Singularities of the quad curl problem

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September 19, 2017

Abstract

We consider the quad curl problem in smooth and non smooth domains of the space. We first give an augmented variational formulation equivalent to the one from [25] if the datum is divergence free. We describe the singularities of the variational space which correspond to the ones of the Maxwell system with perfectly conducting boundary conditions. The edge and corner singularities of the solution of the corresponding boundary value problem with smooth data are also characterized. We finally obtain some regularity results of the variational solution.

AMS (MOS) subject classification 35Q60, 35B65
Key Words Fourth order problem, Maxwell system, singularities

1 Introduction

On a bounded domain Ω of $\mathbb{R}^3$, we consider the following system, called the quad curl problem in [21, 25]

$$
\text{(1.1)} \qquad \left\{ \begin{array}{ll} \operatorname{curl}^4 \mathbf{u} = \mathbf{f} & \text{in } \Omega, \\ \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} \times \mathbf{n} = (\operatorname{curl} \mathbf{u}) \times \mathbf{n} = 0 & \text{on } \partial\Omega, \end{array} \right.
$$

where $\mathbf{f}$ belongs to $\mathbf{H}(\operatorname{div} = 0, \Omega) = \{\mathbf{u} \in \mathbf{H}(\operatorname{div}, \Omega) : \operatorname{div} \mathbf{u} = 0\}$.

This model problem arises in different applications, such as in inverse electromagnetic scattering theory [5, 21, 25] or in magnetohydrodynamics [27]. Some numerical methods are proposed in [25, 27, 13] and some error estimates are proved under some regularity assumptions. The $\mathbf{H}^3$ regularity of the weak solution is even assumed in [25, p. 190]. As mentioned in [25], such regularity results are not available in the literature. Hence our goal is to prove such results in the case of smooth and non smooth domains. For that purpose, in the spirit of [6], we first transform the variational formulation from [25] into an augmented one. We show that this augmented problem is well-posed and give its equivalence with the original one when the datum is divergence free. The advantage of this augmented formulation is that its associated boundary value problem is an elliptic system in the Agmon-Douglis-Nirenberg sense (but with unusual boundary conditions). The drawback is, as for the variational formulation from [25], that the variational space is *a priori* not a closed subspace of $H^2(\Omega)^3$ (except if the domain is smooth or under some very restricted geometrical conditions, see below). Nevertheless in the case of a domain with point singularities or

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with corners and edges, we show that the singularities of this variational space are the ones of the Maxwell system with the perfect conductor boundary conditions. According to the results from [6], they are then fully described with the help of the singularities of the Laplace equation with Dirichlet or Neumann boundary conditions.

In the case of smooth domains, even if the boundary conditions of our augmented problem are unusual, they enter in the framework of elliptic systems and therefore, owing to the surjectivity of a trace operator and an appropriate Green formula, we can deduce that the variational solution is smooth if the datum is smooth. Conversely, for non-smooth domains, regularity results developed for instance in [14, 11, 7, 16] cannot be directly applied to our system. Hence we here describe the edge and corner singularities of the augmented problem by adapting the cascade method from [6] (reduction to a lower triangular system). As in that reference, for the two-dimensional and three-dimensional corner singularities, we find three types of singularities. Type 1 are exactly those of the Maxwell system with perfect conductor boundary conditions, while those of type 2 or 3 are fully different from the ones of [6]. The two-dimensional corner singularities are also characterized with the help of the singularities of the Laplace equation with Dirichlet boundary conditions, while in the three-dimensional case, new sets of singular exponents appear which are related to the singularities of the Stokes system with Dirichlet boundary conditions. The three-dimensional edge singularities are obtained with the help of two-dimensional singularities of the biharmonic operator with Dirichlet boundary conditions, which are also well-known [24, 11] and of the two-dimensional corner singularities.

The characterization of the edge singularities allows us to show that in polyhedral domains the variational solution of the augmented problem is not in $H^3(\Omega)^3$ in general. But since our problem is not easily localizable, we are not able to show that this variational solution admits a decomposition into a regular part and an explicit singular one. We believe that such a result holds but it requires more investigations. Conversely, in the case of domains with point singularities, we can apply global regularity results in weighted Sobolev spaces described in section 8.2 of [16] for instance. Such a result combined with our characterization of our variational space allows to obtain a decomposition of the weak solution into a regular part and an explicit singular one. Unfortunately similar global regularity results in weighted Sobolev spaces are not available for fourth order operators in polyhedral domains. This is again a question that merits to be investigated in the future.

The paper is organized as follows: In section 2 we introduce some notations and some function spaces. The augmented variational formulation is introduced in section 3, where its well-posedness and its equivalence with the original problem are analyzed. Some regularity results of the variational space are also presented. The next section 4 is devoted to regularity results in the case of a domain with a smooth boundary. In sections 5 and 6, we describe the edge and corner singularities. Finally section 7 concerns regularity results in domains with point singularities.

2 Some notations and function spaces

In the whole paper Ω will be a bounded and simply connected domain of $\mathbb{R}^3$ with a connected and Lipschitz boundary. Three cases will retain our attention:
Case 1. the domain has a smooth boundary;
Case 2. the domain has point singularities, i.e., the boundary is smooth, except at a finite number of points $c \in \mathcal{C}$, for which Ω coincides with a three-dimensional cone Γ_c in a sufficiently small neighborhood of c. Such points will be called the corners of Ω;
Case 3. the domain has corner points and edges, namely the boundary is piecewise plane (i.e., its

boundary is a finite union of polygons). In such a situation, the boundary of Ω is smooth, except at the corner points and along the edges. Hence we will denote by $\mathcal{C}$ the (finite) set of corners of Ω, and by $\mathcal{E}$ the (finite) set of edges. This means that in a neighborhood of a corner c, Ω behaves like a three-dimensional cone Γ_c, while near an interior point of an edge e, Ω behaves like a dihedral cone $C_e \times \mathbb{R}$. In this case we also say that Ω is a polyhedral domain.

For shortness, in the case 1, we set $\mathcal{C} = \mathcal{E} = \emptyset$, while in the case 2, we set $\mathcal{E} = \emptyset$.

For a subset $\mathcal{O}$ of Ω or of the unit sphere and a real number s, $H^s(\mathcal{O})$ is the usual Sobolev space defined in $\mathcal{O}$ and for shortness we denote $\mathbf{H}^s(\mathcal{O}) = H^s(\mathcal{O})^3$ and $\mathbf{L}^2(\mathcal{O}) = L^2(\mathcal{O})^3$. The norm (resp. semi-norm) of $H^s(\mathcal{O})$ (or $\mathbf{H}^s(\mathcal{O})$) will be denoted by $\|\cdot\|_{s,\mathcal{O}}$ (resp. $|\cdot|_{s,\mathcal{O}}$); for $s = 0$, we drop the index 0 and for $\mathcal{O} = \Omega$, we also drop the index Ω. As usual $H_0^1(\Omega)$ is the subspace of $H^1(\Omega)$ with a zero trace on the boundary. We further recall that

$$\begin{aligned}
\mathbf{H}(\mathrm{div}, \Omega) &= \{\mathbf{u} \in \mathbf{L}^2(\Omega) : \mathrm{div}\ \mathbf{u} \in L^2(\Omega)\}, \\
\mathbf{H}(\mathrm{curl}, \Omega) &= \{\mathbf{u} \in \mathbf{L}^2(\Omega) : \mathrm{curl}\ \mathbf{u} \in \mathbf{L}^2(\Omega)\}, \\
\mathbf{H}_0(\mathrm{curl}, \Omega) &= \{\mathbf{u} \in \mathbf{H}(\mathrm{curl}, \Omega) : \mathbf{u} \times \mathbf{n} = 0 \text{ on } \partial\Omega\}, \\
X_N(\Omega) &= \mathbf{H}_0(\mathrm{curl}, \Omega) \cap \mathbf{H}(\mathrm{div}, \Omega),
\end{aligned}$$

are Hilbert spaces with their natural norm. For further purposes, we set

$$\mathbf{H}_0^2(\mathrm{curl}, \Omega) = \{\mathbf{u} \in \mathbf{H}_0(\mathrm{curl}, \Omega) : \mathrm{curl}\ \mathbf{u} \in \mathbf{H}_0(\mathrm{curl}, \Omega)\},$$

which is again a Hilbert space with the norm

$$\|\mathbf{u}\|_{2,\mathrm{curl},\Omega}^2 := \|\mathbf{u}\|^2 + \|\operatorname{curl} \mathbf{u}\|^2 + \|\operatorname{curl}\operatorname{curl} \mathbf{u}\|^2.$$

Let us denote by $-\Delta_{\mathrm{Dir}}$ the Laplace operator with Dirichlet boundary conditions, defined by

$$D(-\Delta_{\mathrm{Dir}}) := \{u \in H_0^1(\Omega) : \Delta u \in L^2(\Omega)\},$$

and

$$-\Delta_{\mathrm{Dir}} u = -\Delta u, \forall u \in D(-\Delta_{\mathrm{Dir}}).$$

It is well-known that this operator is a positive self adjoint operator, and that it is also an isomorphism from $D((-\Delta_{\mathrm{Dir}})^s)$ to $D((-\Delta_{\mathrm{Dir}})^{s-1})$, for all $s \in \mathbb{R}$. In particular for $s = \frac{1}{2}$, as $D((-\Delta_{\mathrm{Dir}})^{\frac{1}{2}}) = H_0^1(\Omega)$ and $D((-\Delta_{\mathrm{Dir}})^{-\frac{1}{2}}) = H^{-1}(\Omega)$, for any $g \in H^{-1}(\Omega)$, $w_g = (-\Delta_{\mathrm{Dir}})^{-1} g$ belongs to $H_0^1(\Omega)$ and is the unique solution of

$$\int_\Omega \nabla w_g \cdot \nabla \bar{y} = \langle g, y \rangle, \forall y \in H_0^1(\Omega).$$

For further uses, let us recall the following Green formulas

$$\int_\Omega (\nabla u \cdot \mathbf{v} + u \,\mathrm{div}\ \mathbf{v}) = 0, \forall u \in H_0^1(\Omega), \mathbf{v} \in H(\mathrm{div}, \Omega), \tag{2.1}$$

$$\int_\Omega (\mathrm{curl}\ \mathbf{u} \cdot \mathbf{v} - \mathbf{u} \cdot \mathrm{curl}\ \mathbf{v}) = 0, \forall \mathbf{u} \in H_0(\mathrm{curl}, \Omega), \mathbf{v} \in H(\mathrm{curl}, \Omega), \tag{2.2}$$

see formula (I.2.17) of [9] for the first one and Remark 3.28 in [20] for the second one.

By Leibniz's rule, one can show that for any (smooth enough) vector fields $\mathbf{a}$ and scalar field q, we have

$$\operatorname{curl}(q\mathbf{a}) = q \operatorname{curl} \mathbf{a} + \nabla q \times \mathbf{a}, \tag{2.3}$$

$$\nabla(\mathbf{a}\cdot\mathbf{x}) = (r\partial_r + 1)\mathbf{a} + \mathbf{x} \times \operatorname{curl} \mathbf{a}, \tag{2.4}$$

$$\operatorname{curl}(\mathbf{a}\times\mathbf{x}) = (r\partial_r + 2)\mathbf{a} - \mathbf{x} \operatorname{div} \mathbf{a}, \tag{2.5}$$

$$\operatorname{curl}(q\mathbf{x}) = \nabla q \times \mathbf{x}, \tag{2.6}$$

$$\operatorname{curl}\operatorname{curl}(q\mathbf{x}) = (r\partial_r + 2)\nabla q - \mathbf{x}\Delta q. \tag{2.7}$$

Here and below, points of $\mathbb{R}^3$ will be denoted by $\mathbf{x}$ (in the Cartesian coordinate system centred at the origin) and $r = |\mathbf{x}|$ the distance from $\mathbf{x}$ to the origin, which is also the radial variable. Clearly, ∂_r will then mean the partial derivative with respect to r.

Note that the identity (2.7) is a direct consequence of (2.5) and (2.6).

Finally in the whole paper, the notation $A \lesssim B$ is used for the estimate $A \leq C\ B$, where C is a generic constant which does not depend on A and B. The notation $A \sim B$ means that both $A \lesssim B$ and $B \lesssim A$ hold.

3 Augmented variational formulation

The variational formulation of (1.1) from [25] consists in looking for $\mathbf{u} \in V_0 := \{\mathbf{u} \in \mathbf{H}_0^2(\operatorname{curl}, \Omega) : \operatorname{div} \mathbf{u} = 0\}$ solution of

$$\int_\Omega \operatorname{curl}^2 \mathbf{u} \cdot \operatorname{curl}^2 \bar{\mathbf{v}} = \int_\Omega \mathbf{f} \cdot \bar{\mathbf{v}}, \forall \mathbf{v} \in V_0. \tag{3.1}$$

This problem is well posed since the left-hand side of (3.1) is a coercive form on V_0 (see Theorem 1 of [25]).

In order to relax the divergence free constraint in V_0, we propose the following alternative formulation. Introduce the space

$$V := \{\mathbf{u} \in \mathbf{H}_0^2(\operatorname{curl}, \Omega) : \operatorname{div} \mathbf{u} \in H_0^1(\Omega)\}, \tag{3.2}$$

which is a Hilbert space with the norm

$$\|\mathbf{u}\|_V^2 = \|\mathbf{u}\|_{2,\operatorname{curl},\Omega}^2 + |\operatorname{div} \mathbf{u}|_1^2.$$

The augmented variational formulation of (1.1) consists in looking for $\mathbf{u} \in V$ solution of

$$a(\mathbf{u}, \mathbf{v}) = \int_\Omega \mathbf{f} \cdot \bar{\mathbf{v}}, \forall \mathbf{v} \in V, \tag{3.3}$$

where

$$a(\mathbf{u}, \mathbf{v}) = \int_\Omega (\operatorname{curl}\operatorname{curl} \mathbf{u} - \nabla \operatorname{div} \mathbf{u}) \cdot (\operatorname{curl}\operatorname{curl} \bar{\mathbf{v}} - \nabla \operatorname{div} \bar{\mathbf{v}}), \forall \mathbf{u}, \mathbf{v} \in V. \tag{3.4}$$

Let us notice that a admits the equivalent expression

$$a(\mathbf{u}, \mathbf{v}) = \int_\Omega \Delta\mathbf{u} \cdot \Delta\bar{\mathbf{v}},$$

since for $\mathbf{u} \in V$, we have curl curl $\mathbf{u} - \nabla \operatorname{div} \mathbf{u} = -\Delta \mathbf{u}$ in $\mathcal{D}'(\Omega)^3$ (and hence $\Delta \mathbf{u}$ belongs to $\mathbf{L}^2(\Omega)$ as the sum of two elements of $\mathbf{L}^2(\Omega)$).

Before any further considerations, let us show that the space V and the variational problem (3.4) are invariant by translation and rotation.

Theorem 3.1 *Let R be an orthogonal matrix with determinant 1 and let $c \in \mathbb{R}^3$ be fixed. Let F be the transformation from $\mathbb{R}^3$ into itself defined by $F(\hat{\mathbf{x}}) = R\hat{\mathbf{x}} + \mathbf{c}$, which corresponds to a translation of $\mathbf{c}$ and a rotation of R. Let $\hat{\Omega} = F^{-1}(\Omega)$. Then for any $\mathbf{u} \in V$, the vector field $\hat{\mathbf{u}}$ defined by the covariant transformation*

$$\hat{\mathbf{u}} = R^{-1}(\mathbf{u} \circ F) \tag{3.5}$$

belongs to

$$\hat{V} = \{\mathbf{v} \in \mathbf{H}^2_0(\mathrm{curl}, \hat{\Omega}) : \operatorname{div} \mathbf{v} \in H^1_0(\hat{\Omega})\}.$$

Furthermore if $\mathbf{u} \in V$ is a solution of (3.3), then $\hat{\mathbf{u}}$ given by (3.5) is solution of

$$\hat{a}(\hat{\mathbf{u}}, \hat{\mathbf{v}}) = \int_{\hat{\Omega}} R^{-1}(\mathbf{f} \circ F) \cdot \overline{\hat{\mathbf{v}}}, \forall \hat{\mathbf{v}} \in \hat{V},$$

where $\hat{a}$ is defined as a with $\hat{\Omega}$ instead of Ω.

Proof. It turns out that for a rotation matrix R the covariant and contravariant transformations coincide, hence by the chain rule, see for instance [4, p. 59-63], we have

$$\operatorname{div} \mathbf{u} = \widehat{\operatorname{div}}\, \hat{\mathbf{u}}, \ \operatorname{curl} \mathbf{u} = R\widehat{\operatorname{curl}}\, \hat{\mathbf{u}},$$
$$\nabla \operatorname{div} \mathbf{u} = R\hat{\nabla}\widehat{\operatorname{div}}\, \hat{\mathbf{u}}, \ \operatorname{curl} \operatorname{curl} \mathbf{u} = R\widehat{\operatorname{curl}}\, \widehat{\operatorname{curl}}\, \hat{\mathbf{u}}.$$

The two assertions follow from these identities and the fact that $\mathbf{u} \times \mathbf{n} = \mathbf{0}$ on $\partial\Omega$ if and only if $\hat{\mathbf{u}} \times \hat{\mathbf{n}} = \mathbf{0}$ on $\partial\hat{\Omega}$. ■

We now show that problem (3.3) is well posed and that it is equivalent to (3.1) is $\mathbf{f}$ is divergence free.

Theorem 3.2 *For any $\mathbf{f} \in \mathbf{L}^2(\Omega)$, there exists a unique solution $\mathbf{u} \in V$ of (3.3). The divergence of this solution is the unique solution in $H^1_0(\Omega)$ of*

$$\int_\Omega \nabla \operatorname{div} \mathbf{u} \cdot \nabla \bar{g} = \int_\Omega (-\Delta_{\mathrm{Dir}})^{-1} \operatorname{div} \mathbf{f} \bar{g}, \forall g \in H^1_0(\Omega). \tag{3.6}$$

Hence in particular, it satisfies

$$-\Delta \operatorname{div} \mathbf{u} = (-\Delta_{\mathrm{Dir}})^{-1} \operatorname{div} \mathbf{f} \in H^1_0(\Omega). \tag{3.7}$$

Consequently if $\mathbf{f}$ is divergence free, then $\mathbf{u}$ is divergence free and is also the unique solution of (3.1).

Proof. From (3.4), for any $\mathbf{u} \in V$, we have

$$\begin{aligned} a(\mathbf{u}, \mathbf{u}) &= \int_\Omega |\operatorname{curl} \operatorname{curl} \mathbf{u} - \nabla \operatorname{div} \mathbf{u}|^2 \\ &= \int_\Omega |\operatorname{curl} \operatorname{curl} \mathbf{u}|^2 + |\nabla \operatorname{div} \mathbf{u}|^2 - 2\Re \int_\Omega \operatorname{curl} \operatorname{curl} \mathbf{u} \cdot \nabla \operatorname{div} \bar{\mathbf{u}}. \end{aligned}$$

As curl curl $\mathbf{u}$ belongs to $H(\text{div}, \Omega)$ and div $\mathbf{u}$ is in $H_0^1(\Omega)$, by Green's formula (2.1), we find that

$$\int_\Omega \text{curl curl } \mathbf{u} \cdot \nabla \,\text{div } \bar{\mathbf{u}} = 0.$$

This shows the identity

$$a(\mathbf{u}, \mathbf{u}) = \int_\Omega (|\text{ curl curl } \mathbf{u}|^2 + |\nabla \,\text{div } \mathbf{u}|^2), \forall \mathbf{u} \in V.$$

Hence by Poincaré's inequality in $H_0^1(\Omega)$, we find that

$$a(\mathbf{u}, \mathbf{u}) \gtrsim \int_\Omega (|\text{ curl curl } \mathbf{u}|^2 + |\nabla \,\text{div } \mathbf{u}|^2 + |\text{ div } \mathbf{u}|^2), \forall \mathbf{u} \in V. \tag{3.8}$$

From Friedrichs' inequality in $X_N(\Omega)$ (see Corollary 3.19 in [1]), we get

$$\int_\Omega |\text{ curl curl } \mathbf{u}|^2 \gtrsim \int_\Omega (|\text{ curl curl } \mathbf{u}|^2 + |\text{ curl } \mathbf{u}|^2),$$

because curl $\mathbf{u}$ belongs to $X_N(\Omega)$. As $\mathbf{u}$ itself belongs to $X_N(\Omega)$, applying again Friedrichs' inequality in $X_N(\Omega)$, we find that

$$\int_\Omega (|\text{ curl } \mathbf{u}|^2 + |\text{ div } \mathbf{u}|^2) \gtrsim \int_\Omega (|\text{ curl } \mathbf{u}|^2 + |\text{ div } \mathbf{u}|^2 + |\mathbf{u}|^2).$$

The two previous estimates in the estimate (3.8) lead to the coerciveness of a on V, namely

$$a(\mathbf{u}, \mathbf{u}) \gtrsim \|\mathbf{u}\|_V^2.$$

By Lax-Milgram lemma, we deduce the well-posedness of (3.3).

For the second assertion, it suffices to take in (3.3) a test function $\mathbf{v} = \nabla w_g$, with an arbitrary $g \in H_0^1(\Omega)$, where $w_g = (-\Delta_{\text{Dir}})^{-1} g$. Note that w_g also satisfies

$$-\Delta w_g = g,$$

in the distributional sense and hence div $\mathbf{v} = -g \in H_0^1(\Omega)$. Then we get

$$\int_\Omega (\text{curl curl } \mathbf{u} - \nabla \,\text{div } \mathbf{u}) \cdot \nabla \bar{g} = \int_\Omega \mathbf{f} \cdot \nabla \bar{w}_g.$$

As curl curl $\mathbf{u}$ belongs to $H(\text{div}, \Omega)$ and is divergence free, by Green's formula (2.1) in the left-hand side, we find that

$$\int_\Omega \nabla \,\text{div } \mathbf{u} \cdot \nabla \bar{g} = \langle \text{div } \mathbf{f}; w_g \rangle_{H^{-1}(\Omega) - H_0^1(\Omega)},$$

recalling that $\mathbf{f}$ is assumed to be in $\mathbf{L}^2(\Omega)$. Since $L^2(\Omega)$ is the pivot space between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$, we have

$$\langle \text{div } \mathbf{f}; w_g \rangle_{H^{-1}(\Omega) - H_0^1(\Omega)} = \langle \text{div } \mathbf{f}; (-\Delta_{\text{Dir}})^{-1} g \rangle_{H^{-1}(\Omega) - H_0^1(\Omega)} = \int_\Omega (-\Delta_{\text{Dir}})^{-1} (\text{div } \mathbf{f}) \, \bar{g}.$$

The two previous identities lead to (3.6). As this identity implies that

$$-\Delta \operatorname{div} u = (-\Delta_{\mathrm{Dir}})^{-1} \operatorname{div} \mathbf{f},$$

in the distributional sense, we directly obtain (3.7) (since div $\mathbf{f} \in H^{-1}(\Omega)$).

Finally if $\mathbf{f}$ is divergence free, the right-hand side of (3.6) is zero, consequenctly div $\mathbf{u} = 0$, and therefore $\mathbf{u} \in V_0$. Reducing (3.3) to test functions in V_0 leads to the fact that $\mathbf{u}$ is also solution of (3.1). ■

Corollary 3.3 *For any* $\mathbf{f} \in \mathbf{L}^2(\Omega)$*, the unique solution* $\mathbf{u} \in V$ *of* (3.3) *satisfies the boundary value problem*

$$\left\{ \begin{array}{ll} \Delta^2 \mathbf{u} = \mathbf{f} & \text{in } \Omega, \\ \operatorname{div} \mathbf{u} = \Delta \operatorname{div} \mathbf{u} = 0 & \text{on } \partial\Omega, \\ \mathbf{u} \times \mathbf{n} = (\operatorname{curl} \mathbf{u}) \times \mathbf{n} = 0 & \text{on } \partial\Omega, \end{array} \right. \tag{3.9}$$

Proof. By taking test functions $\mathbf{v} \in \mathcal{D}(\Omega)^3$ in (3.3), we directly find that

$$\Delta^2 \mathbf{u} = \mathbf{f} \text{ in } \mathcal{D}'(\Omega)^3.$$

The boundary conditions

$$\begin{aligned} \operatorname{div} \mathbf{u} = 0 \quad & \text{on } \partial\Omega, \\ \mathbf{u} \times \mathbf{n} = (\operatorname{curl} \mathbf{u}) \times \mathbf{n} = 0 \quad & \text{on } \partial\Omega, \end{aligned}$$

come from the fact that $\mathbf{u}$ is in V. Finally the boundary condition

$$\Delta \operatorname{div} \mathbf{u} = 0 \text{ on } \partial\Omega.$$

follows from the property (3.7). ■

Before going on, let us notice that the boundary conditions

$$\mathbf{u} \times \mathbf{n} = (\operatorname{curl} \mathbf{u}) \times \mathbf{n} = 0 \text{ on } \partial\Omega,$$

in (3.9) implies

$$\operatorname{curl} \mathbf{u} = 0 \text{ on } \partial\Omega. \tag{3.10}$$

Indeed, the first condition $\mathbf{u} \times \mathbf{n} = 0$ implying

$$(\operatorname{curl} \mathbf{u}) \cdot \mathbf{n} = 0 \text{ on } \partial\Omega,$$

combining this boundary condition with the second boundary condition $(\operatorname{curl} \mathbf{u}) \times \mathbf{n} = 0$, we obtain that (3.10) holds.

It is easy to check that the system (3.9) is an elliptic system in the Agmon-Douglis-Nirenberg sense, but its variational formulation is, *a priori*, not based on a closed subspace of $\mathbf{H}^2(\Omega)$. Nevertheless, let us show that its variational space V (and hence V_0) is indeed a subspace of $\mathbf{H}^2(\Omega)$ if the boundary of Ω is smooth and admits a decomposition into regular fields in $\mathbf{H}^2(\Omega)$ and singular ones in the two other cases.

Lemma 3.4 *If the boundary of* Ω *is smooth, then* V *is a closed subspace of* $\mathbf{H}^2(\Omega)$*. In the case 2 or 3, with the notations of sections 5 and 6 below (see also [6, §4]), we assume that*

$$\left\{ \begin{array}{l} \forall c \in \mathcal{C},\ \frac{3}{2} \notin \Lambda_{Dir}(\Gamma_c) \text{ and } \frac{1}{2} \notin \Lambda_{Neu}(\Gamma_c), \\ \forall e \in \mathcal{E},\ \omega_e \notin \{\frac{\pi}{2}, \frac{3\pi}{2}\}. \end{array} \right. \tag{3.11}$$

Then any $\mathbf{u} \in V$ *admits the decomposition*

$$\mathbf{u} = \mathbf{u}_{\mathrm{reg}} + \mathbf{u}_{\mathrm{sing}}, \tag{3.12}$$

where $\mathbf{u}_{\mathrm{reg}} \in \mathbf{H}^2(\Omega)$ *while* $\mathbf{u}_{\mathrm{sing}} \in V \setminus \mathbf{H}^2(\Omega)$ *is a sum of corner and edge singularities.*

Proof. Fix $\mathbf{u} \in V$ and $\mathbf{v} \in X_N(\Omega)$. As div $\mathbf{u}$ belongs to $H^1_0(\Omega)$ and $\mathbf{v}$ is in $H(\mathrm{div}, \Omega)$, Green's formula (2.1) yields

$$\int_\Omega \mathrm{div}\ \mathbf{u}\, \mathrm{div}\ \bar{\mathbf{v}} = -\int_\Omega \nabla\, \mathrm{div}\ \mathbf{u} \cdot \bar{\mathbf{v}}.$$

Similarly as curl $\mathbf{u}$ is in $H_0(\mathrm{curl}, \Omega)$ and $\mathbf{v}$ is in $H(\mathrm{curl}, \Omega)$, Green's formula (2.2) leads to

$$\int_\Omega \mathrm{curl}\ \mathbf{u} \cdot \mathrm{curl}\ \bar{\mathbf{v}} = \int_\Omega \mathrm{curl}^2\, \mathbf{u} \cdot \bar{\mathbf{v}}.$$

Taking the sum of these two identities, we deduce that $\mathbf{u} \in V$ satisfies

$$\int_\Omega (\mathrm{curl}\ \mathbf{u} \cdot \mathrm{curl}\ \bar{\mathbf{v}} + \mathrm{div}\ \mathbf{u}\, \mathrm{div}\ \bar{\mathbf{v}}) = \int_\Omega (\mathrm{curl}^2\, \mathbf{u} - \nabla\, \mathrm{div}\ \mathbf{u}) \cdot \bar{\mathbf{v}}, \forall \mathbf{v} \in X_N(\Omega). \tag{3.13}$$

Therefore we can see $\mathbf{u} \in V$ as the unique solution in $X_N(\Omega)$ of (3.13). As $\mathrm{curl}^2\, \mathbf{u} - \nabla\, \mathrm{div}\ \mathbf{u}$ belongs to $\mathbf{L}^2(\Omega)$, we can apply Theorem 4.7 of [6] to obtain a decomposition of $\mathbf{u}$ into a regular field $\mathbf{u}_{\mathrm{reg}} \in \mathbf{H}^2(\Omega)$ and a singular one $\mathbf{u}_{\mathrm{sing}}$. Obviously in the case 1, this singular part does not exist. ■

Remark 3.5 *If the assumption* (3.11) *does not hold, then a similar decomposition holds but with* $\mathbf{u}_{\mathrm{reg}} \in \mathbf{H}^{2-\varepsilon}(\Omega)$*, for* $\varepsilon > 0$ *small enough.*

A direct consequence of the previous result is that in the case 2 or 3, V is embedded into $\mathbf{H}^2(\Omega)$ if and only if the singular part is zero, and therefore if and only if the set of corner (resp. edge) singular exponents $\Lambda_{N,1}(\Gamma_c)$ (resp. $\Lambda_{N,1}(W_e)$) of problem (3.13) described in [6, §4] is empty. But before stating such a result, let us show that if no edge singular exponent appears, then the contribution of $\Lambda_{Neu}(\Gamma_c)$ in the corner singular part disappears.

Lemma 3.6 *In the case 2 or 3, with the notations of sections 5 and 6 below, assume that*

$$\left\{ \begin{array}{l} \forall c \in \mathcal{C},\ \frac{3}{2} \notin \Lambda_{Dir}(\Gamma_c) \text{ and } \frac{1}{2} \notin \Lambda_{Neu}(\Gamma_c), \\ \forall e \in \mathcal{E},\ \omega_e < \frac{\pi}{2}. \end{array} \right. \tag{3.14}$$

Then $\mathbf{u} \in V$ *admits the decomposition* (3.12) *where* $\mathbf{u}_{\mathrm{reg}} \in \mathbf{H}^2(\Omega)$*,*

$$\mathbf{u}_{\mathrm{sing}} = \sum_{c \in \mathcal{C}} \eta_c \sum_{\lambda \in (-\frac{3}{2}, \frac{1}{2}) : \lambda + 1 \in \Lambda_{\mathrm{Dir}}(\Gamma_c)} \sum_p d_{c,\lambda,p} \nabla u^{\lambda+1,p}_{\mathrm{Dir}}, \tag{3.15}$$

η_c *being a radial cut-off function equal to 1 near* c *and zero in a neighbourhood of the other corner points of* Ω*, and for* $\lambda \in (-\frac{3}{2}, \frac{1}{2})$ *such that* $\lambda + 1 \in \Lambda_{\mathrm{Dir}}(\Gamma_c)$*,* $d_{c,\lambda,p} \in \mathbb{C}$ *and*

$$\|\mathbf{u}_{\mathrm{reg}}\|_{2,\Omega} + \sum_{c \in \mathcal{C}} \sum_{\lambda \in (-\frac{3}{2}, \frac{1}{2}) : \lambda + 1 \in \Lambda_{\mathrm{Dir}}(\Gamma_c)} \sum_p |d_{c,\lambda,p}| \lesssim \|\mathbf{u}\|_V. \tag{3.16}$$

Proof. Using Lemmas 4.3 and 4.6 of [6] and since $\omega_e < \frac{\pi}{2}$, for all $e \in \mathcal{E}$, in the decomposition (3.12), only corner singularities contribute to $\mathbf{u}_{\text{sing}}$, namely (again with the notations from section 6)

$$\begin{aligned} \mathbf{u}_{\text{sing}} &= \sum_{c\in\mathcal{C}} \eta_c \sum_{\lambda\in(-\frac{3}{2},\frac{1}{2}):\lambda+1\in\Lambda_{\text{Dir}}(\Gamma_c)} \sum_p d^{(1)}_{c,\lambda,p} \nabla u^{\lambda+1,p}_{\text{Dir}} \\ &+ \sum_{c\in\mathcal{C}} \eta_c \sum_{\lambda\in(-\frac{1}{2},\frac{1}{2})\cap\Lambda_{\text{Neu}}(\Gamma_c)} \sum_p d^{(2)}_{c,\lambda,p} (\nabla u^{\lambda,p}_{\text{Neu}} \times \mathbf{x}_c), \end{aligned} \tag{3.17}$$

where $\mathbf{x}_c$ denotes any point of $\mathbb{R}^3$ corresponding to the Cartesian coordinate system centred at c, for $\lambda \in (-\frac{3}{2}, \frac{1}{2})$ such that $\lambda + 1 \in \Lambda_{\text{Dir}}(\Gamma_c)$, $d^{(1)}_{c,\lambda,p} \in \mathbb{C}$ and for $\lambda \in (-\frac{1}{2}, \frac{1}{2}) \cap \Lambda_{\text{Neu}}(\Gamma_c)$, $d^{(2)}_{c,\lambda,p} \in \mathbb{C}$.

But the essential boundary condition (3.10) implies that

$$\operatorname{curl} \mathbf{u}_{\text{sing}} \in \mathbf{H}^{\frac{1}{2}}(\partial\Omega). \tag{3.18}$$

From the previous splitting of $\mathbf{u}_{\text{sing}}$, we have

$$\begin{aligned} \operatorname{curl} \mathbf{u}_{\text{sing}} &= \sum_{c\in\mathcal{C}} \sum_{\lambda\in(-\frac{3}{2},\frac{1}{2}):\lambda+1\in\Lambda_{\text{Dir}}(\Gamma_c)} \sum_p d^{(1)}_{c,\lambda,p} \operatorname{curl} \left(\eta_c \nabla u^{\lambda+1,p}_{\text{Dir}}\right) \\ &+ \sum_{c\in\mathcal{C}} \sum_{\lambda\in(-\frac{1}{2},\frac{1}{2}):\lambda\in\Lambda_{\text{Neu}}(\Gamma_c)} \sum_p d^{(2)}_{c,\lambda,p} \operatorname{curl} \left(\eta_c \nabla u^{\lambda,p}_{\text{Neu}} \times \mathbf{x}_c\right). \end{aligned}$$

By (2.3) and (2.5), we have

$$\begin{aligned} \operatorname{curl} \left(\eta_c \nabla u^{\lambda+1,p}_{\text{Dir}}\right) &= \nabla\eta_c \times \nabla u^{\lambda+1,p}_{\text{Dir}}, \\ \operatorname{curl} \left(\eta_c \nabla u^{\lambda,p}_{\text{Neu}} \times \mathbf{x}_c\right) &= \eta_c(\lambda+1)\nabla u^{\lambda,p}_{\text{Neu}} + \nabla\eta_c \times (\nabla u^{\lambda,p}_{\text{Neu}} \times \mathbf{x}_c). \end{aligned}$$

As $u^{\lambda+1,p}_{\text{Dir}}$ is in H^2 far from c, we deduce that $\nabla\eta_c \times \nabla u^{\lambda+1,p}_{\text{Dir}}$ belongs to $\mathbf{H}^{\frac{1}{2}}(\partial\Omega)$. For the same reason $\nabla\eta_c \times (\nabla u^{\lambda,p}_{\text{Neu}} \times \mathbf{x}_c)$ belongs to $\mathbf{H}^{\frac{1}{2}}(\partial\Omega)$. Hence the condition (3.18) implies that

$$\sum_{c\in\mathcal{C}} \sum_{\lambda\in(-\frac{1}{2},\frac{1}{2}):\lambda\in\Lambda_{\text{Neu}}(\Gamma_c)} \eta_c d^{(2)}_{c,\lambda,p} (\lambda+1)\nabla u^{\lambda,p}_{\text{Neu}} \in \mathbf{H}^{\frac{1}{2}}(\partial\Omega).$$

Since for each $\lambda \in (-\frac{1}{2}, \frac{1}{2}) \cap \Lambda_{\text{Neu}}(\Gamma_c)$, $\nabla u^{\lambda,p}_{\text{Neu}}$ belongs to $\mathbf{H}^s(\partial\Omega)$ if and only if $s < \lambda$, we deduce that $d^{(2)}_{c,\lambda,p} = 0$ for all $\lambda \in (-\frac{1}{2}, \frac{1}{2}) \cap \Lambda_{\text{Neu}}(\Gamma_c)$, all p, and all c. This means that (3.17) reduces to (3.15) with $d_{c,\lambda,p} = d^{(1)}_{c,\lambda,p}$.

The estimate (3.16) is relatively standard and is based on the open mapping theorem. For completeness let us give its proof. Introduce the space

$$W = \{(u_{\text{reg}}, d_{c,\lambda,p}) \in \mathbf{H}^2(\Omega) \times \mathbb{C}^N : u_{\text{reg}} + \sum_{c\in\mathcal{C}} \eta_c \sum_{\lambda\in(-\frac{3}{2},\frac{1}{2}):\lambda+1\in\Lambda_{\text{Dir}}(\Gamma_c)} \sum_p d_{c,\lambda,p} \nabla u^{\lambda+1,p}_{\text{Dir}} \in V\},$$

where N is the cardinal of the set of triples (c, λ, p) satisfying the above constraints. This space is a Banach space with the norm

$$\|(u_{\text{reg}}, d_{c,\lambda,p})\|_W = \|u_{\text{reg}}\|_{2,\Omega} + \sum_{c\in\mathcal{C}} \sum_{\lambda\in(-\frac{3}{2},\frac{1}{2}):\lambda+1\in\Lambda_{\text{Dir}}(\Gamma_c)} \sum_p |d_{c,\lambda,p}|,$$

because

$$\|u_{\rm reg} + \sum_{c\in\mathcal{C}} \eta_c \sum_{\lambda\in(-\frac{3}{2},\frac{1}{2}):\lambda+1\in\Lambda_{\rm Dir}(\Gamma_c)} \sum_p d_{c,\lambda,p} \nabla u_{\rm Dir}^{\lambda+1,p}\|_V \lesssim \|(u_{\rm reg}, d_{c,\lambda,p})\|_W.$$

Now introduce the continuous mapping

$$T : W \to V : (u_{\rm reg}, d_{c,\lambda,p}) \to u_{\rm reg} + \sum_{c\in\mathcal{C}} \eta_c \sum_{\lambda\in(-\frac{3}{2},\frac{1}{2}):\lambda+1\in\Lambda_{\rm Dir}(\Gamma_c)} \sum_p d_{c,\lambda,p} \nabla u_{\rm Dir}^{\lambda+1,p}.$$

Our first assertion means that T is surjective. Furthermore, T is injective because the functions $\nabla u_{\rm Dir}^{\lambda+1,p}$ are linearly independent and do not belong to $\mathbf{H}^2(\Omega)$. Hence T is bijective and by the open mapping theorem its inverse is also continuous, which proves (3.16). ■

This result directly yields the

Corollary 3.7 *With the notations of sections 5 and 6 below, assume that*

$$\left\{ \begin{array}{l} \forall c \in \mathcal{C},\ (-\frac{1}{2}, \frac{3}{2}] \cap \Lambda_{Dir}(\Gamma_c) = \emptyset \text{ and } \frac{1}{2} \notin \Lambda_{Neu}(\Gamma_c), \\ \forall e \in \mathcal{E},\ \omega_e < \frac{\pi}{2}. \end{array} \right. \tag{3.19}$$

Then V is continuously embedded into $\mathbf{H}^2(\Omega)$.

From point (b) of subsection 4.4.2 in [6], the assumption (3.19) holds if Ω is a convex polyheral domain and $\omega_e < \frac{\pi}{2}$ for all edges e (recalling that the convexity of Ω implies that $\lambda > \frac{1}{2}$ for all $\lambda \in \Lambda_{Neu}(\Gamma_c)$). As a consequence, the convexity of Ω is far from being sufficient to guarantee the $\mathbf{H}^2$ regularity of the solution of (3.3) (or (3.1)). Nevertheless, we always get the inclusion $V \subset \mathbf{H}^{\frac{1}{2}+\varepsilon}(\Omega)$, with $\varepsilon > 0$ depending on the geometry of Ω.

4 Regularity results in the smooth case

Even if system (3.9) is an elliptic system in the Agmon-Douglis-Nirenberg sense, it is a non standard one and, to our best knowledge, the regularity $\mathbf{H}^4(\Omega)$ of its weak solution with $\mathbf{f} \in \mathbf{L}^2(\Omega)$ is not directly available in the literature. Nevertheless, a priori error estimates are available owing to the general theory of elliptic systems, leading to Fredholm property of the associated operator (see [16, Thm 4.2.4]). Hence we are mainly reduced to study the cokernel of the associated operator. This will be made via the surjectivity of the trace mapping from Lemma 4.1 below and the use of an adapted Green formula.

In order to formulate the Fredholm property, we introduce the following operators: for any $\ell \in \mathbb{N}$ and $\mathbf{u} \in \mathbf{H}^{4+\ell}(\Omega)$, if γ is the standard trace operator, let us set

$$B\mathbf{u} = ((\gamma\mathbf{u}) \times \mathbf{n}, \gamma(\text{curl } \mathbf{u}) \times \mathbf{n}, \gamma \,\text{div } \mathbf{u}, \gamma\Delta \,\text{div } \mathbf{u})^\top, \tag{4.1}$$

which clearly belongs to $R^\ell := \mathbf{H}_T^{\frac{7}{2}+\ell}(\partial\Omega) \times \mathbf{H}_T^{\frac{5}{2}+\ell}(\partial\Omega) \times H^{\frac{5}{2}+\ell}(\partial\Omega) \times H^{\frac{1}{2}+\ell}(\partial\Omega)$, where

$$\mathbf{L}_T^2(\partial\Omega) = \{\mathbf{v} \in \mathbf{L}^2(\partial\Omega) : \mathbf{v}\cdot\mathbf{n} = \mathbf{0}\},$$

and for any $s > 0$,

$$\mathbf{H}_T^s(\partial\Omega) = \mathbf{H}^s(\partial\Omega) \cap \mathbf{L}_T^2(\partial\Omega).$$

Similarly for any $\mathbf{u} \in \mathbf{H}^4(\Omega)$ we denote by

$$T\mathbf{u} = (\mathbf{n} \times (\gamma \operatorname{curl} \Delta \mathbf{u}) \times \mathbf{n}, \mathbf{n} \times (\gamma \Delta \mathbf{u}) \times \mathbf{n}, \gamma \Delta \mathbf{u} \cdot \mathbf{n}, \gamma \mathbf{u} \cdot \mathbf{n})^\top,$$
$$D\mathbf{u} = (\gamma \mathbf{u}, \gamma \partial_{\mathbf{n}} \mathbf{u}, \gamma \partial_{\mathbf{n}}^2 \mathbf{u}, \gamma \partial_{\mathbf{n}}^3 \mathbf{u})^\top.$$

The system B alone is not a "Dirichlet" system (for each component of $\mathbf{u}$) since $B\mathbf{u} = \mathbf{0}$ on $\partial\Omega$ does not imply $\mathbf{u} \cdot \mathbf{n} = 0$ on $\partial\Omega$, but the full system (B, T) is a "Dirichlet" system as the next Lemma shows. For shortness let us set

$$S^0 := \mathbf{H}_T^{\frac{1}{2}}(\partial\Omega) \times \mathbf{H}_T^{\frac{3}{2}}(\partial\Omega) \times H^{\frac{3}{2}}(\partial\Omega) \times H^{\frac{7}{2}}(\partial\Omega).$$

Lemma 4.1 *The mapping*

$$\mathbf{H}^4(\Omega) \longrightarrow R^0 \times S^0 : \mathbf{u} \to (B\mathbf{u}, T\mathbf{u})^\top \tag{4.2}$$

is surjective.

Proof. We first notice that using a permutation matrix from $\mathbb{C}^{16}$ into itself and for all $\mathbf{x} \in \partial\Omega$, the invertible relation

$$Q_{\mathbf{x}} : \{\mathbf{y} \in \mathbb{C}^3 : \mathbf{y} \cdot \mathbf{n}(\mathbf{x}) = 0\} \times \mathbb{C} \to \mathbb{C}^3 : (\mathbf{y}, z)^\top \mapsto -\mathbf{y} \times \mathbf{n}(\mathbf{x}) + z\mathbf{n}(\mathbf{x}),$$

which yields $Q_{\mathbf{x}}(\mathbf{u} \times \mathbf{n}, \mathbf{u} \cdot \mathbf{n})^\top = \mathbf{u}$, there exists an invertible mapping P from $R^0 \times S^0$ to $\mathbf{R} := \mathbf{H}^{\frac{7}{2}}(\partial\Omega) \times \mathbf{H}^{\frac{5}{2}}(\partial\Omega) \times \mathbf{H}^{\frac{3}{2}}(\partial\Omega) \times \mathbf{H}^{\frac{1}{2}}(\partial\Omega)$ such that

$$P(B\mathbf{u}, T\mathbf{u})^\top = (\gamma \mathbf{u}, \gamma \operatorname{curl} \mathbf{u} \times \mathbf{n}, \gamma \operatorname{div} \mathbf{u}, \gamma \Delta \mathbf{u}, \gamma \operatorname{curl} \Delta \mathbf{u} \times \mathbf{n}, \gamma \operatorname{div} \Delta \mathbf{u})^\top.$$

Hence it suffices to show that the mapping

$$M : \mathbf{H}^4(\Omega) \longrightarrow \mathbf{R} : \mathbf{u} \to P(B\mathbf{u}, T\mathbf{u})^\top$$

is surjective. For that purpose, for all $\mathbf{x} \in \partial\Omega$, if we denote by $(\tau_1(\mathbf{x}), \tau_2(\mathbf{x}))$ two tangential vectors to $\partial\Omega$ at $\mathbf{x}$ such that the triplet $\{\tau_1(\mathbf{x}), \tau_2(\mathbf{x}), \mathbf{n}(\mathbf{x})\}$ forms an orthonormal basis of $\mathbb{R}^3$, then we can notice that

$$\operatorname{curl} \mathbf{u} \times \mathbf{n} = (\mathbf{n} \times \partial_n \mathbf{u}) \times \mathbf{n} + \sum_{i=1}^{2} (\tau_i \times \frac{\partial \mathbf{u}}{\partial \tau_i}) \times \mathbf{n}, \tag{4.3}$$

$$\operatorname{div} \mathbf{u} = \mathbf{n} \cdot \partial_n \mathbf{u} + \sum_{i=1}^{2} (\tau_i \cdot \frac{\partial \mathbf{u}}{\partial \tau_i}). \tag{4.4}$$

This means that the pair $(\operatorname{curl} \mathbf{u} \times \mathbf{n}, \operatorname{div} \mathbf{u})$ is the sum of $\partial_n \mathbf{u}$ and of tangential derivatives of $\mathbf{u}$, namely

$$(\operatorname{curl} \mathbf{u} \times \mathbf{n}, \operatorname{div} \mathbf{u}) = \partial_n \mathbf{u} + H_1 \mathbf{u},$$

where we can identify $((\mathbf{n} \times \partial_n \mathbf{u}) \times \mathbf{n}, \mathbf{n} \cdot \partial_n \mathbf{u})$ with $\partial_n \mathbf{u}$, because $\partial_n \mathbf{u} = (\mathbf{n} \times \partial_n \mathbf{u}) \times \mathbf{n} + (\mathbf{n} \cdot \partial_n \mathbf{u})\mathbf{n}$, and we have set

$$H_1 \mathbf{u} = \sum_{i=1}^{2} (\tau_i \times \frac{\partial \mathbf{u}}{\partial \tau_i}) \times \mathbf{n}, \tau_i \cdot \frac{\partial \mathbf{u}}{\partial \tau_i})$$

which is only made of tangential derivatives of $\mathbf{u}$. This observation implies that

$$M\mathbf{u} = (\mathbf{u}, \partial_n \mathbf{u} + H_1 \mathbf{u}, \Delta \mathbf{u}, \partial_n \Delta \mathbf{u} + H_1 \Delta \mathbf{u}).$$

Since Leibniz's rule yields

$$\partial_n \Delta \mathbf{u} = \Delta \partial_n \mathbf{u} - (\Delta \mathbf{n} \cdot \mathbf{n}) \partial_n \mathbf{u} - \sum_{i=1}^{2} (\Delta \mathbf{n} \cdot \tau_i) \frac{\partial \mathbf{u}}{\partial \tau_i},$$

we get

$$\partial_n \Delta \mathbf{u} = \partial_n^3 \mathbf{u} + H_2 \partial_n \mathbf{u} + H_3 \mathbf{u},$$

where the expressions

$$\begin{aligned} H_2 \mathbf{u} &= (H_4 - (\Delta \mathbf{n} \cdot \mathbf{n}))\, \mathbf{u}, \\ H_3 \mathbf{u} &= -\sum_{i=1}^{2} (\Delta \mathbf{n} \cdot \tau_i) \frac{\partial \mathbf{u}}{\partial \tau_i}, \\ H_4 \mathbf{u} &= \sum_{i=1}^{2} \frac{\partial^2}{\partial \tau_i^2} \mathbf{u}, \end{aligned}$$

are only made of tangential derivatives of $\mathbf{u}$.

In summary, we see that

$$M\mathbf{u} = D\mathbf{u} + (\mathbf{0}, H_1 \mathbf{u}, H_4 \mathbf{u}, (H_3 + H H_4)\mathbf{u} + H_2 \partial_n \mathbf{u} + H_1 \partial_n^2 \mathbf{u})$$

and consequently

$$M\mathbf{u} = (\mathbf{h}_0, \mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3) \in \mathbf{R},$$

if and only if

$$D\mathbf{u} = (\mathbf{g}_0, \mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3) \in \mathbf{R},$$

with the (invertible) relation

$$\begin{aligned} \mathbf{g}_0 &= \mathbf{h}_0, \\ \mathbf{g}_1 &= \mathbf{h}_1 - H_1 \mathbf{h}_0, \\ \mathbf{g}_2 &= \mathbf{h}_2 - H_4 \mathbf{h}_0, \\ \mathbf{g}_3 &= \mathbf{h}_3 - (H_3 + H H_4)\mathbf{h}_0 - H_2 \mathbf{h}_1 - H \mathbf{h}_2. \end{aligned}$$

As a standard trace theorem (see for instance [11, Thm 1.5.1.2]) shows that the mapping D is surjective from $\mathbf{H}^4(\Omega)$ onto $\mathbf{R}$, M inherits the same property. ■

Clearly there exist two 8×12 matrices Q and R of tangential differential operators such that

$$B\mathbf{u} = QD\mathbf{u}, T\mathbf{u} = RD\mathbf{u}, \forall \mathbf{u} \in \mathbf{H}^4(\Omega). \tag{4.5}$$

Let Q^+ (resp. R^+) denote the adjoint operator of Q (resp. R) in the sense that

$$\int_{\partial\Omega} Q\mathbf{u} \cdot \mathbf{v} = \int_{\partial\Omega} \mathbf{u} \cdot Q^+ \mathbf{v}, \int_{\partial\Omega} R\mathbf{u} \cdot \mathbf{v} = \int_{\partial\Omega} \mathbf{u} \cdot R^+ \mathbf{v}, \forall \mathbf{u} \in C^\infty(\partial\Omega)^{12}, \mathbf{v} \in (C^\infty(\partial\Omega))_T^8,$$

where

$$(C^\infty(\partial\Omega))^8_T = \{(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4) \in C^\infty(\partial\Omega)^8 : \mathbf{v}_1 \cdot \mathbf{n} = \mathbf{v}_2 \cdot \mathbf{n} = 0\}.$$

Then we introduce the boundary operator

$$P = R^+ B - Q^+ T. \tag{4.6}$$

We are ready to state the next Green formulas.

Lemma 4.2 *For any* $\mathbf{u} \in \mathbf{H}^4(\Omega)$, $\mathbf{v} \in \mathbf{H}^2(\Omega)$, *we have*

$$\begin{aligned} (4.7)\quad a(\mathbf{u}, \mathbf{v}) &= \int_\Omega \Delta^2 \mathbf{u} \cdot \bar{\mathbf{v}} \\ &+ \int_{\partial\Omega} (\operatorname{div} \Delta \mathbf{u} \bar{\mathbf{v}} \cdot \mathbf{n} - \Delta \mathbf{u} \cdot \mathbf{n} \operatorname{div} \bar{\mathbf{v}} - \operatorname{curl} \Delta \mathbf{u} \cdot (\bar{\mathbf{v}} \times \mathbf{n}) - \Delta \mathbf{u} \cdot (\operatorname{curl} \bar{\mathbf{v}} \times \mathbf{n})) . \end{aligned}$$

Furthermore for any $\mathbf{u}, \mathbf{v} \in \mathbf{H}^4(\Omega)$, *we have*

$$\int_\Omega \Delta^2 \mathbf{u} \cdot \bar{\mathbf{v}} + \int_{\partial\Omega} B\mathbf{u} \cdot T\bar{\mathbf{v}} = \int_\Omega \mathbf{u} \cdot \Delta^2 \bar{\mathbf{v}} + \int_{\partial\Omega} T\mathbf{u} \cdot B\bar{\mathbf{v}}, \tag{4.8}$$

$$\int_\Omega \Delta^2 \mathbf{u} \cdot \bar{\mathbf{v}} = \int_\Omega \mathbf{u} \cdot \Delta^2 \bar{\mathbf{v}} + \int_{\partial\Omega} D\mathbf{u} \cdot P\bar{\mathbf{v}}. \tag{4.9}$$

Proof. The first identity is a simple application of standard Green's formulas (see for instance the identities (I.2.17) and (I.2.22) in [9]). The identity (4.8) follows from the first one by noticing that $a(\mathbf{u}, \mathbf{v}) = \overline{a(\mathbf{v}, \mathbf{u})}$. The identity (4.9) is a re-writting of (4.8) by using the expressions (4.5) and (4.6) and recalling the definition of Q^+ and R^+. ∎

Now for any $\ell \in \mathbb{N}$, we introduce the operator

$$A_\ell : \mathbf{H}^{4+\ell}(\Omega) \longrightarrow \mathbf{H}^\ell(\Omega) \times R^\ell : \mathbf{u} \to (\Delta^2 \mathbf{u}, B\mathbf{u}),$$

which is clearly a linear and continuous operator. By Theorem 4.2.4 of [16], A_ℓ is a Fredholm operator; its kernel consists of smooth functions and its cokernel is made of smooth functions $(\mathbf{v}, \underline{v})$ solution of the homogeneous adjoint problem (4.10) described below. Concerning the kernel of A_ℓ, any $\mathbf{u} \in \ker A_\ell$ clearly belongs to V and owing to (4.7), one obtains

$$a(\mathbf{u}, \mathbf{u}) = 0.$$

By the coerciveness of a, we deduce that $\mathbf{u} = \mathbf{0}$ and consequently the kernel of A_ℓ is reduced to $\{\mathbf{0}\}$.

Let us go on with the characterization of the formal adjoint of A_ℓ. Comparing the identity (4.9) with the identity (4.2.14) from [16], owing to the Definition 4.2.3 of [16], we see that the formal adjoint of system (3.9) consists in $(\mathbf{v}, \underline{v}) \in C^\infty(\Omega)^3 \times (C^\infty(\partial\Omega))^8_T$ solution of

$$\begin{cases} \Delta^2 \mathbf{v} = \mathbf{f} & \text{in } \Omega, \\ P\mathbf{v} + Q^+ \underline{v} = \mathbf{g} & \text{on } \partial\Omega. \end{cases} \tag{4.10}$$

We now make the relationship between the kernel of this adjoint problem with the kernel of A_0.

Lemma 4.3 *$(\mathbf{v}, \underline{v}) \in C^\infty(\Omega)^3 \times (C^\infty(\partial\Omega))^8_T$ is a solution of*

$$\left\{ \begin{array}{ll} \Delta^2 \mathbf{v} = \mathbf{0} & \text{in } \Omega, \\ P\mathbf{v} + Q^+ \underline{v} = \mathbf{0} & \text{on } \partial\Omega, \end{array} \right. \tag{4.11}$$

if and only if

$$\left\{ \begin{array}{ll} \Delta^2 \mathbf{v} = \mathbf{0} & \text{in } \Omega, \\ B\mathbf{v} = \mathbf{0} \text{ and } \underline{v} = T\mathbf{v} & \text{on } \partial\Omega. \end{array} \right.$$

Proof. From the definition of P, for all $\mathbf{u} \in \mathbf{H}^4(\Omega)$ and $(\mathbf{w}, \underline{w}) \in C^\infty(\Omega)^3 \times (C^\infty(\partial\Omega))^8_T$, we have

$$\int_{\partial\Omega} D\mathbf{u} \cdot (P\bar{\mathbf{w}} + Q^+ \overline{\underline{w}}) = \int_{\partial\Omega} T\mathbf{u} \cdot B\bar{\mathbf{w}} + B\mathbf{u} \cdot (\overline{\underline{w}} - T\bar{\mathbf{w}}).$$

Hence for a solution $(\mathbf{v}, \underline{v}) \in C^\infty(\Omega)^3 \times (C^\infty(\partial\Omega))^8_T$ of (4.11), we get

$$0 = \int_{\partial\Omega} D\mathbf{u} \cdot (P\bar{\mathbf{v}} + Q^+ \overline{\underline{v}}) = \int_{\partial\Omega} (B\mathbf{u}, T\mathbf{u}) \cdot (\overline{\underline{v}} - T\bar{\mathbf{v}}, B\bar{\mathbf{v}}) = 0, \forall \mathbf{u} \in \mathbf{H}^4(\Omega).$$

The conclusion follows, as Lemma 4.1 implies that the range of the mapping (4.2) is dense in $\left(\mathbf{L}^2_T(\partial\Omega) \times \mathbf{L}^2_T(\partial\Omega) \times L^2(\partial\Omega) \times L^2(\partial\Omega)\right)^2$. ■

Corollary 4.4 *If $(\mathbf{v}, \underline{v}) \in C^\infty(\Omega)^3 \times (C^\infty(\partial\Omega))^8_T$ is a solution of* (4.11)*, then $(\mathbf{v}, \underline{v}) = \mathbf{0}$.*

In conclusion we have proved the next result.

Theorem 4.5 *For all $\ell \in \mathbb{N}$, the operator A_ℓ is an isomorphism.*

Corollary 4.6 *Let $\mathbf{f} \in \mathbf{H}^\ell(\Omega)$, with $\ell \in \mathbb{N}$, then the weak solution $\mathbf{u} \in V$ of* (3.3) *belongs to $\mathbf{H}^{4+\ell}(\Omega)$.*

Proof. By Theorem 4.5, there exists a unique solution $\mathbf{u} \in \mathbf{H}^{4+\ell}(\Omega)$ of (3.9). But due to (4.7), we have

$$a(\mathbf{u}, \mathbf{v}) = \int_\Omega \mathbf{f} \cdot \bar{\mathbf{v}}, \forall \mathbf{v} \in V.$$

Since $\mathbf{u}$ belongs to V, this means that $\mathbf{u}$ is the unique weak solution of (3.3). ■

5 Edge singularities

In this section, we assume that Ω is a polyhedral domain. Our goal is to describe the edge singularities of problem (3.9). Let us then fix an edge e of Ω, hence near an interior point of e, up to a translation and a rotation, Ω behaves like $W_e = C_e \times \mathbb{R}$ where C_e is a two-dimensional cone centred at $(0,0)$ of opening $\omega_e \in (0, 2\pi)$, with $\omega_e \neq \pi$. Below we will also use the polar coordinates (r, θ) in C_e centred at $(0,0)$. Let us recall that the set $\Lambda_{Dir}(C_e)$ of singular exponents of the Laplace operator with Dirichlet boundary conditions in C_e is defined by

$$\Lambda_{Dir}(C_e) = \{\frac{k\pi}{\omega_e} : k \in \mathbb{Z} \setminus \{0\}\}.$$

For convenience, when no confusion is possible, we will drop the index e. As usual, for $\lambda \in \mathbb{C}$, the edge singularities are obtained by looking for a non-polynomial solution $\mathbf{u} = r^\lambda \sum_{q=0}^{Q} (\ln r)^q \varphi_q(\theta)$ (hence independent of the x_3 variable) of (cfr. Theorem 3.1 and (3.9))

$$
\left\{
\begin{array}{ll}
\Delta^2 \mathbf{u} = \mathbf{F} & \text{in } C \times \mathbb{R}, \\
\text{div } \mathbf{u} = \Delta \, \text{div } \mathbf{u} = 0 & \text{on } \partial C \times \mathbb{R}, \\
\mathbf{u} \times \mathbf{n} = (\text{curl } \mathbf{u}) \times \mathbf{n} = 0 & \text{on } \partial C \times \mathbb{R},
\end{array}
\right.
\tag{5.1}
$$

$\mathbf{F} = (\mathbf{f}, f_3)$ being a polynomial in the x_1, x_2 variables. In this way, we see that the third component u_3 satisfies

$$
\left\{
\begin{array}{ll}
\Delta^2 u_3 = f_3 & \text{in } C, \\
u_3 = \frac{\partial u_3}{\partial n} = 0 & \text{on } \partial C,
\end{array}
\right.
\tag{5.2}
$$

while the pair $\mathbf{v} = (u_1, u_2)$, made of the first two components of $\mathbf{u}$, satisfies

$$
\left\{
\begin{array}{ll}
\Delta^2 \mathbf{v} = \mathbf{f} & \text{in } C, \\
\text{div } \mathbf{v} = \Delta \, \text{div } \mathbf{v} = 0 & \text{on } \partial C, \\
\mathbf{v} \cdot \mathbf{t} = \text{curl } \mathbf{v} = 0 & \text{on } \partial C,
\end{array}
\right.
\tag{5.3}
$$

which is the two-dimensional version of (5.1) [1].

The singularities of problem (5.2) are described in [11, Chap. 7] for instance (see also [24]), where it is shown that a function u_3 of the form $r^\lambda \varphi(\theta)$ is a solution of (5.2) with $f_3 = 0$ if and only if λ is a root of

$$
\sin^2((\lambda - 1)\omega) = (\lambda - 1)^2 \sin^2 \omega.
\tag{5.4}
$$

It is shown in [8, §5.1] (see also [11, Lemma 7.3.2.4]) that the strip $\Re\lambda \in [\frac{1}{2}, \frac{3}{2}]$ does not contain roots of (5.4) except $\lambda = 1$ if $\omega \in (0, 2\pi)$, while the strip $\Re\lambda \in [1 - \frac{\pi}{\omega}, 1 + \frac{\pi}{\omega}]$ does not contain roots of (5.4) except $\lambda = 1$ if $\omega \in (0, \pi)$. The case when f_3 is a non-zero polynomial corresponds to integer exponents and will be treated below.

As in [6], the singularities of system (5.3) are obtained by introducing the scalar variables $q = \text{div } \mathbf{v}$ and $\psi = \text{curl } \mathbf{v}$. In this way, we find the equivalent system

$$
\Delta^2 q = \text{div}\, \mathbf{f} \text{ in } C, \text{ with } q = \Delta q = 0 \text{ on } \partial C,
\tag{5.5a}
$$

$$
\Delta \,\text{curl } \psi = -\nabla \Delta q \text{ in } C, \text{ with } \psi = 0 \text{ on } \partial C,
\tag{5.5b}
$$

$$
\text{curl } \mathbf{v} = \psi, \text{div } \mathbf{v} = q \text{ in } C, \text{ with } \mathbf{v} \cdot \mathbf{t} = 0 \text{ on } \partial C.
\tag{5.5c}
$$

5.1 Non-integer exponents

If λ is not an integer, then $\mathbf{f} = \mathbf{0}$ and three types of singularities appear for system (5.5):
Type 1: $q = 0$, $\psi = 0$ and $\mathbf{v}$ general non-zero solution of (5.5c).
Type 2: $q = 0$, ψ general non-zero solution of (5.5b) and $\mathbf{u}$ particular solution of (5.5c).
Type 3: q general non-zero solution of (5.5a), $\boldsymbol{\psi}$ particular solution of (5.5b) and $\mathbf{u}$ particular solution of (5.5c).

The singularities of type 1 correspond exactly to the singularities of type 1 in [6, §5.b], where it is shown that λ is such that $\lambda + 1 \in \Lambda_{Dir}(C)$ and $\mathbf{v} = \nabla\Phi$ with $\Phi = r^{\lambda+1} \sin((\lambda+1)\theta)$, as $\lambda \notin \mathbb{Z}$.

Let us now analyze the singularities of types 2 and 3.

[1] in the whole section, except in the last Theorem 5.8, when the operators ∇, div , curl and Δ are applied to functions of the variables x_1 and x_2, they correspond to the two-dimensional operators

Lemma 5.1 *If $\omega \neq \frac{3\pi}{2}$, $\lambda \notin \mathbb{Z}$ is a singular exponent of type 2 of the system* (5.3) *if and only if $\lambda - 1 \in \Lambda_{Dir}(C)$. If $\omega = \frac{3\pi}{2}$, there is no non-integer singular exponent of type 2 for system* (5.3).

Proof. For a singularity of type 2, as $q = 0$, by (5.5b), ψ satisfies

$$\Delta \operatorname{curl} \psi = 0 \text{ in } C, \text{ with } \psi = 0 \text{ on } \partial C.$$

Since $\Delta \operatorname{curl} \psi = \operatorname{curl} \Delta\psi$, we deduce that there exists a constant c such that

$$\Delta\psi = c \text{ in } C.$$

As $\Delta\psi$ behaves like $r^{\lambda-3}$ and we look for a non-integer exponent λ, c has to be zero. Therefore ψ is solution of

$$\Delta\psi = 0 \text{ in } C, \text{ with } \psi = 0 \text{ on } \partial C. \tag{5.6}$$

This means that a non-zero solution ψ exists if and only if $\lambda - 1 \in \Lambda_{Dir}(C)$ and

$$\psi = r^{\lambda-1} \sin((\lambda - 1)\theta).$$

Now by (5.5c), $\mathbf{v}$ has to satisfy

$$\operatorname{curl} \mathbf{v} = r^{\lambda-1} \sin((\lambda - 1)\theta), \operatorname{div} \mathbf{v} = 0 \text{ in } C, \text{ with } \mathbf{v} \cdot \mathbf{t} = 0 \text{ on } \partial C. \tag{5.7}$$

But we readily check that the function $\mathbf{v}_0 = \frac{1}{\lambda+1}(-x_2\psi, x_1\psi)$ satisfies

$$\operatorname{curl} \mathbf{v}_0 = \psi.$$

Therefore by setting $\mathbf{v}_1 := \mathbf{v} - \mathbf{v}_0$, (5.7) is equivalent to

$$\operatorname{curl} \mathbf{v}_1 = 0, \operatorname{div} \mathbf{v}_1 = -\operatorname{div} \mathbf{v}_0 \text{ in } C, \text{ with } \mathbf{v}_1 \cdot \mathbf{t} = 0 \text{ on } \partial C. \tag{5.8}$$

As C is simply connected, there exists a scalar function Φ such that $\mathbf{v}_1 = \nabla\Phi$ so that (5.7) reduces to

$$\Delta\Phi = -\operatorname{div} \mathbf{v}_0 \text{ in } C, \text{ with } \Phi = 0 \text{ on } \partial C. \tag{5.9}$$

As $\operatorname{div} \mathbf{v}_0 = r^{\lambda-1}\kappa(\theta)$ for some smooth function κ, we deduce that this problem has a solution Φ in the form $r^{\lambda+1}\varphi(\theta)$ if and only if $\lambda + 1$ does not belong to $\Lambda_{Dir}(C)$ or $\lambda + 1$ belongs to $\Lambda_{Dir}(C)$ but

$$\int_0^\omega \kappa(\theta) \sin((\lambda + 1)\theta)\, d\theta = 0. \tag{5.10}$$

We now notice that $\lambda - 1$ and $\lambda + 1$ both belong to $\Lambda_{Dir}(C)$ if and only if $\omega = \frac{\pi}{2}$ or $\frac{3\pi}{2}$. Hence if $\omega \notin \{\frac{\pi}{2}, \frac{3\pi}{2}\}$, we find a solution Φ and hence $\mathbf{v}$. Otherwise, if $\omega = \frac{3\pi}{2}$, then easy calculations show that (5.10) does not hold, and therefore λ is not a singular exponent. Note that the case $\omega = \frac{\pi}{2}$ leads to integer exponents and is here excluded. ■

Lemma 5.2 *If $\omega \neq \frac{3\pi}{2}$, $\lambda \notin \mathbb{Z}$ is a singular exponent of type 3 of the system* (5.3) *if and only if*

$$\lambda - 1 \in \Lambda_{Dir}(C) \text{ or } \lambda - 3 \in \Lambda_{Dir}(C). \tag{5.11}$$

If $\omega = \frac{3\pi}{2}$, $\lambda \notin \mathbb{Z}$ is a singular exponent of type 3 of the system (5.3) *if and only if $\lambda - 1 \in \Lambda_{Dir}(C)$.*

Proof. For a singularity of type 3, $q = r^{\lambda-1}Q(\theta)$ is a general solution of (5.5a), i.e.,

$$\Delta^2 q = 0 \text{ in } C, \text{ with } q = \Delta q = 0 \text{ on } \partial C. \tag{5.12}$$

By the results from [22, §3.2.2], a non zero solution q exists if and only if either $\lambda - 1$ belongs to $\Lambda_{Dir}(C)$ and $Q(\theta) = \sin((\lambda-1)\theta)$ or $\lambda - 3$ belongs to $\Lambda_{Dir}(C)$ and $Q(\theta) = \sin((\lambda-3)\theta)$.

Let us now treat the two cases separately.

a) If $\lambda - 1 \in \Lambda_{Dir}(C)$, then q is harmonic and therefore, we can take $\psi = 0$. Hence (5.5c) becomes

$$\operatorname{curl} \mathbf{v} = 0, \operatorname{div} \mathbf{v} = r^{\lambda-1}\sin((\lambda-1)\theta) \text{ in } C, \text{ with } \mathbf{v}\cdot\mathbf{t} = 0 \text{ on } \partial C.$$

As before this means that $\mathbf{v} = \nabla\Phi$ with

$$\Delta\Phi = r^{\lambda-1}\sin((\lambda-1)\theta) \text{ in } C, \text{ with } \Phi = 0 \text{ on } \partial C. \tag{5.13}$$

A solution $\Phi = r^{\lambda+1}\varphi(\theta)$ always exists. Indeed either $\lambda + 1 \notin \Lambda_{Dir}(C)$ and then it is direct, or $\lambda + 1 \in \Lambda_{Dir}(C)$ and its existence follows from the orthogonality property:

$$\int_0^\omega \sin((\lambda-1)\theta)\sin((\lambda+1)\theta)\,d\theta = 0.$$

b) If $\lambda - 3 \in \Lambda_{Dir}(C)$, then q is no more harmonic, but

$$\Delta q = 4(\lambda-1)S_{\text{Dir}},$$

with $S_{\text{Dir}} = r^{\lambda-3}\sin((\lambda-3)\theta)$, hence (5.5b) becomes

$$\operatorname{curl} \Delta\psi = -4(\lambda-1)\nabla S_{\text{Dir}} \text{ in } C, \text{ with } \psi = 0 \text{ on } \partial C.$$

But simple calculations show that

$$\nabla S_{\text{Dir}} = -\operatorname{curl} S_{\text{Neu}},$$

with $S_{\text{Neu}} = r^{\lambda-3}\cos((\lambda-3)\theta)$. Consequently

$$\operatorname{curl} \Delta\psi = 4(\lambda-1)\operatorname{curl} S_{\text{Neu}} \text{ in } C,$$

and since λ is not an integer, ψ is solution of

$$\Delta\psi = 4(\lambda-1)S_{\text{Neu}} \text{ in } C, \text{ with } \psi = 0 \text{ on } \partial C.$$

If $\lambda - 1 \notin \Lambda_{Dir}(C)$, we find ψ in the form $r^{\lambda-1}\Psi(\theta)$, otherwise (which holds if and only if $\omega \in \{\frac{\pi}{2}, \frac{3\pi}{2}\}$) we do not find a solution in this form since

$$\int_0^\omega \cos((\lambda-3)\theta)\sin((\lambda-1)\theta)\,d\theta \neq 0.$$

In the first case, by (5.5c), $\mathbf{v}$ has to satisfy

$$\operatorname{curl} \mathbf{v} = \psi, \operatorname{div} \mathbf{v} = q \text{ in } C, \text{ with } \mathbf{v}\cdot\mathbf{t} = 0 \text{ on } \partial C. \tag{5.14}$$

As $\mathbf{v}_0 = \frac{1}{\lambda+1}(-x_2\psi, x_1\psi)$ satisfies $\operatorname{curl}\mathbf{v}_0 = \psi$, by setting $\mathbf{v}_1 := \mathbf{v} - \mathbf{v}_0$, (5.14) is equivalent to

$$\operatorname{curl} \mathbf{v}_1 = 0, \operatorname{div} \mathbf{v}_1 = q - \operatorname{div}\mathbf{v}_0 \text{ in } C, \text{ with } \mathbf{v}_1\cdot\mathbf{t} = 0 \text{ on } \partial C. \tag{5.15}$$

As C is simply connected, there exists a scalar function Φ such that $\mathbf{v}_1 = \nabla\Phi$ so that (5.15) reduces to

$$\Delta\Phi = q - \operatorname{div}\mathbf{v}_0 \text{ in } C, \text{ with } \Phi = 0 \text{ on } \partial C.$$

As before, we deduce that this problem has a solution in the form $r^{\lambda+1}\varphi(\theta)$ if either $\lambda+1$ does not belong to $\Lambda_{Dir}(C)$ or if $\lambda+1$ belongs to $\Lambda_{Dir}(C)$ with the constraint

$$\int_0^\omega \partial_\theta\Psi(\theta)\sin((\lambda+1)\theta)\,d\theta = 0, \tag{5.16}$$

since an easy calculation shows that

$$q - \operatorname{div}\mathbf{v}_0 = r^{\lambda-1}\left(\sin((\lambda-3)\theta) - \frac{\lambda-1}{\lambda+1}\partial_\theta\Psi(\theta)\right).$$

The second case holds if and only if both $\lambda-3$ and $\lambda+1$ belong to $\Lambda_{Dir}(C)$, which is possible only if $\omega = \frac{j\pi}{4}$, for $j = 1,\cdots,7$ and in that case, $\lambda = 3 + \frac{4k}{j}$, with $k \in \mathbb{Z}^\star$. In such a situation, we have to check if (5.16) holds or not. To do so, as $\lambda-1$ is not in $\Lambda_{Dir}(C)$, we notice that

$$\Psi(\theta) = \alpha\Big[(\cos((\lambda-3)\theta) - \cos((\lambda-1)\theta)) + \frac{\sin((\lambda-1)\theta)}{\sin((\lambda-1)\omega)}(\cos((\lambda-3)\omega) - \cos((\lambda-1)\omega))\Big],$$

with $\alpha = \frac{1}{4(\lambda-2)}$. This expression and simple calculations allow to show that (5.16) holds, if $\omega = \frac{j\pi}{4}$, for $j = 3, 5$ or 7. Since the other cases are excluded, the proof is complete. ■

5.2 Integer exponents

If λ is a non negative integer, again three types of singularities appear for system (5.5):
Type 1: q, ψ polynomial and $\mathbf{v}$ general non-polynomial solution of (5.5c).
Type 2: q polynomial, ψ general non-polynomial solution of (5.5b) and $\mathbf{u}$ particular solution of (5.5c).
Type 3: q general non-polynomial solution of (5.5a), $\boldsymbol{\psi}$ particular solution of (5.5b) and $\mathbf{u}$ particular solution of (5.5c).

The singularities of type 1 are treated in [6, §5c], where it is shown that λ is such that $\lambda+1 \in \Lambda_{Dir}(C)\setminus\{2\}$ and $\mathbf{v} = \nabla\Phi$ with $\Phi = r^{\lambda+1}(\ln r\sin((\lambda+1)\theta) + \theta\cos((\lambda+1)\theta)) + p_\lambda$, where p_λ is a polynomial of degree λ.

Let us go on with the other singularities. First for any $n \in \mathbb{N}$, we define Q^n as the space of homogeneous polynomials of degree n, and set $Q^n = \{0\}$ if n is a negative integer.

Lemma 5.3 *$\lambda \in \mathbb{N}\setminus\{0,1,2\}$ is a singular exponent of type 2 of the system (5.3) if and only if $\lambda - 1 \in \Lambda_{Dir}(C)$. $\lambda \in \{0,1,2\}$ is not a singular exponent of type 2 of the system (5.3).*

Proof. For a singularity of type 2, as $q \in Q^{\lambda-1}$, by (5.5b), ψ satisfies

$$\Delta\operatorname{curl}\psi \in Q^{\lambda-4} \text{ with } \psi = 0 \text{ on } \partial C.$$

Since $\Delta\operatorname{curl}\psi = \operatorname{curl}\Delta\psi$, we deduce that

$$\Delta\psi \in Q^{\lambda-3} \text{ with } \psi = 0 \text{ on } \partial C,$$

if $\lambda \geq 3$ and

$$\Delta\psi = 0 \text{ with } \psi = 0 \text{ on } \partial C,$$

if $\lambda < 3$.

In the first case, a non-polynomial solution ψ exists if and only if $\lambda - 1 \in \Lambda_{Dir}(C)$ (see [7, §13.C]). Now by (5.5c), $\mathbf{v}$ has to satisfy

$$\operatorname{curl} \mathbf{v} = \psi, \operatorname{div} \mathbf{v} = q \text{ in } C, \text{ with } \mathbf{v} \cdot \mathbf{t} = 0 \text{ on } \partial C,$$

and the arguments of the end of the proof of Lemma 5.1 allow to conclude the existence of such a $\mathbf{v}$.

The case $\lambda = 1$ yields $\psi = 0$, while in the case $\lambda = 0$ (resp. 2), a non-zero solution ψ exists if and only if $-1 \in \Lambda_{Dir}(C)$ (resp. $1 \in \Lambda_{Dir}(C)$), which is excluded by the assumption $\omega_e \neq \pi$. ∎

Lemma 5.4 *$\lambda \in \{0, 1\}$ is a singular exponent of type 3 of system* (5.3) *if and only if $\lambda - 3 \in \Lambda_{Dir}(C)$. 2, 3 and 4 are not a singular exponent of type 3 of system* (5.3)*. $\lambda \in \mathbb{N}$ with $\lambda \geq 5$ is a singular exponent of type 3 of system* (5.3) *if and only if* (5.11) *holds.*

Proof. For a singularity of type 3, q has to be a general non-polynomial solution of (5.5a), i.e.,

$$\Delta^2 q \in Q^{\lambda-5}, \text{ with } q = \Delta q = 0 \text{ on } \partial C.$$

For $\lambda \leq 4$, this reduces to (5.12) and by the proof of Lemma 5.2, a non-zero solution q exists if and only if $\lambda - 1 \in \Lambda_{Dir}(C)$ or $\lambda - 3 \in \Lambda_{Dir}(C)$. The case $\lambda - 1 \in \Lambda_{Dir}(C)$ is either excluded or gives rise to a polynomial solution q. The case $\lambda - 3 \in \Lambda_{Dir}(C)$ gives a non polynomial solution for $\lambda = 0$ or 1.

If λ is ≥ 5, we consider the mapping

$$\Delta^2 : \{q \in Q^{\lambda-1} : q = \Delta q = 0 \text{ on } \partial C\} \to Q^{\lambda-5} : q \to \Delta^2 q.$$

As $\{q \in Q^{\lambda-1} : q = \Delta q = 0 \text{ on } \partial C\}$ and $Q^{\lambda-5}$ have the same dimension, this mapping is onto if and only if it is injective. Since the injectivity holds if and only if (5.11) does not hold, we find a non-polynomial solution q if and only if (5.11) holds.

The existence of ψ and $\mathbf{v}$ is obtained with the help of the arguments of the proof of Lemma 5.2. ∎

Let us finish this subsection by looking at the integer singular exponents of problem (5.2).

Lemma 5.5 *$\lambda \in \mathbb{N}$ is a singular exponent of problem* (5.2) *if and only if λ is a root of* (5.4)*.*

Proof. For $\lambda \in \mathbb{N}$ with $\lambda \leq 3$, the right-hand side f_3 of (5.2) is a polynomial if and only if $f_3 = 0$. Hence λ is a root of (5.4) and the question is whether u_3 is a polynomial or not.

For $\lambda = 0$ or 2, u_3 is in the form

$$u_3(r, \theta) = r^\lambda(c_1 + c_2\theta + c_3 \sin(2\theta) + c_4 \cos(2\theta)),$$

with $c_i \in \mathbb{C}$, $i = 1, 2, 3, 4$. But c_2 is different from zero, otherwise the boundary conditions in (5.2) would imply that u_3 is zero. Consequently u_3 is not a polynomial.

For $\lambda = 1$, u_3 is in the form

$$u_3(r, \theta) = r(c_1 \sin\theta + c_2\theta \sin\theta + c_3 \cos\theta + c_4\theta \cos\theta), \tag{5.17}$$

with $c_i \in \mathbb{C}$, $i = 1, 2, 3, 4$. For the same reason as before, we can check that $(c_2, c_4) \neq (0, 0)$, whence u_3 is not a polynomial.

Finally 3 is not a root of (5.4), because ω is different from π and 2π.

In the case $\lambda \in \mathbb{N}$ with $\lambda \geq 4$, one easily shows that the set

$$Q^\lambda_{\Delta^2,\mathrm{Dir}} := \{v \in Q^\lambda : v = \partial_n v = 0 \text{ on } \partial C\}$$

has the same dimension than $Q^{\lambda-4}$. Therefore the operator

$$Q^\lambda_{\Delta^2,\mathrm{Dir}} \to Q^{\lambda-4} : v \to \Delta^2 u,$$

is bijective if and only if λ is not a root of (5.4). As the kernel of the above operator is not reduced to zero, a non-polynomial solution of (5.2) with $f_3 \in Q^{\lambda-4}$ always exists. ■

5.3 Conclusion

Up to now we did not take into account the regularity constraints on $\mathbf{u}$ coming from the augmented variational formulation. For that purpose, we introduce the sets

$$L^2_{loc}(\bar{C}^*) := \{v \in L^2_{loc}(C) : v \in L^2(C \cap B(0, R)), \forall R > 0\},$$
$$H^1_{loc}(\bar{C}^*) := \{v \in H^1_{loc}(C) : v \in H^1(C \cap B(0, R)), \forall R > 0\},$$

their vectorial form being defined in the same way. Hence the additional constraints on the solution $\mathbf{u}$ of (5.1) are

$$\mathbf{u} \in \mathbf{L}^2_{loc}(\bar{C}^*), \text{curl } \mathbf{u} \in \mathbf{L}^2_{loc}(\bar{C}^*), \text{curl curl } \mathbf{u} \in \mathbf{L}^2_{loc}(\bar{C}^*) \text{ and } \text{div } \mathbf{u} \in H^1_{loc}(\bar{C}^*).$$

In terms of $\mathbf{v}$ and u_3, this means that

$$u_3 \in L^2_{loc}(\bar{C}^*), \Delta u_3 \in L^2_{loc}(\bar{C}^*), \tag{5.18}$$

and

$$\mathbf{v} \in \mathbf{L}^2_{loc}(\bar{C}^*), \text{curl } \mathbf{v} \in H^1_{loc}(\bar{C}^*), \text{div } \mathbf{v} \in H^1_{loc}(\bar{C}^*). \tag{5.19}$$

These constraints eliminate some of the singular exponents highlighted before. Indeed the first constraint in (5.18) or (5.19) yields $\Re\lambda > -1$. The consequences of the other constraints are summarised in the next Theorem. Before, let us set

$$S_b := \{\lambda \in \mathbb{C} : \lambda \text{ is a root of (5.4) with } \Re\lambda > 1\},$$
$$S_1 := \{\lambda \in \mathbb{R} \setminus \{1\} : \lambda > -1 \text{ and } \lambda + 1 \in \Lambda_{Dir}(C)\},$$
$$S_2 := \{\lambda \in \mathbb{R} \setminus \{2\} : \lambda > 1 \text{ and } \lambda - 1 \in \Lambda_{Dir}(C)\},$$
$$S_3(\omega) := \{\lambda \in \mathbb{R} \setminus \{2, 3, 4\} : \lambda > 1 \text{ and } \lambda - 3 \in \Lambda_{Dir}(C)\}, \text{ if } \omega \neq \frac{3\pi}{2},$$

while $S_3(\frac{3\pi}{2}) = \emptyset$.

Theorem 5.6 *The set of singular exponents of system* (5.1) *with the constraints* (5.18) *and* (5.19) *is*

$$S_b \cup S_1 \cup S_2 \cup S_3(\omega).$$

Proof. For u_3, we have to see the consequence of the second constraint in (5.18). But we notice that Δu_3 behaves like $r^{\lambda-2}$ hence it is in $L^2_{loc}(\bar{C}^*)$ if either $\Re\lambda > 1$ or ($\lambda = 1$ with $\Delta u_3 = 0$). In this second case, easy calculations from (5.17) yield

$$\Delta u_3 = 2r^{-1}(c_2 \cos\theta - c_4 \sin\theta).$$

Since the functions $\cos\theta$ and $\sin\theta$ are linearly independent (in $L^2(0,\omega)$), we have

$$\int_0^\omega |c_2 \cos\theta - c_4 \sin\theta|^2 \, d\theta > 0,$$

because $(c_2, c_4) \neq (0,0)$. Consequently $r^{-1}(c_2 \cos\theta - c_4 \sin\theta)$ cannot be in $L^2_{loc}(\bar{C}^*)$, hence this singularity has to be excluded.

For $\mathbf{v}$, the second and third constraints in (5.19) mean that

$$q \in H^1_{loc}(\bar{C}^*) \text{ and } \boldsymbol{\psi} \in H^1_{loc}(\bar{C}^*).$$

This yields no more constraint than $\lambda > -1$ for the singularities of type 1. Otherwise, for singularities of types 2 or 3, this implies that $\lambda > 1$, hence the conclusion. ■

Corollary 5.7 *For any $s \geq 2$, there is no edge singular exponent associated with $e \in \mathcal{E}$ in the strip $\Re\lambda \in (-1, s-1]$ if and only if $\omega_e < \frac{\pi}{s}$.*

Proof. For the set S_b, this follows from [8, §5.1] which shows that for $\omega_e \in (0,\pi)$, any root λ of (5.4) such that $\Re\lambda > 1$ satisfies $\Re\lambda > 1 + \frac{\pi}{\omega_e}$. Now any $\lambda \in S_1$ is given by

$$\lambda = \frac{k\pi}{\omega_e} - 1,$$

with $k \in \mathbb{N}^*$. Hence we find the condition

$$\frac{\pi}{\omega_e} > s.$$

Similarly any $\lambda \in S_2$ is given by

$$\lambda = \frac{k\pi}{\omega_e} + 1,$$

with $k \in \mathbb{N}^*$ and we find here the condition

$$\frac{\pi}{\omega_e} > s - 2.$$

Finally, if $\omega_e \neq \frac{3\pi}{2}$, $\lambda \in S_3$ is given by

$$\lambda = \frac{k\pi}{\omega_e} + 3,$$

with $k \in \mathbb{Z}^*$ such that $k > -\frac{2\omega_e}{\pi}$. Since we have already found the constraint $\omega_e < \frac{\pi}{s}$, the quantity $-\frac{2\omega_e}{\pi}$ is larger than $-\frac{2}{s} > -1$. Hence only positive integers k have to be considered and for such k, we have

$$\lambda = \frac{k\pi}{\omega_e} + 3 \geq \frac{\pi}{\omega_e} + 3 > s + 3.$$

■

From this result we will deduce that in the case of a polyhedral domain, the variational solution of problem (3.3) does not belong to $\mathbf{H}^3(\Omega)$ in general, namely we show the next result.

Theorem 5.8 *Assume that Ω is a polyhedral domain. Then there exists $\mathbf{F} \in \mathbf{H}^{-1}(\Omega) \cap V'$ such that the solution $\mathbf{u} \in V$ of*

$$a(\mathbf{u}, \mathbf{v}) = \mathbf{F}(\mathbf{v}), \forall \mathbf{v} \in V, \tag{5.20}$$

does not belong to $\mathbf{H}^3(\Omega)$.

Proof. We first show that there always exists an edge e such that $\omega_e \geq \frac{\pi}{3}$. Indeed for a fixed corner c of Ω, consider the section G_c of the cone Γ_c which coincides with Ω near c (see section 6 below). This section has N_c corners which correspond to the edges e_i, $i = 1, \cdots, N_c$, of Ω having c as extremity. Then by the local Gauss-Bonnet theorem, we have

$$\sum_{i=1}^{N_c} \omega_{e_i} = (N_c - 2)\pi + |G_c|,$$

where $|G_c|$ is the area of G_c. Hence $\omega_{\max} = \max_{i=1,\cdots,N_c} \omega_{e_i}$ satisfies

$$N_c \omega_{\max} \geq (N_c - 2)\pi,$$

and since $N_c \geq 3$, we get $\omega_{\max} \geq \frac{\pi}{3}$, which proves the assertion.

If one edge e has an opening $\omega_e \geq \frac{\pi}{2}$, by Lemma 3.4 (see also Remark 3.5), any element of V is even not in $\mathbf{H}^2(\Omega)$. Hence we can now assume that Ω has one $e \in \mathcal{E}$ such that $\frac{\pi}{3} \leq \omega_e < \frac{\pi}{2}$ and, for shortness, set $\lambda = \frac{\pi}{\omega_e} - 1$, which belongs to $(1, 2]$. From our previous considerations, we know that the function $\mathbf{U}^\lambda = \left(\nabla_2 \left(r^{\lambda+1} \sin((\lambda+1)\theta)\right), 0\right)^\top$ is an edge singularity of our problem along e [2]. To localize it, we fix a cut-off function η_0 depending only on r and another cut-off function η_1 depending only on the x_3-variable (the edge one) such that $\eta_1 = 1$ near an interior point of e. Both are fixed with a sufficiently small support such that $\eta_0\eta_1$ is zero on all faces of Ω except the two ones having the edge e in common.

In that way we consider $\eta_0\eta_1\mathbf{U}^\lambda$ which does not belong to V because by (2.3)

$$\operatorname{curl}(\eta_0\eta_1\mathbf{U}^\lambda) = (\eta_1\nabla_2\eta_0, \eta_0\partial_3\eta_1)^\top \times \mathbf{U}^\lambda \tag{5.21}$$

which is not zero on the boundary of Ω. Hence we need to correct it appropriately. Therefore we look for $\mathbf{r} \in \mathbf{H}^3(\Omega)$ such that

$$\begin{cases} \mathbf{r} = \mathbf{0} & \text{on } \partial\Omega, \\ \operatorname{div} \mathbf{r} = 0 & \text{on } \partial\Omega, \\ \operatorname{curl} \mathbf{r} \times \mathbf{n} = \operatorname{curl}(\eta_0\eta_1\mathbf{U}^\lambda) \times \mathbf{n} & \text{on } \partial\Omega. \end{cases}$$

But using the expressions (4.3) and (4.4) on each face, this system is equivalent to

$$\begin{cases} \mathbf{r} = \mathbf{0} & \text{on } \partial\Omega, \\ \mathbf{n} \cdot \partial_n \mathbf{r} = 0 & \text{on } \partial\Omega, \\ (\mathbf{n} \times \partial_n \mathbf{r}) \times \mathbf{n} = \operatorname{curl}(\eta_0\eta_1\mathbf{U}^\lambda) \times \mathbf{n} & \text{on } \partial\Omega. \end{cases}$$

This means that it suffices to require that

$$\begin{cases} \mathbf{r} = \mathbf{0} & \text{on } \partial\Omega, \\ \partial_n \mathbf{r} = \operatorname{curl}(\eta_0\eta_1\mathbf{U}^\lambda) \times \mathbf{n} & \text{on } \partial\Omega. \end{cases} \tag{5.22}$$

[2]In this proof, the index $_2$ means the two-dimensional version of the differential operator, if no index is used then it is the standard three-dimensional operator

But the identity (5.21) implies that the restriction of $\mathrm{curl}\,(\eta_0\eta_1\mathbf{U}^\lambda)\times\mathbf{n}$ on each face F of Ω belongs to $\mathbf{H}^{\frac{3}{2}}(F)\cap\mathbf{H}^1_0(F)$ and its tangential derivatives are in $\tilde{H}^{\frac{1}{2}}(F)^3$. Consequently by the trace theorem from [10], there exists $\mathbf{r}\in\mathbf{H}^3(\Omega)$ satisfying the boundary conditions (5.22).

Therefore the function

$$\mathbf{u}^\lambda = \eta_0\eta_1\mathbf{U}^\lambda - \mathbf{r}$$

belongs to V but not in $\mathbf{H}^3(\Omega)$ because $\eta_0\eta_1\mathbf{U}^\lambda$ belongs to $\mathbf{H}^2(\Omega)$ but not to $\mathbf{H}^3(\Omega)$ since $\lambda\in(1,2]$. It then remains to show that it is the solution of problem (5.20) with an appropriate right-hand side. But Leibniz's rule and the fact that $\mathbf{U}^\lambda$ is harmonic imply that

$$\Delta(\eta_0\eta_1\mathbf{U}^\lambda) = 2\eta_1\partial_r\eta_0\partial_r\mathbf{U}^\lambda + \eta_1\Delta_2\eta_0\mathbf{U}^\lambda + \eta_0\partial_3^2\eta_1\mathbf{U}^\lambda.$$

The assumption $\lambda > 1$ guarantees that $\eta_0\partial_3^2\eta_1\mathbf{U}^\lambda$ belongs to $\mathbf{H}^2(\Omega)$ and since $\mathbf{U}^\lambda$ is smooth far away from the edge, we deduce that $\Delta(\eta_0\eta_1\mathbf{U}^\lambda)$ belongs to $\mathbf{H}^2(\Omega)$. With the regularity of $\mathbf{r}$, $\Delta\mathbf{u}^\lambda$ then belongs to $\mathbf{H}^1(\Omega)$. Now for $\mathbf{v}\in V$, $\mathrm{curl}\,\mathbf{v}$ belongs to $X_N(\Omega)$ and $\mathrm{div}\,\mathbf{v}$ is in $H^1_0(\Omega)$, and Green's formulas (2.2) and (2.1) lead to

$$\int_\Omega \Delta\mathbf{u}^\lambda\cdot\mathrm{curl}\,\mathrm{curl}\,\mathbf{v} = \int_\Omega \mathrm{curl}\,\Delta\mathbf{u}^\lambda\cdot\mathrm{curl}\,\mathbf{v},$$

$$\int_\Omega \Delta\mathbf{u}^\lambda\cdot\nabla\,\mathrm{div}\,\mathbf{v} = -\int_\Omega \mathrm{div}\,\Delta\mathbf{u}^\lambda\;\mathrm{div}\,\mathbf{v}.$$

The difference of these two identities directly furnishes (5.20) with

$$\mathbf{F}(\mathbf{v}) = -\int_\Omega\left(\mathrm{curl}\,\Delta\mathbf{u}^\lambda\cdot\mathrm{curl}\,\mathbf{v} + \mathrm{div}\,\Delta\mathbf{u}^\lambda\;\mathrm{div}\,\mathbf{v}\right),$$

which is indeed in $\mathbf{H}^{-1}(\Omega)\cap V'$. ■

Remark 5.9 For $\mathbf{F}\in\mathbf{H}^{-1}(\Omega)\cap V'$ the maximal regularity that the variational solution $\mathbf{u}$ of (5.20) can have is $\mathbf{H}^3(\Omega)$ because $\Delta^2\mathbf{u}=\mathbf{F}$ in $\mathcal{D}'(\Omega)^3$. Theorem 5.8 asserts that such a maximal regularity does not hold in general. Furthermore if Ω has an edge e with an opening $\omega_e\in[\frac{\pi}{4},\frac{\pi}{3})$, then similar arguments show that there exists $\mathbf{f}\in\mathbf{L}^2(\Omega)$ such that the solution $\mathbf{u}\in V$ of problem (3.3) does not belong to $\mathbf{H}^4(\Omega)$.

6 Corner singularities

In the case 2 or 3, let c be a corner of Ω, Γ_c be the three-dimensional cone which coincides with Ω in a neighbourhood of c and let G_c be its section with the unit sphere. For shortness, if no confusion is possible, we will drop the index c. As usual we denote by (r,ϑ) the spherical coordinates centred at c. Then we look for corner singularities $\mathbf{u}$ in the form $\mathbf{u}=r^\lambda\mathbf{U}(\vartheta)$, with $\lambda\in\mathbb{C}$ such that $\Re\lambda > -1$ and $\mathbf{U}\in\mathbf{L}^2(G)$, which is solution of

$$\begin{cases} \Delta^2\mathbf{u}=\mathbf{0} & \text{in } \Gamma,\\ \mathrm{div}\,\mathbf{u}=\Delta\,\mathrm{div}\,\mathbf{u}=0 & \text{on } \partial\Gamma,\\ \mathbf{u}\times\mathbf{n}=(\mathrm{curl}\,\mathbf{u})\times\mathbf{n}=0 & \text{on } \partial\Gamma. \end{cases} \tag{6.1}$$

As in [6], we introduce the auxiliary variables $q = \operatorname{div} \mathbf{u}$ and $\boldsymbol{\psi} = \operatorname{curl} \mathbf{u}$ and can re-write the above system in the equivalent form

$$\Delta^2 q = 0 \text{ in } \Gamma, \text{ with } q = \Delta q = 0 \text{ on } \partial\Gamma, \tag{6.2a}$$

$$\Delta \operatorname{curl} \boldsymbol{\psi} = -\nabla \Delta q, \operatorname{div} \boldsymbol{\psi} = 0 \text{ in } \Gamma, \text{ with } \boldsymbol{\psi} = 0 \text{ on } \partial\Gamma, \tag{6.2b}$$

$$\operatorname{curl} \mathbf{u} = \boldsymbol{\psi}, \operatorname{div} \mathbf{u} = q \text{ in } \Gamma, \text{ with } \mathbf{u} \times \mathbf{n} = 0 \text{ on } \partial\Gamma. \tag{6.2c}$$

Then as in section 5, three types of singularities appear:
Type 1: $q = 0$, $\boldsymbol{\psi} = 0$ and $\mathbf{u}$ general non-zero solution of (6.2c). This case corresponds to singularities of type 1 in [6] and are described in Lemma 6.4 of [6].
Type 2: $q = 0$, $\boldsymbol{\psi}$ general non-zero solution of (6.2b) and $\mathbf{u}$ particular solution of (6.2c).
Type 3: q general non-zero solution of (6.2a), $\boldsymbol{\psi}$ particular solution of (6.2b) and $\mathbf{u}$ particular solution of (6.2c).

Remark 6.1 The general case where the right-hand side in the first identity of (6.1) is replaced by a polynomial $\mathbf{F}$ of degree $\lambda - 4$ is not treated here because for $\lambda \leq 4$, $\operatorname{div} \mathbf{F} = 0$ (which corresponds to (6.2)) and the knowledge of the corner singular exponents in the strip $\Re\lambda \in (-\frac{3}{2}, 5)$ allows to analyze the regularity $\mathbf{H}^{s+2}(\Omega)$ of our solution up to $s + 2 < 6.5$, which is more than the expected maximal regularity with a datum in $\mathbf{L}^2(\Omega)$. Furthermore the knowledge of the corner singular exponents of (6.1) allows to state regularity results in weighted Sobolev spaces (see section 7).

The singularities of types 2 and 3 are fully different from those from [6] and are described below. For that purpose, we recall the corner singularities of the Laplace operator with Dirichlet (resp. Neumann) boundary conditions in Γ, see [11, 7, 6] for instance; as well as the corner singularities of the Stokes system with Dirichlet boundary conditions in Γ, see [8, 18, 17, 19]. We first denote by L_G^{Dir} (resp. L_G^{Neu}) the positive (resp. non-negative) Laplace-Beltrami operator with Dirichlet (resp. Neumann) boundary conditions on G. Recall that L_G^{Dir} and L_G^{Neu} are self-adjoint operators with a compact resolvent in $L^2(G)$, hence we denote by $\sigma(L_G^{\mathrm{Dir}})$ and $\sigma(L_G^{\mathrm{Neu}})$ their respective spectrum. Then we make the following definition.

Definition 6.2 *The set $\Lambda_{\mathrm{Dir}}(\Gamma)$ of corner singular exponents of the Laplace operator with Dirichlet boundary conditions in Γ is defined as the set of $\lambda \in \mathbb{C}$ such that there exists a non-trivial solution $\varphi \in H_0^1(G)$ of*

$$\Delta(r^\lambda \varphi(\vartheta)) = 0. \tag{6.3}$$

We denote by Z_{Dir}^λ the (finite) set of such solutions and fix a basis $\{u_{\mathrm{Dir}}^{\lambda,p}\}_{p=1}^{N_{\mathrm{Dir}}(\lambda)}$, with $N_{\mathrm{Dir}}(\lambda) \in \mathbb{N} \setminus \{0\}$. Similarly, the set $\Lambda_{\mathrm{Neu}}(\Gamma)$ of corner singular exponents of the Laplace operator with Neumann boundary conditions in Γ is defined as the set of $\lambda \in \mathbb{C}$ different from -1, such that there exists a solution $\varphi \in H^1(G)$ of (6.3) with Neumann boundary conditions:

$$\partial_n(r^\lambda \varphi) = 0 \text{ on } \partial\Gamma.$$

We denote by Z_{Neu}^λ the set of such solutions and fix a basis $\{u_{\mathrm{Neu}}^{\lambda,p}\}_{p=1}^{N_{\mathrm{Neu}}(\lambda)}$, with $N_{\mathrm{Neu}}(\lambda) \in \mathbb{N} \setminus \{0\}$.

Due to the relation

$$r^2 \Delta = (r\partial_r)^2 + (r\partial_r) + \Delta_G,$$

for any $\lambda \in \mathbb{C}$ and $\varphi \in H^1(G)$, we have

$$\Delta(r^\lambda \varphi) = r^{\lambda-2} \mathcal{L}(\lambda)\varphi, \tag{6.4}$$

where

$$\mathcal{L}(\lambda)\varphi = \Delta_G\varphi + \lambda(\lambda+1)\varphi, \tag{6.5}$$

with Δ_G the Laplace-Beltrami operator on G. Consequently, the sets $\Lambda_{\mathrm{Dir}}(\Gamma)$ and $\Lambda_{\mathrm{Neu}}(\Gamma)$ are related to the spectrum of L_G^{Dir} and L_G^{Neu} as follows (see [6, Lemma 2.4]):

$$\begin{aligned}
\Lambda_{\mathrm{Dir}}(\Gamma) &= \{-\frac{1}{2} \pm \sqrt{\mu + \frac{1}{4}} : \mu \in \sigma(L_G^{\mathrm{Dir}})\},\\
\Lambda_{\mathrm{Neu}}(\Gamma) &= \{-\frac{1}{2} \pm \sqrt{\mu + \frac{1}{4}} : \mu \in \sigma(L_G^{\mathrm{Neu}}) \setminus \{0\}\}.
\end{aligned}$$

For $\lambda \in \Lambda_{\mathrm{Dir}}(\Gamma)$, the elements of Z^λ_{Dir} are related to the set $V_{\mathrm{Dir}}(\lambda)$ of eigenvectors of L_G^{Dir} associated with $\mu = \lambda(\lambda+1)$ via the relation

$$Z^\lambda_{\mathrm{Dir}} = \{r^\lambda\varphi : \varphi \in V_{\mathrm{Dir}}(\lambda)\}.$$

The same holds for $\lambda \in \Lambda_{\mathrm{Neu}}(\Gamma)$, namely

$$Z^\lambda_{\mathrm{Neu}} = \{r^\lambda\varphi : \varphi \in V_{\mathrm{Neu}}(\lambda)\},$$

where $V_{\mathrm{Neu}}(\lambda)$ is the set of eigenvectors of L_G^{Neu} associated with $\mu = \lambda(\lambda+1)$.

Remark that for $\lambda \notin \Lambda_{\mathrm{Dir}}(\Gamma)$, the operator $\mathcal{L}(\lambda)$ is an isomorphism from $H_0^1(G)$ into $H^{-1}(G)$, we then denote its inverse by $\mathcal{L}_{\mathrm{Dir}}(\lambda)^{-1}$.

Now let us recall that ∇_T is the tangential component of the gradient on the unit sphere, while div_T is the adjoint of $-\nabla_T$, namely for a distribution $\mathbf{u}$, we define

$$\langle \mathrm{div}_T\, \mathbf{u}, \varphi\rangle = -\int_G \mathbf{u}\cdot\nabla_T\varphi\, d\sigma, \forall\varphi \in \mathcal{D}(G).$$

Further for a vector field $\boldsymbol{\psi}$ defined on G, we denote by $\boldsymbol{\psi}_r = \boldsymbol{\psi}\cdot\vartheta$ its radial component, while $\boldsymbol{\psi}_T = \boldsymbol{\psi} - \boldsymbol{\psi}_r\vartheta$ is its angular component.

For any $\varphi \in H_0^1(G)$ and any $\lambda \in \mathbb{C}$, we recall that

$$\nabla(r^\lambda\varphi) = r^{\lambda-1}g(\lambda)\varphi, \tag{6.6}$$

where for shortness, we have set

$$g(\lambda)\varphi = \nabla_T\varphi + \lambda\varphi\vartheta.$$

Similarly for $\mathbf{v} \in \mathbf{H}_0^1(G)$ and any $\lambda \in \mathbb{C}$, we notice that

$$\mathrm{div}\,(r^\lambda\mathbf{v}) = r^{\lambda-1}d(\lambda)\mathbf{v}, \tag{6.7}$$

$$\mathrm{curl}\,(r^\lambda\mathbf{v}) = r^{\lambda-1}c(\lambda)\mathbf{v}, \tag{6.8}$$

where we have set

$$d(\lambda)\mathbf{v} = \mathrm{div}_T\,\mathbf{v}_T + (\lambda+2)\mathbf{v}\cdot\vartheta,\ c(\lambda)\mathbf{v} = \nabla_T\times\mathbf{v} + \lambda\vartheta\times\mathbf{v}.$$

Let us go on with the description of the corner singularities of the Stokes system with Dirichlet boundary conditions in Γ.

Definition 6.3 *The set $\Lambda_S(\Gamma)$ of corner singular exponents of the Stokes system with Dirichlet boundary conditions in Γ is defined as the set of $\lambda \in \mathbb{C}$ such that there exists a non-trivial pair $(\mathbf{v}, p) \in \mathbf{H}_0^1(G) \times L^2(G)$ solution of*

$$\text{(6.9)} \qquad \begin{cases} \Delta(r^\lambda \mathbf{v}(\vartheta)) + \nabla(r^{\lambda-1}p) = \mathbf{0} & \text{in } \Gamma, \\ \operatorname{div}(r^\lambda \mathbf{v}(\vartheta)) = 0 & \text{in } \Gamma. \end{cases}$$

For $\lambda \in \Lambda_S(\Gamma)$, we denote by $V_S(\lambda)$, the space of non-trivial solutions $(\mathbf{v}, p) \in \mathbf{H}_0^1(G) \times L^2(G)$ of (6.9).

Due to the relations (6.4), (6.6), and (6.7), we see that (6.9) is equivalent to

$$\mathcal{L}_S(\lambda)(\mathbf{v}, p) = (\mathbf{0}, 0),$$

where we have set

$$\mathcal{L}_S(\lambda)(\mathbf{v}, p) = (\Delta_G \mathbf{v} + \lambda(\lambda+1)\mathbf{v} + g(\lambda-1)p, d(\lambda)\mathbf{v}).$$

As before, for $\lambda \notin \Lambda_S(\Gamma)$, the operator $\mathcal{L}_S(\lambda)$ is an isomorphism from $\mathbf{H}_0^1(G) \times L^2(G)$ into $\mathbf{H}^{-1}(G) \times L^2(G)$, see [17, Thm 5.2.1], hence we will denote its inverse by $\mathcal{L}_S(\lambda)^{-1}$.

We also need to introduce two subsets of $\Lambda_S(\Gamma)$, namely

$$\begin{aligned} \Lambda_{Sg}(\Gamma) &= \{\lambda \in \Lambda_S(\Gamma) : \lambda + 2 \notin \Lambda_{\text{Dir}}(\Gamma)\}, \\ \Lambda_{Se}(\Gamma) &= \{\lambda \in \Lambda_S(\Gamma) : \lambda + 2 \in \Lambda_{\text{Dir}}(\Gamma), \text{ and satisfying} \\ &\quad \exists (\boldsymbol{\psi}_0, p_0) \in V_S(\lambda) \setminus \{\mathbf{0}\} : (\vartheta \cdot c(\lambda)\boldsymbol{\psi}_0, \tau)_G = 0, \forall \tau \in V_{\text{Dir}}(\lambda+2)\}. \end{aligned}$$

The first case is a generic one, while the second one is an exceptional one.

We are now ready to characterize the corner singularities of type 2.

Theorem 6.4 *A complex number λ, with $\Re\lambda > 1/2$ is a corner singular exponent of type 2 if and only if $\lambda - 1 \in \Lambda^{(2)}(\Gamma) := \Lambda_{Sg}(\Gamma) \cup \Lambda_{Se}(\Gamma)$.*

Proof. For a singularity of type 2, as $q = 0$, by (6.2b), $\boldsymbol{\psi} = r^{\lambda-1}\boldsymbol{\psi}_0$ with $\boldsymbol{\psi}_0 \in \mathbf{L}^2(G)$ satisfies

$$\text{(6.10)} \qquad \Delta \operatorname{curl} \boldsymbol{\psi} = 0, \operatorname{div} \boldsymbol{\psi} = 0 \text{ in } \Gamma, \text{ with } \boldsymbol{\psi} = 0 \text{ on } \partial\Gamma.$$

This implies that $\operatorname{curl} \boldsymbol{\psi}$ is regular in Γ. As $\Delta \operatorname{curl} \boldsymbol{\psi} = \operatorname{curl} \Delta\boldsymbol{\psi}$, we deduce that

$$\operatorname{curl} \Delta\boldsymbol{\psi} = 0.$$

Therefore there exists p such that

$$\text{(6.11)} \qquad \Delta\boldsymbol{\psi} = -\nabla p \text{ in } \Gamma,$$

with $p = \frac{1}{\lambda-2}(\Delta\boldsymbol{\psi}) \cdot \mathbf{x}$ if $\lambda \neq 2$, due to (2.4), otherwise (see the identity [6, (6.11)])

$$p(r, \vartheta) = r^{-1}\varphi(\vartheta),$$

with $\varphi \in L^2_{\text{loc}}(G)$. As $\Delta\boldsymbol{\psi} = -\operatorname{curl}\operatorname{curl}\boldsymbol{\psi}$, p is regular in Γ and since $\Delta\boldsymbol{\psi}$ belongs to $\mathbf{H}^{-1}(\Sigma)$ (where $\Sigma = \{x \in \Gamma : |x| \in (1,2)\}$), by Corollary I.2.2 of [9], we deduce that $p \in L^2(\Sigma)$. Hence, in both cases, we have

$$p = r^{\lambda-2}p_0(\vartheta)$$

with $p_0 \in L^2(G)$.

By (6.10) and (6.11), we deduce that the pair $(\boldsymbol{\psi}, p) = (r^{\lambda-1}\boldsymbol{\psi}_0, r^{\lambda-2}p_0)$ is solution of the Stokes problem

$$\text{(6.12)} \qquad \begin{cases} \Delta\boldsymbol{\psi} + \nabla p = 0 & \text{in } \Gamma, \\ \text{div } \boldsymbol{\psi} = 0 & \text{in } \Gamma, \\ \boldsymbol{\psi} = 0 & \text{on } \partial\Gamma. \end{cases}$$

Hence a non-trivial solution exists if and only if $\lambda - 1$ belongs to $\Lambda_S(\Gamma)$.

Once $\boldsymbol{\psi}$ is known, it remains to find $\mathbf{u}$ solution of (6.2c) with $q = 0$, namely

$$\text{curl } \mathbf{u} = \boldsymbol{\psi}, \text{div } \mathbf{u} = 0 \text{ in } \Gamma, \text{ with } \mathbf{u} \times \mathbf{n} = 0 \text{ on } \partial\Gamma.$$

But (2.5) yields

$$\text{curl}\,(\boldsymbol{\psi} \times \mathbf{x}) = (\lambda + 1)\boldsymbol{\psi},$$

and therefore

$$\text{curl}\,(\mathbf{u} - \frac{1}{\lambda+1}\boldsymbol{\psi} \times \mathbf{x}) = 0.$$

Hence there exists a scalar field $\Xi = r^{\lambda+1}\xi$ with $\xi \in H^1_0(G)$ such that

$$\mathbf{u} - \frac{1}{\lambda+1}\boldsymbol{\psi} \times \mathbf{x} = \nabla\Xi.$$

The divergence free property of $\mathbf{u}$ then gives

$$\Delta\Xi = -\frac{1}{\lambda+1}\,\text{div}(\boldsymbol{\psi} \times \mathbf{x}) = -\frac{1}{\lambda+1}(\mathbf{x} \cdot \text{curl } \boldsymbol{\psi}),$$

which is equivalent to

$$\mathcal{L}(\lambda+1)\xi = -\frac{1}{\lambda+1}h,$$

where h is given by (recalling (6.8) and the definition of $\boldsymbol{\psi}_0$)

$$h = \vartheta \cdot c(\lambda - 1)\boldsymbol{\psi}_0,$$

with $\mathbf{0} \neq (\boldsymbol{\psi}_0, p_0) \in V_S(\lambda - 1)$. Therefore we need to distinguish between the case $\lambda + 1$ in $\Lambda_{\text{Dir}}(\Gamma)$ or not. In the case $\lambda + 1 \notin \Lambda_{\text{Dir}}(\Gamma)$, corresponding to the case $\lambda - 1 \in \Lambda_{Sg}(\Gamma)$, there is no condition on h, and no additional condition on λ is needed to find ξ and then $\mathbf{u}$. In the case $\lambda + 1 \in \Lambda_{\text{Dir}}(\Gamma)$, ξ exists if and only if h satisfies the orthogonality condition

$$(h, \tau)_G = 0, \forall \tau \in V_{\text{Dir}}(\lambda + 1),$$

which corresponds to the condition $\lambda - 1$ in $\Lambda_{Se}(\Gamma)$. ■

To describe the corner singular exponents of type 3, we clearly need to characterize the non trivial solutions of (6.2a).

Lemma 6.5 *A non zero solution $q = r^\nu Q(\vartheta)$ with $\nu \in \mathbb{C}$ and a function Q defined on G of*

$$\text{(6.13)} \qquad \Delta^2 q = 0 \text{ in } \Gamma, \text{ with } q = \Delta q = 0 \text{ on } \partial\Gamma$$

exists if and only if either $\nu \in \Lambda_{\text{Dir}}(\Gamma)$ or $\nu - 2 \in \Lambda_{\text{Dir}}(\Gamma)$. In the first case, Q belongs to $V_{\text{Dir}}(\nu)$, while in the second case $\mathcal{L}(\nu)Q$ belongs to $V_{\text{Dir}}(\nu - 2)$ if $\nu \neq \frac{1}{2}$.

Proof. If q is a solution of (6.13), by setting $s = \Delta q$, we get the equivalent lower triangular system

(6.14a) $$\Delta s = 0 \text{ in } \Gamma, \text{ with } s = 0 \text{ on } \partial\Gamma,$$

(6.14b) $$\Delta q = s \text{ in } \Gamma, \text{ with } q = 0 \text{ on } \partial\Gamma.$$

Hence two types of singularities appear:
Type 1: $s = 0$ and we have to find a general non-zero solution q of

$$\Delta q = 0 \text{ in } \Gamma, \text{ with } q = 0 \text{ on } \partial\Gamma.$$

Therefore ν belongs to $\Lambda_{\mathrm{Dir}}(\Gamma)$ and Q belongs to $V_{\mathrm{Dir}}(\nu)$.

Type 2: s is a general non-zero solution of (6.14a) and q is a particular solution of (6.14b). Since $s = r^{\nu-2}S$, with a function S defined in G, we find that $\nu - 2$ belongs to $\Lambda_{\mathrm{Dir}}(\Gamma)$ and $S \in V_{\mathrm{Dir}}(\nu - 2)$. Now q is solution of (6.14b) if and only if

(6.15) $$\mathcal{L}(\nu)Q = S \text{ in } G, \text{ with } Q = 0 \text{ on } \partial G.$$

If $\nu \neq \frac{1}{2}$, a solution Q of (6.15) always exists since either $\nu \notin \Lambda_{\mathrm{Dir}}(\Gamma)$ and then $\mathcal{L}_{\mathrm{Dir}}(\nu)$ is invertible or $\nu \in \Lambda_{\mathrm{Dir}}(\Gamma)$ and a solution exists since S is orthogonal to any element of $V_{\mathrm{Dir}}(\nu)$.

Notice that

$$\mathcal{L}(\nu)S = (\mathcal{L}(\nu) - \mathcal{L}(\nu - 2))S = 2(2\nu - 1)S,$$

hence if $\nu \neq \frac{1}{2}$, a solution Q of (6.15) is given by $\frac{1}{2(2\nu-1)}S$.

If $\nu = \frac{1}{2}$, as by assumption $\nu - 2 = -\frac{3}{2}$ belongs to $\Lambda_{\mathrm{Dir}}(\Gamma)$, then $\frac{1}{2}$ is also in $\Lambda_{\mathrm{Dir}}(\Gamma)$. Since in that case, S does not satisfy the orthogonality relation, q exists in the form $r^{\frac{1}{2}}(\ln rS + \psi)$, with $\psi \in H_0^1(G)$, see Theorem 4.22 of [23]. ∎

Before stating our result about corner singular exponents of type 3, let us introduce the following sets:

$$\begin{aligned}
\Lambda_{3g}(\Gamma) &= \{\lambda \in \mathbb{C} \setminus \Lambda_S(\Gamma) : \lambda - 2 \in \Lambda_{\mathrm{Dir}}(\Gamma)\},\\
\Lambda_{3e}(\Gamma) &= \{\lambda \in \Lambda_S(\Gamma) : \lambda - 2 \in \Lambda_{\mathrm{Dir}}(\Gamma) \text{ such that } \exists S \in V_{\mathrm{Dir}}(\lambda - 2) \setminus \{0\} :\\
&\quad (\nabla_T S \times \vartheta, \boldsymbol{\psi})_G = 0, \forall (\boldsymbol{\psi}, p) \in V_S(-(\bar{\lambda} + 1)\}.
\end{aligned}$$

These sets will be used in the construction of $\boldsymbol{\psi}$ in case of singular exponent of type 3, for the construction of $\mathbf{u}$, we further need the following subsets:

$$\begin{aligned}
\Lambda_{3g,g}(\Gamma) &= \{\lambda \in \Lambda_{3g}(\Gamma) : \lambda + 2 \notin \Lambda_{\mathrm{Dir}}(\Gamma)\},\\
\Lambda_{3e,g}(\Gamma) &= \{\lambda \in \Lambda_{3e}(\Gamma) : \lambda + 2 \notin \Lambda_{\mathrm{Dir}}(\Gamma)\}.
\end{aligned}$$

As in the case of singularities of type 2, if λ in $\Lambda_{3g}(\Gamma)$ (resp. $\Lambda_{3e}(\Gamma)$) is such that $\lambda + 2 \in \Lambda_{\mathrm{Dir}}(\Gamma)$, the situation is more delicate and we need to define

$$\Lambda_{3g,e}(\Gamma) = \{\lambda \in \Lambda_{3g}(\Gamma) : \lambda + 2 \in \Lambda_{\mathrm{Dir}}(\Gamma) \text{ satisfying (6.16) below }\},$$
$$\Lambda_{3e,e}(\Gamma) = \{\lambda \in \Lambda_{3e}(\Gamma) : \lambda + 2 \in \Lambda_{\mathrm{Dir}}(\Gamma) \text{ satisfying (6.17) below }\}.$$

(6.16) $\exists S \in V_{\mathrm{Dir}}(\lambda - 2) \setminus \{0\} : (\boldsymbol{\psi}_0, p_0) = \mathcal{L}_S(\lambda)^{-1} (\nabla_T S \times \vartheta, 0)$ satisfies
$(\vartheta \cdot c(\lambda)\boldsymbol{\psi}_0, \tau)_G = 0, \forall \tau \in V_{\mathrm{Dir}}(\lambda + 2)$.

(6.17) $\exists (S, (\boldsymbol{\chi}, p)) \in V_{\mathrm{Dir}}(\lambda - 2) \times V_S(\lambda) \setminus \{\mathbf{0}\} : (\boldsymbol{\psi}_0, p_0) = \mathcal{L}_S(\lambda)^{-1} (\nabla_T S \times \vartheta, 0) + (\boldsymbol{\chi}, p)$ satisfies
$(\vartheta \cdot c(\lambda)\boldsymbol{\psi}_0, \tau)_G = 0, \forall \tau \in V_{\mathrm{Dir}}(\lambda + 2)$.

Here is our result about corner singular exponents of type 3.

Theorem 6.6 *A complex number λ with $\Re\lambda > \frac{1}{2}$ is a corner singular exponent of type 3 if and only if* $\lambda - 1 \in \Lambda^{(3)}(\Gamma) := \Lambda_{\mathrm{Dir}}(\Gamma) \cup \Lambda_{3g,g}(\Gamma) \cup \Lambda_{3g,e}(\Gamma) \cup \Lambda_{3e,g}(\Gamma) \cup \Lambda_{3e,e}(\Gamma)$.

Proof. Let $q = r^{\lambda-1}Q$ be a solution of (6.2a). Then owing to Lemma 6.5 either $\lambda - 1 \in \Lambda_{\mathrm{Dir}}(\Gamma)$ and $Q \in V_{\mathrm{Dir}}(\lambda - 1)$ or $\lambda - 3 \in \Lambda_{\mathrm{Dir}}(\Gamma)$ with $\Delta q = r^{\lambda-3}S$ and $S \in V_{\mathrm{Dir}}(\lambda - 3)$ if $\lambda - 1 \neq \frac{1}{2}$.

In the first case, $\Delta q = 0$ and therefore as particular solution of (6.2b) we can chose $\boldsymbol{\psi} = \mathbf{0}$. Consequently (6.2c) reduces to

$$\operatorname{curl} \mathbf{u} = 0, \operatorname{div} \mathbf{u} = q \text{ in } \Gamma, \text{ with } \mathbf{u} \times \mathbf{n} = 0 \text{ on } \partial\Gamma.$$

The first condition and the boundary condition allow to write $\mathbf{u} = \nabla(r^{\lambda+1}\varphi)$, with $\varphi \in H_0^1(G)$, and the divergence constraint $\operatorname{div} \mathbf{u} = q$ becomes

$$\mathcal{L}(\lambda + 1)\varphi = Q \text{ in } G.$$

This problem has always a solution because in the case $\lambda + 1 \in \Lambda_{\mathrm{Dir}}(\Gamma)$, Q is orthogonal to any element of $V_{\mathrm{Dir}}(\lambda + 1)$.

Let us go on with the second case, namely, when $\lambda - 3 \in \Lambda_{\mathrm{Dir}}(\Gamma)$. First we notice that we can assume that $\lambda - 1 \neq \frac{1}{2}$. Indeed if $\lambda - 1 = \frac{1}{2}$, then $\lambda - 3 = -\frac{3}{2}$ which by assumption belongs to $\Lambda_{\mathrm{Dir}}(\Gamma)$, but then $\frac{1}{2}$ belongs to $\Lambda_{\mathrm{Dir}}(\Gamma)$ as well, and by the first case, we have previously shown that it generates the corner singular exponent $\frac{3}{2}$.

As $\lambda - 1 \neq \frac{1}{2}$, problem (6.2b) becomes

$$\Delta \operatorname{curl} \boldsymbol{\psi} = -\nabla(r^{\lambda-3}S), \operatorname{div} \boldsymbol{\psi} = 0 \text{ in } \Gamma, \text{ with } \boldsymbol{\psi} = 0 \text{ on } \partial\Gamma, \tag{6.18}$$

with $S \in V_{\mathrm{Dir}}(\lambda - 3)$. But due to (2.5), we have

$$\operatorname{curl}(\nabla(r^{\lambda-3}S) \times \mathbf{x}) = (r\partial_r + 2)(\nabla(r^{\lambda-3}S)) = (\lambda - 2)\nabla(r^{\lambda-3}S).$$

Since -1 does not belong to $\Lambda_{\mathrm{Dir}}(\Gamma)$, λ cannot be equal to 2, therefore replacing S by $-(\lambda - 2)S$, which is still in $V_{\mathrm{Dir}}(\lambda - 3)$, we get

$$\operatorname{curl}\big(\Delta\boldsymbol{\psi} - \nabla(r^{\lambda-3}S) \times \mathbf{x}\big) = \mathbf{0}.$$

Consequently there exists $p_0 \in L^2(G)$ such that

$$\Delta\boldsymbol{\psi} = \nabla(r^{\lambda-3}S) \times \mathbf{x} - \nabla p. \tag{6.19}$$

where $p = -r^{\lambda-2}p_0$. This property and (6.18) imply that the pair $(\boldsymbol{\psi}, p)$ is solution of the non-homogeneous Stokes system

$$\begin{cases} \Delta\,\boldsymbol{\psi} + \nabla p = \nabla(r^{\lambda-3}S) \times \mathbf{x} = r^{\lambda-3}\nabla_T S \times \vartheta & \text{in } \Gamma, \\ \operatorname{div}\,\boldsymbol{\psi} = 0 & \text{in } \Gamma, \\ \boldsymbol{\psi} = 0 & \text{on } \partial\Gamma. \end{cases} \tag{6.20}$$

Again we need to distinguish between the case $\lambda - 1 \in \Lambda_S(\Gamma)$ or not.
1. If $\lambda - 1 \notin \Lambda_S(\Gamma)$, then $(\boldsymbol{\psi}, p)$, in the form described above, exists and is unique. This precisely means that $\lambda - 1$ belongs to $\Lambda_{3g}(\Gamma)$.
2. In the case when $\lambda - 1 \in \Lambda_S(\Gamma)$, the right-hand side of (6.20) has to be in the range of the Stokes

system, which means that $(\nabla_T S \times \vartheta, 0)$ belongs to $\ker \mathcal{L}_S(\lambda-1)^*$. As $\mathcal{L}_S(\lambda-1)^* = \mathcal{L}_S(-\bar{\lambda})$, see [17, p. 149], we get the condition

$$(\nabla_T S \times \vartheta, \boldsymbol{\psi})_G = 0, \forall (\boldsymbol{\psi}, p) \in V_S(-\bar{\lambda}),$$

which precisely means that $\lambda - 1$ belongs to $\Lambda_{3e}(\Gamma)$. As in the previous theorem, we then get

$$(\boldsymbol{\psi}_0, p_0) = \mathcal{L}_S(\lambda-1)^{-1} (\nabla_T S \times \vartheta, 0) + (\boldsymbol{\chi}_0, q_0),$$

with $(\boldsymbol{\chi}_0, q_0) \in V_S(\lambda-1)$.

Once we have $(\boldsymbol{\psi}, p)$ in hand, we look for $\mathbf{u}$ solution of (6.2c). As in the proof of Theorem 6.4, we then get

$$\mathbf{u} - \frac{1}{\lambda+1}\boldsymbol{\psi} \times \mathbf{x} = \nabla(r^{\lambda+1}\xi),$$

with $\xi \in H_0^1(G)$. The divergence constraint div $\mathbf{u} = q$ becomes

$$\Delta(r^{\lambda+1}\xi) = q - \frac{1}{\lambda+1}\operatorname{div}(\boldsymbol{\psi} \times \mathbf{x}) = q - \frac{1}{\lambda+1}(\mathbf{x} \cdot \operatorname{curl} \psi),$$

which is equivalent to

$$\mathcal{L}(\lambda+1)\xi = Q - \frac{1}{\lambda+1}h,$$

where, in its full generality, we have

$$h = \vartheta \cdot c(\lambda-1)\boldsymbol{\psi}_0,$$

where

$$(\boldsymbol{\psi}_0, p_0) = \mathcal{L}_S(\lambda-1)^{-1} (\nabla_T S \times \vartheta, 0) + (\boldsymbol{\chi}_0, q_0).$$

Consequently if $\lambda+1 \notin \Lambda_{\mathrm{Dir}}(\Gamma)$, no more constraint is needed (corresponding to the case $\lambda - 1 \in \Lambda_{3g,g}(\Gamma)$ or to the case $\lambda - 1 \in \Lambda_{3e,g}(\Gamma)$). On the contrary if $\lambda+1 \in \Lambda_{\mathrm{Dir}}(\Gamma)$, $Q - \frac{1}{\lambda+1}h$ has to be orthogonal to the elements of $V_{\mathrm{Dir}}(\lambda+1)$. But since we have assumed that $\lambda - 1 \neq \frac{1}{2}$, Q is already orthogonal to that space, therefore it remains to impose this orthogonality property on h, which leads to the additional constraint $\lambda - 1 \in \Lambda_{3g,e}(\Gamma)$ or $\lambda - 1 \in \Lambda_{3e,e}(\Gamma)$. ■

Finally as in subsection 5.3 we have to take into account the constraint that the singular functions have to be locally in V. This leads to the following set of singular exponents (see Theorems 6.4 and 6.6)

$$\begin{aligned}\Lambda &= \{\lambda \in \mathbb{R} : \lambda+1 \in \Lambda_{\mathrm{Dir}}(\Gamma) \text{ with } \lambda > -\frac{3}{2}\} \\ &\cup \{\lambda \in \mathbb{C} : \lambda - 1 \in \Lambda^{(2)}(\Gamma) \cup \Lambda^{(3)}(\Gamma) \text{ with } \Re\lambda > \frac{1}{2}\}.\end{aligned}$$

For each $\lambda \in \Lambda$, we will fix a basis $\{\mathbf{S}^{\lambda,p}\}_{p=1}^{N(\lambda)}$, with $N(\lambda) \in \mathbb{N} \setminus \{0\}$, of the set of linearly independent solutions of system (6.2).

Remark 6.7 As usual, the minimal regularity near the corner c is related to the minimal value of the real part of the elements from Λ. Analytical and/or numerical results about the set $\Lambda_{\mathrm{Dir}}(\Gamma)$ (resp. $\Lambda_S(\Gamma)$) are available in [26, 2, 7, 15, 3] (resp. [8, 18, 17, 19]). Such results can be used to obtain informations on the sets $\Lambda^{(2)}(\Gamma)$ and $\Lambda^{(3)}(\Gamma)$.

7 Some regularity results for domains with point singularities

To end this paper we want to prove some regularity results for domains with point singularities (case 2). Note that standard localization procedures do not work in our setting since our differential operator is of order 4 and since the multiplication by a cut-off function is not stable in V, in the sense that if $\mathbf{u} \in V$ and η is a smooth function, then we do not automatically have $\eta\mathbf{u} \in V$. Hence again we use global regularity results in weighted Sobolev spaces for domains with point singularities described in section 8.2 of [16]. We restrict ourselves to this case, because such results are not available for fourth order operators in polyhedral domains.

We first recall the usual weighted Sobolev spaces of Kondratiev type. For any $\ell \in \mathbb{N}$ and $\beta > 0$, we set

$$V_\alpha^\ell(\Omega) = \{u \in L^2_{\text{loc}}(\Omega) : r^{\alpha-\ell+|\beta|} D^\beta u \in L^2(\Omega), \forall |\beta| \leq \ell\},$$

which is a Hilbert space with its natural inner product and norm $\|\cdot\|_{\ell,\alpha}$. The vectorial version will be denoted by $\mathbf{V}_\alpha^\ell(\Omega)$. We directly check that if $u \in V_\alpha^\ell(\Omega)$, then $\partial_j u$ belongs to $V_\alpha^{\ell-1}(\Omega)$, for all $j = 1, 2$ or 3, with the estimate

$$\|\partial_j u\|_{\ell-1,\alpha} \lesssim \|u\|_{\ell,\alpha}. \tag{7.1}$$

Furthermore owing to Lemma 6.1.2 of [16], if $u \in V_\alpha^\ell(\Omega)$, then its trace γu on $\partial\Omega$ satisfies $r^{\alpha-(\ell-\frac{1}{2})}\gamma u \in L^2(\partial\Omega)$ with the estimate

$$\|r^{\alpha-(\ell-\frac{1}{2})}\gamma u\|_{\partial\Omega} \lesssim \|u\|_{\ell,\alpha}. \tag{7.2}$$

Now for a corner c of Ω, we introduce the operator pencil $\mathcal{C}(\lambda)$ defined by

$$\mathcal{C}(\lambda) : \mathbf{H}^4(G) \to \mathbf{L}^2(G) \times R(G) : \mathbf{u} \to \mathcal{C}(\lambda)\mathbf{u} = (\mathcal{L}(\lambda-2)\mathcal{L}(\lambda)\mathbf{u}, \mathcal{B}(\lambda)\mathbf{u}),$$

where $R(G) = \mathbf{H}_T^{\frac{7}{2}}(\partial G) \times \mathbf{H}_T^{\frac{5}{2}}(\partial G) \times H^{\frac{5}{2}+\ell}(\partial G) \times H^{\frac{1}{2}+\ell}(\partial G)$, and

$$\mathcal{B}(\lambda)\mathbf{u} = ((\gamma\mathbf{u}) \times \mathbf{n}, \gamma(c(\lambda)\,\mathbf{u}) \times \mathbf{n}, \gamma d(\lambda)\,\mathbf{u}, \gamma\mathcal{L}(\lambda-2)d(\lambda)\mathbf{u})^\top,$$

recalling the definition (4.1) of B and the relations (6.4), (6.7), (6.8) and

$$\Delta^2(r^\lambda\mathbf{u}) = r^{\lambda-4}\mathcal{L}(\lambda-2)\mathcal{L}(\lambda)\mathbf{u}.$$

Since our starting system is elliptic, according to the considerations from section 8.2 of [16], the operator $\mathcal{C}(\lambda)$ has the following properties.

Theorem 7.1 *The operator $\mathcal{C}(\lambda)$ is a Fredholm operator for all $\lambda \in \mathbb{C}$ and is an isomorphism except for a countable number of isolated points which are the corner singular exponents described in the previous section. Further in any double sector*

$$\{\lambda \in \mathbb{C} : |\Re\lambda| < \delta|\Im\lambda|\}, \delta > 0,$$

$\mathcal{C}(\lambda)$ is an isomorphism except for a finite number of points.

Now we can state our first regularity result.

Theorem 7.2 *Assume that the line $\Re\lambda = \frac{1}{2}$ is free of corner singular exponents. Then the operator A_0 is an isomorphism from $\mathbf{V}_2^4(\Omega)$ into $\mathbf{V}_2^0(\Omega) \times B\mathbf{V}_2^4(\Omega)$.*

Proof. Owing to Theorem 8.2.1 of [16], the operator A_0 is Fredholm from $\mathbf{V}_2^4(\Omega)$ into $\mathbf{V}_2^0(\Omega) \times B\mathbf{V}_2^4(\Omega)$; its kernel consists of functions only in $\mathbf{V}_2^4(\Omega)$ and its cokernel is made of functions $(\mathbf{v}, \underline{v})$ solution of the homogeneous adjoint problem (4.10) such that $\mathbf{v} \in \mathbf{V}_2^4(\Omega)$ and $\underline{v}$ is smooth far from the corners. But owing to a local version of Lemma 4.3, if $(\mathbf{v}, \underline{v})$ is in the cokernel of A_0, $\mathbf{v}$ belongs to the kernel of A_0.

Hence the conclusion follows if we show that the kernel of A_0 (as operator from $\mathbf{V}_2^4(\Omega)$ into $\mathbf{V}_2^0(\Omega) \times B\mathbf{V}_2^4(\Omega)$) is reduced to zero. So let us fix $\mathbf{u} \in \mathbf{V}_2^4(\Omega)$ such that

$$\begin{cases} \Delta^2 \mathbf{u} = \mathbf{0} & \text{in } \Omega, \\ B\mathbf{u} = \mathbf{0} & \text{on } \partial\Omega. \end{cases}$$

Since

$$C_0^\infty(\bar{\Omega} \setminus \mathcal{C}) = \{v \in C^\infty(\bar{\Omega}) : v = 0 \text{ in a neighborhood of the corners}\}$$

is dense in $V_2^4(\Omega)$, we can fix a sequence of functions $\mathbf{u}_n \in C_0^\infty(\bar{\Omega} \setminus \mathcal{C})^3$, $n \in \mathbb{N}$ such that

$$\mathbf{u}_n \to \mathbf{u} \text{ in } \mathbf{V}_2^4(\Omega), \text{ as } n \to \infty. \tag{7.3}$$

Hence applying Green's formula (4.7) we get

$$\begin{aligned} a(\mathbf{u}_n, \mathbf{u}_n) &= \int_\Omega \Delta^2 \mathbf{u}_n \cdot \bar{\mathbf{u}}_n \\ &+ \int_{\partial\Omega} \Big(\operatorname{div} \Delta \mathbf{u}_n \bar{\mathbf{u}}_n \cdot \mathbf{n} - \Delta \mathbf{u}_n \cdot \mathbf{n} \operatorname{div} \bar{\mathbf{u}}_n - \operatorname{curl} \Delta \mathbf{u}_n \cdot (\bar{\mathbf{u}}_n \times \mathbf{n}) - \Delta \mathbf{u}_n \cdot (\operatorname{curl} \bar{\mathbf{u}}_n \times \mathbf{n}) \Big). \end{aligned} \tag{7.4}$$

As $\mathbf{V}_2^4(\Omega)$ is continuously embedded into $\mathbf{H}^2(\Omega)$, the left-hand side of this identiy tends to $a(\mathbf{u}, \mathbf{u})$ as n goes to ∞. Hence it remains to pass to the limit in each term of the right-hand side. For the first term, we notice that (7.3) implies that

$$\begin{aligned} r^{-2} \mathbf{u}_n &\to r^{-2} \mathbf{u} \text{ in } \mathbf{L}^2(\Omega), \text{ as } n \to \infty, \\ r^2 \Delta^2 \mathbf{u}_n &\to r^2 \Delta^2 \mathbf{u} \text{ in } \mathbf{L}^2(\Omega), \text{ as } n \to \infty, \end{aligned}$$

and consequently, by Cauchy-Schwarz's inequality we deduce that

$$\int_\Omega \Delta^2 \mathbf{u}_n \cdot \bar{\mathbf{u}}_n \to \int_\Omega r^2 \Delta^2 \mathbf{u} \cdot (r^{-2} \bar{\mathbf{u}}) = 0, \text{ as } n \to \infty.$$

Similarly, using (7.1) and (7.2), (7.3) implies that

$$\begin{aligned} r^{-\frac{3}{2}} \mathbf{u}_n &\to r^{-\frac{3}{2}} \mathbf{u} \text{ in } \mathbf{L}^2(\partial\Omega), \text{ as } n \to \infty, \\ r^{\frac{3}{2}} \operatorname{div} \Delta \mathbf{u}_n &\to r^{\frac{3}{2}} \operatorname{div} \Delta \mathbf{u} \text{ in } \mathbf{L}^2(\partial\Omega), \text{ as } n \to \infty, \end{aligned}$$

and as before we deduce that

$$\int_{\partial\Omega} \operatorname{div} \Delta \mathbf{u}_n \bar{\mathbf{u}}_n \cdot \mathbf{n} \to \int_{\partial\Omega} \operatorname{div} \Delta \mathbf{u} \bar{\mathbf{u}} \cdot \mathbf{n} = 0, \text{ as } n \to \infty.$$

The same argument applies to the other boundary terms and yields

$$\int_{\partial\Omega} \Big(\operatorname{div} \Delta \mathbf{u}_n \bar{\mathbf{u}}_n \cdot \mathbf{n} - \Delta \mathbf{u}_n \cdot \mathbf{n} \operatorname{div} \bar{\mathbf{u}}_n - \operatorname{curl} \Delta \mathbf{u}_n \cdot (\bar{\mathbf{u}}_n \times \mathbf{n}) - \Delta \mathbf{u}_n \cdot (\operatorname{curl} \bar{\mathbf{u}}_n \times \mathbf{n}) \Big) \to 0, \text{ as } n \to \infty.$$

This means that we have shown that

$$a(\mathbf{u},\mathbf{u}) = 0,$$

and since $\mathbf{u}$ belongs to V, we deduce that $\mathbf{u} = \mathbf{0}$. ■

With this result in hand we can use the comparison Theorem 8.2.2 of [16] which directly leads to the

Theorem 7.3 *Let $\ell \in \mathbb{N}$ and $\beta \in \mathbb{R}$ such that $\beta - \ell < 2$. Assume that the lines $\Re\lambda = \frac{1}{2}$ and $\Re\lambda = \ell - \beta + \frac{5}{2}$ are free of corner singular exponents. Let $\mathbf{u}_0 \in \mathbf{V}_2^4(\Omega)$ be the unique solution of*

$$\left\{ \begin{array}{ll} \Delta^2 \mathbf{u}_0 = \mathbf{f} & \text{in } \Omega, \\ B\mathbf{u}_0 = \mathbf{g} & \text{on } \partial\Omega, \end{array} \right. \tag{7.5}$$

with $\mathbf{f} \in \mathbf{V}_\beta^\ell(\Omega)$ and $\mathbf{g} \in B\mathbf{V}_\beta^{\ell+4}(\Omega)$. Then $\mathbf{u}_0$ admits the decomposition

$$\mathbf{u}_0 = \mathbf{u}_R + \sum_{c\in\mathcal{C}} \sum_{\lambda\in\Lambda_c : \frac{1}{2} < \Re\lambda < \ell-\beta+\frac{5}{2}} \sum_p k_{c,\lambda,p} \mathbf{S}^{\lambda,p}, \tag{7.6}$$

where $\mathbf{u}_R \in \mathbf{V}_\beta^{\ell+4}(\Omega)$ and for all $c \in \mathcal{C}, \lambda \in \Lambda_c : \frac{1}{2} < \Re\lambda < \ell - \beta + \frac{5}{2}$ and p, $k_{c,\lambda,p} \in \mathbb{C}$.

We now exploit this result to get a decomposition of the variational solution $\mathbf{u} \in V$ of (3.9) (with $\mathbf{f} \in \mathbf{L}^2(\Omega)$) into a regular part and a singular one. Before we need a variant of Lemma 3.6 where the regular and the singular parts are orthogonal for the inner product induced by the sesquilinear form a. In a first step as $\eta_c \nabla u_{\text{Dir}}^{\lambda+1,p}$ are not in V, we need to correct them.

Lemma 7.4 *For all $c \in \mathcal{C}, \lambda \in (-\frac{3}{2}, \frac{1}{2}) : \lambda + 1 \in \Lambda_{\text{Dir}}(\Gamma_c)$ and all p, there exists $\mathbf{r}_{c,\lambda,p} \in \mathbf{V}_0^2(\Omega)$ such that*

$$\mathbf{U}_{c,\lambda,p} = \eta_c \nabla u_{\text{Dir}}^{\lambda+1,p} - \mathbf{r}_{c,\lambda,p} \tag{7.7}$$

belongs to V.

Proof. For shortness, set $\mathbf{S} = \eta_c \nabla u_{\text{Dir}}^{\lambda+1,p}$ and drop the indices c, λ and p. As η is a radial function, we see that

$$\begin{aligned} \mathbf{S} \times \mathbf{n} &= \mathbf{0} \text{ on } \partial\Omega, \\ \operatorname{div} \mathbf{S} &= 0 \text{ on } \partial\Omega. \end{aligned}$$

Unfortunately $(\operatorname{curl} \mathbf{S}) \times \mathbf{n}$ is not zero on the boundary, because (2.3) yields

$$\operatorname{curl} \mathbf{S} = \nabla\eta \times \nabla u_{\text{Dir}}.$$

But as $\nabla\eta$ is zero near c and far from c and as u_{Dir} is smooth far from c, we deduce that $\operatorname{curl} \mathbf{S}$ belongs to $C_0^\infty(\bar{\Omega} \setminus \mathcal{C})^3$.

Therefore we look for $\mathbf{r} \in \mathbf{V}_0^2(\Omega)$ such that

$$\left\{ \begin{array}{ll} \mathbf{r} = \mathbf{0} & \text{on } \partial\Omega, \\ \operatorname{div} \mathbf{r} = 0 & \text{on } \partial\Omega, \\ \operatorname{curl} \mathbf{r} \times \mathbf{n} = (\operatorname{curl} \mathbf{S}) \times \mathbf{n} & \text{on } \partial\Omega. \end{array} \right.$$

Indeed by the expressions (4.3) and (4.4), this system is equivalent to

$$\begin{cases} \mathbf{r} = \mathbf{0} & \text{on } \partial\Omega, \\ \mathbf{n} \cdot \partial_n \mathbf{r} = 0 & \text{on } \partial\Omega, \\ (\mathbf{n} \times \partial_n \mathbf{r}) \times \mathbf{n} = (\operatorname{curl} \mathbf{S}) \times \mathbf{n} & \text{on } \partial\Omega. \end{cases}$$

This means that it suffices to require that

$$\begin{cases} \mathbf{r} = \mathbf{0} & \text{on } \partial\Omega, \\ \partial_n \mathbf{r} = (\operatorname{curl} \mathbf{S}) \times \mathbf{n} & \text{on } \partial\Omega. \end{cases} \tag{7.8}$$

By the property $\operatorname{curl} \mathbf{S} \in C_0^\infty(\bar{\Omega} \setminus \mathcal{C})^3$, the existence of $\mathbf{r} \in \mathbf{V}_0^2(\Omega)$ satisfying the two boundary conditions (7.8) follows. Indeed applying Theorem 1.5.1.2 of [11] in a smooth domain $\tilde{\Omega}$ which coincides with Ω except in a small neighborhood of the corners we get a function $\tilde{\mathbf{r}} \in \mathbf{H}^2(\tilde{\Omega})$ satisfying (7.8) on the boundary of $\tilde{\Omega}$. We get the desired function by multiplying $\tilde{\mathbf{r}}$ by a cut-off function which is equal to 1 on the support of $\operatorname{curl} \mathbf{S}$ and equal to 0 near the corners. ■

Now we notice that the estimate (3.16) implies that $\mathbf{H}^2(\Omega) \cap V$ is a closed subspace of V, hence we can define the projection P on $\mathbf{H}^2(\Omega) \cap V$ with respect to the inner product a. Let us further set $K = (I - P)V$.

Lemma 7.5 *Under the assumptions of Lemma 3.6, any $\mathbf{u} \in V$ admits the decomposition* (3.12) *where $\mathbf{u}_{\text{reg}} \in \mathbf{H}^2(\Omega) \cap V$, and $\mathbf{u}_{\text{sing}} \in K$ is given by*

$$\mathbf{u}_{\text{sing}} = \sum_{c \in \mathcal{C}} \sum_{\lambda \in (-\frac{3}{2}, \frac{1}{2}): \lambda + 1 \in \Lambda_{\text{Dir}}(\Gamma_c)} \sum_p d_{c,\lambda,p} (I - P) \mathbf{U}_{c,\lambda,p}, \tag{7.9}$$

where for $\lambda \in (-\frac{3}{2}, \frac{1}{2})$ such that $\lambda + 1 \in \Lambda_{\text{Dir}}(\Gamma_c)$, $d_{c,\lambda,p} \in \mathbb{C}$. Consequently $\mathbf{u}_{\text{reg}}$ and $\mathbf{u}_{\text{sing}}$ are orthogonal, namely

$$a(\mathbf{u}_{\text{reg}}, \mathbf{u}_{\text{sing}}) = 0.$$

Proof. Let $\mathbf{u}$ be fixed in V. According to Lemma 3.6 $\mathbf{u}$ admits the decomposition

$$\mathbf{u} = \mathbf{u}_{\text{reg}} + \sum_{c \in \mathcal{C}} \eta_c \sum_{\lambda \in (-\frac{3}{2}, \frac{1}{2}): \lambda + 1 \in \Lambda_{\text{Dir}}(\Gamma_c)} \sum_p d_{c,\lambda,p} \nabla u_{\text{Dir}}^{\lambda+1,p},$$

with $\mathbf{u}_{\text{reg}} \in \mathbf{H}^2(\Omega)$ and $d_{c,\lambda,p}$ in $\mathbb{C}$. Hence by the previous Lemma, we have

$$\mathbf{u} = \mathbf{u}_{\text{reg}}^{(1)} + \sum_{c \in \mathcal{C}} \sum_{\lambda \in (-\frac{3}{2}, \frac{1}{2}): \lambda + 1 \in \Lambda_{\text{Dir}}(\Gamma_c)} \sum_p d_{c,\lambda,p} \mathbf{U}_{c,\lambda,p},$$

with

$$\mathbf{u}_{\text{reg}}^{(1)} = \mathbf{u}_{\text{reg}} + \sum_{c \in \mathcal{C}} \sum_{\lambda \in (-\frac{3}{2}, \frac{1}{2}): \lambda + 1 \in \Lambda_{\text{Dir}}(\Gamma_c)} \sum_p d_{c,\lambda,p} \mathbf{r}_{c,\lambda,p}$$

which clearly belongs to $\mathbf{H}^2(\Omega)$. But it also belongs to V because $\mathbf{u}$ is in V as well as all $\mathbf{U}_{c,\lambda,p}$. We get the orthogonal decomposition by splitting $\mathbf{U}_{c,\lambda,p}$ into the sum of $P\mathbf{U}_{c,\lambda,p}$ and of $(I - P)\mathbf{U}_{c,\lambda,p}$. ■

We are now ready to state the main result of this section.

Theorem 7.6 *Let $\ell \in \mathbb{N}$ and $\beta \in \mathbb{R}$ such that $\beta - \ell \leq 0$. Assume that the lines $\Re\lambda = \frac{1}{2}$ and $\Re\lambda = \ell - \beta + \frac{5}{2}$ are free of corner singular exponents. Assume further that $\frac{1}{2} \notin \Lambda_{Neu}(\Gamma_c)$, for all $c \in \mathcal{C}$. Let $\mathbf{u} \in V$ be the unique solution of (3.3) with $\mathbf{f} \in \mathbf{V}^{\ell}_{\beta}(\Omega)$. Then it admits the next decomposition*

$$\begin{aligned}\mathbf{u} &= \mathbf{u}_R + \sum_{c\in\mathcal{C}} \Big(\sum_{\lambda\in\Lambda_c:\frac{1}{2}<\Re\lambda<\ell-\beta+\frac{5}{2}} \sum_p k_{c,\lambda,p}\mathbf{S}^{\lambda,p} \\ &\quad + \sum_{\lambda\in(-\frac{3}{2},\frac{1}{2}):\lambda+1\in\Lambda_{\rm Dir}(\Gamma_c)} \sum_p d_{c,\lambda,p}(I-P)\mathbf{U}_{c,\lambda,p}\Big),\end{aligned} \tag{7.10}$$

where $\mathbf{u}_R \in \mathbf{V}^{\ell+4}_{\beta}(\Omega)$, $k_{c,\lambda,p} \in \mathbb{C}$ and $d_{c,\lambda,p} \in \mathbb{C}$.

Proof. The assumption $\beta - \ell \leq 0$ implies that $\mathbf{V}^{\ell}_{\beta}(\Omega)$ is embedded into $\mathbf{L}^2(\Omega)$, therefore problem (3.3) has indeed a unique solution $\mathbf{u} \in V$. Then according to Lemma 7.5, $\mathbf{u}$ admits the decomposition (3.12) where $\mathbf{u}_{\rm reg} \in \mathbf{H}^2(\Omega) \cap V$, and $\mathbf{u}_{\rm sing} \in K$ (given by (7.9)). Using a similar decomposition for $\mathbf{v} = \mathbf{v}_{\rm reg} + \mathbf{v}_{\rm sing}$ with $\mathbf{v}_{\rm reg} \in \mathbf{H}^2(\Omega) \cap V$ and $\mathbf{v}_{\rm sing} \in K$, (3.3) is equivalent to

$$a(\mathbf{u}_{\rm reg}, \mathbf{v}_{\rm reg}) = \int_\Omega \mathbf{f} \cdot \bar{\mathbf{v}}_{\rm reg}, \forall \mathbf{v}_{\rm reg} \in \mathbf{H}^2(\Omega) \cap V, \tag{7.11}$$

and the finite linear system

$$a(\mathbf{u}_{\rm sing}, \mathbf{v}_{\rm sing}) = \int_\Omega \mathbf{f} \cdot \bar{\mathbf{v}}_{\rm sing}, \forall \mathbf{v}_{\rm sing} \in K. \tag{7.12}$$

Now by Theorem 7.3, there exists a unique solution $\mathbf{u}_0 \in \mathbf{V}^4_2(\Omega)$ of

$$\left\{ \begin{array}{ll} \Delta^2 \mathbf{u}_0 = \mathbf{f} & \text{in } \Omega, \\ B\mathbf{u}_0 = \mathbf{0} & \text{on } \partial\Omega, \end{array} \right. \tag{7.13}$$

which admits the decomposition (7.6) with $\mathbf{u}_R \in \mathbf{V}^{\ell+4}_{\beta}(\Omega)$ and $k_{c,\lambda,p} \in \mathbb{C}$.

As $\mathbf{u}_0$ belongs to $\mathbf{H}^2(\Omega)$ and satisfies $B\mathbf{u}_0 = \mathbf{0}$ on $\partial\Omega$, it belongs to $\mathbf{H}^2(\Omega) \cap V$. Hence it remains to show that

$$a(\mathbf{u}_0, \mathbf{v}_{\rm reg}) = \int_\Omega \mathbf{f} \cdot \mathbf{v}_{\rm reg}, \forall \mathbf{v}_{\rm reg} \in \mathbf{H}^2(\Omega) \cap V. \tag{7.14}$$

Indeed since the solution of (7.11) is unique, we will deduce that $\mathbf{u}_0 = \mathbf{u}_{\rm reg}$, whence the decomposition (7.10) for $\mathbf{u}$.

For an arbitrary $\mathbf{v}_{\rm reg} \in \mathbf{H}^2(\Omega) \cap V$, we transform the left-hand side of (7.14) by using the decomposition (7.6). First for the term $a(\mathbf{u}_R, \mathbf{v}_{\rm reg})$, as $\mathbf{u}_R$ belongs to $\mathbf{H}^4(\Omega)$, we can apply Green's formula (4.7) to get

$$\begin{aligned} a(\mathbf{u}_R, \mathbf{v}_{\rm reg}) &= \int_\Omega \Delta^2 \mathbf{u}_R \cdot \bar{\mathbf{v}}_{\rm reg} \\ &+ \int_{\partial\Omega} \Big(\operatorname{div} \Delta \mathbf{u}_R \bar{\mathbf{v}}_{\rm reg} \cdot \mathbf{n} - \Delta \mathbf{u}_R \cdot \mathbf{n} \operatorname{div} \bar{\mathbf{v}}_{\rm reg} - \operatorname{curl} \Delta \mathbf{u}_R \cdot (\bar{\mathbf{v}}_{\rm reg} \times \mathbf{n}) - \Delta \mathbf{u}_R \cdot (\operatorname{curl} \bar{\mathbf{v}}_{\rm reg} \times \mathbf{n}) \Big). \end{aligned} \tag{7.15}$$

For the term $a(\mathbf{S}^{\lambda,p}, \mathbf{v}_{\rm reg})$, for one $\lambda \in \Lambda_c$ and p, we denote by $\mathbf{h}_{\lambda,p} = -\Delta \mathbf{S}^{\lambda,p}$, and recall that $\mathbf{h}_{\lambda,p}$ is in $\mathbf{H}^2$ far from c and $\Delta \mathbf{h}_{\lambda,p} = -\Delta^2 \mathbf{S}^{\lambda,p}$ is zero near the corner c. Consequently one has

$$\int_\Omega (\operatorname{curl}^2 - \nabla \operatorname{div}) \mathbf{h}_{\lambda,p} \cdot \mathbf{v}_{\rm reg} = \lim_{\varepsilon\to 0} \int_{\Omega_\varepsilon} (\operatorname{curl}^2 - \nabla \operatorname{div}) \mathbf{h}_{\lambda,p} \cdot \mathbf{v}_{\rm reg},$$

where $\Omega_\varepsilon = \{\mathbf{x} \in \Omega : r(\mathbf{x}) > \varepsilon\}$. Now applying Green's formula in Ω_ε, we find that

$$\begin{aligned}\int_\Omega (\mathrm{curl}^2 - \nabla\,\mathrm{div})\mathbf{h}_{\lambda,p} \cdot \mathbf{v}_{\mathrm{reg}} &= \lim_{\varepsilon\to 0}\Big(\int_{\Omega_\varepsilon} (\mathrm{curl}\;\mathbf{h}_{\lambda,p} \cdot \mathrm{curl}\;\mathbf{v}_{\mathrm{reg}} + \mathrm{div}\;\mathbf{h}_{\lambda,p}\,\mathrm{div}\;\mathbf{v}_{\mathrm{reg}}) \\ &+ \int_{\partial\Omega_\varepsilon} (\mathrm{curl}\;\mathbf{h}_{\lambda,p} \cdot (\mathbf{v}_{\mathrm{reg}} \times \mathbf{n}) - \mathrm{div}\;\mathbf{h}_{\lambda,p}\,\mathbf{v}_{\mathrm{reg}} \cdot \mathbf{n})\Big).\end{aligned}$$

But $\mathbf{v}_{\mathrm{reg}}$ being in $\mathbf{H}^2(\Omega)$ by the Sobolev embedding theorem, $\mathbf{v}_{\mathrm{reg}}$ belongs to $C^{0,\frac{1}{2}}(\bar{\Omega})$ and since $\mathbf{v}_{\mathrm{reg}} \times \mathbf{n} = \mathbf{0}$, we deduce that $\mathbf{v}_{\mathrm{reg}}(c) = \mathbf{0}$ and

$$|\mathbf{v}_{\mathrm{reg}}(\mathbf{x})| \lesssim r^{\frac{1}{2}}.$$

Since curl $\mathbf{h}_{\lambda,p}$ and div $\mathbf{h}_{\lambda,p}$ behaves like $r^{\lambda-3}$ near c, we deduce that

$$\Big|\int_{\partial\Omega_\varepsilon \cap \{r=\varepsilon\}} (\mathrm{curl}\;\mathbf{h}_{\lambda,p} \cdot (\mathbf{v}_{\mathrm{reg}} \times \mathbf{n}) - \mathrm{div}\;\mathbf{h}_{\lambda,p}\,\mathbf{v}_{\mathrm{reg}} \cdot \mathbf{n})\Big| \lesssim \varepsilon^{\Re\lambda - \frac{1}{2}},$$

and since $\Re\lambda - \frac{1}{2} > 0$, we get

$$\lim_{\varepsilon\to 0}\int_{\partial\Omega_\varepsilon \cap \{r=\varepsilon\}} (\mathrm{curl}\;\mathbf{h}_{\lambda,p} \cdot (\mathbf{v}_{\mathrm{reg}} \times \mathbf{n}) - \mathrm{div}\;\mathbf{h}_{\lambda,p}\,\mathbf{v}_{\mathrm{reg}} \cdot \mathbf{n}) = 0.$$

Further as $\mathrm{div}\,\mathbf{h}_{\lambda,p} = -\,\mathrm{div}\,\Delta\mathbf{S}^{\lambda,p}$ is zero on $V \cap \partial\Omega$ with a sufficiently small neighborhood V of the corners, $\mathbf{v}_{\mathrm{reg}} \times \mathbf{n} = \mathbf{0}$ on $\partial\Omega$ and with the previous property, we deduce that

$$\lim_{\varepsilon\to 0}\int_{\partial\Omega_\varepsilon} (\mathrm{curl}\;\mathbf{h}_{\lambda,p} \cdot (\mathbf{v}_{\mathrm{reg}} \times \mathbf{n}) - \mathrm{div}\;\mathbf{h}_{\lambda,p}\,\mathbf{v}_{\mathrm{reg}} \cdot \mathbf{n}) = -\int_{\partial\Omega} \mathrm{div}\;\mathbf{h}_{\lambda,p}\,\mathbf{v}_{\mathrm{reg}} \cdot \mathbf{n}.$$

On the other hand as div $\mathbf{v}_{\mathrm{reg}} \in H_0^1(\Omega)$ and curl $\mathbf{v}_{\mathrm{reg}} \in H_0^1(\Omega)^3$, by Hardy's inequality we deduce

$$r^{-1}\,\mathrm{div}\;\mathbf{v}_{\mathrm{reg}} \in L^2(\Omega) \text{ and } r^{-1}\,\mathrm{curl}\;\mathbf{v}_{\mathrm{reg}} \in \mathbf{L}^2(\Omega).$$

This implies that

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} (\mathrm{curl}\;\mathbf{h}_{\lambda,p} \cdot \mathrm{curl}\;\mathbf{v}_{\mathrm{reg}} + \mathrm{div}\;\mathbf{h}_{\lambda,p}\,\mathrm{div}\;\mathbf{v}_{\mathrm{reg}}) = \int_\Omega (\mathrm{curl}\;\mathbf{h}_{\lambda,p} \cdot \mathrm{curl}\;\mathbf{v}_{\mathrm{reg}} + \mathrm{div}\;\mathbf{h}_{\lambda,p}\,\mathrm{div}\;\mathbf{v}_{\mathrm{reg}}),$$

and therefore we have proved that

$$\begin{aligned}\int_\Omega (\mathrm{curl}^2 - \nabla\,\mathrm{div})\mathbf{h}_{\lambda,p} \cdot \mathbf{v}_{\mathrm{reg}} &= \int_\Omega (\mathrm{curl}\;\mathbf{h}_{\lambda,p} \cdot \mathrm{curl}\;\mathbf{v}_{\mathrm{reg}} + \mathrm{div}\;\mathbf{h}_{\lambda,p}\,\mathrm{div}\;\mathbf{v}_{\mathrm{reg}}) \\ &- \int_{\partial\Omega} \mathrm{div}\;\mathbf{h}_{\lambda,p}\,\mathbf{v}_{\mathrm{reg}} \cdot \mathbf{n}.\end{aligned} \tag{7.16}$$

At this stage we again have to integrate by parts in the first term of this right-hand side. For that purpose, we fix a cut-off function η with a support included in V and such that $\eta = 1$ near the corners. The previous considerations show that $\eta\,\mathrm{div}\;\mathbf{v}_{\mathrm{reg}} \in H_0^1(\Omega)$ and $\eta\,\mathrm{curl}\;\mathbf{v}_{\mathrm{reg}} \in H_0^1(\Omega)^3$, since $\mathcal{D}(\Omega)$ is dense in $H_0^1(\Omega)$, there exists a sequence $d_n \in \mathcal{D}(\Omega)$ (resp. $\mathbf{w}_n \in \mathcal{D}(\Omega)^3$), $n \in \mathbb{N}$ such that

$$\begin{aligned} d_n &\to \eta\,\mathrm{div}\;\mathbf{v}_{\mathrm{reg}} \text{ in } H_0^1(\Omega), \text{ as } n \to \infty, \\ \mathbf{w}_n &\to \eta\,\mathrm{curl}\;\mathbf{v}_{\mathrm{reg}} \text{ in } H_0^1(\Omega)^3, \text{ as } n \to \infty.\end{aligned}$$

Hence

$$\int_\Omega (\text{curl } \mathbf{h}_{\lambda,p} \cdot (\eta \,\text{curl } \mathbf{v}_{\text{reg}}) + \text{div } \mathbf{h}_{\lambda,p}(\eta \,\text{div } \mathbf{v}_{\text{reg}})) = \lim_{n\to\infty} \int_\Omega (\text{curl } \mathbf{h}_{\lambda,p} \cdot \mathbf{w}_n + \text{div } \mathbf{h}_{\lambda,p}\, d_n),$$

and by Green's formula, we deduce that

$$\int_\Omega (\text{curl } \mathbf{h}_{\lambda,p} \cdot (\eta \,\text{curl } \mathbf{v}_{\text{reg}}) + \text{div } \mathbf{h}_{\lambda,p}(\eta \,\text{div } \mathbf{v}_{\text{reg}})) = \lim_{n\to\infty} \int_\Omega \mathbf{h}_{\lambda,p} \cdot (\text{curl } \mathbf{w}_n - \nabla d_n).$$

As $\mathbf{h}_{\lambda,p}$ belongs to $\mathbf{L}^2(\Omega)$, we can pass to the limit in this right-hand side and deduce that

$$\begin{aligned} \int_\Omega &(\text{curl } \mathbf{h}_{\lambda,p} \cdot (\eta \,\text{curl } \mathbf{v}_{\text{reg}}) + \text{div } \mathbf{h}_{\lambda,p}(\eta \,\text{div } \mathbf{v}_{\text{reg}})) \\ &= \int_\Omega \mathbf{h}_{\lambda,p} \cdot (\text{curl}\,(\eta \,\text{curl } \mathbf{v}_{\text{reg}}) - \nabla(\eta \,\text{div } \mathbf{v}_{\text{reg}})). \end{aligned} \tag{7.17}$$

As $(1-\eta)\,\text{div } \mathbf{v}_{\text{reg}}$ and $(1-\eta)\,\text{curl } \mathbf{v}_{\text{reg}}$ are in H^1 and is zero near the corners, we can directly apply Green's formula to get

$$\begin{aligned} \int_\Omega &(\text{curl } \mathbf{h}_{\lambda,p} \cdot ((1-\eta)\,\text{curl } \mathbf{v}_{\text{reg}}) + \text{div } \mathbf{h}_{\lambda,p}((1-\eta)\,\text{div } \mathbf{v}_{\text{reg}})) \\ &= \int_\Omega \mathbf{h}_{\lambda,p} \cdot (\text{curl}\,((1-\eta)\,\text{curl } \mathbf{v}_{\text{reg}}) - \nabla((1-\eta)\,\text{div } \mathbf{v}_{\text{reg}})). \end{aligned}$$

This identity with (7.17) implies that

$$\int_\Omega (\text{curl } \mathbf{h}_{\lambda,p} \cdot \text{curl } \mathbf{v}_{\text{reg}} + \text{div } \mathbf{h}_{\lambda,p}\, \text{div } \mathbf{v}_{\text{reg}}) \quad = \quad \int_\Omega \mathbf{h}_{\lambda,p} \cdot (\text{curl curl } \mathbf{v}_{\text{reg}}) - \nabla\, \text{div } \mathbf{v}_{\text{reg}}).$$

Inserting this identity in (7.16), we have shown that

$$\int_\Omega (\text{curl}^2 - \nabla\, \text{div})\mathbf{h}_{\lambda,p} \cdot \mathbf{v}_{\text{reg}} = a(\mathbf{S}^{\lambda,p}, \mathbf{v}_{\text{reg}}) - \int_{\partial\Omega} \text{div } \mathbf{h}_{\lambda,p}\, \mathbf{v}_{\text{reg}} \cdot \mathbf{n}.$$

This identity with (7.15) and the expression (7.6) lead to

$$\begin{aligned} a(\mathbf{u}_0, \mathbf{v}_{\text{reg}}) \quad &= \quad \int_\Omega \Delta^2 \mathbf{u}_0 \cdot \bar{\mathbf{v}}_{\text{reg}} + \int_{\partial\Omega} \text{div}\, \Delta \mathbf{u}_0 \bar{\mathbf{v}}_{\text{reg}} \cdot \mathbf{n} \\ &= \quad \int_\Omega \mathbf{f} \cdot \bar{\mathbf{v}}_{\text{reg}}, \end{aligned} \tag{7.18}$$

due to (7.13). This proves (7.14). ■

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