with respect to the spherical metric is equivalent to convergence with respect to the ball-truncated metric

$$\delta_{\text{bt}}(P, Q) := \text{haus}(P \cap \mathbb{B}_n, Q \cap \mathbb{B}_n).$$

In fact, one has

$$\delta_{\text{bt}}(P, Q) \leq \delta(P, Q) \leq 2\delta_{\text{bt}}(P, Q)$$

for all P, $Q \in \Xi_n$. Convergence with respect to the spherical metric is also equivalent to convergence in the Painlevé-Kuratowski sense.

A nontrivial closed convex cone is said to be solid if it has nonempty interior. Since the sets

$$\Xi_n^{\text{ptd}} := \{K \in \Xi_n : K \text{ is pointed}\},$$
$$\Xi_n^{\text{sol}} := \{K \in \Xi_n : K \text{ is solid}\}$$

are open in the metric space (Ξ_n, δ), so does their intersection $\Pi_n = \Xi_n^{\text{ptd}} \cap \Xi_n^{\text{sol}}$. A natural question to ask is whether $\text{lpv} : \Pi_n \to \mathbb{R}$ and $\varrho : \Pi_n \to \mathbb{R}^n$ are continuous functions. The situation is well understood in what concerns ball-truncated volumes, inradiuses, and incenters. For instance, Gourion and Seeger (2012, [Theorem 2.3]) proved that $\text{btv} : \Xi_n \to \mathbb{R}$ is Lipschitzian.

Proposition 5.1 *There exists a constant ℓ_n such that*

$$|\text{btv}(P) - \text{btv}(Q)| \leq \ell_n \delta(P, Q)$$

for all P, $Q \in \Xi_n$.

The real valued function $\text{btv}(\cdot)$ is well defined not just on Π_n, but also on the larger set Ξ_n. The following Lipschitzness result is a direct consequence of (Iusem and Seeger 2008, Proposition 13).

Proposition 5.2 *For all $P, Q \in \Xi_n$, one has*

$$|r(P) - r(Q)| \leq 2\delta(P, Q).$$

The vector $\xi(K)$ is well defined as long as K belongs to Ξ_n^{sol}. The continuity of the incenter function $\xi : \Xi_n^{\mathrm{sol}} \to \mathbb{R}^n$ has been established in (Henrion and Seeger 2010a, Theorem 2.9). Recently, Seeger (2014, Theorem 7) proved that the incenter function is not merely continuous, but also locally Hölderian.

Proposition 5.3 *For all $P, Q \in \Xi_n^{\mathrm{sol}}$, one has*

$$\|\xi(P) - \xi(Q)\| \leq \frac{4}{\sqrt{r(P) + r(Q)}} [\delta(P, Q)]^{1/2}.$$

In particular, $\xi : \Xi_n^{\mathrm{sol}} \to \mathbb{R}^n$ is locally Hölderian with Hölder exponent $1/2$.

The next theorem is the main result of this section. It shows that $\mathrm{lpv}(K)$ and $\varrho(K)$ behave in a continuous manner with respect to perturbations in the argument $K \in \Pi_n$. We state first three useful lemmas. The first lemma concerns n-dimensional volumes of symmetric differences of compact convex sets. Recall that the circumradius of a convex body $\Omega \subseteq \mathbb{R}^n$ is defined as the radius of the smallest closed ball containing Ω.

Lemma 5.4 *Let $\Omega \subseteq \mathbb{R}^n$ be a convex body with circumradius equal to ρ. Then*

$$\mathrm{vol}_n(C \Delta D) \leq \frac{(3\rho)^n - 1}{\rho} \mathrm{vol}_n(\mathbb{B}_n)\ \mathrm{haus}(C, D)$$

for any pair C, D of nonempty compact convex subsets of Ω.

Proof There exists a unique $z \in \mathbb{R}^n$, called the circumcenter of Ω, such that $\Omega \subseteq z + \rho\mathbb{B}_n$. Let $t = \mathrm{haus}(C, D)$. Since $D \subseteq C + t\mathbb{B}_n$, one has

$$\mathrm{vol}_n(D \backslash C) \ \leq\ \mathrm{vol}_n((C + t\mathbb{B}_n) \backslash C) \ =\ \mathrm{vol}_n(C + t\mathbb{B}_n) - \mathrm{vol}_n(C).$$

But

$$\mathrm{vol}_n(C + t\mathbb{B}_n) \ =\ \sum_{k=0}^{n} \Phi_k(C)\, t^k,$$

where $\Phi_0(C), \ldots, \Phi_n(C)$ are the coefficients in the Steiner polynomial associated to C. The coefficients $\Phi_k(C)$ are nonnegative and depend not just on C, but also on the dimension n of the ambient space. In fact,

$$\Phi_k(C) = \frac{n!}{k!(n-k)!}\, V(\underbrace{C, \ldots, C}_{n-k}, \underbrace{\mathbb{B}_n, \ldots, \mathbb{B}_n}_{k}),$$

where $V(C_1, \ldots, C_n)$ stands for the mixed volume of the convex bodies $C_1, \ldots, C_n$. In particular, $\Phi_0(C) = \mathrm{vol}_n(C)$. As mentioned in (Gardner 1995, Appendix A.3), the

function V is nondecreasing in each argument with respect to set inclusion. It follows that

$$\begin{aligned}\mathrm{vol}_n(D\backslash C) &\leq \sum_{k=1}^{n} \Phi_k(C)\, t^k \;\leq\; \left[\sum_{k=1}^{n} \Phi_k(z+\rho\mathbb{B}_n)\, t^{k-1}\right] t \\ &\leq \frac{1}{2\rho}\left[\sum_{k=1}^{n} \Phi_k(z+\rho\mathbb{B}_n)\,(2\rho)^k\right] t. \end{aligned} \tag{25}$$

The last equality is due to the fact that $t \leq 2\rho$, because C and D are subsets of Ω. By equating coefficients in the polynomials

$$\begin{aligned}\mathrm{vol}_n(z+\rho\mathbb{B}_n+t\mathbb{B}_n) &= (\rho+t)^n\,\mathrm{vol}_n(\mathbb{B}_n), \\ \mathrm{vol}_n(z+\rho\mathbb{B}_n+t\mathbb{B}_n) &= \sum_{k=0}^{n} \Phi_k(z+\rho\mathbb{B}_n)\, t^k, \end{aligned}$$

one gets

$$\Phi_k(z+\rho\mathbb{B}_n) = \frac{n!}{k!(n-k)!}\,\rho^{n-k}\,\mathrm{vol}_n(\mathbb{B}_n).$$

Substituting this information into (25) and simplifying, one obtains

$$\mathrm{vol}_n(D\backslash C) \leq \frac{(3\rho)^n - 1}{2\rho}\,\mathrm{vol}_n\,(\mathbb{B}_n)\, t.$$

The same estimate holds for the n-dimensional volume of $C\backslash D$. □

The second lemma is a technical result concerning the distance from a point to the boundary of a convex cone.

Lemma 5.5 *Let $K \in \Xi_n$. Then, for all $x \in \mathbb{R}^n$ and $r \geq \|x\|$, one has*

$$\mathrm{dist}(x, r\mathbb{B}_n \cap \partial K) = \mathrm{dist}(x, \partial K) \leq \|x\|.$$

Proof Take any $y_0 \in \partial K$ such that $\mathrm{dist}(x, \partial K) = \|x - y_0\|$. Since ∂K is stable under multiplication by positive scalars, one has $\langle x - y_0, y_0\rangle = 0$. By using Pythagoras rule

$$\|x - y_0\|^2 + \|y_0\|^2 = \|x\|^2,$$

one gets $\|y_0\| \leq \|x\|$ and $\mathrm{dist}(x, \partial K) \leq \|x\|$. One also obtains

$$\mathrm{dist}(x, r\mathbb{B}_n \cap \partial K) = \mathrm{dist}(x, \partial K),$$

because $y_0 \in r\mathbb{B}_n \cap \partial K$. □

The third and last lemma concerns the convergence of boundaries of converging cones. As a consequence of Lemma 5.6, one sees that:

$$\begin{cases} \text{If } \{K_\nu\}_{\nu\in\mathbb{N}} \subseteq \Pi_n \text{ converges to } K_\infty \in \Pi_n, \\ \text{then } \lim_{\nu\to\infty} \operatorname{haus}(\mathbb{S}_n \cap \partial K_\nu, \mathbb{S}_n \cap \partial K_\infty) = 0. \end{cases}$$

Lemma 5.6 *For all $P, Q \in \Pi_n$, one has*

$$\operatorname{haus}(\mathbb{S}_n \cap \partial P, \mathbb{S}_n \cap \partial Q) \leq 2\delta_{\mathrm{bt}}(P, Q). \tag{26}$$

Proof It suffices to proves that $\operatorname{dist}(x, \partial Q) \leq 2\delta_{\mathrm{bt}}(P, Q)$ for all $x \in \mathbb{S}_n \cap \partial P$. Let x be as just mentioned. Note that $C := 2\mathbb{B}_n \cap P$ and $D := 2\mathbb{B}_n \cap Q$ are convex bodies in $\mathbb{R}^n$. Since

$$\partial D = (2\mathbb{B}_n \cap \partial Q) \cup (2\mathbb{S}_n \cap Q),$$

one has

$$\operatorname{dist}(x, \partial D) \;=\; \min\left\{\operatorname{dist}(x, 2\mathbb{B}_n \cap \partial Q), \operatorname{dist}(x, 2\mathbb{S}_n \cap Q)\right\}.$$

Clearly,

$$\begin{aligned} \operatorname{dist}(x, 2\mathbb{S}_n \cap Q) \;&\geq\; \operatorname{dist}(x, 2\mathbb{S}_n) = 1, \\ \operatorname{dist}(x, 2\mathbb{B}_n \cap \partial Q) &= \operatorname{dist}(x, \partial Q) \leq 1, \end{aligned}$$

the second line being a consequence of Lemma 5.5. It follows that

$$\begin{aligned} \operatorname{dist}(x, \partial Q) = \operatorname{dist}(x, \partial D) \;&\leq\; \max_{z\in\mathbb{S}_n\cap\partial P} \operatorname{dist}(z, \partial D) \\ &\leq \max_{z\in\partial C} \operatorname{dist}(z, \partial D) \;\leq\; \operatorname{haus}(C, D), \end{aligned}$$

where the last inequality is obtaining by applying (Wills 2007, Theorem 14). Finally, by using a simple positive homogeneity argument, one sees that $\operatorname{haus}(C, D) = 2\delta_{\mathrm{bt}}(P, Q)$. □

We do not know if the inequality (26) remains true for P, Q in the larger set Ξ_n. Anyway, we now are ready to state:

Theorem 5.7 *The functions* $\operatorname{lpv} : \Pi_n \to \mathbb{R}$ *and* $\varrho : \Pi_n \to \mathbb{R}^n$ *are continuous.*

Proof Let $\{K_\nu\}_{\nu\in\mathbb{N}}$ be a sequence in Π_n converging to a certain $K_\infty \in \Pi_n$, i.e.,

$$\lim_{\nu\to\infty} \delta(K_\nu, K_\infty) = 0. \tag{27}$$

We must prove that

$$\lim_{\nu\to\infty} \mathrm{lpv}(K_\nu) = \mathrm{lpv}(K_\infty) \tag{28}$$

$$\lim_{\nu\to\infty} \varrho(K_\nu) = \varrho(K_\infty). \tag{29}$$

For the sake of clarity in the exposition, we divide the proof into several steps.

Step 1 We prove that $\{\mathrm{lpv}(K_\nu)\}_{\nu\in\mathbb{N}}$ is bounded. For each ε positive, let $\Gamma_\varepsilon(K_\infty)$ denote the smallest closed convex cone containing the compact set $K_\infty \cap \mathbb{S}_n + \varepsilon\mathbb{B}_n$. We take ε sufficiently small in order to ensure that $\Gamma_\varepsilon(K_\infty)$ is proper. One has

$$\begin{aligned} K_\nu \cap \mathbb{S}_n &\subseteq K_\infty \cap \mathbb{S}_n + \delta(K_\nu, K_\infty)\mathbb{B}_n \\ &\subseteq K_\infty \cap \mathbb{S}_n + \varepsilon\mathbb{B}_n \\ &\subseteq \Gamma_\varepsilon(K_\infty) \end{aligned}$$

for all ν large enough. In particular, $K_\nu \subseteq \Gamma_\varepsilon(K_\infty)$ and $\mathrm{lpv}(K_\nu) \leq \mathrm{lpv}(\Gamma_\varepsilon(K_\infty))$.

Step 2 We prove that for each sequence $\{x_\nu\}_{\nu\in\mathbb{N}}$ satisfying

$$x_\nu \in \mathbb{S}_n \cap \mathrm{int}(K_\nu^*), \tag{30}$$

$$x_\infty := \lim_{\nu\to\infty} x_\nu \text{ exists and belongs to } \mathrm{int}(K_\infty^*), \tag{31}$$

there exists a constant $s \in]0, 1[$ such that

$$K_\nu \cup K_\infty \subseteq \mathrm{rev}(x_\nu, s) \tag{32}$$

for all ν large enough. Thanks to the Walkup-Wets isometry theorem (cf. [Walkup and Wets 1967, Theorem 1]), the condition (27) is equivalent to

$$\lim_{\nu\to\infty} \delta(K_\nu^*, K_\infty^*) = 0. \tag{33}$$

Since $x_\infty \in \mathrm{int}(K_\infty^*)$, there exists a small positive s such that

$$x_\infty + 2s\mathbb{B}_n \subseteq \mathrm{int}(K_\infty^*).$$

By combining (33) and a standard stability result for Painlevé-Kuratowski limits (cf. [Rockafellar and Wets 1998, Proposition 4.15]), one gets

$$x_\infty + 2s\mathbb{B}_n \subseteq \mathrm{int}(K_\nu^*)$$

for all ν greater than a certain integer ν_0. Since $\{x_\nu\}_{\nu\in\mathbb{N}}$ converges to x_∞, it follows that

$$x_\nu + s\mathbb{B}_n \subseteq K_\infty^* \quad \text{and} \quad x_\nu + s\mathbb{B}_n \subseteq K_\nu^*$$

for all ν greater than another integer $\nu_1 \geq \nu_0$. By passing to dual cones in

$$\mathbb{R}_+(x_\nu + s\mathbb{B}_n) \subseteq K_\infty^* \quad \text{and} \quad \mathbb{R}_+(x_\nu + s\mathbb{B}_n) \subseteq K_\nu^*,$$

one gets the reverse inclusions $\mathrm{rev}(x_\nu, s) \supseteq K_\infty$ and $\mathrm{rev}(x_\nu, s) \supseteq K_\nu$. This proves (32).

Step 3 We show that, for any $\{x_\nu\}_{\nu\in\mathbb{N}}$ as in Step 2, one has

$$\lim_{\nu\to\infty} v_{K_\nu}(x_\nu) = v_{K_\infty}(x_\infty). \tag{34}$$

As we saw already, $x_\nu \in \mathrm{int}(K_\infty^*)$ for all ν large enough. Hence, one can write

$$|v_{K_\nu}(x_\nu) - v_{K_\infty}(x_\infty)| \;\leq\; |v_{K_\nu}(x_\nu) - v_{K_\infty}(x_\nu)| + |v_{K_\infty}(x_\nu) - v_{K_\infty}(x_\infty)|.$$

Since $v_{K_\infty} : \mathrm{int}(K_\infty^*) \to \mathbb{R}$ is continuous, one has $\lim_{\nu\to\infty} v_{K_\infty}(x_\nu) = v_{K_\infty}(x_\infty)$. On the other hand, thanks to (5), one gets

$$n!\,|v_{K_\nu}(x_\nu) - v_{K_\infty}(x_\nu)| \;\leq\; a_\nu$$

with

$$a_\nu := \int_{D_\nu} e^{-\langle x_\nu, y\rangle} dy \quad \text{and} \quad D_\nu := K_\nu \Delta K_\infty.$$

By using (32), one obtains

$$\begin{aligned} a_\nu &\leq \int_{D_\nu} e^{-s\|y\|} dy \;=\; \frac{1}{s^n}\int_{D_\nu} e^{-\|y\|} dy \;=\; \frac{1}{s^n}\int_0^\infty \left(\int_{D_\nu \cap t\mathbb{S}_n} e^{-t} dy\right) dt \\ &= \frac{1}{s^n}\int_0^\infty e^{-t}\,\mathrm{vol}_{n-1}(D_\nu \cap t\mathbb{S}_n) dt \;=\; \frac{(n-1)!}{s^n}\,\mathrm{vol}_{n-1}(D_\nu \cap \mathbb{S}_n). \end{aligned}$$

But

$$\mathrm{vol}_{n-1}(D_\nu \cap \mathbb{S}_n) \;=\; n\,\mathrm{vol}_n(D_\nu \cap \mathbb{B}_n) \;=\; n\,\mathrm{vol}_n\left((K_\nu \cap \mathbb{B}_n)\Delta(K_\infty \cap \mathbb{B}_n)\right).$$

By applying Lemma 5.4 with $\Omega = \mathbb{B}_n$, one sees that

$$\lim_{\nu\to\infty} \mathrm{vol}_{n-1}(D_\nu \cap \mathbb{S}_n) = 0.$$

Hence, $\lim_{\nu\to\infty} a_\nu = 0$ and $\lim_{\nu\to\infty} v_{K_\nu}(x_\nu) = v_{K_\infty}(x_\infty)$. This completes the proof of (34).

Step 4 We show that, for any $\{x_\nu\}_{\nu\in\mathbb{N}}$ as in Step 2, one has

$$\lim_{\nu\to\infty} \nabla v_{K_\nu}(x_\nu) = \nabla v_{K_\infty}(x_\infty).$$

We start by writing

$$\|\nabla v_{K_\nu}(x_\nu)-\nabla v_{K_\infty}(x_\infty)\| \le \|\nabla v_{K_\nu}(x_\nu)-\nabla v_{K_\infty}(x_\nu)\|+\|\nabla v_{K_\infty}(x_\nu)-\nabla v_{K_\infty}(x_\infty)\|.$$

The continuity of $\nabla v_{K_\infty} : \mathrm{int}(K_\infty^*) \to \mathbb{R}^n$ yields

$$\lim_{\nu\to\infty} \|\nabla v_{K_\infty}(x_\nu) - \nabla v_{K_\infty}(x_\infty)\| = 0.$$

On the other hand,

$$n!\,\|\nabla v_{K_\nu}(x_\nu) - \nabla v_{K_\infty}(x_\nu)\| \;=\; \left\| \int_{K_\nu} y e^{-\langle x_\nu, y\rangle} dy - \int_{K_\infty} y e^{-\langle x_\nu, y\rangle} dy \right\| \;\le\; b_\nu$$

with

$$b_\nu := \int_{D_\nu} \|y\| e^{-\langle x_\nu, y\rangle} dy.$$

One proves that $\lim_{\nu\to\infty} b_\nu = 0$ by using a similar technique as in Step 3. Indeed,

$$\begin{aligned} b_\nu &\le \int_{D_\nu} \|y\| e^{-s\|y\|} dy = \frac{1}{s^{n+1}} \int_{D_\nu} \|y\| e^{-\|y\|} dy = \frac{1}{s^{n+1}} \int_0^\infty \left(\int_{D_\nu \cap t\mathbb{S}_n} t e^{-t} dy \right) dt \\ &= \frac{1}{s^{n+1}} \int_0^\infty t e^{-t} \mathrm{vol}_{n-1}(D_\nu \cap t\mathbb{S}_n)\, dt \;=\; \frac{n!}{s^{n+1}} \mathrm{vol}_{n-1}(D_\nu \cap \mathbb{S}_n). \end{aligned}$$

Step 5 We prove a technical inequality relating $v_K(x)$ and $\mathrm{dist}(x, \partial K^*)$. To be more precise, we show that

$$v_K(x) \ge \frac{\mathrm{vol}_{n-1}(\mathbb{B}_{n-1})}{n!} \frac{[r(K)]^{n-1} e^{-2\|x\|-1}}{\mathrm{dist}(x, \partial K^*)} \tag{35}$$

for all $K \in \Pi_n$ and $x \in \mathrm{int}(K^*)$. The term $\mathrm{dist}(x, \partial K^*)$ is clearly positive and equal to

$$\alpha_x := \min_{u \in K \cap \mathbb{S}_n} \langle u, x\rangle.$$

Let $u_x \in K \cap \mathbb{S}_n$ be such that $\langle u_x, x\rangle = \alpha_x$. The set

$$M := \xi(K) + r(K)\mathbb{B}_n + \alpha_x^{-1} \mathrm{co}\{0, u_x\}$$

is contained in K, and

$$\max_{y\in M} \langle y, x\rangle \;=\; \langle \xi(K), x\rangle + r(K)\|x\| + 1 \;\le\; 2\|x\| + 1.$$

Hence,

$$n!\, v_K(x) \geq \int_M e^{-\langle y,x\rangle} dy \geq e^{-2\|x\|-1} \mathrm{vol}_n(M).$$

One obtains (35) by combining above inequality and the fact that

$$\begin{aligned} \mathrm{vol}_n(M) &= \mathrm{vol}_n\left(\xi(K) + r(K)\mathbb{B}_n\right) + \alpha_x^{-1}\mathrm{vol}_{n-1}\left(r(K)\mathbb{B}_{n-1}\right) \\ &\geq \alpha_x^{-1}[r(K)]^{n-1}\mathrm{vol}_{n-1}\left(\mathbb{B}_{n-1}\right). \end{aligned}$$

Step 6 Finally, we prove (28) and (29). To do this, we work with the sequence $\{x_\nu\}_{\nu\in\mathbb{N}}$ given by

$$x_\nu := \varrho(K_\nu^*). \tag{36}$$

The condition (30) is clearly in force. One may assume that $x_\infty := \lim_{\nu\to\infty} x_\nu$ exists, otherwise one works with a convergent subsequence $\{x_{\varphi(\nu)}\}_{\nu\in\mathbb{N}}$ and the corresponding subsequence $\{K_{\varphi(\nu)}\}_{\nu\in\mathbb{N}}$. Clearly, $x_\infty \in K_\infty^*$. We claim that $x_\infty \in \mathrm{int}(K_\infty^*)$. Suppose, to the contrary, that $x_\infty \in \partial K_\infty^*$. One has

$$\mathrm{dist}(x_\nu, \partial K_\nu^*) \;\leq\; \|x_\nu - x_\infty\| + \mathrm{dist}(x_\infty, \partial K_\nu^*) \;\leq\; \|x_\nu - x_\infty\| + \;\zeta_\nu$$

with

$$\zeta_\nu := \max_{x\in\mathbb{S}_n\cap\partial K_\infty^*} \mathrm{dist}(x, \partial K_\nu^*).$$

By exploiting the convergence condition (33) and Lemma 5.6, one sees that $\lim_{\nu\to\infty}\zeta_\nu = 0$. Hence, $\lim_{\nu\to\infty}\mathrm{dist}(x_\nu, \partial K_\nu^*) = 0$. But, thanks to Step 5, one has

$$\mathrm{lpv}(K_\nu) = v_{K_\nu}(x_\nu) \geq \frac{\mathrm{vol}_{n-1}(\mathbb{B}_{n-1})}{n!e^3}\,\frac{[r(K_\nu)]^{n-1}}{\mathrm{dist}(x_\nu, \partial K_\nu^*)}\,.$$

The inradius function $r : \Pi_n \to \mathbb{R}$ is known to be continuous. Since $\lim_{\nu\to\infty} r(K_\nu) = r(K_\infty) \neq 0$, it follows that $\lim_{\nu\to\infty}\mathrm{lpv}(K_\nu) = \infty$, contradicting the result of Step 1. Summarizing, the special sequence (36) satisfies (30) and (31). By passing to the limit in the optimality condition

$$\nabla v_{K_\nu}(x_\nu) = \langle \nabla v_{K_\nu}(x_\nu), x_\nu\rangle\, x_\nu,$$

one gets

$$\nabla v_{K_\infty}(x_\infty) = \langle \nabla v_{K_\infty}(x_\infty), x_\infty\rangle\, x_\infty.$$

This and $x_\infty \in \mathrm{int}(K_\infty^*)$ imply that $x_\infty = \varrho(K_\infty^*)$. We have shown in this way that

$$\lim_{\nu\to\infty}\varrho(K_\nu^*) = \varrho(K_\infty^*).$$

The equality (29) is then obtained by exchanging the roles of K_ν and K_ν^*. Finally, observe that

$$\lim_{\nu\to\infty} \operatorname{lpv}(K_\nu) = \lim_{\nu\to\infty} v_{K_\nu}(x_\nu) = v_{K_\infty}(x_\infty) = \operatorname{lpv}(K_\infty).$$

This completes the proof of the theorem. □

The inequality (35) has further consequences. For instance, it can be used to derive the following generalization of Proposition 2.2. A sequence $\{K_\nu\}_{\nu\in\mathbb{N}}$ in Π_n is said to be *uniformly solid* if

$$\inf_{\nu\in\mathbb{N}} r(K_\nu) > 0. \tag{37}$$

Uniform solidity is an assumption arising naturally in a number of situations, cf. (Henrion and Seeger 2011, Theorem 3.4).

Proposition 5.8 *Let $\{K_\nu\}_{\nu\in\mathbb{N}}$ be a sequence in Π_n satisfying the uniform solidity condition* (37)*. Let $\{x_\nu\}_{\nu\in\mathbb{N}}$ be a bounded sequence in $\mathbb{R}^n$ such that $x_\nu \in K_\nu^*$. Then*

$$\lim_{\nu\to\infty} v_{K_\nu}(x_\nu) = \infty \quad \textit{if and only if} \quad \lim_{\nu\to\infty} \operatorname{dist}(x_\nu, \partial K_\nu^*) = 0.$$

Proof The "only if" part is a consequence of the inequality

$$v_{K_\nu}(x_\nu) \;\le\; \left(\frac{1}{\operatorname{dist}(x_\nu, \partial K_\nu^*)}\right)^n \operatorname{btv}(K_\nu)$$

and the boundedness of $\{\operatorname{btv}(K_\nu)\}_{\nu\in\mathbb{N}}$. The "if" part follows from

$$v_{K_\nu}(x_\nu) \;\ge\; \frac{\operatorname{vol}_{n-1}(\mathbb{B}_{n-1})}{n!}\,\frac{[r(K_\nu)]^{n-1} e^{-2\|x_\nu\|-1}}{\operatorname{dist}(x_\nu, \partial K_\nu^*)}$$

and the hypothesis (37). □

We do not know whether $\varrho : \Pi_n \to \mathbb{R}^n$ is locally Hölderian or not. This difficult question is left open for the time being. What is clear however is that $\operatorname{lpv} : \Pi_n \to \mathbb{R}$ is not Lipschitzian. The next example illustrates this point.

Example 5.9 Let $K = \mathbb{L}_n$ and $K_\nu = \operatorname{rev}(e_n, \cos\theta_\nu)$ with $\theta_\nu := (\pi/2) - (1/\nu)$. Then

$$\begin{aligned} \operatorname{lpv}(K_\nu) - \operatorname{lpv}(K) &= \left(\tan^2\theta_\nu - 1\right)\operatorname{lpv}(\mathbb{L}_n), \\ \delta(K_\nu, K) &= \sin(\theta_\nu - (\pi/4)). \end{aligned}$$

Hence,

$$\lim_{\nu\to\infty} \frac{\operatorname{lpv}(K_\nu) - \operatorname{lpv}(K)}{\delta(K_\nu, K)} = \infty.$$

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