A class of robust numerical schemes to compute front propagation

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ABSTRACT. In this work a class of finite volume schemes is proposed to numerically solve equations involving propagating fronts. They fall into the class of Hamilton-Jacobi equations. Finite volume schemes based on staggered grids, and initially developed to compute fluid flows, are adapted to the G-equation, using the Hamilton-Jacobi theoretical framework. The designed scheme has a maximum principle property and is consistent an monotonous on Cartesian grids. A convergence property is then obtained for the scheme on Cartesian grids and numerical experiments evidence the convergence of the scheme on more general meshes.

1. INTRODUCTION

The work presented here falls into a larger thematic undertaken for several years, which is the development of staggered schemes to simulate all Mach flows. Numerical schemes were proposed for the Navier-Stokes equations [10] , and Euler equations [11, 12]. Adaptations of these schemes to more complex models, such as reactive mixture flows, is underway. In this context, equations describing reactive front propagation are involved and need to be discretized using natural extensions of the staggered schemes.

We focus on a particular equation, used in the combustion science to simulate flame front propagation, the so called G-equation, which reads :

$$\partial_t(\rho G) + \operatorname{div}(\rho \boldsymbol{u} G) + \rho u_f |\boldsymbol{\nabla} G| = 0, \tag{1}$$

where ρ is the density of the fluid, G stands for the front indicator, $\boldsymbol{u}$ is a convective velocity and u_f is a front propagation speed. The challenging issue is to adapt staggered discretization to the last term $\rho u_f|\boldsymbol{\nabla} G|$ as the convective part of the equation has already been handled previously. When combined with the mass balance equation of the system

$$\partial_t \rho + \operatorname{div}(\rho \boldsymbol{u}) = 0,$$

the convective part of the equation is a transport operator and we get:

$$\partial_t G + \boldsymbol{u} \cdot \boldsymbol{\nabla} G + u_f |\boldsymbol{\nabla} G| = 0, \tag{2}$$

provided that the density never vanish. This is a particular Hamilton-Jacobi equation. The theory of such equations is well known and was vastly developed by J.-L. Lions in [9, 14]. More precisely, consider the following Cauchy problem:

$$\begin{cases} \partial_t G + H(\boldsymbol{\nabla} G) = 0, \\ G(0, \boldsymbol{x}) = G_0(\boldsymbol{x}), \end{cases} \tag{3}$$

defined on $[0, T] \times \mathbb{R}^d$, with $H \in C(\mathbb{R}^d)$ and $G_0 \in \mathrm{BUC}(\mathbb{R}^d)$ ($\mathrm{BUC}(\Omega)$ stands for the set of bounded uniformly continuous functions on Ω). There exists exactly one viscosity solution $G \in \mathrm{BUC}([0, T] \times \mathbb{R}^d)$ such that $G(0, \boldsymbol{x}) = G_0(\boldsymbol{x})$ and G satisfies a particular weak formulation based on the maximum principle (we refer to [14] for

2010 *Mathematics Subject Classification.* 35F21, 65N08, 65N12.
Key words and phrases. Finite volumes, Hamilton-Jacobi, Stability, Convergence.

more details). Various numerical methods exist to approach such viscosity solutions. A first converging finite difference scheme was developed in [8]. From this point high order extensions to this scheme were given by S.Osher and James A. Sethian in [15], and a simple finite volume scheme was derived in [13], inspired from a unstructured finite difference scheme based on triangular meshes developed by R. Abgrall in [1]. The convergence theory of numerical approximations of Hamilton Jacobi equations, was first proposed for finite difference scheme in [8] and a generalized formulation was given in [3, 21]. Since then, various schemes were presented for Hamilton-Jacobi equations; high-order finite difference schemes in [6, 19, 16] and schemes for unstructured meshes [5, 20, 22, 2]. These methods are difficult to adapt to our problem. Besides, all the existing schemes proposed in the literature are designed to solve very generic Hamilton-Jacobi equations. In this paper, we only deal with a very particular operator, namely, $H(\boldsymbol{x}) = \boldsymbol{u} \cdot \boldsymbol{x} + u_f|\boldsymbol{x}|$. Consequently, we propose a finite volume discretization of $u_f|\boldsymbol{\nabla} G|$ that is compatible with the staggered discretization of the transport operator $\boldsymbol{u} \cdot \boldsymbol{\nabla} G$.

For the sake of clarity, we focus on key elements of the discretization and we suppose that $\boldsymbol{u} = 0$ and $u_f = 1$, so the problem considered here is the unsteady eikonal equation,

$$\partial_t G + |\boldsymbol{\nabla} G| = 0, \tag{4a}$$

$$G(0, \boldsymbol{x}) = G_0(\boldsymbol{x}), \quad \forall \boldsymbol{x} \in \mathbb{R}^d. \tag{4b}$$

$G_0 \in \mathrm{BUC}(\mathbb{R}^d)$. The choice of such a simplified model is also convenient as its analytical solutions can be computed easily (see appendix A for more details). The scheme proposed to approximate this problem can be defined on unstructured meshes. On Cartesian grids, The scheme is consistent and monotonous and the L^∞ convergence is proved thanks to the theory developed in [3]. Numerical results are given to highlight this convergence results as well as the numerical convergence of the scheme on unstructured discretizations.

The paper is organized as follows. We start by the description of the spatial discretization and the corresponding notations that are used throughout the paper. We present the scheme and its properties in the second part. We finish with some convergence and numerical results.

2. Spatial discretization

In this section, we focus on the discretization of a multi-dimensional domain (*i.e.* $d = 2$ or $d = 3$); the extension to the one-dimensional case is straightforward.

Let $\mathcal{M}$ be a mesh of the domain Ω (which is an open bounded connected subset of $\mathbb{R}^d$ or $\mathbb{R}^d$ itself), supposed to be regular in the usual sense of the finite element literature (*e.g.* [7]). The cells of the mesh are assumed to be:

- for a general domain Ω, either non-degenerate quadrilaterals ($d = 2$) or hexahedra ($d = 3$) or simplices, both types of cells being possibly combined in a same mesh,
- for a domain whose boundaries are hyperplanes normal to a coordinate axis, rectangles ($d = 2$) or rectangular parallelepipeds ($d = 3$) (the faces of which, of course, are then also necessarily normal to a coordinate axis).

By $\mathcal{E}$ and $\mathcal{E}(K)$ we denote the set of all $(d-1)$-faces σ of the mesh and of the element $K \in \mathcal{M}$ respectively. The set of faces included in the boundary of Ω is denoted by $\mathcal{E}_{\text{ext}}$ and the set of internal faces (*i.e.* $\mathcal{E} \setminus \mathcal{E}_{\text{ext}}$) is denoted by $\mathcal{E}_{\text{int}}$; a face $\sigma \in \mathcal{E}_{\text{int}}$ separating the cells K and L is denoted by $\sigma = K|L$. The outward normal vector to a face σ of K is denoted by $\boldsymbol{n}_{K,\sigma}$. For $K \in \mathcal{M}$ and $\sigma \in \mathcal{E}$, we denote by $|K|$ the measure of K and by $|\sigma|$ the $(d-1)$-measure of the face σ. The mass center of a face is denoted by $\boldsymbol{x}_\sigma$.

Finally we denote by d_σ the measure of $\overrightarrow{\boldsymbol{x}_K\boldsymbol{x}_L}$.

The unknown discrete function G is piecewise constant on the cells K. We denote by $H_\mathcal{M}$ the space of such piecewise constant functions.

$$G_\mathcal{M} \in H_\mathcal{M} \Longleftrightarrow G_\mathcal{M} = \sum_{K\in\mathcal{M}} G_K \mathcal{X}_K,$$

where $\mathcal{X}_O$ stands for the characteristic function of the set O.

3. The scheme

The problem eq. (4) is posed over $\mathbb{R}^d \times (0,T)$, where $(0,T)$ is a finite time interval. Concerning the initial data, we have $G_0 \in \mathrm{BUC}(\mathbb{R}^d)$. According to the known results at the continuous level, the problem has a unique viscosity solution in $\mathrm{BUC}([0,T]\times\mathbb{R}^d)$, that we denote $\bar{G}$. In order to be able to perform computations, the domain can be reduced to an open bounded connected subset Ω of $\mathbb{R}^d$ with zero-flux boundary conditions. We propose three versions of the scheme depending on the regularity of the mesh. The finite volume scheme is written on an alternative form of Equation eq. (4a) :

$$\partial_t G + \left(\frac{\boldsymbol{\nabla} G}{|\boldsymbol{\nabla} G|}\right)\cdot\boldsymbol{\nabla} G = 0, \tag{5}$$

and makes use of the classical identity:

$$\boldsymbol{u}.\boldsymbol{\nabla}\phi = \mathrm{div}(\phi\boldsymbol{u}) - \phi\,\mathrm{div}(\boldsymbol{u}). \tag{6}$$

Let us consider a partition $0 = t_0 < t_1 < \ldots < t_N = T$ of the time interval $(0,T)$, which we suppose uniform for the sake of simplicity, and let $\delta t = t_{n+1} - t_n$ for $n = 0, 1, \ldots, N-1$ be the (constant) time step. We consider an explicit-in-time scheme, which reads, for $0 \le n \le N-1$ and $K \in \mathcal{M}$:

$$\eth_t G^n + F_\mathcal{M}(G^n) = 0, \tag{7}$$

with,

$$\eth_t G^n = \sum_{K\in\mathcal{M}} \frac{G_K^{n+1} - G_K^n}{\delta t}\mathcal{X}_K, \tag{8}$$

and

$$F_\mathcal{M}(G^n) = \mathrm{div}\left(\frac{\boldsymbol{\nabla}_\mathcal{E} G^n}{|\boldsymbol{\nabla}_\mathcal{E} G^n|}G^n\right)_K - G_K^n\,\mathrm{div}\left(\frac{\boldsymbol{\nabla}_\mathcal{E} G^n}{|\boldsymbol{\nabla}_\mathcal{E} G^n|}\right)_K. \tag{9}$$

The discrete divergence operator is given by:

$$\text{for } K\in\mathcal{M}, \qquad (\mathrm{div}\boldsymbol{u})_K = \frac{1}{|K|}\sum_{\sigma=K|L\in\mathcal{E}(K)} \kappa_{K,\sigma}^\mathcal{M}|\sigma|\ \boldsymbol{u}_\sigma.\boldsymbol{n}_{K,\sigma}, \tag{10}$$

where $\kappa_{K,\sigma}^\mathcal{M}$ is a coefficient equal to 1 for unstructured meshes, and equal to $\kappa_{K,\sigma}^\mathcal{M} = \dfrac{2\,|K|}{|K|+|L|}$ on Cartesian grids. Likewise

$$\text{for } K\in\mathcal{M}, \qquad (\mathrm{div}\,G\boldsymbol{u})_K = \frac{1}{|K|}\sum_{\sigma=K|L\in\mathcal{E}(K)} \kappa_{K,\sigma}^\mathcal{M}|\sigma|\ G_\sigma\boldsymbol{u}_\sigma.\boldsymbol{n}_{K,\sigma}, \tag{11}$$

where G_σ denotes an interpolation of G on the edge σ that is:

$$\text{for } \sigma = K|L \in \mathcal{E}_{\mathrm{int}}, \quad G_\sigma = \left|\begin{array}{ll} G_K & \text{if } \boldsymbol{u}_\sigma.\boldsymbol{n}_{K,\sigma} \ge 0, \\ G_L & \text{otherwise.}\end{array}\right.$$

For a face $\sigma \in \mathcal{E}_{\text{ext}}$ one simply take $G_\sigma = G_K$ so that

$$\boldsymbol{\nabla} G \cdot \boldsymbol{n}_{K,\sigma} = \frac{|\sigma|}{|K|}(G_\sigma - G_K) = 0.$$

The expression of the discrete spatial operator eq. (9) becomes

$$F_{\mathcal{M}}(G^n_{\mathcal{M}}) = \sum_{K\in\mathcal{M}} \left[\sum_{\sigma=K|L\in\mathcal{E}(K)} \kappa^{\mathcal{M}}_{K,\sigma} \frac{|\sigma|}{|K|} \frac{(\boldsymbol{\nabla}_{\mathcal{E}} G^n)_\sigma}{|(\boldsymbol{\nabla}_{\mathcal{E}} G^n)_\sigma|} \cdot \boldsymbol{n}_{K,\sigma}(G^n_\sigma - G^n_K) \right] \mathcal{X}_K, \tag{12}$$

where $\boldsymbol{\nabla}_{\mathcal{E}}$ refers to a discrete gradient operator defined on every $\sigma \in \mathcal{E}_{\text{int}}$.

3.1. **Unstructured meshes.** For $\sigma = K|L \in \mathcal{E}_{\text{int}}$, we take:

$$(\boldsymbol{\nabla}_{\mathcal{E}} G)_\sigma = \sum_{\sigma\in\partial(K\cup L)} \frac{|\sigma|}{|K\cup L|} \tilde{G}_\sigma \boldsymbol{n}_{K\cup L,\sigma}, \tag{13}$$

with $\tilde{G}_\sigma$ a second order approximation of G at the barycenter of the face σ.

3.2. **Cartesian meshes.** When the scheme is based on Cartesian grids, we have for $\sigma = \overrightarrow{K|L}$ (which means the flow goes from K to L) :

$$\text{For } \sigma \in \mathcal{E}_{\text{int}}, \qquad (\boldsymbol{\nabla}_{\mathcal{E}} G)_\sigma = \left[\frac{G_L - G_K}{d_\sigma} \boldsymbol{n}_{K,\sigma} + \boldsymbol{\nabla}_{//\sigma} G \right], \tag{14}$$

where $\boldsymbol{\nabla}^C_{//\sigma}$ is defined by:

$$(\boldsymbol{\nabla} G)^C_{//\sigma} = \sum_{\substack{i=1,\\ \boldsymbol{e}^{(i)}\cdot\boldsymbol{n}_{K,\sigma}=0}}^{d} \frac{(G_{K_i^+} - G_K)^+}{d_{\sigma_i^+}} - \frac{1}{2}\left(1 - \operatorname{sgn}(G_{K_i^+} - G_K)^+\right) \frac{(G_K - G_{K_i^-})^-}{d_{\sigma_i^-}} \boldsymbol{e}^{(i)}, \tag{15}$$

with $\sigma = \overrightarrow{K|L}$. For a cell $K \in \mathcal{M}$, σ_i^+ and σ_i^- stand for the two faces of K normal to $\boldsymbol{e}^{(i)}$. Superscripts $-$ and $+$ refer to the up and down faces of K respectively. We set $\sigma_i^+ = K|K_i^+$ and $\sigma_i^- = K|K_i^-$. We illustrate these notations in the following figure. We recall that $a^+ = \max(a, 0)$ and $a^- = \max(-a, 0)$, for $a \in \mathbb{R}$.

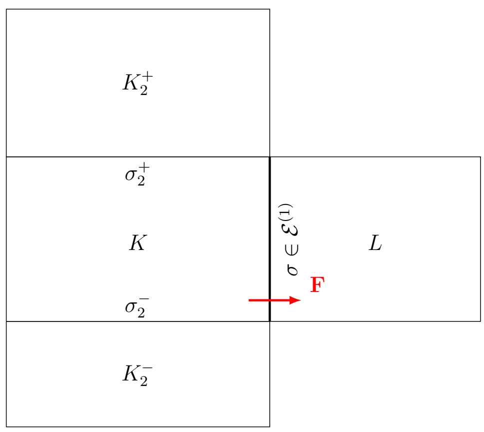


FIGURE 1. Notations for the alternative gradient definition on Cartesian grids with $\mathbf{F} = (G_L - G_K)\boldsymbol{n}_{K,\sigma}$.

3.3. **High order extension.** It is possible to replace the upwind interpolation by a higher order interpolation based on a MUSCL reconstruction. Adopting the same notations as in eq. (11), its important property, based on [18] is stated below. For any $K \in \mathcal{M}$, and for any $\sigma \in \mathcal{E}(K) \cap \mathcal{E}_{\text{int}}$, there exists $\alpha_{K,\sigma} \in [0,1]$ such that :

$$G_\sigma - G_K = \left| \begin{array}{ll} \beta_{K,\sigma}(G_K - G_{M_\sigma^K}) & \text{if } \dfrac{\boldsymbol{\nabla}_{\mathcal{E}} G_\sigma^n}{|\boldsymbol{\nabla}_{\mathcal{E}} G_\sigma^n|} \cdot \boldsymbol{n}_{K,\sigma} \geq 0, \\ \beta_{K,\sigma}(G_{M_\sigma^K} - G_K) & \text{otherwise.} \end{array} \right. \tag{16}$$

The procedure is the following:

- We define a tentative value $\tilde{G}_\sigma$ based on a high order geometric interpolation.
- The next step is to create a limitation procedure for ρ_σ and e_σ. Let $\sigma \in \mathcal{E}_{\text{int}}$, $\sigma = \overrightarrow{K|L}$ and V_K a set of neighboring cells to K. We make the two following assumptions :

$$\begin{array}{ll} (H1) & G_\sigma - G_K \in |[0, \frac{\zeta^+}{2}(G_L - G_K)]| \\ (H2) & \exists M \in V_K,\, G_\sigma - G_K \in |[0, \frac{\zeta^-}{2}\frac{d_\sigma}{d_{K|M}}(G_K - G_M)]|, \end{array} \tag{17}$$

  where, for $a, b \in \mathbb{R}$, we denote by $|[a, b]|$ the convex hull of a and b and $\overrightarrow{K|L}$ means that the flow is going from K to L ($\dfrac{\boldsymbol{\nabla}_{\mathcal{E}} G_\sigma^n}{|\boldsymbol{\nabla}_{\mathcal{E}} G_\sigma^n|}.\boldsymbol{n}_{K,\sigma} \geq 0$). The parameters ζ^+ and ζ^- lie in $[0, 2]$.
- We compute G_σ as the nearest point to $\tilde{G}_\sigma$ in the limitation interval.

Whenever it is possible (*i.e.* with a mesh obtained by Q_1 mappings from the $(0,1)^d$ reference element), V_K may be chosen as the opposite cells to σ in K. Otherwise V_K is defined as the set of "upstream cells" to K. Note that, for a structured mesh, the first choice allows to recover the usual minmod limiter.

Remark 3.1 (Cartesian grids). *We impose $\zeta^+ = \zeta^- = 1$ for the Cartesian version of the scheme. This particular choice of parameters is the only one possible if we want to get consistency properties for the discrete spatial operator of the scheme.*

4. Properties of the scheme

We expose in this section the properties of the scheme. Specific paragraph is devoted to its additional properties on Cartesian grids, derived from the convergence theory [3, 21]. This ensures that the given discretization behaves like usual finite difference methods for Hamilton-Jacobi equations.

4.1. **Stability.** Thanks to the definition of the discrete convective operator, we have the following property:

Proposition 4.1 (Maximum principle). *Let $G_{\mathcal{M}}^n \in H_{\mathcal{M}}$, $n \in [0, N]$, be the solution of the scheme eq. (7). For all $K \in \mathcal{M}$ and $n \in [0, N-1]$, we have:*

$$\min_{L \in \mathcal{M}} G_L^n \leq G_K^{n+1} \leq \max_{L \in \mathcal{M}} G_L^n,$$

under the CFL condition:

$$\delta t \leq \min_{K \in \mathcal{M}} \frac{|K|}{\displaystyle\sum_{\sigma \in \mathcal{E}(K)} |\sigma|} \tag{18}$$

Proof. We have, for $K \in \mathcal{M}$ and $n \in [0, N-1]$:

$$G_K^{n+1} = \left(1 - \delta t \sum_{\sigma \in \mathcal{E}(K)} \frac{|\sigma|}{|K|} \left(\frac{\boldsymbol{\nabla}_{\mathcal{E}} G_\sigma^n}{|\boldsymbol{\nabla}_{\mathcal{E}} G_\sigma^n|} \cdot \boldsymbol{n}_{K,\sigma} \right)^- \right) G_K^n + \delta t \sum_{\sigma = K|L \in \mathcal{E}(K)} \frac{|\sigma|}{|K|} \left(\frac{\boldsymbol{\nabla}_{\mathcal{E}} G_\sigma^n}{|\boldsymbol{\nabla}_{\mathcal{E}} G_\sigma^n|} \cdot \boldsymbol{n}_{K,\sigma} \right)^- G_L^n.$$

Consequently, G_K^{n+1} is a convex combination of its neighbors at time n if eq. (18) is verified, which completes the proof. □

Remark 4.1 (Cartesian grids)**.** *The property remains the same with the scheme on Cartesian grids, only the CFL is modified. One must replace* $|K|$ *by* $\frac{|K|+|L|}{2}$ *in eq.* (18).

Remark 4.2 (MUSCL interpolation)**.** *Concerning the MUSCL interpolation, we use the property eq.* (16) *and use it in the scheme to get:*

$$G_K^{n+1} = \left(1 - \delta t \sum_{\sigma \in \mathcal{E}(K)} \frac{|\sigma|}{|K|} \beta_{K,\sigma} \left| \frac{\boldsymbol{\nabla}_{\mathcal{E}} G_\sigma^n}{|\boldsymbol{\nabla}_{\mathcal{E}} G_\sigma^n|} \cdot \boldsymbol{n}_{K,\sigma} \right| \right) G_K^n + \delta t \sum_{\sigma \in \mathcal{E}(K)} \frac{|\sigma|}{|K|} \beta_{K,\sigma} \left| \frac{\boldsymbol{\nabla}_{\mathcal{E}} G_\sigma^n}{|\boldsymbol{\nabla}_{\mathcal{E}} G_\sigma^n|} \cdot \boldsymbol{n}_{K,\sigma} \right| G_{M_K^\sigma}^n.$$

The maximum principle is still satisfied with the same CFL condition.

5. Invariance under translation

Proposition 5.1 (Invariance under Translation with constants)**.**
$\forall \lambda \in \mathbb{R}$, *and* $\forall \phi_{\mathcal{M}} \in H_{\mathcal{M}}$,

$$F_{\mathcal{M}}(\phi_{\mathcal{M}} + \lambda) = F_{\mathcal{M}}(\phi_{\mathcal{M}}). \tag{19}$$

Proof. Let $\lambda \in \mathbb{R}$ and $\phi_{\mathcal{M}} \in H_{\mathcal{M}}$. Looking at eq. (12), we need to check that $\boldsymbol{\nabla}_{\mathcal{E}} (\phi_{\mathcal{M}} + \lambda) = \boldsymbol{\nabla}_{\mathcal{E}} \phi_{\mathcal{M}}$. We remind that:

$$\boldsymbol{\nabla}_{\mathcal{E}} (\phi_{\mathcal{M}} + \lambda) = \sum_{\sigma \in \partial K \cup L} \frac{|\sigma|}{|K \cup L|} (\phi_\sigma + \lambda)\, \boldsymbol{n}_{K \cup L, \sigma}$$

We have:

$$\boldsymbol{\nabla}_{\mathcal{E}} (\phi_{\mathcal{M}} + \lambda) = \boldsymbol{\nabla}_{\mathcal{E}} \phi_{\mathcal{M}} + \lambda \sum_{\sigma \in \partial K \cup L} \frac{|\sigma|}{|K \cup L|} \boldsymbol{n}_{K \cup L, \sigma}.$$

Using the divergence theorem, we get that:

$$\sum_{\sigma \in \partial K \cup L} \frac{|\sigma|}{|K \cup L|} \boldsymbol{n}_{K \cup L, \sigma} = \int_{K \cup L} \boldsymbol{\nabla}(1) = 0,$$

which concludes the proof. □

On Cartesian meshes, the result is immediate.

5.1. Properties of the Cartesian scheme.

5.1.1. *Consistency.* We need to define interpolates of test functions on the mesh. Let $\phi \in C_c^\infty(\Omega)$. We set:

$$\phi_{\mathcal{M}} = \sum_{K \in \mathcal{M}} \phi_K \mathcal{X}_K, \qquad \phi_K = \phi(\boldsymbol{x}_K). \tag{20}$$

We now give the definition of the consistency property.

Definition 5.1 (Consistency)**.** *Let $F(G)$ be an operator approximated by $F_{\mathcal{M}}(G_{\mathcal{M}})$. Let $h_{\mathcal{M}} = \max_{K \in \mathcal{M}} diam(K)$. Let $\mathcal{D}^{(m)} = \left\{\mathcal{M}^{(m)}, \mathcal{E}^{(m)}, \mathcal{P}^{(m)}\right\}$ be a sequence of discretizations such that the size $h_{\mathcal{M}}^{(m)}$ tends to zero as $m \to \infty$. The discrete spatial operator $H_{\mathcal{M}}$ is said to be consistent with H if for every $\phi \in C_c^\infty(\Omega)$:*

$$\lim_{m \to \infty} \| F_{\mathcal{M}^{(m)}}(\phi_{\mathcal{M}^{(m)}}) - F(\phi) \|_{L^\infty(\Omega)} = 0.$$

The next proposition states

Proposition 5.2. *The spatial operator in the Cartesian case, given by, for $G_{\mathcal{M}} \in H_{\mathcal{M}}$:*

$$F_{\mathcal{M}}(G_{\mathcal{M}}) = \sum_{K \in \mathcal{M}} \left[\sum_{\sigma = K|L \in \mathcal{E}(K)} \frac{1}{d_\sigma} \frac{(G_L - G_K)}{\sqrt{(G_L - G_K)^2 + d_\sigma^2 |\boldsymbol{\nabla}_{//\sigma} G_{\mathcal{M}}|^2}} (G_\sigma - G_K) \right] \mathcal{X}_K, \tag{21}$$

is consistent with $|\boldsymbol{\nabla} G|$.

Proof. Let $\phi \in C_c^\infty(\Omega)$ and $\phi_{\mathcal{M}} \in H_{\mathcal{M}}$ its interpolation on the mesh. Consider $K \in \mathcal{M}$ and $\boldsymbol{v}$ a constant vector. Let $\tilde{F}_K(\phi_{\mathcal{M}}, \boldsymbol{v})$ be:

$$\tilde{F}_K(\phi_{\mathcal{M}}, \boldsymbol{v}) = \sum_{\sigma = K|L \in \mathcal{E}(K)} \frac{1}{d_\sigma} (\boldsymbol{v} \cdot \boldsymbol{n}_{K,\sigma})(\phi_\sigma - \phi_K).$$

With the upwind interpolation, we get that:

$$\tilde{F}_K(\phi_{\mathcal{M}}, \boldsymbol{v}) = - \sum_{\sigma = K|L \in \mathcal{E}(K)} \frac{1}{d_\sigma} (\boldsymbol{v} \cdot \boldsymbol{n}_{K,\sigma})^- (\phi_L - \phi_K).$$

A simple Taylor expansion leads to:

$$\tilde{F}_K(\phi_{\mathcal{M}}, \boldsymbol{v}) = - \sum_{\sigma = K|L \in \mathcal{E}(K)} (\boldsymbol{v} \cdot \boldsymbol{n}_{K,\sigma})^- \boldsymbol{\nabla}\phi(\boldsymbol{x}_K).\boldsymbol{n}_{K,\sigma} + \mathcal{O}(h_{\mathcal{M}}),$$

so

$$\tilde{F}_K(\phi_{\mathcal{M}}, \boldsymbol{v}) = \boldsymbol{\nabla}\phi(\boldsymbol{x}_K) \cdot \sum_{\sigma = K|L \in \mathcal{E}(K)} (\boldsymbol{v} \cdot \boldsymbol{n}_{L,\sigma})^+ \boldsymbol{n}_{L,\sigma} + \mathcal{O}(h_{\mathcal{M}}).$$

Thanks to the Cartesian grid, we have:

$$\sum_{\sigma = K|L \in \mathcal{E}(K)} (\boldsymbol{v} \cdot \boldsymbol{n}_{L,\sigma})^+ \boldsymbol{n}_{L,\sigma} = \sum_{i=1}^d (\boldsymbol{v} \cdot \boldsymbol{e}^{(i)}) \boldsymbol{e}^{(i)} = \boldsymbol{v},$$

so we have:

$$\tilde{F}_K(\phi_{\mathcal{M}}, \boldsymbol{v}) = \boldsymbol{v} \cdot \boldsymbol{\nabla}\phi(\boldsymbol{x}_K) + \mathcal{O}(h_{\mathcal{M}}).$$

Concerning the MUSCL interpolation, we have:

$$
\begin{aligned}
\tilde{F}_K(\phi_{\mathcal{M}}, \boldsymbol{v}) = \frac{1}{2} \sum_{\sigma=K|L\in\mathcal{E}(K)} & \frac{1}{d_\sigma}(\boldsymbol{v}\cdot\boldsymbol{n}_{K,\sigma})^+ \min\left(\phi_K - \phi_{M_K^\sigma}\frac{d_\sigma}{d_{K|M_K^\sigma}}, \phi_L - \phi_K\right) \\
& - \sum_{\sigma=K|L\in\mathcal{E}(K)} (\boldsymbol{v}\cdot\boldsymbol{n}_{K,\sigma})^-(\phi_L - \phi_K) \\
-\frac{1}{2} \sum_{\sigma=K|L\in\mathcal{E}(K)} & \frac{1}{d_\sigma}(\boldsymbol{v}\cdot\boldsymbol{n}_{K,\sigma})^- \min\left(\phi_L - \phi_{M_L^\sigma}\frac{d_\sigma}{d_{L|M_L^\sigma}}, \phi_K - \phi_L\right),
\end{aligned}
$$

where M_K^σ refers to the opposite cell to σ in K. It is easy to see that:

$$
\frac{1}{d_\sigma}\min\left(\phi_K - \phi_{M_K^\sigma}\frac{d_\sigma}{d_{K|M_K^\sigma}}, \phi_L - \phi_K\right) = \boldsymbol{\nabla}\phi(\boldsymbol{x}_K)\cdot\boldsymbol{n}_{K,\sigma} + \mathcal{O}(h_{\mathcal{M}}),
$$

and,

$$
\frac{1}{d_\sigma}\min\left(\phi_L - \phi_{M_L^\sigma}\frac{d_\sigma}{d_{L|M_L^\sigma}}, \phi_K - \phi_L\right) = \boldsymbol{\nabla}\phi(\boldsymbol{x}_K)\cdot\boldsymbol{n}_{L,\sigma} + \mathcal{O}(h_{\mathcal{M}}).
$$

Therefore,

$$
\begin{aligned}
\tilde{F}_K(\phi_{\mathcal{M}}, \boldsymbol{v}) = \frac{1}{2}\sum_{\sigma\in\mathcal{E}(K)} (\boldsymbol{v}\cdot\boldsymbol{n}_{K,\sigma})^+ \boldsymbol{\nabla}\phi(\boldsymbol{x}_K)\cdot\boldsymbol{n}_{K,\sigma} + \\
\frac{1}{2}\sum_{\sigma\in\mathcal{E}(K)} (\boldsymbol{v}\cdot\boldsymbol{n}_{L,\sigma})^+ \boldsymbol{\nabla}\phi(\boldsymbol{x}_K)\cdot\boldsymbol{n}_{L,\sigma} + \mathcal{O}(h_{\mathcal{M}}),
\end{aligned}
$$

which leads to:

$$
\begin{aligned}
\tilde{F}_K(\phi_{\mathcal{M}}, \boldsymbol{v}) &= \boldsymbol{\nabla}\phi(\boldsymbol{x}_K)\cdot \sum_{\sigma=K|L\in\mathcal{E}(K)} \frac{1}{2}\left((\boldsymbol{v}\cdot\boldsymbol{n}_{K,\sigma})^+\boldsymbol{n}_{K,\sigma} + (\boldsymbol{v}\cdot\boldsymbol{n}_{L,\sigma})^+\boldsymbol{n}_{L,\sigma}\right) \\
&\quad + \mathcal{O}(h_{\mathcal{M}}) \\
&= \boldsymbol{\nabla}\phi(\boldsymbol{x}_K)\cdot\boldsymbol{v} + \mathcal{O}(h_{\mathcal{M}}).
\end{aligned}
$$

Noticing, thanks to the consistency of $\boldsymbol{\nabla}_{\mathcal{E}}$, that:

$$
F_{\mathcal{M}}(\phi_{\mathcal{M}}) = \sum_{K\in\mathcal{M}} \tilde{F}_K\left(\phi_{\mathcal{M}}, \frac{\boldsymbol{\nabla}\phi(\boldsymbol{x}_K)}{|\boldsymbol{\nabla}\phi(\boldsymbol{x}_K)|}\right)\mathcal{X}_K + \mathcal{O}(h_{\mathcal{M}}),
$$

we can conclude that:

$$
\lim_{m\to\infty} F_{\mathcal{M}}(\phi_{\mathcal{M}}) = |\boldsymbol{\nabla}\phi|,
$$

which concludes the proof. □

5.1.2. *Monotonicity.* Let $(\phi_{\mathcal{M}}, \psi_{\mathcal{M}}) \in H_{\mathcal{M}}$. Let us define the following partial order

$$
\phi_{\mathcal{M}} \le \psi_{\mathcal{M}} \iff \forall K\in\mathcal{M}, \quad \phi_K \le \psi_K. \tag{22}
$$

Then we get the following result with the Cartesian upwind scheme only.

Proposition 5.3 (Monotonicity of the upwind Cartesian scheme)**.**
Suppose that the following CFL condition is satisfied

$$
\delta t \le \frac{1}{\sum_{\sigma\in\mathcal{E}(K)} \frac{1+\frac{1}{2}\sqrt{1+r^2}}{d_\sigma}}, \qquad r = \max_{(\sigma,\sigma')\in\mathcal{E}(K)} \frac{d_\sigma}{d_{\sigma'}}. \tag{23}
$$

Then he have the following result:

$$
\forall(\phi_{\mathcal{M}}, \psi_{\mathcal{M}}) \in H_{\mathcal{M}}, \qquad \phi_{\mathcal{M}} \le \psi_{\mathcal{M}} \implies \phi_{\mathcal{M}} + \delta t\ F_{\mathcal{M}}(\phi_{\mathcal{M}}) \le \psi_{\mathcal{M}} + \delta t\ F_{\mathcal{M}}(\psi_{\mathcal{M}}).
$$

Proof. For the sake of clarity we prove the result in 2D. The extension to all dimension can be done at the cost of heavier notations and CFL conditions. We can equivalently check that SCH is a non decreasing function of each variable. Let $K \in \mathcal{M}$ and $\phi_{\mathcal{M}} \in H_{\mathcal{M}}$. We have:

$$\phi_{\mathcal{M}} + \delta t\ F_{\mathcal{M}}(\phi_{\mathcal{M}})|_K = \phi_K + \delta t \sum_{\sigma=K|L\in\mathcal{E}(K)} \frac{1}{d_\sigma} f_{K,\sigma}\left(\phi_{\mathcal{M}}\right),$$

with,

$$f_{K,\sigma}\left(\phi_{\mathcal{M}}\right) = \frac{(\phi_L-\phi_K)^-}{\sqrt{(\phi_L-\phi_K)^2 + d_\sigma^2|\boldsymbol{\nabla}_{//\sigma}\phi_{\mathcal{M}}|^2}}(\phi_L-\phi_K)$$

The monotonicity of $f_{K,\sigma}$ in ϕ_L is equivalent to the monotonicity of the function:

$$f : x \longmapsto \frac{x^- x}{|x|} = -x^-, \quad \forall x \in \mathbb{R}$$

because $\boldsymbol{\nabla}_{//\sigma}\phi_{\mathcal{M}}$ does not depend on ϕ_L in the Cartesian case (see eq. (15)). We can conclude that $f_{K,\sigma}$ is a non decreasing function of ϕ_L. Concerning the monotonicity in ϕ_{K^-} and ϕ_{K^+} it is equivalent to the variations of:

$$f : x \longmapsto -\frac{1}{x^+},$$

which is a non decreasing function. We can conclude that $SCH(\phi_{\mathcal{M}})|_K$ is an increasing function of each $(\phi_M)_{\substack{M\in\mathcal{M}\\ M\neq K}}$. Concerning ϕ_K, we have:

$$SCH(\phi_{\mathcal{M}})|_K = g(\phi_K) = \\ \phi_K - \delta t \sum_{\sigma=K|L\in\mathcal{E}(K)} \frac{1}{d_\sigma} \frac{(\phi_K-\phi_L)^+(\phi_K-\phi_L)}{\sqrt{(\phi_L-\phi_K)^2 + d_\sigma^2|\boldsymbol{\nabla}_{//\sigma}\phi_{\mathcal{M}}|^2}}$$

The analysis of this function can be split into three cases. If, $\forall\sigma \in \mathcal{E}(K)$, $\phi_K \leq \phi_L$, then $g(\phi) = \phi_K$ which is non decreasing. The second case is when, $\forall\sigma \in \mathcal{E}(K)$, $\phi_K \geq \max(\phi_{K^+},\ \phi_{K^-},\ \phi_L)$. We have:

$$g(\phi_K) = \phi_K - \sum_{\sigma=K|L\in\mathcal{E}(K)} \frac{\delta t}{d_\sigma}\left(\phi_K - \phi_L\right).$$

which is non decreasing if,

$$\delta t \leq \frac{1}{\sum_{\sigma\in\mathcal{E}(K)} d_\sigma^{-1}}.$$

Finally, suppose that $\forall\sigma \in \mathcal{E}(K)$, $\phi_L \leq \phi_K \leq \phi_{K^+}$ (or ϕ_{K^-}), we have, denoting by $r_\sigma = \frac{d_\sigma}{d_{\sigma^+}}$:

$$g(\phi_K) = \phi_K - \sum_{\sigma=K|L\in\mathcal{E}(K)} \frac{1}{d_\sigma} \frac{\phi_K-\phi_L}{\sqrt{(\phi_K-\phi_L)^2 + r_\sigma^2(\phi_K-\phi_{K^+})^2}}(\phi_K-\phi_L).$$

Let us derive this function:

$$g'(\phi_K) = 1 - \sum_{\sigma=K|L\in\mathcal{E}(K)} \frac{1}{d_\sigma} \frac{\phi_K-\phi_L}{\sqrt{(\phi_K-\phi_L)^2 + r_\sigma^2(\phi_K-\phi_{K^+})^2}} \\ - \sum_{\sigma=K|L\in\mathcal{E}(K)} \frac{1}{d_\sigma} \frac{r_\sigma^2(\phi_{K^+}-\phi_K)(\phi_K-\phi_L)(\phi_{K^+}-\phi_K)}{\left((\phi_K-\phi_L)^2 + r_\sigma^2(\phi_K-\phi_{K^+})^2\right)^{3/2}}$$

One can notice directly that:

$$\sum_{\sigma=K|L\in\mathcal{E}(K)} \frac{1}{d_\sigma} \frac{\phi_K-\phi_L}{\sqrt{(\phi_K-\phi_L)^2 + r_\sigma^2(\phi_K-\phi_{K^+})^2}} \leq 1.$$

In order to upper-bound the second sum, we analyze the function

$$h : x \longmapsto \frac{r^2 x(a-x)a}{(x^2 + d^2(a-x)^2)^{3/2}},$$

where a, r are strictly positive constants. We split the function in two parts $h(x) = h_1(x)h_2(x)$ with:

$$\begin{aligned} h_1(x) =& \frac{r^2 x(a-x)}{x^2 + r^2(a-x)^2}, \\ h_2(x) =& \frac{a}{\sqrt{x^2 + r^2(a-x)^2}}. \end{aligned}$$

Concerning h_1 we can equivalently consider the function defined on $\mathbb{R}^+$ by:

$$y \longmapsto \frac{r^2}{y + \frac{r^2}{y}} = \frac{r^2 y}{y^2 + r^2}.$$

A quick study of the function shows that,

$$\max_{y \in \mathbb{R}^+} \frac{r^2 y}{y^2 + r^2} = \frac{r}{2} = \max_{x \in [0,a]} h_1(x).$$

The same work is performed with h_2 and leads to:

$$\max_{x \in [0,a]} h_2(x) = \frac{\sqrt{1 + r^2}}{r}$$

Gathering the results, we get that:

$$\forall x \in [0, a], \quad h(x) \leq \frac{1}{2}\sqrt{1 + r^2}$$

As a result, writing out $r = \max_{(\sigma,\sigma') \in \mathcal{E}(K)} \frac{d_\sigma}{d_{\sigma'}}$, we get that $g'(\phi_K) \geq 0$ provided that eq. (23) is satisfied. This CFL condition ensures that $\phi_{\mathcal{M}} + \delta t\ F_{\mathcal{M}}(\phi_{\mathcal{M}})|_K$ is a non decreasing function of ϕ_K, which concludes the proof. □

Remark 5.1. *All the results proved here can be extended with a transport velocity $\boldsymbol{u} \neq 0$ and a front propagation speed $u_f \neq 1$. Only the CFL conditions are modified, the sketch of the proofs is the same. However the monotonicity results cannot be extended to the MUSCL interpolation, and more generally to the non Cartesian case.*

6. A convergence result in the Cartesian case

The previous section ensures that the upwind scheme satisfies the basic properties to seek a convergence result on Cartesian meshes. We first recall the theorem given in [3], adapted to our notations.

Theorem 6.1. *Let $\mathcal{D}^{(m)} = \left\{\mathcal{M}^{(m)}, \mathcal{E}^{(m)}, \mathcal{P}^{(m)}, \delta t^{(m)}\right\}$ be a sequence of discretizations such that the space and time steps tend to zero as $m \to \infty$. Let $\bar{G}$ be the viscosity solution of eq. (4). Consider the following explicit scheme, for $n \in [0, N-1]$:*

$$\eth t G_m^n + F_{\mathcal{M}}(G_m^n) = 0,$$

and the complete solution defined by $G_m^{(T)} = \sum_{n=0}^{N-1} G_m^{n+1} \mathcal{X}_{[t^n, t^{n+1}]}$. We suppose that:

- *The spatial operator $F_{\mathcal{M}}$ is strongly consistent with the continuous operator $G \longmapsto |\boldsymbol{\nabla} G|$.*
- *The scheme is invariant under translations: $F_{\mathcal{M}}(G_{\mathcal{M}} + v) = F_{\mathcal{M}}(G_{\mathcal{M}})$.*
- *The scheme is monotone.*

Then,

$$G_{\mathcal{M}^{(m)}} \longrightarrow \bar{G} \text{ uniformly as } m \to \infty.$$

Since we have shown the required properties in Theorem 6.1, we can thus conclude to the convergence of the scheme, which we state in the following corollary.

Corollary 6.2. *Let $\mathcal{D}^{(m)} = \left\{\mathcal{M}^{(m)}, \mathcal{E}^{(m)}, \mathcal{P}^{(m)}, \delta t^{(m)}\right\}$ be a sequence of discretizations such that the space and time steps tend to zero as $m \to \infty$. Now suppose there exists $r > 0$, such that $\forall m \in \mathbb{N}$, $\forall (\sigma, \sigma') \in \mathcal{E}^{(m)}$,*

$$\frac{d_\sigma}{d_{\sigma'}} \leq r.$$

Suppose that, for any $m \in \mathbb{N}$,

$$\delta t^{(m)} \leq \max_{K \in \mathcal{M}^{(m)}} \frac{1}{\sum_{\sigma \in \mathcal{E}(K)} \frac{1+\frac{1}{2}\sqrt{1+r^2}}{d_\sigma}}.$$

Then the solution of the upwind Cartesian scheme eq. (7)-eq. (21) $G_m^{(T)}$ converges uniformly towards $\bar{G}$.

7. Numerical results

7.1. **One dimension.** The domain is $\Omega = (0,1)$. We use zero-flux boundary conditions in $x = 0$ and $x = 1$. We suppose that the time and space steps are constant for simplicity. Consider the following initial data:

$$G_0(x) = |\sin(4\pi x)| \tag{24}$$

We give the solution at $T = 0.05s$, with an upwind interpolation for the spatial operator, and a fixed CFL equal to $1/10$.

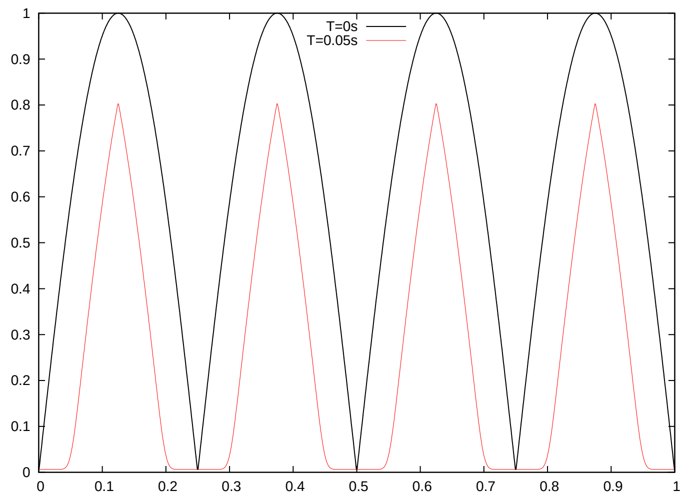


Figure 2. Solution of the G-equation with the upwind scheme at $T = 0.05s$.

It is possible to determine the unique viscosity solution of the eikonal equation for a given bounded uniformly continuous initial data. The expression of the solution is given by eq. (26) and its proof can be found in the appendix A. Consequently we can highlight numerically the theoretical result about the convergence of the solution of our scheme towards the viscosity solution. The figure below gives the error in L^1 norm according to the space step, for a fixed CFL equal to $\frac{1}{10}$.

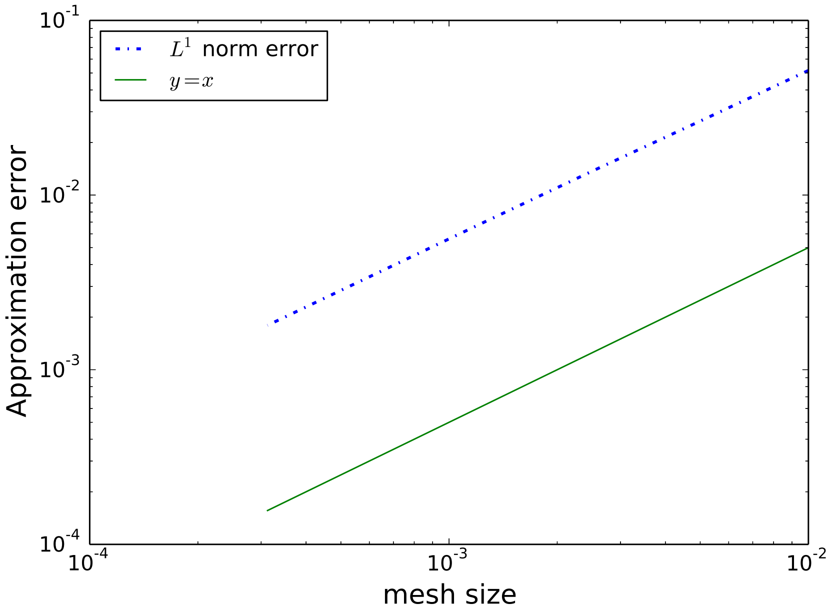


FIGURE 3. L1 norm error at T=0.05s and CFL= $\dfrac{1}{10}$ – upwind interpolation.

We can also see the behavior of the scheme if we use discontinuous initial data. We consider the following:

$$G_0(x) = \left| \begin{array}{ll} 0, & \text{if } x \leq 0.5 \\ 1, & \text{otherwise.} \end{array} \right.$$

The result at time $T = 0.2s$ is given below, for the upwind scheme and the MUSCL scheme.

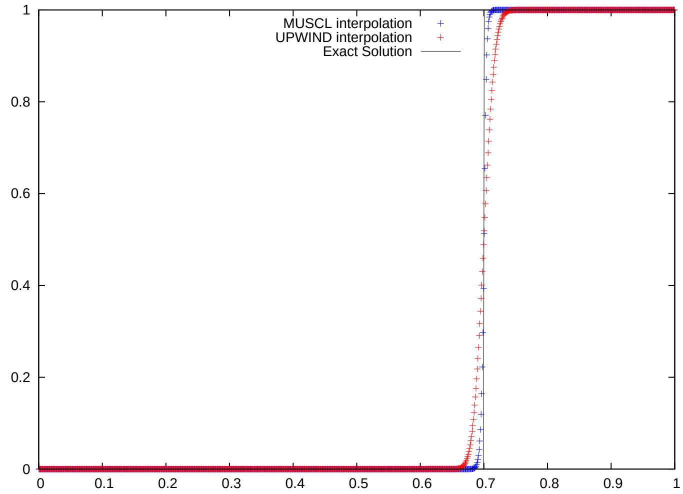


FIGURE 4. solution at $T = 0.05s$ and CFL$=\dfrac{1}{10}$ with $h = 10^{-3}$.

The MUSCL scheme brings less numerical diffusion, as expected. Normally one can not define a viscosity solution for discontinuous initial data. However one expects the solution to be the same as the general viscosity solution given for BUC initial data (see eq. (26) in the Appendix).

7.2. **Two Dimensions.**

7.2.1. *Unstructured grid.* The computational domain is $\Omega = [-\frac{1}{2}, \frac{1}{2}]^2$. The mesh consists in convex quadrilaterals. We give an example of the discretization below. These grids are built from a regular Cartesian grid for which a random displacement

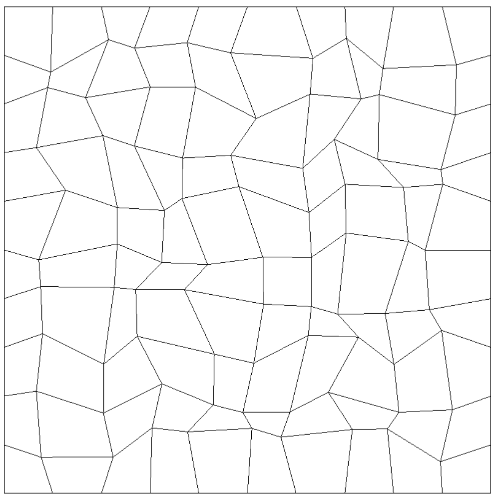
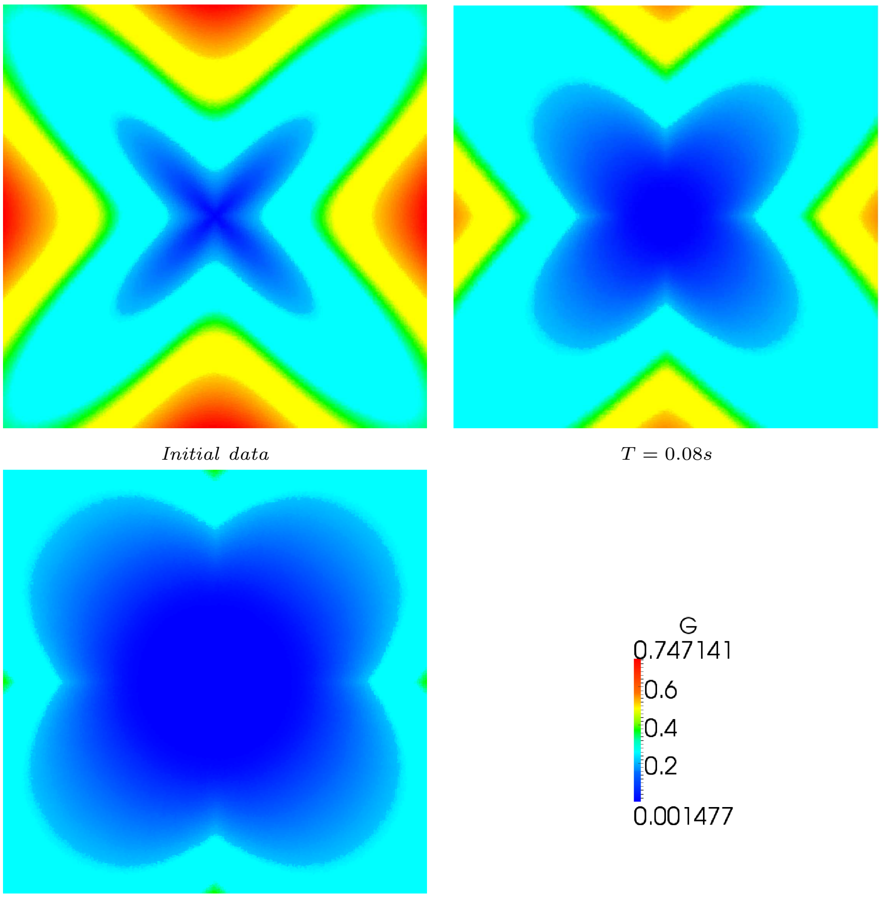


FIGURE 5. Example of a 10×10 unstructured grid

of length ϵh is applied to each node where h is the space step. We consider zero-flux boundary conditions. The initial data are given in the polar coordinates (r, θ):

$$G_0(r, \theta) = r\left(1 + \frac{1}{2}\cos(4\theta)\right).$$

Results obtained at different times are given below. The scheme used is the upwind version for unstructured meshes, with a space step $h = \dfrac{1}{200}$, and a constant CFL equal to $\frac{1}{10}$.



Initial data — $T = 0.08s$ — $T = 0.2s$

Figure 6. G at different times with the upwind scheme on an unstructured mesh – $h = \dfrac{1}{200}$ – $CFL = \dfrac{1}{10}$

Another possible test case is the following one:

$$G_0(r, \theta) = |\sin(4\pi r)|. \tag{25}$$

Results obtained with different meshes are displayed just below. The scheme used is the upwind version for unstructured meshes, with a space step $h = \dfrac{1}{400}$, a constant CFL equal to $\frac{1}{10}$ and a final time equal to $T = 0.04s$.

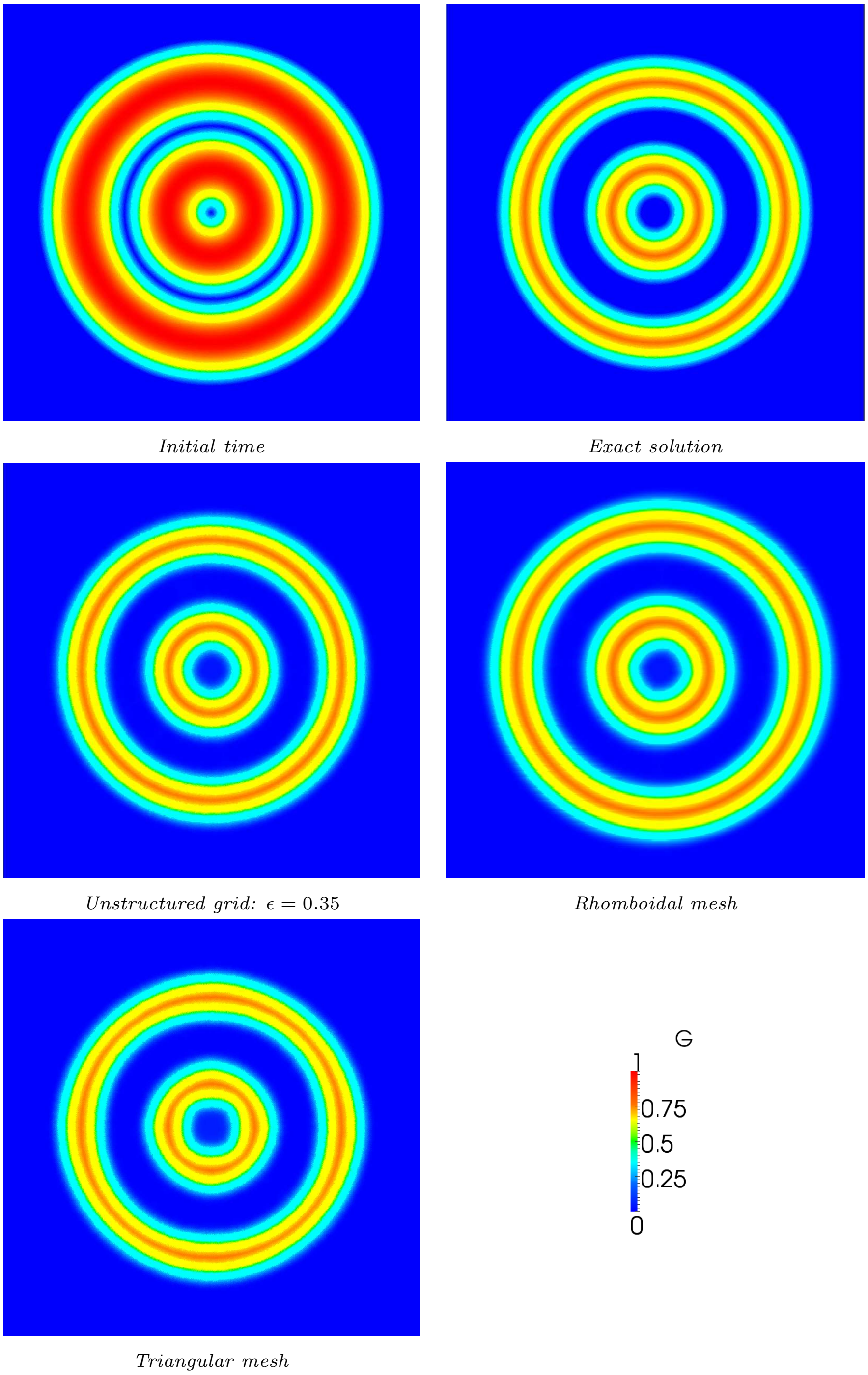


FIGURE 7. G on different meshes $-T = 0.04s - h = \dfrac{1}{400} - CFL = \dfrac{1}{10}$

Finally we plot some convergence results. Let G_{visc} be the viscosity solution associated to the initial data eq. (25) . We take $G_{\text{visc}}(., T = 0.01s)$ as a new initial data. The final time is set to $T = 0.04s$. The results are given below, with a constant CFL equal to $\frac{1}{10}$, using three different meshes : an unstructured mesh with a deformation ratio equal to $\epsilon = 0.1$, a triangular mesh which consists of a square grid where each square is cut in half following the same diagonal, and a Rhomboidal mesh composed of parallelograms with a large angle equal to $\frac{2\pi}{3}$.

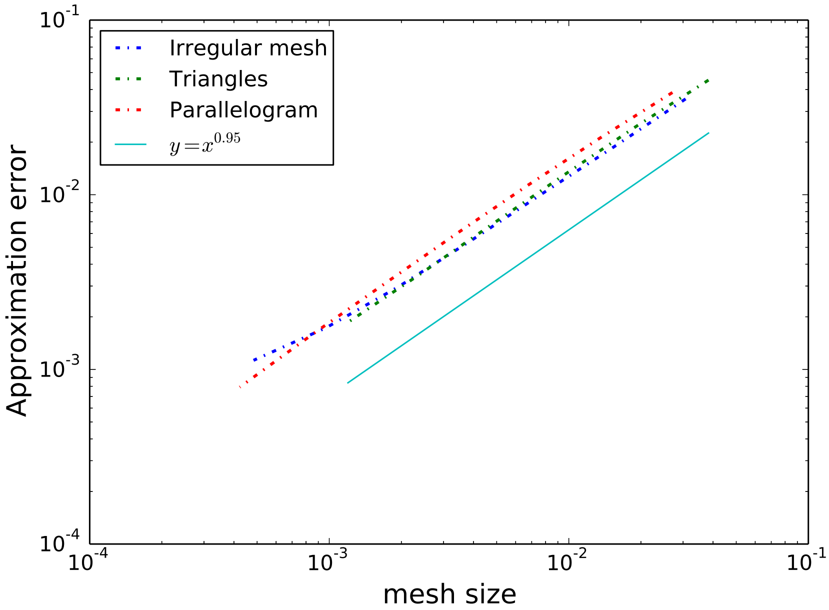


FIGURE 8. L^1 norm error at T=0.05s and CFL= $\dfrac{1}{10}$ – upwind interpolation.

7.2.2. *Cartesian grids.* We use the same test to compare the convergence of the MUSCL scheme, the upwind scheme, and an upwind finite difference scheme described in [8] designed for the Hamilton-Jacobi equations. In order to properly observe a difference in the convergence rate we use a Runge-Kutta time discretization of order two.

To conclude, we introduce a test case with a convective velocity $\boldsymbol{u}$ different from zero. Let the computational domain be $\Omega = (0,1)^2$. Zero-flux boundary conditions are prescribed on the boundary. We consider the following initial data

$$G_0(x) = \left| \begin{array}{ll} 0, & \text{if } \|\boldsymbol{x} - (0.25, 0.8)\| \leq 0.15 \\ 1, & \text{otherwise.} \end{array} \right.$$

The front propagation velocity is equal to $u_f = 0.8$ and the convective velocity corresponds to a vortex centered around $(0,0)$ with a constant angular speed equal to 2π, namely

$$\boldsymbol{u} = 2\pi r \boldsymbol{e}_\theta,$$

in polar coordinates.

The upwind scheme is used on a 400×400 Cartesian grid with a CFL equal to $\frac{1}{20}$. Results are plotted below.

Numerical simulations performed in this section are in good agreement with the properties verified by the scheme. The discretization proposed in this paper has been implemented on a Computational Fluid Dynamics software called P^{2}REMICS [17]. It will be used on a complex model with reactive flows to simulate the flame front propagation in the explosion phenomenon, for nuclear safety issues. Work on this model and its related numerical simulations is underway.

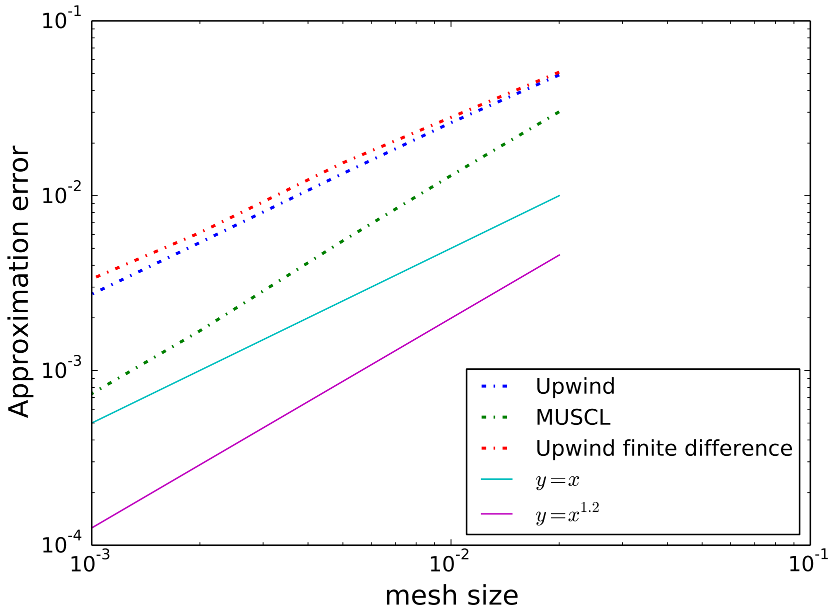


FIGURE 9. L1 norm error at $T = 0.05s$ and CFL$= \dfrac{1}{10}$

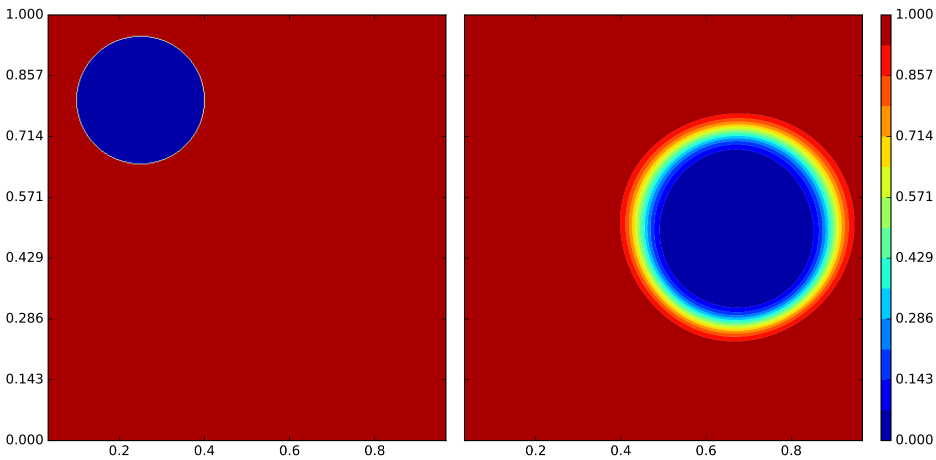


FIGURE 10. G at $T = 0s$ (left) – $T = 0.1s$ (right) with a CFL$= \dfrac{1}{20}$

Appendix A. Viscosity solutions of the eikonal equation

It is possible to compute the viscosity solution of 4 for every $G_O \in BUC(\mathbb{R}^d)$. It is then defined on $\mathbb{R}^d \times (0, +\infty)$ by:

$$G(\boldsymbol{x}, t) = \inf_{|\boldsymbol{x}-\boldsymbol{y}| \leq t} G_0(\boldsymbol{y}). \tag{26}$$

The proof of this result can be found in [4], and it is based on the following lemma

Lemma A.1. *Let us set*

$$S(t)G(\boldsymbol{x}) = \inf_{|\boldsymbol{x}-\boldsymbol{y}| \leq t} G(\boldsymbol{y}).$$

Then S is a monotonous semigroup on $C(\mathbb{R}^d)$.

Proof. The proof is rather simple as

$$S(t) \circ S(s)G(\boldsymbol{x}) = \inf_{|\boldsymbol{x}-\boldsymbol{y}| \leq t} \left(\inf_{|\boldsymbol{z}-\boldsymbol{y}| \leq s} G(\boldsymbol{z}) \right).$$

This computation is equivalent to seek the infimum in the set

$$\{\boldsymbol{z} \text{ such that } \exists \boldsymbol{y} \text{ such that } |\boldsymbol{x} - \boldsymbol{y}| \leq t \text{ and } |\boldsymbol{z} - \boldsymbol{y}| \leq s\}.$$

Now, this set is equal to the set

$$\{\boldsymbol{z}\} \text{ such that } |\boldsymbol{x} - \boldsymbol{z}| \leq t + s,$$

so the infimum are equal and $S(t+s) = S(t) \circ S(s)$. Now consider G_1 and G_2 two functions of $C(\mathbb{R}^d)$ such that $G_1 \leq G_2$ and let $t > 0$. Thanks to the continuity of G_2, $\exists \boldsymbol{y}_{\boldsymbol{x},t} \in B(\boldsymbol{x}, t)$ such that $S(t)G_2(x) = G_2(\boldsymbol{y}_{\boldsymbol{x},t})$. Consequently $G_2(\boldsymbol{y}_{\boldsymbol{x},t}) \geq G_1(\boldsymbol{y}_{\boldsymbol{x},t}) \geq S(t)G_1(x)$, which concludes the proof. □

Now let $\phi \in C^1(\mathbb{R}^d \times (0, +\infty))$ and suppose that $(\boldsymbol{x}, t)$ is a local maximum of $G - \phi$. Thanks to the semigroup property of S we get that:

$$G(\boldsymbol{x}, t) = S(t)G_0(\boldsymbol{x}) = S(h)S(t-h)G_0(\boldsymbol{x}) = S(h)G(\boldsymbol{x}, t-h).$$

Therefore, for all $0 < h < t$, we have

$$G(\boldsymbol{x}, t) = \inf_{|\boldsymbol{x}-\boldsymbol{y}| \leq h} G(\boldsymbol{y}, t-h). \tag{27}$$

$(\boldsymbol{x}, t)$ being a local maximum of $G - \phi$, we have, if h is sufficiently small, and $|\boldsymbol{x} - \boldsymbol{y}| \leq h$:

$$G(\boldsymbol{y}, t-h) - \phi(\boldsymbol{y}, t-h) \leq G(\boldsymbol{x}, t) - \phi(\boldsymbol{x}, t),$$

which is equivalent to

$$G(\boldsymbol{y}, t-h) \leq G(\boldsymbol{x}, t) - \phi(\boldsymbol{x}, t) + \phi(\boldsymbol{y}, t-h).$$

Injecting this in 27 leads to

$$\phi(\boldsymbol{x}, t) \leq \inf_{|\boldsymbol{x}-\boldsymbol{y}| \leq h} \phi(\boldsymbol{y}, t-h).$$

A first order Taylor expansion at the point $(\boldsymbol{x}, t)$ leads to

$$0 \leq \inf_{|\boldsymbol{x}-\boldsymbol{y}| \leq h} \left[-\partial_t \phi(\boldsymbol{x}, t) + \boldsymbol{\nabla} \phi(\boldsymbol{x}, t) \cdot \frac{\boldsymbol{y} - \boldsymbol{x}}{h} + o(1) \right].$$

Using that fact that $-\inf(-) = \sup()$, we have

$$\partial_t \phi(\boldsymbol{x}, t) + \sup_{|\boldsymbol{x}-\boldsymbol{y}| \leq h} \boldsymbol{\nabla} \phi(\boldsymbol{x}, t) \cdot \frac{\boldsymbol{x} - \boldsymbol{y}}{h} + o(1) \leq 0.$$

Thanks to the Cauchy-Schwarz inequality:

$$|\boldsymbol{\nabla} \phi(\boldsymbol{x}, t) \cdot \frac{\boldsymbol{x} - \boldsymbol{y}}{h}| \leq |\boldsymbol{\nabla} \phi(\boldsymbol{x}, t)|.$$

By taking $\boldsymbol{y} = \boldsymbol{x} - \dfrac{\boldsymbol{\nabla}\phi(\boldsymbol{x},t)}{|\boldsymbol{\nabla}\phi(\boldsymbol{x},t)|}h$ we see that the previous upper-bound is reached. Therefore ,

$$\partial_t \phi(\boldsymbol{x},t) + |\boldsymbol{\nabla}\phi(\boldsymbol{x},t)| + o(1) \leq 0,$$

and passing to the limit when $h \to 0$ leads to the desired result.

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