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Numerical approximation schemes for multi-dimensional wave equations in asymmetric spaces

Vincent Lescarret[*], Enrique Zuazua[†‡]

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Abstract

We develop finite difference numerical schemes for a model arising in multi-body structures, previously analyzed by H. Koch and E. Zuazua [13], constituted by two n-dimensional wave equations coupled with a $(n-1)$-dimensional one along a flexible interface.

That model, under suitable assumptions on the speed of propagation in each media, is well-posed in asymmetric spaces in which the regularity of solutions differs by one derivative from one medium to the other.

Here we consider a flat interface and analyze this property at a discrete level, for finite difference and mixed finite element methods on regular meshes parallel to the interface. We prove that those methods are well-posed in such asymmetric spaces uniformly with respect to the mesh-size parameters and we prove the convergence of the numerical solutions towards the continuous ones in these spaces.

In other words, these numerical methods that are well-behaved in standard energy spaces, preserve the convergence properties in these asymmetric spaces too.

These results are illustrated by several numerical experiments.

1 Introduction

This paper is devoted to the analysis of the propagation and convergence properties of numerical schemes approximating the wave equation in multi-dimensional and multi-structured media. This is an important topic in structural engineering (cf. for instance [9, 14]). From a mathematical view point the systems under consideration are given by several wave equations connected on interfaces that also evolve according to wave models. In the simplest case, the n-dimensional space is split in two parts through a $(n-1)$-dimensional interface. Each part (the interface and the two half-spaces) evolve according to the corresponding wave model. They are coupled together so that the overall system preserves the energy.

Various analytical properties of these systems have been analyzed. The first work devoted to this problem was [10], in the $1-d$ case. There, motivated by the so-called controllability problem, the system was shown to be well-posed in an asymmetric space in which the regularity differs by one derivative (in L^2) from one side of the interface to the other. As a consequence of this unexpected property, the system was shown to be exactly controllable in asymmetric spaces but not in the standard energy space that does not distinguish the regularity of solutions from one side of the interface to the other.

Similar issues were discussed in [4]-[7] for $1-d$ models for beams. There, it was proved that for beam models with rotational inertia terms, the same kind of well-posedness results in asymmetric spaces hold, while they do not hold for beam models without rotational inertia term. The second class of models correspond, roughly speaking, to systems of coupled multi-dimensional Schrödinger equations. This issue was later considered in [13] for waves in multi-dimensional domains, proving that the well-posedness in asymmetric spaces depends on the values of the velocities of propagation in the different media.

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The most remarkable property of this model is that incident waves become smoother while crossing the interface provided that they propagate faster in the bulk of the material than on the interface. As shown in [13], when reaching the interface, an incident wave with H^s regularity produces a reflected wave with the same regularity, a smoother transmitted wave with H^{s+1} regularity and a surface wave whose regularity is H^{s+1} with respect to the boundary. This property is also true for other closely related systems such as Schrödinger-like models in the whole space. However the property is untrue for the Schrödinger equation in bounded domains (see the last remark in conclusion). This property fails to hold for parabolic equations as well. We refer to the recent article [15] where the null controllability of the parabolic version of the model under consideration has been proved.

Here we consider the same issue but for semi-discrete and fully discrete approximations of an hybrid system. In particular we consider the semi-discrete finite difference method and the mixed finite element one. For the first method we mainly use Fourier analysis tools which also apply for the finite element and mixed finite element method. This limits the analysis to uniform meshes. However, in $1-d$ and for the mixed finite element scheme we can use a multiplicator identity which is true for irregular meshes. As in [13], we use a plane wave analysis to compute for each scheme the transmission coefficient and examine its behavior for high frequencies, of order $1/h$ where h is the space discretization. Existence in asymmetric spaces is expected when this coefficient behaves like h, a manifestation of the fact that most of the energy bounces back on the interface. Then, as in the continuous case, we prove the well-posedness in asymmetric spaces provided that the speed of propagation at the interface is lower than that in the medium of the incident wave.

Our proofs are given in $1-d$ and in $2-d$ with a straight interface but they extend straightforwardly to higher dimensions for flat interfaces. The Fourier analysis in the transverse space directions does not allow us to treat the case of curved interfaces, that constitutes an interesting open problem. The same can be said for the finite element method in non-uniform meshes.

We conclude this introduction by the plan of the paper. In the next section we recall the main result of [13] in the continuous case. Then, in the third section we consider the semi-discrete finite difference approximation in $1-d$ and $2-d$ with a straight interface. In the fourth section we give analogous results for the mixed finite element discretization, using multiplier techniques instead of the Fourier transform. In the fifth section we consider the $1-d$ and $2-d$ full discrete finite difference scheme for which the existence of solutions in convenient asymmetric spaces holds. We perform numerical calculations which perfectly agree with the theoretical results. More precisely, we consider discrete wave packets of high frequency $1/h$ whose propagation follows the geometric optics laws. By varying the stiffness of the interface around the critical value one sees clearly that the wave packet is either well reflected or well transmitted. We conclude this paper with some remarks on perspectives for future research.

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2 General setting

Let us first present the continuous problem under consideration and the relevant functional setting. As in [13] we consider two elastic media connected by an elastic interface. A continuity condition on the displacement of the three vibrating bodies is prescribed at the interface. This guarantees the system to be connected while evolving. We consider here only flat interfaces.

We denote by $(t, x_1, \dots, x_n) = (t, x)$ the time-space variable. The operator Δ denotes the Laplacian in $\mathbb{R}^n$ and Δ' that in $\mathbb{R}^{n-1}$ (the interface) and u_y denotes the derivative of u with respect to y. With those

notations the equations of motion are:

$$
\begin{aligned}
&u^-_{tt} - \Delta u^- = f^-, \quad \text{in } [0,T]\times\mathbb{R}^{n-1}\times\mathbb{R}^-\\
&u^+_{tt} - (c^+)^2\Delta u^+ = f^+, \quad \text{in } [0,T]\times\mathbb{R}^{n-1}\times\mathbb{R}^+\\
&u^-(x_n=0) = u^+(x_n=0) = w, \quad \text{in } [0,T]\times\mathbb{R}^{n-1}\\
&w_{tt} - c^2\Delta' w = ((c^+)^2 u^+_{x_n}(x_n=0) - u^-_{x_n}(x_n=0)) + f_0, \quad \text{in } [0,T]\times\mathbb{R}^{n-1}
\end{aligned}
\tag{2.1}
$$

We provide these equations with initial values

$$
\begin{aligned}
&u^\pm(t=0) = u_0^\pm, \quad \text{in } \mathbb{R}^{n-1}\times\mathbb{R}^\pm,\\
&u_t^\pm(t=0) = u_1^\pm, \quad \text{in } \mathbb{R}^{n-1}\times\mathbb{R}^\pm\\
&w(t=0) = w_0,\ w_t(t=0) = w_1, \quad \text{in } \mathbb{R}^{n-1}.
\end{aligned}
$$

We would like to draw the attention to the fact that the $1-d$ case is exceptional since the equation at the interface is not a wave equation but an ordinary differential equation. The analysis is simpler, see next section.

We have supposed that the elastic bodies are infinite. This allows to emphasize the mechanism on the interface without mixing with other boundary effects. Nevertheless we need some conditions at infinity. The most natural ones are vanishing displacements.

Let us point out that for numerical purpose we consider finite strings in the last section. While the scenario is more complicated, finite propagation speeds make it possible to perfectly single out each mechanism.

We set $\mathcal{U} = (u^-, w, u^+)$ and $\mathcal{F} = (f^-, f_0, f^+)$. When $\mathcal{F}$ vanishes the previous system preserves the following energy

$$
E(t) = \int_{\mathbb{R}^{n-1}\times\mathbb{R}^-} |u_t^-|^2 + |\nabla u^-|^2 + \int_{\mathbb{R}^{n-1}\times\mathbb{R}^+} |u_t^+|^2 + |c^+|^2|\nabla u^+|^2 + \int_{\mathbb{R}^{n-1}} |w_t|^2 + c^2|\nabla w|^2.
$$

Hence

$$
\begin{aligned}
&u^\pm \in \mathcal{C}([0,T]; H^1(\mathbb{R}^{n-1}\times\mathbb{R}^\pm)) \cap \mathcal{C}^1([0,T]; L^2(\mathbb{R}^{n-1}\times\mathbb{R}^\pm)),\\
&w \in \mathcal{C}([0,T]; H^1(\mathbb{R}^{n-1})) \cap \mathcal{C}^1([0,T]; L^2(\mathbb{R}^{n-1})).
\end{aligned}
\tag{2.2}
$$

Using Hille Yoshida's theorem one can show that the problem is well-posed in the energy space whose elements $\mathcal{U}$ satisfy (2.2). Next, if one takes initial data in the following asymmetric spaces

$$
\mathcal{A}^s = \{\mathcal{U} \in H^{s-1}(\mathbb{R}^{n-1}\times\mathbb{R}^-) \times H^s(\mathbb{R}^{n-1}) \times H^s(\mathbb{R}^{n-1}\times\mathbb{R}^+)\}, \quad s\in\mathbb{N}^*,
$$

one can show the existence (see [13]) of a unique solution in the space

$$
\mathcal{A}^s_T = \bigcap_{a+b=s} \mathcal{C}^a([0,T]; \mathcal{A}^b), \quad a\in\mathbb{N},\ b\in\mathbb{N}^*.
$$

In [13] the authors perform a plane wave analysis which predicts the result of existence in asymmetric spaces. We shortly recall it since we use it for the discretized problem.

Plane wave solutions are of the form $e^{i(\omega t - \xi\cdot x)}$ with $\omega^2 = |\xi|^2$ for $x_n < 0$ and $\omega^2 = (c^+)^2|\xi|^2$ for $x_n > 0$. As in Physics literature, we call such relations between space frequency ξ and pulsation ω *dispersion relation*.

Since the waves travel in an inhomogeneous medium, we expect some scattering. We thus distinguish the normal variable x_n from the tangential one $x' = (x_1, \dots, x_{n-1})$ and we look for the solution of (2.1) under the form

$$
\begin{aligned}
&u^- = e^{i(\omega t - \xi'^-\cdot x' - \xi_n^- x_n)} + C_r e^{i(\omega t - \xi'^-\cdot x' + \xi_n^- x_n)}, \quad x_n < 0\\
&u^+ = C_t e^{i(\omega t - \xi'^+\cdot x' - \xi_n^+ x_n)}, \quad x_n > 0\\
&w = C_t e^{i(\omega t - \xi'\cdot x')}, \quad x_n = 0.
\end{aligned}
$$

Because of the continuity condition at the interface we get $\xi'^- = \xi'^+ = \xi'$ and $1 + C_r = C_t$. Then, using the dispersion relation of the left medium $\omega^2 = |\xi'|^2 + |\xi_n^-|^2$ we get

$$C_t = \frac{2i\xi_n^-}{(c^2-1)|\xi'|^2 - |\xi_n^-|^2 + i\xi_n^- + i\xi_n^+(c^+)^2)}. \tag{2.3}$$

For $0 < c < 1$ and $\xi_n^\pm$ real, one has $|C_t| \leq 2/(1-c^2)|\xi|^{-1}$. This decay in the Fourier space means that a gain of one derivative is expected for u^+. For all other values of $c \geq 1$ the denominator is not bounded below by $|\xi||\xi_n^-|$ and, accordingly, no derivative gain is expected. This is the main observation made in [13].

We next devote the rest of the paper to consider discrete approximations of (2.1). We construct asymmetric spaces $\mathcal{A}_h^s$ which are the discrete counterparts of the spaces $\mathcal{A}^s$. Of course, the definition of the spaces depends on the discrete equation but the same methodology applies for several discrete schemes on uniform rectangular grids. Then, we prove the existence of solutions of the discretized systems in those spaces. Moreover, we give explicit uniform bounds with respect to h and we prove the convergence of the discrete solution towards the solution of (2.1) as h goes to zero. Hence, as in the continuous case, the discrete equations are well behaved both in symmetric and asymmetric spaces.

3 Semi-discrete finite difference approximation

3.1 The $1-d$ case

We first consider the $1-d$ situation where the interface is a point evolving like a forced oscillator. This is the situation studied in [10].

The semi-discrete finite difference approximation of system (2.1) reads:

$$\begin{aligned}
&u_j'' - \frac{u_{j+1} - 2u_j + u_{j-1}}{h^2} = f_j, \quad for\ j \leq -1\\
&u_j'' - (c^+)^2\frac{u_{j+1} - 2u_j + u_{j-1}}{h^2} = f_j, \quad for\ j \geq 1\\
&u_0'' + c^2 u_0 = (c^+)^2\frac{u_1 - u_0}{h} - \frac{u_0 - u_{-1}}{h} + f_0\\
&u_j(t=0) = a_j, \quad u_j'(t=0) = b_j
\end{aligned} \tag{3.1}$$

Let us introduce some notations which will be used further.

Definition and notations 3.1. *Let S be the space of all complex sequences indexed by $\mathbb{Z}$. We set:*

$$U = (u_j)_{j\in\mathbb{Z}}, \quad U_0 = (a_j)_{j\in\mathbb{Z}}, \quad U_1 = (b_j)_{j\in\mathbb{Z}}, \quad F = (f_j)_{j\in\mathbb{Z}}.$$

Let us define the operator $A_h : S \to S$ by

$$(A_h U)_j = -m_j\frac{u_{j+1} - 2u_j + u_{j-1}}{h^2}, \quad where\ \ m_j = \begin{cases} 1 & if\ j < 0,\\ (c^+)^2 & if\ j > 0, \end{cases} \tag{3.2}$$

$$and \quad (A_h U)_0 = c^2 u_0 - (c^+)^2\frac{u_1 - u_0}{h} + \frac{u_0 - u_{-1}}{h}.$$

The system (3.1) *thus reads*

$$U'' + A_h U = F \tag{3.3}$$

$$U(0) = U_0,\ U'(0) = U_1. \tag{3.4}$$

Next, we give a scalar product. For U_1 and U_2 two elements of S, we define

$$(U_1, U_2)_h = h\sum_{j\neq 0} u_{1,j}u_{2,j} + u_{1,0}u_{2,0}, \quad \|U\|_h = \sqrt{(U,U)_h}. \tag{3.5}$$

We denote by $\partial_h U$ *the vector* $\Big((u_j - u_{j-1})/h\Big)_{j\in\mathbb{Z}}$ *and define the so-called discrete Sobolev spaces:*

$$L^2_h = \{U \mid \|U\|_h < \infty\} \quad \textit{and} \quad H^1_h = \{U \in L^2_h \mid h\sum_{j\in\mathbb{Z}} |(\partial_h U)_j|^2 < \infty\}.$$

As expected, A_h is symmetric and positive definite. Indeed

$$(A_h U, V)_h = (c^+)^2 h \sum_{j\geq 1} (\partial_h U)_j (\partial_h V)_j + h\sum_{j\leq 0} (\partial_h U)_j (\partial_h V)_j + c^2 u_0 v_0.$$

So, the quantity $E_{df} := \sqrt{\frac{1}{2}\|U'\|^2_h + (A_h U, U)_h}$ defines an energy and we have the classical h-uniform energy estimate for (3.3),(3.4)

$$E_{df}(T) \leq E_{df}(0) + \int_0^T \|F\|_h. \tag{3.6}$$

In particular, $-A_h$ with domain H^1_h is dissipative in the Hilbert space H^1_h and $Id + A_h$ is H^1_h-coercive. Thus, an easy application of Lax-Milgram's theorem shows that $-A_h$ is maximal. Thus, by Hille-Yoshida's Theorem one gets the existence result:

Lemma 3.2. *Let* $U_0 \in H^1_h$, $U_1 \in L^2_h$ *and* $F \in L^1([0,T]; L^2_h)$. *Then for all* $T \in \mathbb{R}$ *there is a unique solution* U *of* (3.3),(3.4) *in* $\mathcal{C}([0,T]; H^1_h) \cap \mathcal{C}^1([0,T]; L^2_h)$ *satisfying the* h*-uniform energy estimate* (3.6).

As announced in the introduction, we perform a plane wave analysis to compute the transmission coefficient to identify asymmetric spaces stable for the propagator of system (3.1) and in which $(u_j)_{j\leq 0}$ and $(u_j)_{j\geq 0}$ have a different number of discrete derivatives in L^2_h.

3.1.1 Plane wave analysis

In this subsection we look for solutions of the homogeneous system (3.1) (with $F = 0$) under the form of plane waves: $u_j = e^{i(\omega t - \xi j h)}$. As in the continuous case one finds the dispersion relation between ω and ξ, namely (see [17])

$$\omega^2 = \frac{4}{h^2}\sin^2\left(\frac{h\xi}{2}\right), \quad j < 0, \qquad \omega^2 = \left(\frac{2c^+}{h}\right)^2 \sin^2\left(\frac{h\xi}{2}\right), \quad j > 0.$$

For $j < 0$ and for each ω there are two opposite associated wave numbers $\pm\xi^-$ with $\xi^- \geq 0$. Because of the continuity condition at $j = 0$ the pulsation ω remains the same for $j > 0$ and one gets two opposite wave numbers $\pm\xi^+$ with $\xi^+ \geq 0$.

Next, we consider a plane wave in the left medium propagating to the right (j increasing). This wave is then partially transmitted and reflected at the interface:

$$\begin{aligned} u_j &= e^{i(\omega t - jh\xi^-)} + C_r e^{i(\omega t + jh\xi^-)}, \quad j \leq -1 \\ u_j &= C_t e^{i(\omega t - jh\xi^+)}, \quad j \geq 1 \\ u_0 &= C_t e^{i\omega t} \end{aligned}$$

From the Dirichlet conditions at the interface the reflection and transmission coefficients C_r, C_t satisfy $1 + C_r = C_t$. Setting $\alpha_i = \frac{1-e^{-ih\xi_i}}{h}$, $\alpha_t = \frac{1-e^{-ih\xi_t}}{h}$, we get

$$C_t = \frac{\alpha_i - \bar{\alpha}_i}{\alpha_i + (c^+)^2\alpha_t + c^2 - \omega^2}, \quad C_r = -\frac{\bar{\alpha}_i + (c^+)^2\alpha_t + c^2 - \omega^2}{\alpha_i + (c^+)^2\alpha_t + c^2 - \omega^2}\,.$$

As pointed in the introduction, the behavior of C_r and C_t for large ξ tells us how smooth the reflected and transmitted waves are. Here however, the range of frequencies is bounded since $|\xi| \leq \pi/h$. So we

are interested in the behavior of those coefficients in terms of h. Besides, let us observe that a discrete finite difference derivative is transformed in the Fourier space by the multiplication by $\frac{1}{h}e^{ih\xi/2}\sin(h\xi/2)$. So, dividing by h^r in the Fourier space corresponds to differentiating r times in the physical space.

So, consider $\xi_i = \mathcal{O}(1/h)$. From the discrete dispersion relation $\omega = \mathcal{O}(1/h)$. Thus $C_t = \mathcal{O}(h)$ and $C_r = \mathcal{O}(1)$. So, the transmitted wave is expected to gain one degree of regularity while the reflected one does not do it. We next show this fact rigorously.

3.1.2 Existence in asymmetric spaces

In this section we mainly focus on the solution of (3.3) restricted to $j \geq 0$. We thus introduce the following notation

Notations 3.3. *Let $U = (u_j)_{j\in\mathbb{Z}}$. We denote by U^+ the vector $(u_j^+)_{j\in\mathbb{Z}}$ such that $u_j^+ = 0$ for $j \leq 0$ and $u_j^+ = u_j$ for $j > 0$. We also denote by $E_0 \in S$ the unit vector with non vanishing component at $j = 0$.*

According to the previous plane wave analysis we introduce the following so-called asymmetric space whose elements are characterized by a supplementary square integrable discrete derivative in the right-hand side domain:

$$\mathcal{A}_h^2 := \left\{U \in H_h^1 \;\middle|\; \left\|(A_h U)^+\right\|_h < \infty\right\}.$$

Similarly, we introduce the spaces

$$\mathcal{A}_h^1 = \left\{U \in L_h^2 \mid \|(\partial_h U)^+\|_h < \infty\right\} \quad \text{and} \quad \mathcal{A}_h^0 = \left\{U \in S \mid \|u_0 E_0 + U^+\|_h < \infty\right\}.$$

Finally, for $s \in \mathbb{N}$ let us define the space taking into account the regularity in time

$$\mathcal{A}_{h,T}^s = \bigcap_{a+b=s} \mathcal{C}^a([0,T];\mathcal{A}_h^b). \tag{3.7}$$

Theorem 3.4. *Let $U_0 \in \mathcal{A}_h^2$, $U_1 \in \mathcal{A}_h^1$ and $F \in L^1([0,T];\mathcal{A}_h^1)$ with $F' \in L^1([0,T];\mathcal{A}_h^0)$. Then for all $T \in \mathbb{R}$ there is a unique solution U of (3.3),(3.4) in $\mathcal{A}_{h,T}^2$. Moreover, there is a constant C independent of h and T such that*

$$\|U\|_{\mathcal{A}_{h,T}^2} + \|(\partial_h U)_1\|_{H^1([0,T])} \leq C\Big(\|U_0\|_{\mathcal{A}_h^2} + \|U_1\|_{\mathcal{A}_h^1} + \|(F,F')\|_{L^1([0,T];\mathcal{A}_h^1\times\mathcal{A}_h^0)}\Big).$$

Proof. The idea is to show that $(\partial_h U)_0$ and $(\partial_h U)_1$ are in $L^2(0,T)$ which implies from the equation for u_0 that u_0'' is also in $L^2(0,T)$. This allows to win one degree of regularity both in time and space in the right-hand side domain.

Lemma 3.5. *Under the assumptions of the theorem, the solution given by Lemma 3.2 satisfies $(\partial_h U)_1 \in L^2(0,T)$ and $(\partial_h U)_0 \in L^2(0,T)$.*

Proof. It is sufficient to show that $(\partial_h U)_1 \in L^2(0,T)$. First, note that a mere energy estimate unfortunately does not give the result since one is left with the boundary term $(\partial_h U)_1 u_0'$ which has no sign. Instead, we compute $(\partial_h U)_1$ explicitly. Let us introduce χ a smooth function of t which is equal to 1 for $|t| < T$ and vanishes for $|t| > T+1$. Setting $V = \chi U$ we have:

$$\begin{aligned} &v_j'' - (c^+)^2\frac{v_{j+1} - 2v_j + v_{j-1}}{h^2} = \chi f_j + 2\chi' u_j' + \chi'' u_j := g_j, \quad \textit{for } j \geq 1 \\ &v_0 = \chi u_0. \end{aligned} \tag{3.8}$$

Thanks to Lemma 3.2 we already have $u_0 \in H^1([-T,T])$. Since v_j vanishes for $|t| > T$ one can apply the Laplace transform:

$$\tilde{v}_j = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-(i\omega+s)t} v_j(t)dt, \quad s > 0. \tag{3.9}$$

We thus have to solve

$$(i\omega+s)^2\tilde v_j-(c^+)^2\frac{\tilde v_{j+1}-2\tilde v_j+\tilde v_{j-1}}{h^2}=\tilde g_j,\quad for\ j\geq 1,$$
$$\tilde v_0=\widetilde{\chi u_0}.$$

The solutions of the homogeneous equation are linear combination of two plane waves $\tilde v_j^{\pm}=e^{\pm\xi jh}$ where ξ is solution of

$$h^2(i\omega+s)^2=4(c^+)^2\mathrm{sh}^2(\xi h/2). \tag{3.10}$$

This equation has two opposite solutions with non zero real part. We choose ξ the one with $\mathrm{Re}\,\xi>0$. The solution of the inhomogeneous equation is given by the variation of constants formula. We get for $j\geq 1$

$$\tilde v_j=\left(\frac{-h^2}{2\mathrm{sh}(h\xi)}\sum_{n=1}^{j}\tilde g_n e^{nh\xi}+c_1\right)e^{-jh\xi}+\left(\frac{h^2}{2\mathrm{sh}(h\xi)}\sum_{n=1}^{j}\tilde g_n e^{-nh\xi}+c_2\right)e^{jh\xi}, \tag{3.11}$$

and $\tilde v_0=c_1+c_2$. Since $\tilde V$ belongs to L^2_h it implies

$$c_2=-\frac{h^2}{2\mathrm{sh}(h\xi)}\sum_{n\geq 1}\tilde g_n e^{-nh\xi},\quad \text{then}\quad c_1=\tilde v_0+\frac{h^2}{2\mathrm{sh}(h\xi)}\sum_{n\geq 1}\tilde g_n e^{-nh\xi}.$$

Thus

$$\frac{\tilde v_1-\tilde v_0}{h}=\tilde v_0\frac{e^{-\xi h}-1}{h}+2c_2\frac{\mathrm{sh}(h\xi)}{h}.$$

Thanks to Parseval theorem we have

$$\left\|\frac{\tilde v_1-\tilde v_0}{h}\right\|_{L^2(\mathbb{R}_\omega)}=\left\|e^{-s\cdot}\chi\frac{u_1-u_0}{h}\right\|_{L^2([-T,T])}.$$

To estimate this norm, first note that there is some constant $c>0$, independent of ω such that

$$\left|\frac{e^{-\xi h}-1}{h(s+i\omega)}\right|<c.$$

Indeed, for small $h\omega$ (and s small) $\xi=(s+i\omega)/c^+ +\mathcal{O}(h^2)$ so one can perform the Taylor expansion and find that the h factor cancels. Then, for the other values of $h\omega$ the fraction is bounded. Next, setting $G=(g_j)_{j\in\mathbb{Z}}$ we have

$$2c_2\frac{\mathrm{sh}(h\xi)}{h}=-h\sum_{n\geq 1}\tilde g_n e^{-nh\xi}\leq\|\tilde G^+\|_h\Big(h\sum_{n\geq 0}e^{-2nh\mathrm{Re}\,\xi}\Big).$$

The last sum is $h/(1-e^{-2h\mathrm{Re}\,\xi})\sim\frac{1}{2s}$ when $\mathrm{Re}\,\xi\sim s$ is small. Using the Cauchy Schwartz inequality and recalling the definition of G finally leads to

$$\begin{aligned}\|\frac{u_1-u_0}{h}\|_{L^2([-T,T])}\leq e^{sT}\|e^{-st}\chi\frac{u_1-u_0}{h}\|_{L^2([-T,T])}&\leq ce^{sT}\|u_0e^{-st}\|_{H^1([-T,T])}\\&+\frac{c}{s}e^{sT}\left(\|Fe^{-st}\|_{L^2([-T,T];L^2_h)}+\|Ue^{-st}\|_{H^1([-T,T];L^2_h)}\right).\end{aligned}$$

This concludes the proof of Lemma 3.5. □

We now finish the proof of Theorem 3.4. The previous Lemma applied to the equation for u_0 implies that $u_0\in H^2(0,T)$. Next, thanks to the additional regularity of the data at the right-hand side (i.e. $j\geq 0$) we can differentiate the equations indexed by $j\geq 0$ in time. We now prove that $U^{+\prime}$ is of finite energy by

duality, using the method of transposition (see [16], p.46 §4.2). For this, let us take $H \in L^1([0,T];L^2_h)$ and let R be the solution of the transposed problem of (3.8):

$$
\begin{aligned}
& r_j'' - (c^+)^2 \frac{r_{j+1} - 2r_j + r_{j-1}}{h^2} = h_j, \quad for\ j \geq 1 \\
& r_0 = 0 \\
& r_j(t=T) = 0, \quad r_j'(t=T) = 0
\end{aligned} \tag{3.12}
$$

Multiplying the equation by U'' and integrating gives

$$
\int_0^T (U'', H^+)_h = -(F(t=0) - A_h U_0, R^{+\prime}(t=0))_h + ((\partial_h U_1), (\partial_h R)^+(t=0))_h \\
+ \int_0^T \frac{u_0'' r_1}{h} - \int_0^T (F', R^{+\prime})_h.
$$

Then, thanks to the Dirichlet boundary conditions we have the following energy estimate for R (one has to use the discrete version of the Rellich multiplier r_x: $\frac{r_{j+1}-r_{j-1}}{2h}$)

$$
h \left[\sum_{j\geq 1} (r_j')^2 + (c^+)^2 (\partial_h R)_j^2 \right]_0^T + (c^+)^2 \int_0^T \left(\frac{r_1}{h} \right)^2 \leq 4 \int_0^T h \sum_{j\geq 1} f_j^2. \tag{3.13}
$$

We thus get

$$
\int_0^T (U'', H^+)_h \leq C \|H\|_{L^1([0,T];\mathcal{A}_h^0)}.
$$

Thus $\int_0^T (U'', H^+)_h$ is a bounded linear form on $L^1([0,T];\mathcal{A}_h^0)$ so $U'' \in L^\infty([0,T];\mathcal{A}_h^0)$. By a density argument (see [16], p.48 §4.2) we get that $U \in \mathcal{C}^2([0,T];\mathcal{A}_h^0)$. Using the equation allows to conclude that $U \in \mathcal{C}([0,T];\mathcal{A}_h^2)$. Still by transposition one gets $U \in \mathcal{C}^1([0,T];\mathcal{A}_h^1)$. Finally, applying Lemma 3.5 to $U^{+\prime}$ gives a bound of the normal derivative in $H^1(0,T)$. This ends the proof of Theorem 3.4. □

3.1.3 Convergence

Here we prove the convergence of the solution U_h given by Theorem 3.4 towards the solution u of system (2.1) in the continuous asymmetric space $\mathcal{A}_T^2$. This requires to construct an interpolation of U_h in $\mathcal{A}_T^2$. But first recall the lower order interpolation functions as in [18].

Notations 3.6. *Let $q_h U_h$ be the piecewise constant function which takes the value u_j in $[jh - h/2, jh + h/2]$ and $p_h U_h$, the piecewise affine function which is $u_j + (u_{j+1} - u_j)x/h$ for $x \in [jh, (j+1)h]$.*

Lemma 3.7. *There exists a piecewise polynomial interpolated function $\mathcal{U}_h$ of U_h and a constant c such that*

$$
\|\mathcal{U}_h\|_{\mathcal{A}_T^2} \leq c \|U_h\|_{\mathcal{A}_{h,T}^2}.
$$

Next, let $\mathcal{A}_h^3 = \{U \in H_h^2 \mid \|(\partial_h A_h U)^+\|_h < \infty\}$ and define the space $\mathcal{A}_{h,T}^3$ as in (3.7). Then, if U_h solves (3.1) with $F \in \mathcal{A}_{h,T}^1$, there is a positive constant c such that

$$
\|\partial_t^2 \mathcal{U}_h - \partial_x^2 \mathcal{U}_h - p_h F\|_{\mathcal{A}_T^0} \leq ch(\|U_h\|_{\mathcal{A}_{h,T}^3} + \|F\|_{\mathcal{A}_{h,T}^1}).
$$

Proof. Let $j > 0$, set $a_j = (A_h U)_j$ and define on $[0,h]$ the 4^{th} order polynomial function g_j by

$$
g_j(0) = u_j,\ g_j(h) = u_{j+1},\ g_j'(0) = \frac{u_{j+1} - u_{j-1}}{2h},\ g_j'(h) = \frac{u_{j+2} - u_j}{2h},\ g_j''(0) = a_j.
$$

One finds $g_j(x) = u_j + \dfrac{u_{j+1}-u_{j-1}}{2h}x + \dfrac{1}{2}a_j x^2 + \dfrac{a_{j+1}-a_j}{2h}(-x^3 + x^4/h)$. Let us also set $g_0(x) = u_1 + \dfrac{u_2-u_0}{2h}(x-h) + \dfrac{1}{2}a_1(x-h)^2$. Finally, we define $\mathcal{U}_h$ by

$$\mathcal{U}_h(x) = \begin{cases} p_h U_h(x), & x \le 0 \\ \\ g_j(x-jh), & x \in [jh,(j+1)h], \quad j \ge 0\,. \end{cases}$$

From the form of the coefficients one checks that $\mathcal{U}_h \in \mathcal{A}_T^2$ and that the first inequality of the lemma holds. As for the second inequality, by assumption one has $u_j'' - a_j = f_j$ so, for $x \ge h$ one gets

$$\partial_t^2 \mathcal{U}_h - \partial_x^2 \mathcal{U}_h - p_h F_h = -\frac{f_{j+1}-f_j}{h}x - \frac{a_{j+1}-a_j}{h}(-3x + 6x^2/h) \\ + \frac{a_{j+1}-a_{j-1}+f_{j+1}-f_{j-1}}{2h}x - \frac{1}{2}a_j'' x^2 + \frac{a_{j+1}''-a_j''}{2h}(x^3 - x^4/h).$$

For $x \in [0,h]$ one gets a similar expression for $\partial_t^2 g_0 - \partial_x^2 g_0 - p_h F_h$, using that $u_1'' - a_1 = f_1$. Those expressions imply the second estimate of the lemma. □

Theorem 3.8. *(Weak convergence) Let $(\mathcal{U}_0, \mathcal{U}_1) \in \mathcal{A}^2 \times \mathcal{A}^1$. Let $\mathcal{U}$ be the solution of System (2.1) with source term $(\mathcal{F}, \partial_t \mathcal{F}) \in L^1(0,T;\mathcal{A}^1) \times L^1(0,T;\mathcal{A}^0)$.*

Let U_{0h}, U_{1h}, F_h be vectors in $\mathcal{A}_h^2, \mathcal{A}_h^1, L^1(0,T;\mathcal{A}_h^1) \cap W^{1,1}(0,T;\mathcal{A}_h^0)$. Let $\mathcal{U}_{0h}$ be the interpolation of U_{0h} as defined in Lemma 3.7 and let $\mathcal{U}_{1h} = p_h U_{1h}$ and $\mathcal{F}_h = p_h F_h$ be such that, as h goes to zero

$$\begin{aligned} &\mathcal{U}_{0h} \rightharpoonup \mathcal{U}_0 \quad \textit{weakly in } \mathcal{A}^2, \\ &\mathcal{U}_{1h} \rightharpoonup \mathcal{U}_1 \quad \textit{weakly in } \mathcal{A}^1, \\ &(\mathcal{F}_h, \partial_t \mathcal{F}_h) \rightharpoonup (\mathcal{F}, \partial_t \mathcal{F}) \quad \textit{weakly in } L^1([0,T],\mathcal{A}^1) \times L^1([0,T],\mathcal{A}^0). \end{aligned}$$

Let U_h be the solution of (3.3) with initial data U_{0h}, U_{1h} and source term F_h. Then, with $\mathcal{U}_h$ defined in Lemma 3.7, one has

$$\mathcal{U}_h \rightharpoonup \mathcal{U} \quad \textit{in } \mathcal{A}_T^2 \quad \textit{as } h \to 0.$$

Proof. From the assumptions on the data, U_h (resp. $\mathcal{U}$) are well defined in the asymmetric space $\mathcal{A}_{h,T}^2$ (resp. $\mathcal{A}_T^2$).

Then, the main steps of the proof are well-known facts available in [18] (proof of Theorem 1.3) but since we show the convergence in different spaces we recall the main ideas and the changes. The main fact is that $\mathcal{A}^2$ is a reflexive Banach space as a product of Sobolev spaces. Then, from the weak convergence of the data and the bound on U_h (Theorem 3.4), the family $\mathcal{U}_h$ is bounded in $\mathcal{A}_T^2$ so one can extract a sub-sequence (still denoted by $\mathcal{U}_h$) for which there exists $\mathcal{U} = (u^-, w_u, u^+)$ such that

$$\begin{aligned} &\mathcal{U}_h \rightharpoonup \mathcal{U} \quad \text{weakly star in } \mathcal{A}_T^2, \\ &\mathcal{U}_h \to \mathcal{U} \quad \text{locally strongly in } \mathcal{A}_T^1. \end{aligned}$$

The strong convergence is a consequence of the fact that $\mathcal{A}_T^2 \subset \mathcal{A}_T^1$ compactly, which follows from the Rellich's compactness theorem for Sobolev spaces.

Then, we need to show that $\mathcal{U}$ is indeed the solution of system (2.1). This step does not require to work in asymmetric spaces and is done by using the weak formulation of system (3.1). Let us point that the continuity conditions at the interface are included in the weak formulation since $\mathcal{U}(t) \in \mathcal{A}^2 \subset H^1(\mathbb{R})$. Checking that the solution satisfies the initial data and that the whole sequence $\mathcal{U}_h$ converges follows exactly from [18]. □

Next, we show a strong convergence result. As usual, one uses the energy conservation. However, since the energy space is symmetric one needs to use again the supplementary regularity available at the interface to get the convergence in the asymmetric space.

Theorem 3.9. *(Strong convergence) With the notations of the previous theorem let U_{0h}, U_{1h} and F_h be such that*

$$\begin{aligned}
&\mathcal{U}_{0h} \longrightarrow \mathcal{U}_0 \quad \textit{strongly in } \mathcal{A}^2,\\
&\mathcal{U}_{1h} \longrightarrow \mathcal{U}_1 \quad \textit{strongly in } \mathcal{A}^1,\\
&(\mathcal{F}_h, \partial_t \mathcal{F}_h) \longrightarrow (\mathcal{F}_h, \partial_t \mathcal{F}_h) \quad \textit{strongly in } L^1([0,T], \mathcal{A}^1 \times \mathcal{A}^0).
\end{aligned}$$

Let U_h be the solution of (3.3) with the previous data. Then,

$$\mathcal{U}_h \longrightarrow \mathcal{U} \quad \textit{strongly in } \mathcal{A}^2_T.$$

If, moreover $\mathcal{U}_{0h}, \mathcal{U}_{1h}, \mathcal{F}_h$ converge respectively in $\mathcal{A}^3, \mathcal{A}^2, \cap_{j\leq 2} W^{1,j}([0,T]; \mathcal{A}^{2-j})$ then there is a constant c depending only on the data such that

$$\|\mathcal{U}_h - \mathcal{U}\|_{\mathcal{A}^2_T} \leq hc. \tag{3.14}$$

Proof. The proof of the convergence in the (symmetric) energy space $\mathcal{C}(0,T; H^1(\mathbb{R}^-) \times \mathbb{R} \times H^1(\mathbb{R}^+)) \times \mathcal{C}^1(0,T; L^2(\mathbb{R}^-) \times \mathbb{R} \times L^2(\mathbb{R}^+))$ is standard, using the conservation of the energy and the convergence of the initial data.

Next, we claim that $q_h U_h''$ and $\partial_x p_h U_h'$ converge strongly to $\partial_{tt}\mathcal{U}, \partial_{tx}\mathcal{U}$ in $\mathcal{A}^0_T$. To see this, let us consider the energy of $\partial_t u^+$:

$$E'_+(t) := \int_{\mathbb{R}^+} (\partial_{tt} u^+)^2 + (\partial_{tx} u^+)^2.$$

We have the energy identity,

$$E'_+(t) - \int_0^t \partial_{tx} u^+(x=0) \partial_{tt} u^+(x=0) = E'_+(0) + \int_0^t \int_{\mathbb{R}^+} \partial_t \mathcal{F}' \partial_{tt} u^+.$$

Our claim can then be expressed as follows

$$\int_{\mathbb{R}^+} (\partial_{tt} u^+ - q_h U_h'')^2 + (\partial_{tx} u^+ - \partial_x p_h U_h')^2 \underset{h\to 0}{\longrightarrow} 0.$$

We expand this expression. The double products converge to $-2E'_+(t)$ thanks to the weak convergences. Then we show that $\int_{\mathbb{R}^+} (q_h U_h'')^2 + (\partial_x p_h U_h')^2$ converges to $E'_+(t)$. Indeed, let us remark that

$$\int_{\mathbb{R}^+} (q_h U_h'')^2 + (\partial_x p_h U_h')^2 = h \sum_{j>0} (u_j'')^2 + (\partial_h U)_j^2.$$

This quantity which we denote by $E'_{+h}(t)$ is exactly the discrete energy of the solution restricted to the right-hand side. In particular it satisfies the following identity

$$E'_{+h}(t) + \int_0^T \frac{u_1' - u_0'}{h} u_0'' = E'_{+h}(0) + \int_0^T (F_h^{+\prime}, U_h^{+\prime\prime})_h.$$

By assumption $E'_{+h}(0) \to E'_+(0)$ and $F^{+\prime}$ converges strongly to $f^{+\prime}$ in $L^1([0,T]; L^2(\mathbb{R}^+))$. Then, since $(\partial_h U)_1$ converges strongly in $L^2(0,T)$ and

$$u_0'' = -c^2 u_0 + (c^+)^2 (\partial_h U)_1 - (\partial_h U)_0,$$

Theorem 3.4 tells that $(\partial_h U)_1'$ converges weakly in $L^2(0,T)$ and that u_0'' converges strongly in $L^2(0,T)$. This concludes the proof of the claim.

Next, observe that the quantities $\partial_t^2 \mathcal{U}_h - q_h U_h''$ and $\partial_{tx}\mathcal{U}_h - \partial_x p_h U_h'$ converge to zero in $\mathcal{A}^0_T$, and $h\sum_{j>0}(u_{j+1}'' - u_j'')^2$ goes to zero thanks to the strong convergence of $q_h U_h''$ in $\mathcal{A}^0$.

Finally, we need to show that $\partial_x^2 \mathcal{U}_h$ converges strongly to $\partial_x^2 \mathcal{U}$ in $\mathcal{A}^0_T$. But this is done by using the equation for $\mathcal{U}$ and the fact that $\partial_t^2 \mathcal{U}_h - \partial_x^2 \mathcal{U}_h - p_h F_h$ converges to zero as it is seen from the expression given in Lemma 3.7 and using the observation of the previous sentence. Then, the estimate (3.14) comes jointly from the estimates of Theorem 3.4 and Lemma 3.7. □

3.2 The $2-d$ case with straight interface

In this section we show that the result of existence of waves in asymmetric spaces also holds in $2-d$ when the interface is straight and, in fact, this result extends straightforwardly in higher dimensions for flat interfaces. This severe assumption on the geometry of the surface is due to the use of the discrete Fourier transform in the tangential variable. This Fourier analysis fails in the case of curved interfaces or equivalently for equations with variable coefficients. To handle this case one would need to give a discrete version of the pseudo-differential calculus used in [13].

The finite difference discrete version of system (2.1) reads

$$
\begin{aligned}
(3.15)\qquad & u''_{j,k} - \frac{u_{j+1,k}-2u_{j,k}+u_{j-1,k}}{h^2} - \frac{u_{j,k+1}-2u_{j,k}+u_{j,k-1}}{h^2} = f_{j,k}, \quad \textit{for } j\le -1,\ k\in\mathbb{Z}\\
& u''_{j,k} - (c^+)^2\frac{u_{j+1,k}-2u_{j,k}+u_{j-1,k}}{h^2} - (c^+)^2\frac{u_{j,k+1}-2u_{j,k}+u_{j,k-1}}{h^2} = f_{j,k}, \quad \textit{for } j\ge 1,\ k\in\mathbb{Z}\\
& u''_{0,k} - c^2\frac{u_{0,k+1}-2u_{0,k}+u_{0,k-1}}{h^2} = \frac{1}{h}((c^+)^2(u_{1,k}-u_{0,k}) + u_{-1,k}-u_{0,k}), \quad \textit{for } k\in\mathbb{Z}\\
& u_{j,k}(t=0) = a_{j,k},\ u'_{j,k}(t=0) = b_{j,k}, \quad \textit{for } (j,k)\in\mathbb{Z}^2
\end{aligned}
$$

In the following, for simplicity we make some abuse of notation, keeping those introduced in the previous section for similar but different objects:

Notations 3.10.

$$U=(u_{j,k})_{(j,k)\in\mathbb{Z}^2},\quad U_0=(a_{j,k})_{(j,k)\in\mathbb{Z}^2},\quad U_1=(b_{j,k})_{(j,k)\in\mathbb{Z}^2},\qquad F=(f_{j,k})_{(j,k)\in\mathbb{Z}^2}.$$

Let us also denote by $A_{h,h}$ the operator defined by

$$(A_{h,h}U)_{j,k} = -m_j\left(\frac{-4u_{j,k}+u_{j+1,k}+u_{j-1,k}+u_{j,k+1}+u_{j,k-1}}{h^2}\right),\quad m_j=\begin{cases}1 & \textit{if } j<0,\\ (c^+)^2 & \textit{if } j>0,\end{cases}$$

$$(A_{h,h}U)_{0,k} = -c^2\frac{u_{0,k+1}-2u_{0,k}+u_{0,k-1}}{h^2}.$$

The Cauchy problem for the previous equations formally reads:

$$U''-A_{h,h}U = F \tag{3.16}$$

$$U(t=0)=U_0,\ U'(t=0)=U_1. \tag{3.17}$$

For V_1, V_2 two vectors, we define

$$(V_1,V_2)_h = h^2\sum_{j<0}\sum_k v_{1,j,k}v_{2,j,k} + h^2(c^+)^2\sum_{j>0}\sum_k v_{1,j,k}v_{2,j,k} + hc^2\sum_k v_{1,0,k}v_{2,0,k},$$

and we set $\|V\|_h=\sqrt{(V,V)_h}$. We define $L^2_h=\{U\mid \|U\|_h<\infty\}$ and

$$H^1_h = \left\{U\in L^2_h \mid h^2\sum_{j,k}\left|\frac{u_{j+1,k}-u_{j,k}}{h}\right|^2 + \left|\frac{u_{j,k+1}-u_{j,k}}{h}\right|^2 + h\sum_k\left|\frac{u_{0,k+1}-u_{0,k}}{h}\right|^2<\infty\right\}.$$

Setting $E_{df}=\sqrt{\frac{1}{2}\|U'\|_h^2-(A_{h,h}U,U)_h}$ we get the h-uniform energy estimate for (3.16),(3.17)

$$E_{df}(T)\le E_{df}(0)+\int_0^t \|F\|_h\,. \tag{3.18}$$

Exactly as in Lemma 3.2 we can prove the existence of a solution in $\mathcal{C}([0,T];H^1_h)\cap\mathcal{C}^1([0,T];L^2_h)$ satisfying the above uniform energy estimate.

Next, we want to investigate the existence of solutions in asymmetric spaces. The main difference with the $1-d$ case is the presence of waves traveling along the surface with speed c. As in [13] we show that such a result requires $c<1$, i.e. the incoming waves must propagate faster than those at the interface. Before proving this result we first give a formal plane wave analysis which is similar to the continuous case (2.3).

3.2.1 Plane wave analysis

We consider plane wave solutions as in Section 3.1.1.

$$\begin{aligned} u_{j,k} &= e^{i(\omega t - jh\xi_i - kh\xi')} + C_r e^{i(\omega t + jh\xi_i - kh\xi')}, \quad j \leq -1 \\ u_{j,k} &= C_t e^{i(\omega t - jh\xi_t - kh\xi')}, \quad j \geq 1 \\ u_{0,k} &= C_t e^{i(\omega t - kh\xi')} \end{aligned}$$

In view of the continuity of the fields we have $1 + C_r = C_t$. Then, the dispersion relation at the left-hand side (i.e. for $j \leq 0$) is

$$\omega^2 = \frac{4}{h^2}\left(\sin^2\left(\frac{h\xi_i}{2}\right) + \sin^2\left(\frac{h\xi'}{2}\right)\right),$$

while the one at the right-hand side (i.e. for $j \geq 0$) is

$$\omega^2 = \left(\frac{2c^+}{h}\right)^2\left(\sin^2\left(\frac{h\xi_t}{2}\right) + \sin^2\left(\frac{h\xi'}{2}\right)\right) .$$

Setting $\alpha_i = (1 - e^{-ih\xi_i})/h$ and $\alpha_t = (1 - e^{-ih\xi_t})/h$ we get

$$C_t = \frac{\alpha_i - \bar{\alpha}_i}{\alpha_i + (c^+)^2\alpha_t + (c^2-1)\frac{4}{h^2}\sin^2\left(\frac{h\xi'}{2}\right) - \frac{4}{h^2}\sin^2\left(\frac{h\xi_i}{2}\right)},$$

$$C_r = -\frac{\bar{\alpha}_i + (c^+)^2\alpha_t + (c^2-1)\frac{4}{h^2}\sin^2\left(\frac{h\xi'}{2}\right) - \frac{4}{h^2}\sin^2\left(\frac{h\xi_i}{2}\right)}{\alpha_i + (c^+)^2\alpha_t + (c^2-1)\frac{4}{h^2}\sin^2\left(\frac{h\xi'}{2}\right) - \frac{4}{h^2}\sin^2\left(\frac{h\xi_i}{2}\right)} .$$

As in the $1-d$ case, the behavior of those coefficients at high frequency (i.e. ξ', ξ_i of order $1/h$) gives the expected regularity of the waves in each side. Since $C_r \sim 1$ at high frequencies, the regularity of the reflected wave is expected to be the same as the incident one. On the contrary, the transmission coefficient becomes of order $\mathcal{O}(h)$ in some cases. Precisely, if

$$C_t \sim h \quad \text{when} \quad \sqrt{\xi'^2 + \xi_i^2} \sim 1/h, \tag{3.19}$$

then the transmitted wave is expected to have a supplementary discrete derivative in L^2_h. From the form of C_t we need to discuss on the values of the parameter c.

1. When $c < 1$ the asymptotics (3.19) is true. So we expect the existence of solutions of (3.15) in asymmetric spaces.
2. When $c \geq 1$ the asymptotics (3.19) fails and we do not expect existence result in asymmetric spaces.

In the next subsection we tackle the first point. The second one will be considered in Section 5.2 where we present a numerical computation confirming the claim.

3.2.2 Existence in asymmetric spaces when $c < 1$

We want to show a result like Theorem 3.4. We thus first need to show as in Lemma 3.5 that the jump of the derivative across the interface is in $L^2([0,T]; L^{2\prime}_h)$ where

$$L^{2\prime}_h = \{(w_k)_{k\in\mathbb{Z}} \text{ such that } h\sum_{k\in\mathbb{Z}} w_k^2 < \infty\}.$$

We take the Fourier transform of (3.16) with respect to the discrete variable k:

$$\hat{u}_j(\xi') := h\sum_k u_{j,k} e^{ihk\xi'}$$

so to get the system

$$
\begin{aligned}
&\hat{u}_j'' - \frac{\hat{u}_{j+1} - 2\hat{u}_j + \hat{u}_{j-1}}{h^2} + \frac{4}{h^2}\sin^2\left(\frac{h\xi'}{2}\right)\hat{u}_j = \hat{f}_j, \quad for\ j < 0 \\
&\hat{u}_j'' - (c^+)^2\frac{\hat{u}_{j+1} - 2\hat{u}_j + \hat{u}_{j-1}}{h^2} + \frac{4(c^+)^2}{h^2}\sin^2\left(\frac{h\xi'}{2}\right)\hat{u}_j = \hat{f}_j, \quad for\ j > 0 \\
&\hat{u}_0'' + \frac{4c^2}{h^2}\sin^2\left(\frac{h\xi'}{2}\right)\hat{u}_0 = \frac{1}{h}((c^+)^2(\hat{u}_1 - \hat{u}_0) + \hat{u}_{-1} - \hat{u}_0) + \hat{f}_0 \\
&\hat{u}_j(t=0) = \hat{a}_j, \quad \hat{u}_j'(t=0) = \hat{b}_j, \quad for\ j \in \mathbb{Z}
\end{aligned}
\tag{3.20}
$$

Let us set $m_h = \frac{2}{h}\left|\sin\left(\frac{h\xi'}{2}\right)\right|$. We define the tangential Sobolev-like spaces:

$$
H_h^{\beta\prime} = \left\{ w \ \text{ such that } \ \int_{-\pi/h}^{\pi/h} (m_h)^{2\beta}|\hat{w}|^2 < \infty \right\}, \quad \beta \in \{0,1,2\}
$$

$$
H_{h,T}^{s\,\prime} = \bigcap_{\alpha+\beta=s} H^\alpha([0,T]; H_h^{\beta\prime}) \quad s \in \{0,1,2\}.
$$

Denoting by $\partial_{1,h}U$ the partial derivative of U with respect to the first index, i.e. with components $(\partial_{1,h}U)_j = \left(\frac{u_{j,k}-u_{j-1,k}}{h}\right)_{k\in\mathbb{Z}}$, we have:

Lemma 3.11. *Let $U_0 \in H_h^1$, $U_1 \in L_h^2$, $u_0 \in H_{h,T}^{1\,\prime}$ and $F \in L^1([0,T]; L_h^2)$. Then there is a constant C independent of h and T such that*

$$
\|((\partial_{1,h}U)_1, (\partial_{1,h}U)_0)\|_{L^2([0,T];L_h^{2\prime}\times L_h^{2\prime})} \le C\Big(\|U_0\|_{H_h^1} + \|U_1\|_{L_h^2} + \|u_0\|_{H_{h,T}^{1\,\prime}} + \|F\|_{L^1([0,T];L_h^2)}\Big).
$$

Proof. The proof mimics that of Lemma 3.5, working on (3.20). Indeed, to estimate $(\partial_{1,h}U)_1$ one looks for a solution of the problem at the right-hand side (i.e. $j \ge 0$) using the Laplace transform. The solution can still be computed as in Lemma 3.5 but with the change:

$$
(i\omega + s)^2 - \left(\frac{2c^+}{h}\right)^2 \mathrm{sh}(h\xi_t/2)^2 + (c^+ m_h)^2 = 0.
$$

This equation possesses two opposite solutions with non vanishing real part. Let $\xi_t = \xi_{t,1} + i\xi_{t,2}$ be the one with $\xi_{t,1} > 0$. Next, since s is independent of h then $\xi_{t,1} = \mathcal{O}(1)$ and one has

$$
|\mathrm{sh}(h\xi_t/2)|^2 = \mathrm{sh}^2(h\xi_{t,1}/2) + \sin^2(h\xi_{t,2}/2) \sim \frac{h^2}{4}|\xi_t|^2.
$$

By $a \sim b$ we mean that there are two positive constants c_1, c_2 such that $c_1 < a/b < c_2$. We deduce that $|\xi_t|^2 \sim \sqrt{|\xi'^2 - \omega^2 + s^2|^2 + 4s^2\omega^2} \le \omega^2 + s^2 + \xi'^2$. Thus, there is a constant $c > 0$ such that for all $\omega \le \pi/h$, $\xi' \le \pi/h$

$$
\frac{e^{\xi_t h} - 1}{h(1 + \sqrt{\omega^2 + s^2 + \xi'^2})} \le c.
$$

The end of the proof then follows that of Lemma 3.5. □

Next, to get a result like Theorem 3.4 we need to show that the solution of the boundary equation with a right-hand side in $L^2([0,T]; L_h^{2\prime})$ is in $H^2([0,T]; L_h^{2\prime})$. This is not possible since the equation at the interface is a wave equation, thus not elliptic. One gets however this regularity for the waves propagating faster than c. For the others, one just gets a bound in $H^1([0,T]; L_h^{2\prime})$. One thus needs to improve Lemma 3.11 for those waves using the fact that they are more regular with respect to the initial data. This analysis has been done in [13] by means of micro-local analysis. The uniformity of the mesh allows to translate the proofs to our semi-discrete setting. This is done in the next paragraph.

3.2.3 Micro-local estimates

We first consider the equation at the interface

$$\hat{u}_0'' + (c\, m_h)^2 \hat{u}_0 = \hat{g}, \quad \text{in } [0,T]\times\mathbb{R} \tag{3.21}$$

The homogeneous equation possesses the energy:

$$E'_{df}(t) = \sqrt{\int_{-\pi/h}^{\pi/h} \hat{u}_0^{2\prime} + (c\, m_h)^2 |\hat{u}_0|^2 d\xi'} = \sqrt{h\sum_k |u'_{0,k}|^2 + \left(\frac{u_{0,k+1}-u_{0,k}}{h}\right)^2},$$

and we have the energy estimate:

$$E'_{df}(t) \le E'_{df}(0) + \int_0^t \|g(s)\|_{L_h^{2\prime}} ds. \tag{3.22}$$

Next, recalling that $c<1$, let $P_h(\partial_t,\partial_y)$ be a pseudo-differential operator whose symbol is given by $P_h(\omega,\xi') = P(h\omega, h\xi')$ where $P(a,b)$ is a smooth function, independent of h and

$$P(a,b) = 0 \text{ on } \left\{\frac{a}{2} \le c\left|\sin\left(\frac{b}{2}\right)\right|\right\}, \quad P(a,b) = 1 \text{ on } \left\{\frac{a}{2} \ge \left|\sin\left(\frac{b}{2}\right)\right|\right\}.$$

Lemma 3.12. *The equation* (3.21) *with initial value* $\hat{a}_0, \hat{b}_0$ *such that* $a_0 \in H_h^{2\prime}, b_0 \in H_h^{1\prime}$ *and source term* $\hat{g}$ *with* $g \in L^1([0,T]; H_h^{1\prime})$ *possesses a solution* u_0 *in* $H_{h,T}^{2\prime}$ *satisfying*

$$\|u_0\|_{H_{h,T}^{2\prime}} \le \|a_0\|_{H_h^{2\prime}} + \|b_0\|_{H_h^{1\prime}} + \|P_h g\|_{L_{h,T}^{2\prime}} + \|(1-P_h)g\|_{L^1([0,T];H_h^{1\prime})}.$$

Proof. This estimate is local in time so we set $v_0 = \chi u_0$ where χ is a smooth cutoff function depending only on t with compact support say $[-T_1, T_2] \supset [0,T]$. One has

$$\hat{v}_0'' + (c^2 m_h^2 - 1)\hat{v}_0 = \chi\hat{g} + 2\chi'\hat{u}_0' + (\chi'' - \chi)\hat{u}_0 := \hat{q}\,.$$

We have subtracted the equation by $\hat{v}_0$ so as to get a non vanishing elliptic operator. The solution can be written as $\hat{v}_0 = \hat{q}_1 + \hat{q}_2$ where

$$\hat{q}_1 = \frac{-1}{\sqrt{2\pi}} \int_{\mathbb{R}} P_h(\omega,\xi') \frac{\mathcal{F}_t \hat{q}(\omega,\xi')}{1+\omega^2 - (cm_h)^2} e^{i\omega t} d\omega,$$

($\mathcal{F}_t$ denotes the Fourier transform with respect to t) and $\hat{q}_2$ is solution of the same equation but with r.h.s. $(1-P_h)\hat{q}$ and vanishing initial data at time $-T_1$. Since $1+\omega^2-(cm_h)^2 \ge 1+(1-c^2)\omega^2$ whenever $1-P_h \neq 0$, Parseval formula yields

$$\|q_1\|_{H_{h,T}^{2\prime}} \le \frac{1}{1-c^2}\|P_h q\|_{L_{h,T}^{2\prime}} \le \frac{\text{Cst}}{1-c^2}\left(\|P_h g\|_{L_{h,T}^{2\prime}} + \|a_0\|_{H_h^{1\prime}} + \|b_0\|_{L_h^{2\prime}}\right).$$

Then, differentiating the equation for $\hat{q}_2$ and using the energy estimate (3.22) allows to get an estimate in $H_{h,T}^{2\prime}$. This concludes the proof of the estimate of the lemma. □

Using the regularity of the initial data for $j \ge 0$ together with the fact that $u_0 \in H_h^{2\prime}$ allows to apply a version of Lemma 3.11 for smoother data and show that $(\partial_{1,h}U)_1 \in H_h^{1\prime}$. Unfortunately, for $j \le 0$ the initial data are not smooth enough. We thus need to improve Lemma 3.11 as in [13].

Lemma 3.13. *Let* $U_0 \in H_h^1,\ U_1 \in L_h^2,\ F \in L^1([0,T];H_h^1),\ F' \in L^1([0,T];L_h^2)$ *and* $u_0 \in H_{h,T}^{2\prime}$. *Then, the solution of the mixed problem*

$$\hat{u}_j'' - \frac{\hat{u}_{j+1} - 2\hat{u}_j + \hat{u}_{j-1}}{h^2} + m_h^2 \hat{u}_j = \hat{f}_j, \quad for\ j \le -1,$$
$$\hat{u}_j(t=0) = \hat{a}_j, \quad \hat{u}_j'(t=0) = \hat{b}_j, \quad for\ j \le -1,$$

satisfies

$$\|(1-P_h)(\partial_{1,h}U)_0\|_{H_{h,T}^{1\prime}} \le \|U_0\|_{H_h^1} + \|U_1\|_{L_h^2} + \|u_0\|_{H_{h,T}^{2\prime}} + \|F\|_{L^1([0,T];H_h^1)} + \|F'\|_{L^1([0,T];L_h^2)}.$$

Proof. We follow the proof of [13]. We first recast the problem so that u_0 is replaced by 0. Thanks to the regularity of u_0, one can extend this function to $\bar{U} \in \cap_{a+b=2} H^a([0,T]; H_h^b)$. Then, $V = U - \bar{U}$ solves the equations like U with $v_0 = 0$ and with source term $f_j - \bar{u}_j'' + \frac{\bar{u}_{j+1} - 2\bar{u}_j + \bar{u}_{j-1}}{h^2} \in L^2([0,T]; L_h^2)$. Thanks to Lemma 3.11 one already has the estimate for the source term. So we can assume vanishing source terms. Since $v_0 = 0$, we can use the odd Fourier transform

$$\hat{V} = h \sum_{j \leq -1} \hat{v}_j \sin(hj\xi_1).$$

Note that $\hat{V}$ is the Fourier transform of a vector both in j, k. Applying the transform on the equation satisfied by V, we get

$$\hat{V}'' + \omega^2(\xi_1)\hat{V} = 0, \quad \text{where} \quad \omega(\xi_1) = \frac{2}{h}\left(\sin^2\left(\frac{\xi_1 h}{2}\right) + \sin^2\left(\frac{\xi' h}{2}\right)\right)^{1/2},$$

$$\text{so} \qquad \hat{V} = \hat{U}_0 \cos(\omega(\xi_1)t) + \frac{\hat{U}_1}{\omega(\xi_1)} \sin(\omega(\xi_1)t).$$

The discrete left normal derivative at the boundary is given by

$$\frac{\hat{v}_{-1}}{h} = \frac{1}{2h\pi} \int_{-\pi/h}^{\pi/h} \hat{V} \sin(\xi_1 h) d\xi_1.$$

We need to show that $(1 - P_h)\chi \hat{v}_{-1}/h \in {H^1_{h,T}}'$ for any $\mathcal{C}^\infty$ function χ (of the variable t) with compact support in an interval containing $[0,T]$ and such that $\chi(t) = 1$ for $t \in [0,T]$. We first begin to show that $(1 - P_h)\chi \hat{v}'_{-1}/h \in {L^2_{h,T}}'$. By the Parseval theorem it is equivalent to show that the time Fourier transform $(1 - P_h)\mathcal{F}_t \chi \hat{v}'_{-1}/h$ belongs to ${L^2_{h,\mathbb{R}}}'$. One has

$$\left\| (1 - P_h)\frac{\chi \hat{v}'_{-1}}{h} \right\|^2_{{L^2_{h,T}}'} = \int_{-\pi/h}^{\pi/h} \int_{\mathbb{R}} \left| \frac{1}{2h\pi} \int_{-\pi/h}^{\pi/h} (1 - P_h(\omega, \xi'))\mathcal{F}_t \chi \hat{V}' \sin(\xi_1 h) d\xi_1 \right|^2 d\omega d\xi'.$$

$$2\mathcal{F}_t \chi \hat{V}' = i\omega(\xi_1)\hat{U}_0\Big(\hat{\chi}(\omega - \omega(\xi_1)) - \hat{\chi}(\omega + \omega(\xi_1))\Big) + \hat{U}_1\Big(\hat{\chi}(\omega - \omega(\xi_1)) + \hat{\chi}(\omega + \omega(\xi_1))\Big).$$

All the terms in $\mathcal{F}_t \chi \hat{V}'$ are dealt similarly thanks to $1 - P_h$ so we focus on $\omega(\xi_1)\hat{U}_0 \hat{\chi}(\omega - \omega(\xi_1))$. The square of the ${L^2_{h,T}}'$ norm of the corresponding term reads

$$I = \int_{[-\pi/h, \pi/h]^3} \int_{\mathbb{R}} (1 - P_h(\omega, \xi')) \frac{\sin(\xi_1 h)\sin(\xi_2 h)}{h^2} \hat{U}_0(\xi_1, \xi')\hat{U}_0(\xi_2, \xi') \cdot \omega(\xi_1)\omega(\xi_2)\hat{\chi}(\omega - \omega(\xi_1))\hat{\chi}(\omega - \omega(\xi_2)) d\omega d\xi_1 d\xi_2 d\xi'.$$

Putting the integral on ω at front and using the Cauchy-Schwartz inequality w.r.t. ξ_1, ξ_2 gives

$$I \leq \int_{\mathbb{R}} \int_{[-\pi/h, \pi/h]} (1 - P_h(\omega, \xi')) \left\| \hat{U}_0(\cdot_1, \xi')\hat{U}_0(\cdot_2, \xi') \right\|_{L^2([-\pi/h, \pi/h]^2)} \cdot \left\| \frac{\sin(\cdot_1 h)\sin(\cdot_2 h)}{h^2} \omega(\cdot_1)\omega(\cdot_2)\hat{\chi}(\omega - \omega(\cdot_1))\hat{\chi}(\omega - \omega(\cdot_2)) \right\|_{L^2([-\pi/h, \pi/h]^2)} d\xi' d\omega,$$

where $\cdot_1$ means integration w.r.t. ξ_1. The first norm is $\|\hat{U}_0(\cdot, \xi')\|^2_{L^2([-\pi/h, \pi/h])}$ and the second is $1/h^2 \| \sin(\cdot_1 h)\omega(\cdot_1)\hat{\chi}(\omega - \omega(\cdot_1))\|^2_{L^2([-\pi/h, \pi/h])}$. Using that $\sin(x) \leq x$ one gets

$$I \leq \|\hat{U}_0\|^2_{L^2_h{}'} \sup_{\xi'} \left(\int_{-\pi/h}^{\pi/h} \int_{\mathbb{R}} (1 - P_h(\omega, \xi'))\xi_1^2 \omega^2(\xi_1)\hat{\chi}^2(\omega - \omega(\xi_1)) d\omega d\xi_1 \right)^{1/2}. \tag{3.23}$$

Let us prove that the second term in the product (let us call it J) is bounded uniformly in h. For this we split the domain of integration in two parts (we do a splitting according to the direction in Fourier space since the pseudo-differential operator P_h only depends on the direction):

$$\Omega_1 = \left\{(\omega, \xi_1) \quad | \quad |\omega(\xi_1) - \omega| \leq \sqrt{|\xi'|}\right\},$$

and its complementary denoted by Ω_2.

Let us first consider J_{Ω_2} the integral whose integration range is Ω_2. First note that $\hat{\chi}$ belongs to the Schwartz space hence $\sup_\omega (1+t^2)^s |\hat{\chi}(t)| < \infty$ for all $s > 0$. Now, we claim that there is a constant $c > 0$ such that

$$\omega(\xi_1) - \omega \geq \max\left(\sqrt{|\xi'|}, c|\xi_1|, c|\xi_1|^2/|\xi'|\right),$$

from which it follows easily that J_{Ω_2} is bounded uniformly w.r.t. h. Indeed, by definition of Ω_2 the first inequality is clear. For the second one we have $\omega \leq m_h(\xi')$ and so

$$\omega(\xi_1) - \omega \geq \sqrt{m_h^2(\xi') + m_h^2(\xi_1)} - m_h(\xi') = m_h^2(\xi_1) / \left(\sqrt{m_h^2(\xi') + m_h^2(\xi_1)} + m_h(\xi')\right).$$

If $\xi_1 \leq \xi'$ the quantity $m_h^2(\xi_1)/((1+\sqrt{2})m_h(\xi'))$ is a lower bound. Otherwise it is $m_h(\xi_1)/(1+\sqrt{2})$. Finally, recall that $m_h(\xi') \sim \xi'$ and $m_h(\xi_1) \sim \xi_1$.

Next, let us estimate J_{Ω_1} (defined like J_{Ω_2}). Note that in Ω_1 the vector $(\omega, m_h(\xi'))$ lies in a small conical neighborhood of $(1,1)$. Indeed using that $\omega \leq m_h(\xi')$ one gets

$$\left|1 - \frac{\omega}{m_h(\xi')}\right| \leq \frac{|\omega(\xi_1) - \omega| + |\omega(\xi_1) - m_h(\xi')|}{m_h(\xi')} \leq \frac{\pi}{\sqrt{\xi'}}.$$

Since $1 - P_h(\omega, \xi')$ vanishes at infinite order for $\frac{\omega}{m_h(\xi')} \sim 1$, one has

$$1 - P_h(\omega, \xi') \leq \frac{c_p}{|\xi'|^p}, \quad \forall p > 0.$$

Next, using the definition of Ω_1 and $\omega \leq m_h(\xi')$, we get

$$|\xi_1| \leq \frac{\pi}{h}\sin(h\xi_1/2) \leq \frac{\pi}{2}\omega(\xi_1) \leq \sqrt{|\xi'|} + |\xi'|.$$

From this one sees that J_{Ω_1} is a bounded expression of ξ', independent of h.

We have shown that $(1-P_h)\chi\hat{v}'_{-1}/h \in {L^2_{h,T}}'$. To show that $(1-P_h)\chi\hat{v}_{-1}/h \in {H^1_{h,T}}'$ one has to consider $(1-P_h)\chi m_h\hat{v}_{-1}/h$. The previous analysis also applies here since $m_h \leq \omega(\xi_1)$. This concludes the proof of the lemma. □

3.2.4 End of the proof of existence in asymmetric spaces

Let us denote by $\mathcal{A}_h^s$, $s = 0, 1, 2$ the spaces defined in a similar way as in section 3.1.2. For example:

$$\mathcal{A}_h^2 := \left\{U \in H_h^1 \mid \; h\sum_{j\geq 1}\left\|\frac{\hat{u}_{j+1} - 2\hat{u}_j + \hat{u}_{j-1}}{h^2}\right\|^2_{L^{2\prime}_h} + \left\|m_h\frac{\hat{u}_j - \hat{u}_{j-1}}{h}\right\|^2_{L^{2\prime}_h} + \left\|m_h^2\hat{u}_j\right\|^2_{L^{2\prime}_h} + \left\|m_h^2\hat{u}_0\right\|^2_{L^{2\prime}_h} < \infty\right\}.$$

Theorem 3.14. *Let $U_0 \in \mathcal{A}_h^2$, $U_1 \in \mathcal{A}_h^1$, $F \in L^1([0,T];\mathcal{A}_h^1)$ and $F' \in L^1([0,T];\mathcal{A}_h^0)$. Then System (3.16),(3.17) possesses a solution $U \in \mathcal{A}_{h,T}^2$ and there is an h-independent constant C such that*

$$\|U\|_{\mathcal{A}_{h,T}^2} \leq C\left(\|U_0\|_{\mathcal{A}_h^2} + \|U_1\|_{\mathcal{A}_h^1} + \|F\|_{L^1([0,T];\mathcal{A}_h^1)} + \|F'\|_{L^1([0,T];\mathcal{A}_h^0)}\right).$$

Proof. This proof follows closely that of [13] but for sake of completeness we recall it. The idea is to look for a solution of (3.16), (3.17) under the form $U = U^2 + V$ with V solving the same problem with homogeneous initial data and a smooth boundary term so that $V \in \bigcap_{\alpha+\beta=2} \mathcal{C}^\alpha(0,T;H_h^\beta)$ and U^2 is in the asymmetric space. The construction of U^2 is done in a few steps:

1. Let w^1 be a solution of

$$w_k^{1\prime\prime} - c^2 \frac{w_{k+1}^1 - 2w_k^1 + w_{k-1}^1}{h^2} = f_{0,k}, \quad \text{in } [0,T], \quad k \in \mathbb{Z}$$
$$w_k^1(t=0) = a_{0,k}, \quad w_k^{1\prime}(t=0) = b_{0,k}.$$

   Since $a_0 \in H_h^{2\prime}$, $b_0 \in H_h^{1\prime}$ and $(f_{0,k})_{k\in\mathbb{Z}} \in {H_{h,T}^1}'$ it follows that $w^1 \in {H_{h,T}^2}'$ by using an energy estimate.

2. Next let U^1 be the solution of

$$u_{j,k}^{1\prime\prime} - (c^+)^2 \frac{u_{j+1,k}^1 - 2u_{j,k}^1 + u_{j-1,k}^1}{h^2} - (c^+)^2 \frac{u_{j,k+1}^1 - 2u_{j,k}^1 + u_{j,k-1}^1}{h^2} = f_{j,k}, \quad j \geq 1,$$
$$u_{j,k}^{1\prime\prime} - \frac{u_{j+1,k}^1 - 2u_{j,k}^1 + u_{j-1,k}^1}{h^2} - \frac{u_{j,k+1}^1 - 2u_{j,k}^1 + u_{j,k-1}^1}{h^2} = f_{j,k}, \quad j \leq -1,$$
$$u_{0,k}^1 = w_k^1,$$
$$u_{j,k}^1(t=0) = a_{j,k}, \quad u_{j,k}^{1\prime}(t=0) = b_{j,k}, \quad \textit{for } j \neq 0,$$

   Differentiating discretely those equations with respect to each space index and continuously w.r.t. the time and getting an energy estimate in each case shows that $U^1 \in \mathcal{C}(0,T;\mathcal{A}_h^2) \cap \mathcal{C}^1(0,T;\mathcal{A}_h^1) \cap \mathcal{C}^2(0,T;\mathcal{A}_h^0)$. Moreover, from Lemma 3.11 (in fact, a version with smoother data) one has $(\partial_{1,h}U^1)_1 \in {H_{h,T}^2}'$ and from Lemma 3.13 one has $(1-P_h)(\partial_{1,h}U^1)_0 \in {H_{h,T}^1}'$.

3. Now we define w^2 like w^1 but with source term $(c^+)^2(\partial_{1,h}U^1)_1 - (\partial_{1,h}U^1)_0$. The estimate of Lemma 3.12 implies that $w_2 \in {H_{h,T}^2}'$.

4. Finally let us define U^2 as U^1 replacing w_1 by w_2. One has similarly $U^2 \in \mathcal{C}(0,T;\mathcal{A}_h^2) \cap \mathcal{C}^1(0,T;\mathcal{A}_h^1) \cap \mathcal{C}^2(0,T;\mathcal{A}_h^0)$.

We see that $V = U - U^2$ satisfies (3.16), (3.17) with homogeneous initial data and boundary source term $(c^+)^2(\partial_{1,h}(U^2 - U^1))_1 - (\partial_{1,h}(U^2 - U^1))_0$. To estimate V as in the steps 2 and 4 one requires the source term to be in ${H_{h,T}^1}'$. But, in view of Lemma 3.11 (in fact, a version with more regular data), this requires the boundary value $u_0^2 - u_0^1 = w_2 - w_1$ to be in ${H_{h,T}^2}'$ and the initial data of $U^2 - U^1$ in $H_h^2 \times H_h^1$, which is the case. □

Remark 3.15. *Looking for $U = U^1 + V$ is not sufficient because the initial value of U^1 is still in an asymmetric space and thus it does not allow to conclude that $V \in \bigcap_{\alpha+\beta=2} \mathcal{C}^\alpha(0,T;H_h^\beta)$.*

3.2.5 Convergence

Thanks to the estimate of Theorem 3.14 we have a weak and strong convergence result exactly as in $1-d$ since the construction of the interpolated function can be done as in Lemma 3.7 in the coordinate x (index j) and with the Fourier transform in the y coordinate. We refer to [12] for the interpolation estimate.

4 Mixed finite element approximation

4.1 The $1-d$ case

In this section we show a result like Theorem 3.4 in the case of the mixed finite element approximation used in [2]. This method was used in the context of controllability and it was shown that its spectral

properties are in sharp contrast with those of the finite difference and the finite element approximation at high frequencies. Moreover, this method enjoys some symmetries which we exploit here to improve the result of the previous section. Indeed, in Lemma 3.5 we use a Laplace transform to get an estimate of the discrete spatial derivative at the interface. This method also works for the finite element and mixed finite element discretization but it requires to have uniform time discretizations. Here, we can provide an estimate using the discrete version of the Rellich multipliers which applies for the full space-time discretized equations with non-uniform meshes (see [8]).

First, let us introduce the mixed finite element scheme for (2.1):

$$
\begin{aligned}
&\left(\frac{u_{j+1}+2u_j+u_{j-1}}{4}\right)'' - \frac{u_{j+1}-2u_j+u_{j-1}}{h^2} = f_j, \quad \textit{for } j \le -1,\\
&\left(\frac{u_{j+1}+2u_j+u_{j-1}}{4}\right)'' - (c^+)^2\frac{u_{j+1}-2u_j+u_{j-1}}{h^2} = f_j, \quad \textit{for } j \ge 1\\
&u_0'' + c^2 u_0 = \frac{1}{h}((c^+)^2(u_1-u_0)-(u_0-u_{-1})) + f_0.
\end{aligned}
\tag{4.1}
$$

The homogeneous system preserves the energy E_{mfe} defined by

$$
\begin{aligned}
E_{mfe}^2(t) = h\sum_{j\neq 0}\left(\frac{u'_{j+1}+2u'_j+u'_{j-1}}{4}\right)^2 + (c^+)^2 h\sum_{j\ge 1}\left(\frac{u_{j+1}-u_{j-1}}{2h}\right)^2\\
+ h\sum_{j\le -1}\left(\frac{u_{j+1}-u_{j-1}}{2h}\right)^2 + (u_0')^2 + c^2u_0^2.
\end{aligned}
$$

With the notations 3.1 we define the operator M_h by

$$
(M_hU)_j = \left(\left(\frac{u_{j+1}+2u_j+u_{j-1}}{4}\right)_{j<0}, u_0, \left(\frac{u_{j+1}+2u_j+u_{j-1}}{4}\right)_{j>0}\right).
$$

Notations 4.1. *System* (4.1) *reads:*

$$
M_hU'' + A_hU = F, \tag{4.2}
$$

$$
U(t=0) = U_0, \quad U'(t=0) = U_1. \tag{4.3}
$$

The operator A_h is the same as in (3.3). Let us define the new energy space. With the scalar product defined in (3.5) we set

$$
\bar{L}_h^2 := \{U \text{ such that } \|M_hU\|_h^2 < \infty\} \quad \text{and} \quad \bar{H}_h^1 := \{U \in \bar{L}_h^2 \mid \sum_{j\neq 0}\left(\frac{u_{j+1}-u_{j-1}}{2h}\right)^2 < \infty\}.
$$

Since $\bar{L}_h^2$ is a Hilbert space, one can use Hille-Yoshida's theorem to get the following existence result:

Lemma 4.2. *Let $U_0 \in \bar{H}_h^1$, $U_1 \in \bar{L}_h^2$, $F \in L^1([0,T];\bar{L}_h^2)$ be given. Then for all $T \in \mathbb{R}$ there is a unique solution U of* (4.2),(4.3) *in $\mathcal{C}^1([0,T];\bar{L}_h^2)\cap\mathcal{C}([0,T];\bar{H}_h^1)$ satisfying the h–independent estimate*

$$
\bar{E}_{mfe}(t) \le \bar{E}_{mfe}(0) + \int_0^t \|F(s)\|_h ds.
$$

We next perform a formal plane wave analysis as in the previous section to identify the best regularity we can expect for the transmitted/reflected waves propagating in the whole domain.

4.1.1 Plane wave analysis

We look for plane wave solution exactly as in Subsection 3.1.1. The only change is the dispersion relation which becomes (cf. [2], p.7)

$$\omega = \frac{2}{h}\left|\tan\left(\frac{h\xi_i}{2}\right)\right| = \frac{2c^+}{h}\left|\tan\left(\frac{h\xi_t}{2}\right)\right|.$$

Setting $\alpha_i = (1-e^{-ih\xi_i})/h$ and $\alpha_t = (1-e^{-ih\xi_t})/h$, we get

$$C_t = \frac{\alpha_i - \bar{\alpha}_i}{\alpha_i + (c^+)^2\alpha_t + c^2 - \omega^2}, \quad C_r = -\frac{\bar{\alpha}_i + (c^+)^2\alpha_t + c^2 - \omega^2}{\alpha_i + (c^+)^2\alpha_t + c^2 - \omega^2}\,.$$

For the high frequencies $\xi_i = \mathcal{O}(1/h)$, we find that $C_t = \mathcal{O}(h)$ while $C_r = \mathcal{O}(1)$. Note that when $h\xi_i$ is close to π then C_t/h is close to zero thus the reflection is stronger as in the finite difference case.

So, we expect, as in the finite difference case, that the transmitted wave has a supplementary derivative in a L^2_h-like space.

4.1.2 Existence in asymmetric spaces

We define $\bar{\mathcal{A}}^2_h$ ($\bar{\mathcal{A}}^1_h, \bar{\mathcal{A}}^0_h$ resp.) as in Subsection 3.1.2 by restricting the sums in the definition of H^2_h ($\bar{H}^1_h, \bar{L}^2_h$ resp.) to $j \geq 1$ (resp. $j \geq 1,\ j \geq 1$). Finally, we set

$$\bar{\mathcal{A}}^2_{h,T} = \bigcap_{a+b=2} \mathcal{C}^a([0,T];\bar{\mathcal{A}}^b_h).$$

Theorem 4.3. *Let $U_0 \in \bar{\mathcal{A}}^2_h$, $U_1 \in \bar{\mathcal{A}}^1_h$ and $(F,F') \in L^1([0,T];\bar{\mathcal{A}}^1_h \times \bar{\mathcal{A}}^0_h)$. Then for all $T \in \mathbb{R}$, the solution U of (4.2),(4.3) is in the space $\bar{\mathcal{A}}^2_{h,T}$. Moreover we have the h-uniform estimate:*

$$\|U\|_{\bar{\mathcal{A}}^2_{h,T}} + \|(\partial_h U)_1\|_{H^1([0,T])} \leq C\Big(\|U_0\|_{\bar{\mathcal{A}}^2_h} + \|U_1\|_{\bar{\mathcal{A}}^1_h} + \|F\|_{L^1([0,T];\bar{\mathcal{A}}^1_h)} + \|F'\|_{L^1([0,T];\bar{\mathcal{A}}^0_h)}\Big).$$

Proof. As in Theorem 3.4 the key step is to show that $(\partial_h U)_1 \in L^2([0,T])$ and $(\partial_h U)_0 \in L^2([0,T])$. So, we consider

$$\begin{aligned} &\left(\frac{u_{j+1}+2u_j+u_{j-1}}{4}\right)'' - (c^+)^2\frac{u_{j+1}-2u_j+u_{j-1}}{h^2} = f_j, \quad \textit{for } j \geq 1 \\ &u_0 \in H^1([0,T]), \\ &u_j(t=0) = a_j, \quad u_j'(t=0) = b_j \end{aligned} \tag{4.4}$$

Lemma 4.4. *The solution of (4.4) satisfies $(\partial_h U)_1 \in L^2(0,T)$ and $(\partial_h U)_0 \in L^2(0,T)$.*

Proof. Here, contrary to the finite difference case, we have a supplementary symmetry which allows to use a discretized version of the Rellich multiplier (see [16]) in this very simple case where it reduces to multiply the equation by u_x and then integrate w.r.t. x.

For non negative indexes we set

$$u_{j+1/2} = (u_{j+1}+u_j)/2, \quad v_j = \frac{u'_{j+1/2}+u'_{j-1/2}}{2}, \quad w_j = c^+\frac{u_{j+1/2}-u_{j-1/2}}{h}.$$

First, multiplying scalarly the equations by $(v_j)_{j\geq 1}$ gives the energy equality

$$\frac{h}{2}\left(\sum_{j\geq 1} v_j^2\right)' + \frac{h}{2}\left(\sum_{j\geq 1} w_j^2\right)' + (c^+)^2 u'_{1/2}\frac{u_1-u_0}{h} = h\sum_{j\geq 1} f_j v_j.$$

Then, multiplying the equations by $(w_j)_{j\geq 1}$ (discrete Rellich multiplier) gives

$$h\left(\sum_{j\geq 0} v_j w_j\right)' + \frac{c^+}{2}\left((u'_{1/2})^2 + \left(c^+\frac{u_1-u_0}{h}\right)^2\right) = h\sum_{j\geq 0} f_j w_j.$$

Summing, and setting $s_j = v_j + w_j$ we get

$$h\left(\sum_{j\geq 0} s_j^2\right)' \leq 2h\sum_{j\geq 0} f_j s_j, \quad \text{and} \quad \left(u'_{1/2} + c^+\frac{u_1-u_0}{h}\right)^2 \leq \frac{2}{c^+}h\sum_{j\geq 0} f_j s_j.$$

Using the Gronwall Lemma we have $S = (s_j)_j \in \mathcal{A}_h^0$ and then $\int_0^T \left(u'_{1/2} + c^+(u_1-u_0)/h\right)^2$ is bounded independently of h. Next using that $2u'_{1/2}c^+(u_1-u_0)/h \leq \left(u'_{1/2} + c^+(u_1-u_0)/h\right)^2$ in the first identity and integrating the latter over $[0,T]$, we get an estimate for $\|U\|_{\bar{\mathcal{A}}_h^1}$. Then integrating the second identity over $[0,T]$ we get

$$\int_0^T \left((u'_{1/2})^2 + \left(c^+\frac{u_1-u_0}{h}\right)^2\right) \leq \|F\|_h\|U\|_h + \|U_0\|_h^2 + \|U(T)\|_h^2.$$

Thus $(u_1-u_0)/h$ is in $L^2([0,T])$. The same applies for $(u_0-u_{-1})/h$. □

From the previous lemma we get that $u_0 \in H^2([0,T])$. Then, the end of the proof simply repeats the previous lemma for the time derivative of the equations. □

4.1.3 Convergence

As in Section 3.1.3 one needs to construct a function interpolating U. One could think to use the variational formulation as in [2] but this would restrict ourselves to the continuous variational space H^1. Instead, we do as in the semi-discrete, finite difference case, replacing the spaces $\mathcal{A}_{h,T}^s$ by $\bar{\mathcal{A}}_{h,T}^s$. We thus get a result for the weak convergence which reads like Theorem 3.8.

A strong convergence result also holds but the estimate (3.14) does not hold here with the space $\bar{\mathcal{A}}_{h,T}^s$. One needs to work with

$$\tilde{\mathcal{A}}_{h,T}^s = \bar{\mathcal{A}}_{h,T}^s \cap \frac{1}{h}\bar{\mathcal{A}}_{h,T}^{s+1}, \quad \text{with} \quad s \leq 2. \tag{4.5}$$

We claim that Lemma 3.7 still applies with those spaces. But first we need to define $\bar{\mathcal{A}}_{h,T}^3$. This is done by taking the quantities preserved by derivating the equations twice in time. One finds

$$\bar{\mathcal{A}}_{h,T}^3 = \left\{U \in H_{h,T}^2 \mid \sup_{t\leq T}\left(\|(M_hU''')^+\|_h + \|(\partial_h U'')^+\|_h + \|(A_hU')^+\|_h + \|(\partial_h A_h U)^+\|_h\right) < \infty\right\},$$

where $H_{h,T}^2 = \bigcap_{a+b=2}\mathcal{C}^a([0,T];\bar{H}_h^b)$. Next, showing Lemma 3.7 only requires to check that the second estimate of the lemma holds. The only difference is that the leading term of $\partial_t^2\mathcal{U}_h - \partial_x^2\mathcal{U}_h - p_hF_h$ is $u_j'' - a_j - f_j$ which does not vanish since u_j solves (4.1). Instead, we have

$$u_j'' - a_j - f_j = u_j'' - \frac{u_{j+1}'' + 2u_j'' + u_{j-1}''}{4} = \frac{u_j'' - u_{j+1}''}{4} + \frac{u_j'' - u_{j-1}''}{4}.$$

The terms are unfortunately not expressed in terms of the components of $\partial_h U''$. But we can use the equality:

$$\frac{u_{j+1}-u_j}{h} = h\frac{u_{j+1}-2u_j+u_{j-1}}{2h^2} + \frac{u_{j+1}-u_{j-1}}{2h}.$$

Hence the definition of the spaces $\tilde{\mathcal{A}}_{h,T}^s$. With those spaces a strong convergence result holds and reads like Theorem 3.9.

4.2 The $2-d$ case

In $2-d$, the equations can be found in [3]. They are like those of system (3.15) but replacing the terms $u''_{j,k}$ by

$$\frac{1}{16}(4u''_{j,k}+2u''_{j-1,k}+2u''_{j+1,k}+2u''_{j,k+1}+2u''_{j,k-1}+u''_{j-1,k-1}+u''_{j-1,k+1}+u''_{j+1,k+1}+u''_{j+1,k-1}) \quad \text{if } j\neq 0,$$

and replacing $u''_{0,k}$ by $\frac{1}{4}(u''_{0,k+1}+2u''_{0,k}+u''_{0,k-1})$.

For these equations a result like Theorem 3.14 also holds but since the proof is just an adaptation of the $1-d$ mixed finite element and the $2-d$ finite difference method we refrain from doing it. Instead let us indicate the main changes. First, a computation of the transmission coefficient for plane waves gives:

$$C_t=\frac{\alpha_i-\bar{\alpha}_i}{\alpha_i+(c^+)^2\alpha_t+(c^2-1)\frac{4}{h^2}\sin^2\left(\frac{h\xi'}{2}\right)-\frac{4}{h^2}f(h\xi',h\xi_i)}$$

where $f(x,y)=\cos^2(x/2)\tan^2(y/2)+\frac{2}{3}\sin^2(x/2)\tan^2(y/2)$. So, for $c<1$ one expects the existence of solutions in asymmetric spaces as Theorem 3.14. We next make a few comments on how to adapt the Lemma of Section 3.2 and Theorem 3.14.

4.2.1 Comments on the existence result in asymmetric spaces when $c<1$

We first write the full system after Fourier transforming it with respect to the index k. Keeping the notation $m_h=\frac{2}{h}\left|\sin\left(\frac{h\xi'}{2}\right)\right|$ and setting $c_h=\left|\cos\left(\frac{h\xi'}{2}\right)\right|$, we get:

$$\begin{aligned}
&c_h^2\frac{\hat{u}''_{j-1}+2\hat{u}''_j+\hat{u}''_{j+1}}{4}-\frac{\hat{u}_{j+1}-2\hat{u}_j+\hat{u}_{j-1}}{h^2}+m_h^2\hat{u}_j=\hat{f}_j, \quad for\ j<0\\
&c_h^2\frac{\hat{u}''_{j-1}+2\hat{u}''_j+\hat{u}''_{j+1}}{4}-(c^+)^2\frac{\hat{u}_{j+1}-2\hat{u}_j+\hat{u}_{j-1}}{h^2}+(c^+m_h)^2\hat{u}_j=\hat{f}_j, \quad for\ j>0\\
&c_h^2\hat{u}''_0+(cm_h)^2\hat{u}_0=\frac{1}{h}((c^+)^2(\hat{u}_1-\hat{u}_0)+\hat{u}_{-1}-\hat{u}_0)+\hat{f}_0\\
&\hat{u}_j(t=0)=\hat{a}_j, \quad \hat{u}'_j(t=0)=\hat{b}_j, \quad for\ j\in\mathbb{Z}.
\end{aligned} \tag{4.6}$$

We then define

$$\begin{aligned}
&\bar{L}_h^{2\prime}=\{w \ \text{ such that } \ \int_{-\pi/h}^{\pi/h}c_h^2|\hat{w}|^2<\infty\},\\
&\bar{H}_h^{\beta\prime}=\{w \ \text{ such that } \ \int_{-\pi/h}^{\pi/h}(m_h)^{2\beta}|\hat{w}|^2<\infty\},\\
&\bar{H}_{h,T}^{s\ \prime}=\bigcap_{\alpha+\beta=s}H^\alpha([0,T];\bar{H}_h^{\beta\prime}).
\end{aligned}$$

1) Adaptation of Lemma 3.11.

The result is a direct combination of Lemma 3.11 and Theorem 4.3 where the spaces $H_h^s, H_{h,T}^s$ have to be replaced by the spaces $\bar{H}_h^s, \bar{H}_{h,T}^s$.

2) Adaptation of Lemma 3.12 and 3.13.

One needs to replace the definition of the truncation operator P by

$$P(a,b)=0 \text{ on } \left\{\frac{a}{2}\leq c\left|\tan\left(\frac{b}{2}\right)\right|\right\}, \quad P(a,b)=1 \text{ on } \left\{\frac{a}{2}\geq\left|\tan\left(\frac{b}{2}\right)\right|\right\}.$$

Then, the proof of Lemma 3.12 and 3.13 are the same.

Thanks to 1) and 2) one gets a theorem which can be stated just like Theorem 3.14. Finally a convergence result also holds using the spaces (4.5) and the same comment given in paragraph 3.2.5 for the $2-d$ semi-discrete, finite difference.

5 Numerical simulations

In this section we illustrate the theoretical results we obtained for the $1-d$ and $2-d$ finite difference approximation of the hybrid problem (2.1). For obvious reasons we restrict ourselves to bounded domains and we set Dirichlet conditions at the boundary. The analysis we did in the previous sections carries over to this case.

For the numerical results, we could compute the solutions by diagonalising the matrix A_h (resp. $A_{h,h}$) and solve exactly the Cauchy problem. This can be done as in [1], on a discrete level. However, for numerical simulations it is simpler to consider a time discretization of the semi-discrete approximation. We thus introduce fully discrete schemes, quickly explain why the existence of solutions in asymmetric spaces still holds and perform numerical computations.

5.1 The $1-d$ full finite difference approximation

Here, we consider a finite domain $[-L,L]$ and we set $h(2N+1)=2L$ where $2N+1$ is the number of points of discretization and h the distance between two points. We thus now consider finite vectors $U=(u_j)_{-N\le j\le N}$ and we re-define the operator A_h as a finite dimensional operator, including Dirichlet conditions. We keep the notations 3.1.

Next, we consider the finite difference time discretization of (3.3), (3.4). That is

$$\begin{aligned}&\frac{U^{n+1}-2U^n+U^{n-1}}{\delta t^2}+A_hU^n=F^n,\quad for\ n\ge 1\\ &U^0=U_0,\quad U^1=U_0+\delta t\,U_1.\end{aligned}\tag{5.1}$$

This approximation is consistent of order one in time and space and possesses a conserved quantity when $F^n=0$

$$E_{\delta t,h}=\frac12\left\|\frac{U^{n+1}-U^n}{\delta t}\right\|_h^2+(A_hU^n,U^{n+1})_h,$$

which really is an energy when the CFL condition $\max(1,c_+^2)h^2\le\delta t^2$ is satisfied since (cf. [12] p.44)

$$E_{\delta t,h}=\left(1-\frac{\max(1,c_+^2)h^2}{\delta t^2}\right)\left\|\frac{U^{n+1}-U^n}{\delta t}\right\|_h^2+\left(A_h\frac{U^{n+1}+U^n}{2},\frac{U^{n+1}+U^n}{2}\right)_h.\tag{5.2}$$

Then, as in Lemma 3.2, we can show that (5.1) possesses a unique solution of finite energy provided that $F\in L^1_{\delta t}([0,T];L^2_h)$ with

$$\|F\|_{L^1_{\delta t}([0,T];L^2_h)}=\delta t\sum_{n=1}^{[T/\delta t]}\|F^n\|_h.$$

Then, a discrete version of Theorem 3.4 also holds. We just state it:

Theorem 5.1. *Let $U_0\in\mathcal{A}_h^2$, $U_1\in\mathcal{A}_h^1$ and $F\in L^1_{\delta t}([0,T];\mathcal{A}_h^1)$ with $\big((F^{n+1}-F^n)/\delta t\big)_n\in L^1_{\delta t}([0,T];\mathcal{A}_h^0)$. Then for all $T\in\mathbb{R}$ there is a unique solution $(U^n)_{n\le T/\delta t}$ of (5.1). Moreover, setting $\alpha_n=(u_1^n-u_0^n)/h$ there is a constant C independent of h and T such that*

$$\begin{aligned}\sup_{n\le T/\delta t}\left(\|U^n\|_{\mathcal{A}_h^2}+\left\|\frac{U^{n+1}-U^n}{\delta t}\right\|_{\mathcal{A}_h^1}+\left\|\frac{U^{n+1}-2U^n+U^{n-1}}{\delta t^2}\right\|_{\mathcal{A}_h^0}\right)+\left\|\left(\frac{\alpha_{n+1}-\alpha_n}{\delta t}\right)_{n\ge0}\right\|_{L^2_{\delta t}}\\ \le C\Big(\|U_0\|_{\mathcal{A}_h^2}+\|U_1\|_{\mathcal{A}_h^1}+\|F\|_{L^1_{\delta t}([0,T];\mathcal{A}_h^1)}+\delta t\sum_{n=1}^{[T/\delta t]}\left\|\frac{F^{n+1}-F^n}{\delta t}\right\|_{\mathcal{A}_h^0}\Big).\end{aligned}$$

The proof goes along the lines of Theorem 3.4 but one has to replace the Laplace transform (3.9) by a discrete one. If U^n is defined for all $n\in\mathbb{Z}$ and vanishes for n smaller than some value, then

$$\tilde U_j(i\omega+\lambda)=\delta t\sum_{n\in\mathbb{Z}}e^{(i\omega+\lambda)n\delta t}U_j^n.$$

5.1.1 Numerical computation for (5.1)

We chose $L = 10$, $c^+ = c = 1$ took $F^n = 0$ and the following triangular initial datum:

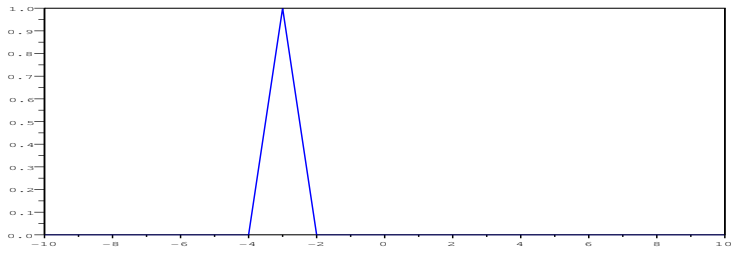


Figure 1: ***Initial datum***

We chose $\delta t = h = 1/20$. We also plotted the following quantities

$$H_+^2(n) := h \sum_{-N<j\leq 0} \left(\frac{u_j^n - u_{j-1}^n}{h} \right)^2 + h \sum_{0<j<N} \left(\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{h^2} \right)^2,$$
$$H_-^2(n) := h \sum_{-N<j<0} \left(\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{h^2} \right)^2.$$

The (square) norm $H_+^2(n)$ measures the H_h^2 regularity of the solution restricted to $[0, 10]$ while the homogeneous (square) norm $H_-^2(n)$ allows to decide if the H_h^1-bounded solution restricted to $[-10, 0]$ is in H_h^2 or not. See Figure 2.

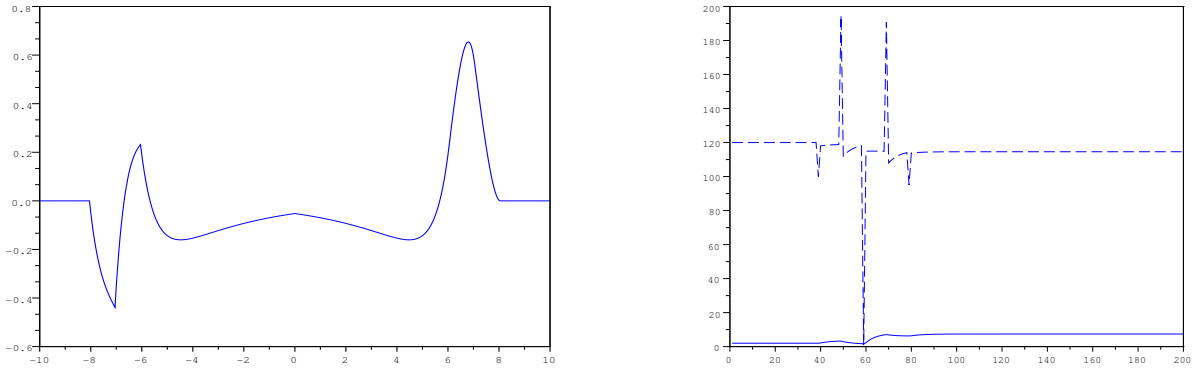


Figure 2: ***Left: the solution at time*** 10***. Right: the evolution of*** H_+^2 ***(solid line) and*** H_-^2 ***(dashed line) with respect to*** n***.***

The quantity $H_+^2(n)$ remains bounded as predicted by Theorem 5.1. Moreover this result is sharp since $H_-^2(n) \approx 1/(3, 3h^2)$ which means that the solution restricted to $[-10, 0]$ is only in H_h^1 and not smoother.

5.1.2 Convergence rates

In this section we develop some numerical experiments that confirm the main results in Theorem 3.9. We perform two different computations. First, we consider smooth (Gaussian) initial data and then rough (triangular) ones as below.

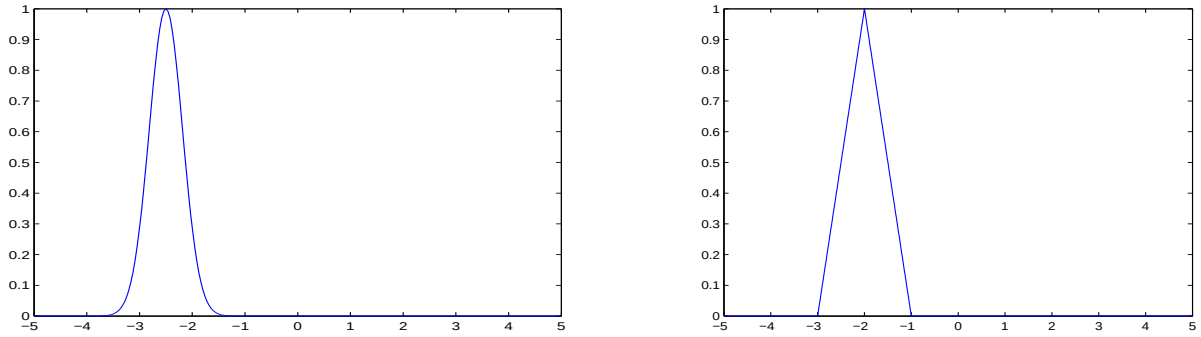


Figure 3: ***Left: Smooth initial datum. Right: rough initial datum.***

The initial velocity is taken to be compatible with the initial datum such that the solution starts propagating towards the interface. We took $c^+ = c = 1$. The solution is computed until the time t in which the wave is reflected on the interface and reaches the boundary.

Denoting by u_0 the initial datum, the exact solution is given at the left-hand side by $u^-_{exact}(t,x) = u_0(x-t) + u_r(x+t)$ and at the right-hand side by $u^+_{exact} = w(t-x)$ where $w(t)$ is the position of the mass at time t. The continuity condition at $x = 0$ shows that $u_r(t) = w(t) - u_i(-t)$ thus the calculation comes down to solving the ode:

$$w''(t) + 2w'(t) + w(t) = -2u_0'(-t), \tag{5.3}$$

with vanishing initial data. We then compared u_{exact} and U^n at time $t = L = 5$, i.e. for $n = N$. Below is the plot of $\ln(\mathcal{N}) = \ln\left(\|u_{exact} - U^N\|_{\mathcal{A}_h^2}\right)$ with respect to $\ln(h)$.

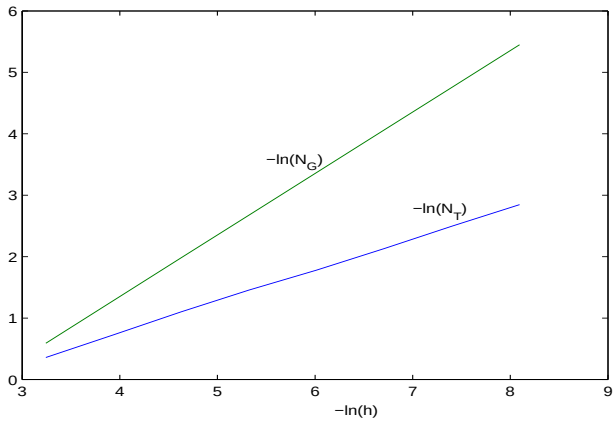


Figure 4: ***Green: the curve*** $-\ln(\mathcal{N})$ ***as a function of*** $-\ln(h)$ ***for the Gaussian initial datum. Blue: the corresponding curve for the triangular initial datum.***

For the Gaussian initial datum the regression coefficients are $a = 1.0014$ and $b = 2.6555$. This is in accordance with Theorem 3.9 which tells us that the worst convergence rate is 1 for smooth enough data. For the triangular initial datum the regression coefficients are $a = 0.5098$ and $b = 1.2778$. This rate of convergence of order $\sqrt{h}$ is probably due to the fact that the discretization of (5.3) is a first order one. Indeed, since we have chosen $\delta t = h$ the numerical scheme is exact outside the interface. The numerical scheme thus comes down to the finite difference equation for the amplitude w^n of the interface:

$$\frac{w^{n+1} - 2w^n + w^{n-1}}{dt^2} + 2\frac{w^n - w^{n-1}}{dt} + w^n = \frac{u_{0,i-n-1} - u_{0,i-n+1}}{dt}.$$

5.1.3 Extension to several mass points

We consider the interval $[-20, 30]$ and three strings: $[-20, 0]$, $[0, 10]$, $[10, 30]$ which are continuously connected at $x = 0$ and $x = 10$ at which there are mass points. We take $h = 1/20$ and start the simulation with a rectangular signal (thus in L^2_h) compactly supported in $[-20, 0]$ and propagating to the right. At time 10 the results are given in Figure 5 and 6.

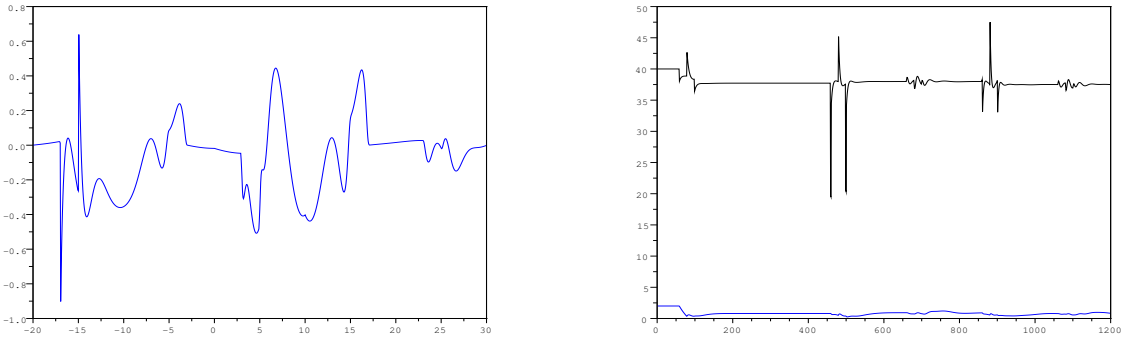


Figure 5: ***Left: the solution at time*** 10 ***for a hybrid system with two mass points. Right: the evolution of the*** L^2_h ***norm (blue) and*** H^1_h ***norm (black) of the solution restricted to*** $[-20, 0]$***.***

Figure 5 shows the complexity of the wave after ten units of time where several reflection/transmission have occurred within $[0, 10]$. The restricted solution to $[-20, 0]$ remains in L^2_h and does not become smoother since its H^1_h norm is big (about $2/h$). Actually the H^1_h norm should be of the order of $1/h^2$ but this is not the case since the initial data only has two points at which the H^1_h-regularity fails.

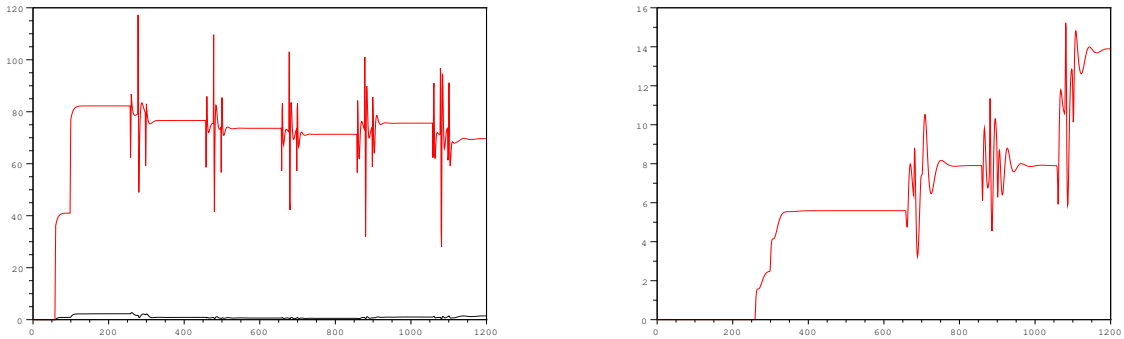


Figure 6: ***Left: the evolution of the*** H^1_h ***norm (black) and*** H^2_h ***(red) of the solution restricted to*** $[0, 10]$***. Right: the evolution of the*** H^2_h ***norm (red) of the solution restricted to*** $[10, 30]$***.***

In Figure 6, from the evolution of the norms, we see that the solution is in H^1_h on $[0, 10]$ (and not in H^2_h since the corresponding norm is about $1/h$) and in H^2_h on $[10, 30]$ though the related norm increases (this is due to multiple transmission coming from the interface at $x = 10$).

5.2 The $2 - d$ full finite difference approximation

Here, we consider the domain $[-L, L] \times [-L, L]$ and define h as in the previous paragraph. We define the operator $A_{h,h}$ as the restriction to $\mathbb{R}^{2N+1}$ of the operator introduced in (3.16) with Dirichlet boundary

conditions. The full discrete scheme reads

$$\frac{U^{n+1}-2U^n+U^{n-1}}{\delta t^2}-A_{h,h}U^n=F^n,\quad for\ n\geq 1 \tag{5.4}$$
$$U^0=U_0,\quad U^1=U_0+\delta t\, U_1.$$

Without source term, the quantity

$$E_{\delta t,h,h}=\frac{1}{2}\left\|\frac{U^{n+1}-U^n}{\delta t}\right\|_h^2+(A_{h,h}U^n,U^{n+1})_h,$$

is conserved and is an energy (cf. (5.2)) as long as the following CFL condition holds

$$\max(1,c^2,c_+^2)\frac{h^2}{\delta t^2}\leq 1/\sqrt{2}.$$

Then, the scheme also describes the propagation of waves in asymmetric spaces and we can state a result exactly as Theorem 5.1 when $c<1$. The proof of this result is the exact translation of Theorem 3.14 since the main tools (Laplace transform, multiplier and odd Fourier transform) also hold in the finite discrete case with Dirichlet boundary conditions.

Let us now give some numerical results illustrating the existence of waves in asymmetric spaces according to the value of the speed c at the interface. We consider successively the cases $c^2=0,9$ and $c^2=1,1$. In order to highlight the radical difference, we take a (wildly) modulated Gaussian initial datum. We choose the frequency of the modulation in such a way that when $c^2=1,1$ (no regularity gain expected) the transmission coefficient C_t of Section 3.2.1 is comparable to the reflection one C_r. For instance, if one takes $\xi'\sim\pi/h$ and $\xi_i\sim 2/h\arcsin(\sqrt{c^2-1})$ then the leading part of the denominator of C_t vanishes and thus $C_t=O(1)$. Choosing $h=1/10$ leads to the initial datum:

$$u_0(x)=e^{-(x+4)^2-(y+7)^2}e^{-i(6x+20y)}.$$

The other initial datum ($\partial_t u(t=0)$) is chosen so that the wave starts propagating along the direction

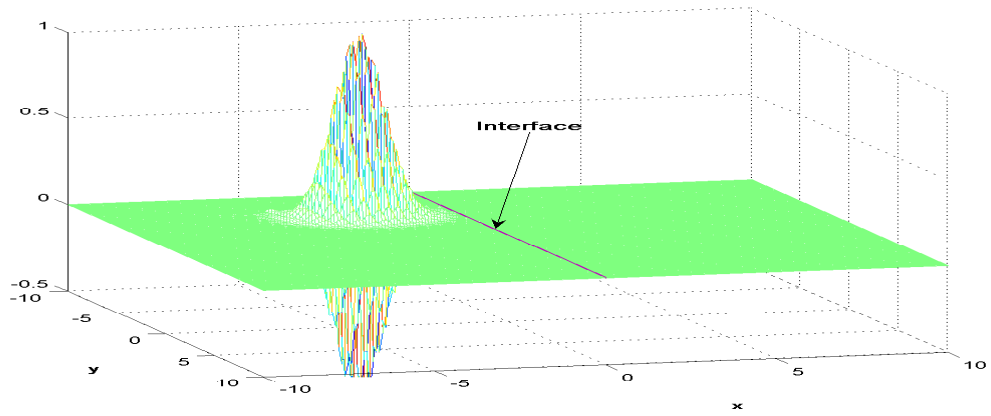


Figure 7: ***Initial datum: Gaussian modulation of a highly oscillating wave***

$(6,20)$. These initial data are used both for $c^2=0,9$ and $c^2=1,1$. The L^2_h-norm of u_0 is about $1,2$ and its H^1_h-norm is about 22 which is of order of $1/h$.

For the computations we took $\delta t=vh$ with $v=0,6$. The scenario is as follows: the initial datum starts propagating in the direction $(6,20)$, thus meets the interface at some time and then produces a reflected and a transmitted waves. We plot the solution at final time $T=18$ (after reflection) and the quantity $H^1_\pm(n)$ corresponding to the L^2_h norm of the gradient of U^n restricted to $[0,\pm 10]$.

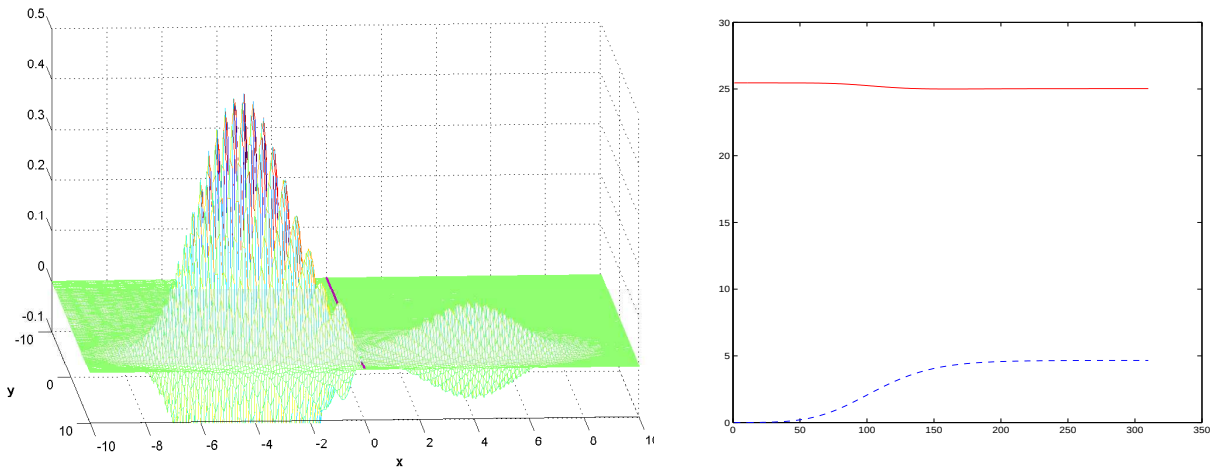


Figure 8: ***Left: the solution at time*** 18 ***with*** $c^2 = 0,9$. ***Right: the evolution of*** H^1_+ ***(dashed blue line),*** H^1_- ***(solid red line) with respect to*** n.

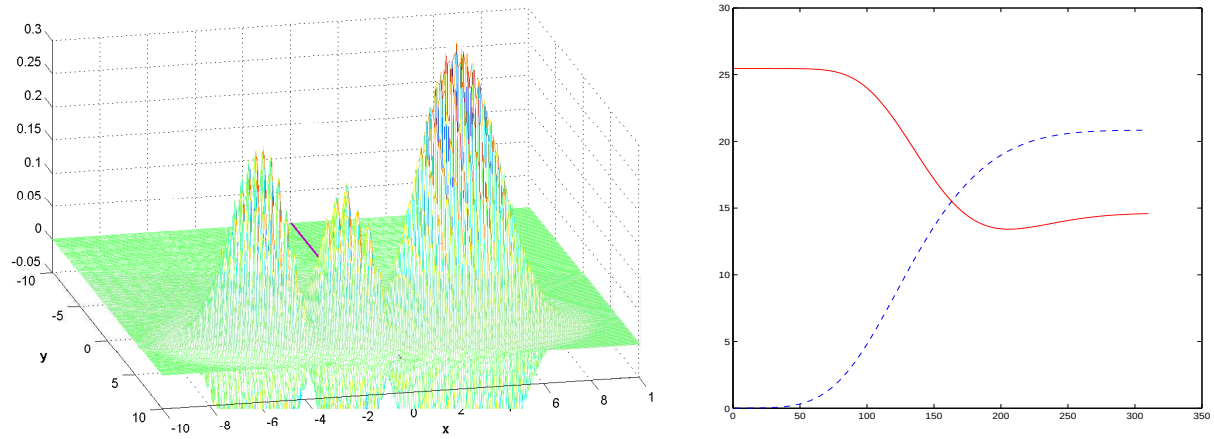


Figure 9: ***Left: the solution at time*** 18 ***with*** $c^2 = 1,1$. ***Right: the evolution of*** H^1_+ ***(dashed blue line),*** H^1_- ***(solid red line) with respect to*** n.

When $c^2 = 0,9$ (see Figure 8), the wave is mostly reflected. At time 18 the value of H^1_+ is about 5 which is roughly equal to $h/2$ times H^1_-. Thus the wave at the right-hand side is smoother by one L^2_h derivative as predicted by Theorem 5.1.

When $c^2 = 1,1$ (see Figure 9) about half of the wave energy goes through the interface as predicted by the formal plane wave analysis. This means that we do not expect any smoothing for c bigger than 1.

Remark 5.2. *When $c^2 = 1,1$ the reflected wave is made of two bumps. This can be described by a first order WKB expansion of the solution. We just give an explanation at the continuous level for ease but this works similarly for the discrete equations. Let us thus consider system* (2.1) *when $n = 2$ with the initial data*

$$\mathcal{U}(t=0) = e^{-(x+4)^2-(y+7)^2} e^{i(\omega t - k_1 x - k_2 y)/h}.$$

The wave numbers ω, k_1, k_2 are chosen of order $\mathcal{O}(1)$ and satisfy the wave dispersion relation. Then, we look for a first order approximate solution of the form

$$\begin{aligned} u^- &= A_i(t,x,y)e^{i(\omega t - k_1 x - k_2 y)/h} + A_r(t,x,y)e^{i(\omega t + k x_1 - k_2 y)/h}, \quad x < 0 \\ u^+ &= A_t(t,x,y)e^{i(\omega t - k_1 x - k_2 y)/h}, \quad x > 0 \\ u_0 &= A_0(t,y)e^{i(\omega t - k_2 y)/h}, \quad x = 0. \end{aligned}$$

We look for amplitudes with gradient independent of h Plugging this ansatz in system (2.1) leads to an equation which looks like a finite Taylor series of h. We solve this equation by canceling the first two terms of this series.

1. *Canceling the terms in $1/h^2$.*

   The corresponding equations give the dispersion relation for each medium: $\omega^2 = k_1^2 + k_2^2$. The equation at the interface reads $(\omega^2 - ck_2^2)A_0 = 0$ so we take $\omega^2 = c^2 k_2^2$.

2. *Canceling the terms in $1/h$.*

$$\omega\partial_t A_i + (k_1\partial_x + k_2\partial_y)A_i = 0, \qquad A_i(t=0) = e^{-(x+3)^2-y^2} \ .$$

   So $A_i(t,x,y) = A_i(t=0, x-k_1/\omega t, y-k_2/\omega t)$. Then, the equation of continuity at the interface reads $A_i(x=0) + A_r(x=0) = A_t(x=0) = A_0$. The equation at the interface is

$$\begin{aligned} &2\omega\partial_t A_0 + 2c^2 k_2 \partial_y A_0 = -k_1(A_t - A_i + A_r)(x=0) = -2k_1(A_0 - A_i(x=0)) \\ &A_0(t=0) \approx 0 \end{aligned}$$

   This is a transport equation with damping. The solution is

$$A_0(t,y) = \frac{k_1}{\omega}\int_0^t e^{-k_1/\omega(t-s)} A_i(s, x=0, y - c^2 k_2/\omega(t-s))ds.$$

   Finally one determines A_r and A_t thanks to the continuity conditions:

$$\begin{array}{ll} \omega\partial_t A_r + (-k_1\partial_x + k_2\partial_y)A_r = 0 & \omega\partial_t A_t + (k_1\partial_x + k_2\partial_y)A_t = 0 \\ A_r(x=0) = A_0 - A_i(x=0) \quad , & A_t(x=0) = A_0 \\ A_r(t=0) = 0 & A_t(t=0) = 0 \end{array}$$

   In particular, for $0 \geq x \geq -k_1/\omega t$, one has

$$\begin{aligned} A_r(t,x,y) = \frac{k_1}{\omega}\int_0^{t+\frac{\omega}{k_1}x} e^{-\frac{k_1}{\omega}(t-s)+x} A_i\left(s, 0, y + (c^2-1)\frac{k_2}{k_1}x - c^2\frac{k_2}{\omega}(t-s)\right) ds \\ - A_i\left(t + \frac{\omega}{k_1}x, 0, y - \frac{k_2}{k_1}x\right). \end{aligned}$$

   The two terms do not propagate at the same speed in the x direction since the first (the integral) propagates with some delay while the second propagates with the constant speed $-k_1/\omega$.

6 Concluding remarks

1. Our results show that the discrete approximations capture accurately the regularity properties of the system. One difficult point at the semi-discrete level is to get an L^2-estimate of the discrete normal derivatives. This was achieved by using a Laplace transform in time for the finite difference scheme while we used an identity involving multiplicators for the mixed finite element scheme. The issue of whether one can also find such a useful identity in the finite difference case is left open.

   Another way of proceeding would have been to perform the spectral analysis of the schemes as in [10].

2. The issue of curved boundaries and non regular meshes is left open. One should consider a finite element approximation but the proof of the micro-local estimate fails since one cannot use the Fourier transform anymore.

   System (2.1) can be written in a weak sense as one equation with some penalization term. Using the delta Dirac surface distribution it is not difficult to see that system (2.1) is equivalent to

$$(1 + \delta_{x_n=0})\partial_t^2 u - (1 + \delta_{x_n=0})\partial_{x'}^2 u - \partial_{x_n}^2 u = 0.$$

For curved interface one has to use the Laplace Beltrami operator Δ_Γ:

$$(1+\delta_\Gamma)\partial_t^2 u - \delta_\Gamma \Delta_\Gamma u - \Delta u = 0.$$

This reminds the Schrödinger or wave equation with delta Dirac potential as considered in [11]. But in our case the Dirac distribution has a much stronger effect since it applies to the principal part of the operator. In particular, by plane wave computations one can check that existence in asymmetric spaces can not be expected for the wave equation with Dirac potential.

3. One could wonder if such a result applies to Schrödinger equations and, more generally, to equations involving fractional powers of the laplacian $i\partial_t u + (-\Delta)^p u = 0$ with $p > 0$. As far as we know, the issue is not yet answered at the continuous level for general situations such as those considered in [13]. A negative answer is given in [7] for $1-d$ Schrödinger-like equations. However a plane wave analysis indicated that an existence result in asymmetric spaces is to be expected for the Cauchy problem in the whole space (the same as the one for the wave equation).

   The apparent contradiction between this computation and the result of [7] can be explained as follows: consider two bounded rectangular domains separated by a straight interface and a wave of frequency ξ and energy 1 in the left domain propagating to the right. Since its speed is $1/\xi$, it bounces back about ξ times during one unit of time and because $1/\xi$ part of the amplitude is transmitted at each reflection, half of the energy has been transmitted after this time. From this we expect in general no result of existence in asymmetric spaces in bounded domains while we do expect such a result for two half spaces connected through a non-trapping interface.

   Finally, for equations with fractional power in bounded domains we can generalize the result of [7] as follows: for $0 < p \leq 1$ there is existence of solutions in asymmetric spaces with a difference of $2(1-p)$ derivatives through the interface.

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