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This paper deals with nonconservative mechanical systems as those subjected to nonconservative positional forces and leading to non-symmetric tangential stiffness matrices. In a previous work, the geometric degree of nonconservativity of such systems, defined as the minimal number ℓ of kinematic constraints necessary to convert the initial system into a conservative one is found to be, in the linear framework, the half of the rank of the skew-symmetric part of the stiffness matrix. In the present paper, news results are reached. First, a more efficient solution of the initial linear problem is proposed. Second, always in the linear framework, the issue of describing the set of all corresponding kinematic constraints is given and reduced to the one of finding all the Lagrangian planes of a symplectic space. Third, the extension to the local non-linear case is solved. A four degree of freedom system exhibiting a maximal geometric degree of nonconservativity ($s = 2$) is used to illustrate our results. The issue of the global non-linear problem is not tackled. Throughout the paper, the issue of the effectiviness of the solution is systematically addressed.

Introduction

Nonconservative elastic mechanical systems exhibit several paradoxical mechanical behaviors. Destabilizing effect by additional friction is certainly the most famous paradox of these mechanical systems and has been deeply investigated (see [1–3] for example). One less reported paradoxical effect is the destabilizing effect by additional kinematical constraints. J.J. Thompson mentioned this effect in [4] but, to the best of our knowledge, this paradoxical effect had never been systematically investigated before recently. This paradoxical effect led to the so-called kinematical structural stability (ki.s.s.) issue: when and how is it possible to destabilize by adding kinematical constraint(s) a given stable system?

During the last five years, in a sequence of papers ([5–9]), we elucidated this kinematical structural stability (ki.s.s.) issue for the linear divergence stability of both conservative and nonconservative elastic systems as well. A big part of these works are also related to the so-called second order work criterion introduced by Hill in the framework of plasticity in 1958 ([10])

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and independently introduced and used in the framework of elastic nonconservative systems in 2004 ([11]). The main result involves the symmetric part K_s of the stiffness matrix and the magnitude of the load parameter as well but it does not depend on the number of the additional kinematic constraints.

By duality to the ki.s.s. issue, we investigated in [12] the issue to convert by (judicious) additional kinematic constraints a nonconservative system Σ into a conservative one. This issue leads to the concept of geometric degree d of nonconservativity of Σ. Calculations show that $d = s$ is the half s of the rank r of the skew-symmetric part $K_a(p)$ (that is always even $r = 2s$). In a second stage, a building of the judicious additional kinematic constraints $C_1, \ldots, C_s \in (\mathbb{R}^n)^s$ has been proposed thanks to the eigenspaces $E_{-\lambda_i^2}, i = 1, \ldots, s$ of the symmetric matrix $K_a^2(p)$ whose the eigenvalues $-\lambda_1^2, \ldots, -\lambda_s^2$ are all double: each C_i may be chosen in each distinct $E_{-\lambda_i^2}$. It is worth noting that, for both issues, the mechanical system Σ is approximated by its linear first order approximation at a given equilibrium configuration q_e. That means that Σ is described by the mass matrix M and the stiffness matrix K. If p is a load parameter, then $K = K(p)$. The non-symmetry of $K(p)$ (namely $K \neq K_s$ or $K_a \neq 0$) is then the signature of the non-conservative nature of the mechanical system Σ. In our previous works, the source of the nonconservativity lies in external forces like follower forces acting on elastic system. Hypoelasticity may also be another mechanical framework leading to a similar mathematical problem. There exists a broad literature covering hypoelasticity (see for example [13–16]).

In this paper we are concerned by finding the complete solution of the linear case and by the generalization and the extension to the non-linear differentiable case about to the latter issue. We then use the language of analytic mechanics. In a first time, we reinvestigate the linear case by using the language of exterior p-forms and especially exterior 1- and 2-forms. That allow us to more deeply highlight the issue of effectiveness of the calculation of the suitable kinematic constraints converting the system into a conservative one. That also allow us to investigate the issue of building the set of all the solutions and to illustrate the geometrical meaning of these solutions. To do it, the language of symplectic geometry is systematically used. That also suggests the good way for tackling the non-linear case.

Thus, in a second time, we tackle the non-linear problem with appropriate notations and especially thanks to the language of differential p-forms. We accurately focus on the link with the linear case. In a third step, the solution is proposed by extending to the nonlinear case the concept of geometric degree of nonconservativity and yielding a geometric meaning to the corresponding non-linear constraints. In the last part, the issue of the calculation of the appropriate non-linear constraints is investigated. The problem of a global solution in relationship with the topology of the configuration manifold is only evoked by just setting the convenient geometric framework of vector bundles. A four degree of freedom system called the Bigoni system (see [12,17]) is continuously used throughout the paper to illustrate the general results.

1. The linear case

In what follows we refer to [12]. We only recall that for the linear framework, dynamic equation of the unconstrained system Σ read:

$$M\ddot{X} + KX = 0, \tag{1}$$

with K any (namely non-symmetric) matrix and M symmetric positive definite. K is the stiffness matrix of the system and M its mass matrix. Because of the nonconservativity of the positional forces acting on σ, K is any. The minimum number of kinematic constraints allowing to convert the system into a conservative one (with a corresponding symmetric stiffness matrix) is the geometric degree of nonconservativity of Σ. (1) is deduced from the Lagrange equation by the usual process of linearization about an equilibrium configuration.

1.1. Effectiviness of the solution proposed in [12]

In introduction, we already recalled the algebraic meaning of the geometric index or degree of nonconservativity: this the half s of the rank $r = 2s$ of K_a and the distinct constraints, viewed as vectors of $\mathbb{R}^n$, can be chosen in the s distinct eigenspaces $E_{-\lambda_i^2}, i = 1, \ldots, s$ of K_a^2. We now question the effectiveness of the building of the constraints as proposed in [12]. To do it, we use the spectral theorem for K_a^2. What does mean the effectiveness for the spectral theorem? The usual proof is done by induction on the dimension of the space. For initializing the induction reasoning, the D'Alembert Gauss theorem is used for finding an eigenvalue of the characteristic polynomial of K_a^2 and this theorem is not effective in the sense where only a numerical method may lead to (an approximation of) the eigenvalues. So, with these tools, the solution of the linear case itself is not effective. Remark however that the constraints are also the critical points of the Rayleigh quotient R associated with K_a^2 and that only the eigenspaces are interesting and not the eigenvalues $-\lambda_i^2, i = 1, \ldots, s$. The use of Rayleigh quotient is then especially relevant and the constraints may be evaluated by successive minimizations of $R(X) = -\frac{X^T K_a^2 X}{X^T X}$. By Minimax theorem, the constraints are also the solutions of

$$\min_{\dim F = k} \max_{X \in F \setminus \{0\}} R(X),$$

for $k = 1, \ldots, n$ avoiding by this way the use of D'alembert–Gauss theorem. However, this minimization process gives no analytic explicit result.

1.2. The language of exterior p-forms

Now, before addressing in a following step the non-linear case, we still focus on the linear case with the help of the exterior p-forms: we look the matrix K_a (skew symmetric part of the stiffness matrix) no longer as the matrix of a linear map of $\mathbb{R}^n$ but as the matrix of an exterior 2-form on $\mathbb{R}^n$. We indifferently note $E = \mathbb{R}^n$ and E^* its dual space, the vector space of the linear forms on E. Thus, let ϕ the exterior 2-form defined on $E = \mathbb{R}^n$ by:

$$\phi(u, v) = u^T K_a v, \tag{2}$$

after identifying a vector $u = (u_1, \ldots, u_n)$ of $\mathbb{R}^n$ with the column vector $u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$ of $\mathcal{M}_{n1}(\mathbb{R})$. Thanks to a basic theorem of linear algebra (see [18] for example), there is a basis $\mathcal{B} = (e_1, \ldots, e_n)$ of $\mathbb{R}^n$ and a number $r = 2s \leq n$ such that $\phi(e_{2i-1}, e_{2i}) = -\phi(e_{2i}, e_{2i-1}) = 1$ for $i \leq s$ and $\phi(e_i, e_j) = 0$ for the other values of i and j. In the dual basis $(e_1^*, \ldots, e_n^*)$ of $(e_1, \ldots, e_n)$, the form ϕ reads:

$$\phi = e_1^* \wedge e_2^* + \ldots + e_{2s-1}^* \wedge e_{2s.}^* \tag{3}$$

According to [12], we have to find out a subspace H of $\mathbb{R}^n$ such that $\phi(u, v) = 0 \;\; \forall u, v \in H$ and the constraints, viewed now as linear forms $C_1, \ldots, C_s \in E^*$, then belong to $H^\perp$, the orthogonality being then understood in the sense of duality. Choosing each constraint C_i in the subspace $< e_{2i-1}^*, e_{2i}^* >$ spanned by e_{2i-1}^* and e_{2i}^* in E^* leads to the wanted result. Indeed, suppose to simplify that $C_i = e_{2i-1}^*$ for all $i = 1, \ldots, s$ and that G is the vector subspace spanned by $(e_{2i-1}^*)_{1 \leq i \leq s}$. Let $u, v \in H = G^\perp$ where the bidual is identified with the space itself. Thus, by use of (3), if $u, v \in H$, $e_{2i-1}^* \wedge e_{2i}^*(u, v) = e_{2i-1}^*(u) e_{2i}^*(v) - e_{2i}^*(u) e_{2i-1}^*(v) = 0 - 0 = 0$ and thus $\phi(u, v) = 0$.

If C_i is any in $< e_{2i-1}^*, e_{2i}^* >$ a similar proof as hereafter for differential forms may be used and is not reproduced.

The effectiveness of the building of the constraints is now brought back to the one of the basis $\mathcal{B} = (e_1, \ldots, e_n)$. The proof is again done by induction on the dimension n of E (see for example [18] pp 30–31). This proof is effective and the following paragraph will highlight how it is performing on an example. Before dealing with the example, an interesting issue is to characterize all the solutions.

1.3. Set of solutions

For describing the set of solutions, usual concepts of symplectic geometry are used. To do it, we first brought back the issue in the usual framework of symplectic geometry. The exterior 2-form ϕ does not necessarily define a symplectic structure on $\mathbb{R}^n$ because it has not necessarily a maximal rank namely ϕ may be degenerate. For instance, that necessarily occurs when n is odd. Let then F be the kernel of ϕ. Then $(\mathbb{R}^n/F, \tilde{\phi})$ is a $2s$-dimensional symplectic vector space where $\tilde{\phi}$ is canonically defined by $\tilde{\phi}(\bar{u}, \bar{v}) = \phi(x, y)$ with x (resp. y) any vector of the class $\bar{u}$ (resp. $\bar{v}$). Remark that thanks to the canonical scalar product $(.|.)$ on $\mathbb{R}^n$, one could choose the orthogonal $F^\perp$ of F for the scalar product as "canonical" supplementary space of F in $\mathbb{R}^n$ and $\phi_{F^\perp}$ the restriction of ϕ to $F^\perp$. Then $(F^\perp, \phi_{F^\perp})$ is also a $2s$-dimensional symplectic vector spaces and there are three possible meanings for orthogonality in $F^\perp$: duality, scalar product, and ϕ-orthogonality. However the scalar product has no meaning on $(\mathbb{R}^n/F, \tilde{\phi})$ and only ϕ-orthogonality and orthogonality for duality keep useful on $(\mathbb{R}^n/F, \tilde{\phi})$. Moreover,when generalizing the reasoning from vector spaces to manifolds, the natural Euclidean structure of $\mathbb{R}^n$ does not exist in the tangent and cotangent spaces. It is then more judicious to avoid the use of this structure. The orthogonal for the duality of any subspace G will the denoted by $G^\perp$.

If W is any subspace of $\mathbb{R}^n/F$ the ϕ-orthogonal or symplectic orthogonal of W is the vector subspace of $\mathbb{R}^n/F$ noted $W^{\perp\phi}$ defined by: $\bar{u} \in W^{\perp\phi} \iff \tilde{\phi}(\bar{u}, \bar{v}) = 0 \;\; \forall \bar{v} \in W$. Because $\tilde{\phi}$ is non-degenerate, the map $\phi^\flat : \mathbb{R}^n/F \to (\mathbb{R}^n/F)^*$ defined by $\phi^\flat(\bar{u})(\bar{v}) = \tilde{\phi}(\bar{u}, \bar{v}) \;\; \forall \bar{u}, \bar{v} \in \mathbb{R}^n/F$ is an isomorphism (canonical) and then $\phi^\flat(W^\perp) = W^{\perp\phi}$ for any W subspace of $\mathbb{R}^n/F$.

A subspace L of $\mathbb{R}^n/F$ is called Lagrangian if $L = L^{\perp\phi}$. It is also often called a Lagrangian plane even though $\dim L = s$. A straightforward calculation shows that the dual basis of any basis of any Lagrangian subspace is a solution of the initial problem. More accurately, if L_1 is a Lagrangian plane then there is a Lagrangian supplementary space L_2 of L_1 in $\mathbb{R}^n/F$. If $(e_{2i-1})_{1 \leq i \leq s}$ (resp. $(e_{2i})_{1 \leq i \leq s}$) is any basis of L_1 (resp. L_2), then the dual basis $(e_{2i-1}^*)_{1 \leq i \leq s}$ of $(e_{2i-1})_{1 \leq i \leq s}$ in L_1^* is a family of constraints solution of the mechanical issue. In fact this process realizes any solution. Thus, by this process , the set of all the solutions of the problem is in a bijective relationship with the set of the bases of all Lagrangian planes of $\mathbb{R}^n/F$ but the geometrical meaning of the solutions lies in the set of Lagrangian spaces noted $\Lambda(\phi)$ of $(\mathbb{R}^n/F, \tilde{\phi})$. $\Lambda(\phi)$ has been deeply investigated especially in relationship with the theory of Maslov index (see [19] for example for a highlighting presentation of the construction of this index). $\Lambda(\phi)$ is a $\frac{s(s+1)}{2}$ dimensional submanifold of the Grassmanniann manifold of all s-planes of $\mathbb{R}^n/F$ and called the Lagrangian Grassmanniann of $(\mathbb{R}^n/F, \tilde{\phi})$.

Let $\Lambda(s)$ the Lagrangian Grassmanniann manifold of the usual $\mathbb{R}$-symplectic vector space $(\mathbb{C}^s, \omega)$ with its canonical symplectic structure. This manifold may be explicitly described by and identified with the set $U_s(s)$ of unitary symmetric complex matrices of $\mathcal{M}_s(\mathbb{C})$. A Lagrangian plane $L \in \Lambda(s)$ is identified with the matrix $U_L \in U_s(s)$ by the following way: $x \in L \longleftrightarrow x = U_L c(x)$ where $c(x)$ is the conjugate column vector of $x \in \mathbb{C}^s$. If u is a symplectomorphism from $(\mathbb{C}^s, \omega)$ onto $(\mathbb{R}^n/F, \tilde{\phi})$, then $\Lambda(\phi) = u(\Lambda(s))$ which achieves the complete and explicit description of the set of solutions namely $\Lambda(\phi)$. Note also that there is an explicit representation of matrices of $U_s(s)$. If U belongs to $U_s(s)$, then $U = X + iY$ with X, Y two

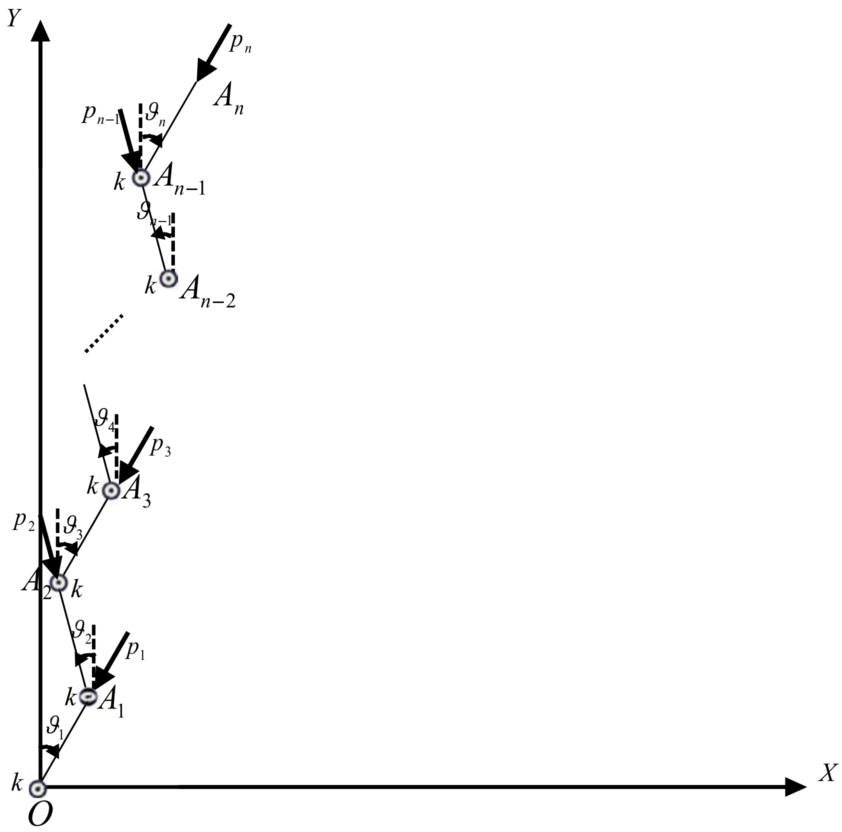

Fig. 1. n d.o.f. Bigoni system.

real symmetric matrices of size s. Because of U is unitary, then $X^2+Y^2=I_s$ and $XY=YX$. Thus there is a common basis of diagonalization of X and Y and a $\mathbb{R}$- orthogonal matrix O so that $X=O^T Diag(x_i)O$ and $Y=O^T Diag(y_i)O$. We deduce that $x_i^2+y_i^2=1$ for all $i=1,\ldots,s$ and we may parametrize the problem by $x_i=r_i\cos\alpha_i$ and $y_i=r_i\sin\alpha_i$. We deduce that $U=O^T RVO$ with $R=diag(r_i)$ and $V=diag(e^{i\alpha_j})$. Remark also that similar tools like the Lagrangian planes may be used to investigate other objects like the spectrum of Hamiltonian systems (see for example [20]). Although some developments seem to be close to each other, the original and unexpected fact of our own investigations lies in the use of these tools of symplectic geometry and algebra in a non-hamiltonian framework.

1.4. The example

As previously announced in the introduction, we illustrate these results with a four degree of freedom system. We tackle the four degree of freedom non-linear Ziegler system with complete follower forces at each joint like in Fig. 1 with $n=4$. This system is called Bigoni system in [12] because of the experimental device proposed by this author in [17]. This same case has been handled in [12] in the linear framework and is also used here in order to compare both linear approaches (and also to illustrate hereafter the non-linear case). In this section, namely in the linear case, the linearization is done about the unique equilibrium position $\theta=0_{\mathbb{R}^4}$.

The usual case of a unique follower force at the extremity (namely the usual Ziegler system) is not very interesting since the geometric degree of nonconservativity is then reduced to 1 (see [12]) with an obvious solution $\theta_4=0$. In this case, the direction of the external force remains constant and this force becomes conservative!. According to the previous notations, it means that $\dim F=2$, $s=1$ and that the Lagrangian spaces are one dimensional subspace of the two dimensional symplectic space $(\mathbb{R}^4/F,\tilde{\phi})$. Finding the one dimensional Lagrangian subspaces of this symplectic space is equivalent to find the set of linear kinematic constraints such that when the system undergoes one of these constraints, it becomes conservative.

The force system is then now set up by $p=(p_1,\ldots,p_4)$ (see Fig. 1). The skew symmetric matrix $K_a(p)$ reads (see [12]) (obviously all the elastic terms having a symmetric input in the system are not involved in this matrix):

$$K_a(p)=\frac{1}{2}\begin{pmatrix}0 & p_2 & p_3 & p_4\\ -p_2 & 0 & p_3 & p_4\\ -p_3 & -p_3 & 0 & p_4\\ -p_4 & -p_4 & -p_4 & 0\end{pmatrix},$$

meaning that if $(\epsilon_1, \dots, \epsilon_4)$ is the canonical basis of $\mathbb{R}^4$ then

$$\phi = \frac{1}{2}(p_2\epsilon_1^* \wedge \epsilon_2^* + p_3\epsilon_1^* \wedge \epsilon_3^* + p_3\epsilon_2^* \wedge \epsilon_3^* + p_4\epsilon_1^* \wedge \epsilon_4^* + p_4\epsilon_2^* \wedge \epsilon_4^* + p_4\epsilon_3^* \wedge \epsilon_4^*).$$

Here, $\det(K_a(p)) = p_2^2 p_4^2$ showing that ϕ is not degenerate when $p_2p_4 \neq 0$ which is now supposed. Thus, with the previous notations, $F = \{0\}$ and $\mathbb{R}^4/F = \mathbb{R}^4$ and $(\mathbb{R}^4, \phi)$ becomes a four dimensional symplectic space.

We want to find a basis $(e_1, \dots, e_4)$ so that

$$\phi = e_1^* \wedge e_2^* + e_3^* \wedge e_4^*. \tag{4}$$

Let us choose

$$e_1^* = \epsilon_1^*, e_2^* = \frac{p_2}{2}\epsilon_2^*, e_3^* = \epsilon_1^* + \epsilon_2^* + \epsilon_3^*, e_4^* = \frac{p_3}{2}\epsilon_3^* + \frac{p_4}{2}\epsilon_4^*. \tag{5}$$

Then (4) holds and for example the constraints $x_1 = 0, x_1 + x_2 + x_3 = 0$ convert the system into a conservative one.

We focus now on the set of all solutions namely here on the set of Lagrangian planes.

Let

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

the matrix of the $\mathbb{R}$-symplectic four dimensional vector space $(\mathbb{C}^2, \omega)$ in its canonical basis $v_1 = (1,0), v_2 = (i,0), v_3 = (0,1), v_4 = (0,i)$. Let A be the matrix of a symplectomorphism u from $(\mathbb{C}^2, \omega)$ onto $(\mathbb{R}^4, \phi)$ in their respective canonical basis. Then the relation $\omega(x,y) = \phi(u(x), u(y))$ for all $x, y \in \mathbb{C}^2$ leads to the usual relation $J = A^T K_a A$. But if Q denotes the change-of-basis matrix to pass from (ϵ_i^*) to (e_i^*) then $P = (Q^T)^{-1}$ is the corresponding change-of-basis matrix to pass from (ϵ_i) to (e_i) and the formula (4) then reads $J = P^T K_a P$. (5) means that

$$Q = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & \frac{p_2}{2} & 1 & 0 \\ 0 & 0 & 1 & \frac{p_3}{2} \\ 0 & 0 & 0 & \frac{p_4}{2} \end{pmatrix}.$$

It follows that $A = (Q^T)^{-1}$ and calculations give:

$$A = 2\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & \frac{1}{p_2} & 0 & 0 \\ -1 & -\frac{1}{p_2} & 1 & 0 \\ \frac{p_3}{p_4} & \frac{p_3}{p_2p_4} & -\frac{p_3}{p_4} & \frac{1}{p_4} \end{pmatrix}. \tag{6}$$

It now remains to parametrize the 3 $(= \frac{2\times 3}{2})$ dimensional Grassmaniann $\Lambda(2)$ of Lagrangian planes of $\mathbb{C}^2$ which is done as above in the general case through a parametrization of $U_s(2)$.

Let

$$U = \begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix} \in U_s(2).$$

Because U is symmetric, $u_2 = u_3$ and because U is unitary, the following three independent relations hold: $|u_1|^2 + |u_2|^2 = 1$, $|u_4|^2 + |u_2|^2 = 1$, $u_1c(u_2) + u_2c(u_4) = 0$. Then $|u_1| = |u_4|$ and we parametrize the problem by $u_1 = \cos\alpha e^{i\alpha_1}, u_4 = \cos\alpha e^{i\alpha_4}$, $u_2 = u_3 = \sin\alpha e^{i\alpha_2}$. The third relation $u_1c(u_2) + u_2c(u_4) = 0$ then reads $\cos\alpha\sin\alpha(e^{i(\alpha_1-\alpha_2)} + e^{i(\alpha_2-\alpha_4)}) = 0$. Generically that leads to $\alpha_2 = \frac{\alpha_1+\alpha_4}{2} + (2k+1)\frac{\pi}{2}$: the parametrization is given by α, α_1, α_4 and the corresponding matrix $U(\alpha, \alpha_1, \alpha_4)$ reads:

$$U(\alpha, \alpha_1, \alpha_4) = \begin{pmatrix} \cos\alpha e^{i\alpha_1} & \sin\alpha \, ie^{i\frac{\alpha_1+\alpha_4}{2}} \\ \sin\alpha \, ie^{i\frac{\alpha_1+\alpha_4}{2}} & \cos\alpha e^{i\alpha_4} \end{pmatrix}.$$

A vector $v = (x, y) \in \Lambda(2)$ if and only if

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\alpha e^{i\alpha_1} & (-1)^k \sin\alpha \; ie^{i\frac{\alpha_1+\alpha_4}{2}} \\ (-1)^k \sin\alpha \; ie^{i\frac{\alpha_1+\alpha_4}{2}} & \cos\alpha e^{i\alpha_4} \end{pmatrix} \begin{pmatrix} c(x) \\ c(y) \end{pmatrix}, \tag{7}$$

which gives

$$\begin{cases} x = \cos\alpha e^{i\alpha_1} c(x) + (-1)^k \sin\alpha \; ie^{i\frac{\alpha_1+\alpha_4}{2}} c(y) \\ y = (-1)^k \sin\alpha \; ie^{i\frac{\alpha_1+\alpha_4}{2}} c(x) + \cos\alpha e^{i\alpha_4} c(y). \end{cases} \tag{8}$$

Writing $v = (x, y) = \sum_{i=1}^{4} x_i v_i$ in the canonical basis of the $\mathbb{R}$-symplectic four dimensional vector space $(\mathbb{C}^2, \omega)$, then $c(v) = (c(x), c(y)) = \sum_{i=1}^{4} (-1)^{i+1} x_i v_i$ and (8) then reads:

$$X = (x_1 \; x_2 \; x_3 \; x_4)^T \in \ker B(\alpha, \alpha_1, \alpha_4),$$

with

$$B = \begin{pmatrix} \cos\alpha\cos\alpha_1 - 1 & \cos\alpha\sin\alpha_1 & (-1)^{k+1}\sin\alpha\sin\frac{\alpha_1+\alpha_4}{2} & (-1)^k\sin\alpha\cos\frac{\alpha_1+\alpha_4}{2} \\ \cos\alpha\sin\alpha_1 & -\cos\alpha\cos\alpha_1 - 1 & (-1)^k\sin\alpha\cos\frac{\alpha_1+\alpha_4}{2} & (-1)^k\sin\alpha\sin\frac{\alpha_1+\alpha_4}{2} \\ (-1)^{k+1}\sin\alpha\sin\frac{\alpha_1+\alpha_4}{2} & (-1)^k\sin\alpha\cos\frac{\alpha_1+\alpha_4}{2} & \cos\alpha\cos\alpha_4 - 1 & \cos\alpha\sin\alpha_4 \\ (-1)^k\sin\alpha\cos\frac{\alpha_1+\alpha_4}{2} & (-1)^k\sin\alpha\sin\frac{\alpha_1+\alpha_4}{2} & \cos\alpha\sin\alpha_4 & -\cos\alpha\cos\alpha_4 - 1 \end{pmatrix}. \tag{9}$$

These equations define a plane $P = P(\alpha, \alpha_1, \alpha_4)$ of $\mathbb{R}^4$ and $L(\alpha, \alpha_1, \alpha_4) = A(P(\alpha, \alpha_1, \alpha_4))$ with A given by (6) is then the Lagrangian plane of $(\mathbb{R}^4, K_a)$ defined by α, α_1, α_4. It is the parametrization of the set of all solutions of our problem.

This "symplectic" solution is simple in comparison with the one proposed in [12] that involved tedious calculations for the calculation of only one solution. This method allows us moreover to get the set of all solutions as well. Thus it shows that this way is strongly more efficient than the way using the spectral theorem for K_a^2 by avoiding the calculations of the eigenvalues and the eigenvectors of K_a^2. Such a more efficient method of building the set of convenient constraints converting the nonconservative system into a conservative system is certainly a good (at least a better!) way for the extension to the nonlinear case which is now tackled in the following section.

2. Set up of the non-linear issue. Notations. Link with the linear framework

2.1. Non-linear issue and notations

We then consider now a mechanical discrete system Σ so that its configuration manifold is a $\mathcal{C}^\infty$ real n dimensional manifold and $(q = (q^1, \dots, q^n), U)$ denotes generically a local coordinate system. That means that U is an open set of M and there is a function $\phi : U \subset M \to \mathbb{R}^n$ $(\mathcal{C}^\infty)$ with for all $m \in M, \phi(m) = q$. We suppose that the system is subjected to a positional force system Π so that Π is described by a differential 1-form Q_Π on M whose local expression in (q, U) is:

$$Q_\Pi = \sum_{k=1}^{n} Q_{\Pi,k}(q) dq^k. \tag{10}$$

According to [18], the exact vocable for describing Π is a semi-basic 1-form. That means that Π is described by a 1-form (or a Pfaff form) ω_Π on the total space TM of the tangent bundle τM such that ω is on the image of the canonical vertical operator. The local expression of a such semi-basic form is then

$$\omega_\Pi = \sum_{k=1}^{n} \omega_{\Pi,k}(q, \dot{q}) dq^k. \tag{11}$$

Thanks to the positional property of the forces, $\omega_{\Pi,k}(q, \dot{q})$ only depends on q meaning only of the projection $TM \to M$ of a point of TM and may be viewed as a function on the basis M of the tangent bundle. Then, (11) takes the form (10).

If there is no ambiguity, we omit the force system Π and write Q instead Q_Π. Any p dimensional ($\mathcal{C}^\infty$ real) submanifold N of M may locally in (q, U) be described by a family $(f^1, \dots, f^{n-p})$ of $n - p$ independent ($\mathcal{C}^\infty$ real) functions defined on $\phi(U)$ so that for all $q \in \phi(U)$, $m = \phi^{-1}(q) \in N \cap \phi^{-1}(U) \Leftrightarrow f^1(q) = \ldots = f^{n-p}(q) = 0$. The mechanical system whose N is the configuration space is called a subsystem $\Sigma_\mathcal{C}$ of Σ and functions $(f^1, \dots, f^{n-p})$ are called the local (non-linear) expressions of the constraints $\mathcal{C}$. We indifferently note $\Sigma_\mathcal{C}$ or Σ_N.

The positional force system Π acting on Σ is said conservative if there is a function h on M so that in any coordinate system (q, U):

$$Q_{\Pi,k}(q) = \frac{\partial h(\phi^{-1}(q))}{\partial q^k},$$

which is equivalent to $Q = dh$ on M or that Q is locally exact on M. It implies that Q is a closed differential 1-form and then $dQ = 0$ on M. Locally, by use of Poincaré's theorem and after having chosen an appropriate coordinate system, we may suppose the reciprocal property true on each U of the atlas covering M. As mentioned above, the global issue involving the topology of M is out of the scope of this paper. In [18], the vocable "conservative" means only that the semi basic Pfaff form Q is closed. The word "Lagrangian system" is reserved for the case where this form is exact in the framework of positional force system. In this paper, only the local extension to the non-linear case is investigated.

The issue is then the following: is there a submanifold N of M so that the physical action Π on Σ_N is conservative? If so, find all the possible N with the highest possible dimension p so that the number of constraints is the smallest possible (because every subsystem built by adding kinematical constraints to a conservative system is again a conservative system!!!).

2.2. Link with the linear framework

We focus now on the link with the linear framework of [12]. Suppose that there is a configuration q_e (equilibrium configuration of Σ) such that the linear approximation is used. Let $x = q - q_e = (x_1, \ldots, x_n)$. That means that Q_Π is approximated by its Taylor expansion to the first order $\tilde{Q}_\Pi$ leading to:

$$\tilde{Q}_\Pi = \tilde{Q}_\Pi(x) = \sum_{k=1}^{n} \left(Q_{\Pi,k}(q_e) + \sum_{\ell=1}^{n} \frac{\partial Q_{\Pi,k}(q_e)}{\partial q^\ell} x^\ell \right) dx^k. \tag{12}$$

That means that if $u = (u^1, \ldots, u^n) \in \mathbb{R}^n \approx T_{q_e}M$

$$\tilde{Q}_\Pi(x)(u) = \sum_{k=1}^{n} \left(Q_{\Pi,k}(q_e) + \sum_{\ell=1}^{n} \frac{\partial Q_{\Pi,k}(q_e)}{\partial q^\ell} x^\ell \right) u^k,$$

and the exterior derivative of $\tilde{Q}_\Pi(x)$ reads:

$$d\tilde{Q}_\Pi(x) = d\tilde{Q}_\Pi(0) = \sum_{k=1}^{n} \sum_{\ell<k} \left(\frac{\partial Q_{\Pi,k}(q_e)}{\partial q^\ell} - \frac{\partial Q_{\Pi,\ell}(q_e)}{\partial q^k} \right) dx^\ell \wedge dx^k.$$

Because the matrix $(\frac{\partial Q_{\Pi,k}(q_e)}{\partial q^\ell})_{k,\ell}$ is the stiffness matrix $K = K(q_e)$, then $d\tilde{Q}_\Pi(x) = d\tilde{Q}_\Pi(0)$ is the linear form ϕ of the previous section.

To conclude this paragraph, it may be noted that the issue of the extension to the non-linear framework of the decomposition of the stiffness matrix into its symmetric and skew symmetric part is investigated in [21]. The proposed solution does not use the language of differential forms even though the trick of Poincaré's Lemma is strongly used in the paper. The proposed decomposition aims to extend to the non-linear case the remarkable (but obvious) following property of the skew symmetric matrix K_a: $x^T K_a x = \sum_{k,\ell=1}^{n} K_{a,k\ell} x_k x_\ell = 0$. In our context, this interesting extension is not usable.

3. Solution of the non-linear problem

3.1. The solution

If there is a such p dimensional submanifold N of M, then the restriction $Q_{\Pi,N}$ of Q_Π to N must be closed. Locally both conditions are equivalent and the local condition in the local coordinate system (q, U) of M reads:

$$\forall q \in \phi(U \cap N) \;\; \forall X, Y \in TN \;\; dQ_\Pi(q)(X(q), Y(q)) = 0, \tag{13}$$

where the differential 2-form $dQ_\Pi \in \Lambda^2(M)$. Straightforward calculations give:

$$dQ_\Pi(q) = \sum_{\ell<k} \left(\frac{\partial Q_{\Pi,k}(q)}{\partial q^\ell} - \frac{\partial Q_{\Pi,\ell}(q)}{\partial q^k} \right) dq^\ell \wedge dq^k,$$

and the vector fields X, Y belong to TN if and only if (locally and with above notations)

$$df^i(X) = df^i(Y) = 0,$$

for all $i = 1, \ldots, n - p$ (dependency of q is omitted).

We suppose now that the form dQ_Π is regular on M meaning that its class r is constant on M. Then here, since the form dQ is itself a closed form ($d^2 = 0$), its class is also equal to its rank and is even: $r = 2s$. s is the unique number such that $(dQ_\Pi)^s \neq 0$ and $(dQ_\Pi)^{s+1} = 0$. We then deduce that $2s \leq n$.

Darboux's theorem gives the local modeling of dQ_Π on an open set U of M and reads:

$$dQ_\Pi = \sum_{k=1}^{s} dy^k \wedge dy^{k+s}, \tag{14}$$

where $y^1, \dots, y^{2s}$ are $2s$ independent functions on U. Choose now the s functions $f^k = y^k$ on U for all $k = 1, \dots, s$. Then if X and $Y \in TN$ then $X(f^i) = df^i(X) = 0 = dy^i(X)$ and $Y(f^i) = df^i(Y) = 0 = dy^i(Y)$ for all $i = 1, \dots, s$. Thus,

$$dQ_\Pi(X,Y) = \sum_{k=1}^{s} dy^k \wedge dy^{k+s}(X,Y) = \sum_{k=1}^{s} dy^k(X)dy^{k+s}(Y) - dy^k(Y)dy^{k+s}(X) = 0.$$

It is the proof of the following

Proposition 1. *Suppose that the class of dQ_Π is constant (namely maximal). The (non-linear) degree of non-conservativity of Π is then the half s of the class $2s$ of dQ_Π and thus $p = n - s$. The local definition of the submanifold N is given by the family $f^1 = 0, \dots, f^s = 0$ of equations on M where f^i is any linear combination (in the vector space on $\mathbb{R}$ and not in the modulus on the ring on the functions on $\mathbb{R}$) of y^i and y^{i+s} for all $i = 1, \dots, s$.*

Proof. The proof has been given for $f^k = y^k$ on U for all $k = 1, \dots, s$. Suppose now that f^i is any linear combination of y^i and y^{i+s} for all $i = 1, \dots, s$ meaning that $f^i = \alpha_i y^i + \beta_i y^{i+s}$ for $i = 1, \dots, s$ with $(\alpha_i,\ \beta_i) \neq (0,\ 0)$. Then, choosing $g^i = -\frac{\beta_i}{\alpha_i^2+\beta_i^2} y^i + \frac{\alpha_i}{\alpha_i^2+\beta_i^2} y^{i+s}$ we deduce $dy^k \wedge dy^{k+s} = df^k \wedge dg^k$ which allow to conclude by a same argument as above. □

Build a family of non-linear constraints converting the system into a conservative one is then brought back to find (thanks to Darboux's theorem) the decomposition (14). This decomposition comes itself (by exterior derivative) from Darboux's decomposition for the differential 1-form Q_Π. There are several proofs of the existence of such a decomposition. Before examining an example, we first ask the effectiveness of the solution. In the following paragraph, the issue of the set of all non-linear solutions is only formalized in the language of fiber bundles.

3.2. *Effectiveness of the solution*

The issue of the effectiveness of the solution is a significant problem for the physicist. Here, the effectiveness of the calculation of the degree of nonconservativity is clear at least under the assumption of maximal rank (or class) of dQ_Π. Calculating the successive powers of dQ_Π leads to the value of s. The example in Section 3.4 illustrates this point.

On the contrary, the effectiveness of the calculation of the constraints defining an appropriate submanifold N of M is brought back to the one of the canonical expression of dQ_Π thanks to Darboux's theorem. In Section 1, we investigated the effectiveness of both solutions of the same issue within the linear framework: the one proposed in [12] and the one of this paper. The former is constructive but not really effective whereas the latter is effective.

We tackle now the corresponding non-linear issue of the effective calculations of family of constraints $f_i, i = 1, \dots, s$ which is equivalent to Darboux's theorem for dQ_Π. There are several proofs of Darboux's theorem. Roughly speaking, one may find two different kinds of proof of Darboux's theorem. With our notations, the first one is done by induction on s and we may find such a type of proofs in several books of Analytic Mechanics like in [18] or in [22]. The non-effective step then lies in the calculation of the flow of a no time-depending vector field at each step of the induction namely here the calculations of the flows of s no time-depending vector fields (these vectors fields have to be calculated at each step of the induction reasoning). The second kind of proofs of Darboux's theorem uses a Moser's lemma which is a reasoning by homotopy without induction on s but involving the calculation of the flow of a time-depending vector field (see for example [23]–[25]). Both kinds of proofs have its own advantages and disadvantages but both need to calculate the flow of non-linear vector fields or at least to integrate these vector fields which cannot be done generally analytically but only numerically. That is the main obstacle to an analytic solution of this issue. The following example of the four degree of freedom Bigoni system leads to such a situation.

3.3. *Set of solutions*

They are as many solutions as functions $y_k, k = 1, \dots, s$ involved in (14). But this point of view is not geometric. It is a similar issue as the one met in the linear case: a solution is a family of linear forms but the set of solutions is described by (the manifold of) Lagrangian spaces. Here to get an analogous object for the differentiable non-linear case, one has to use the language of vector bundle. The 2-differential form dQ_π may be viewed as a section of the vector bundle $\Lambda^2(M)$ of the 2-differential forms on M which is itself a vector bundle associated to the tangent bundle $\tau(M) = (T(M), p_M, M)$ the fiber being $\mathbb{R}^n$. Supposing that the class $2s$ of dQ_π is constant on M then the field $q \to \ker dQ_\pi(q)$ defines a vector subbundle $\tau_0(M)$ over M whose the fiber is $\mathbb{R}^k$ with $k = n - 2s$. The total space of this vector bundle is $\bigcup_{q\in M}\{q\} \times \ker dQ_\pi(q)$. On the quotient vector bundle $\tilde{\tau}(M)$ over M of $\tau(M)$ by $\tau_0(M)$, the 2-differential form $d\tilde{Q}_\pi$ induced by $dQ_\pi(q)$ on each fiber $T_qM/\ker dQ_\pi(q)$ defines a structure of symplectic vector bundle over M. The set of solutions is then built by the set of all Lagrangian manifolds L of this symplectic vector bundle. The very difficult issue to built it is let to further investigations.

3.4. The example

We now tackle the same four degree of freedom non-linear Bigoni system as in 1.4 with total follower force at each joint like in Fig. 1 with $n = 4$. As already noted, the most usual case of a unique follower force at the extremity (usual Ziegler system) is not interesting in our context because the geometric degree of nonconservativity is then reduced to 1 with the obvious solution $\theta_4 = 0$ in the linear case and $\theta_4 = cte$ in the non-linear case. In this last case, the direction of the external force remains constant and this force becomes conservative!. We however deal with this case as a particular case in the paragraph below. We begin by doing calculations for any value of n and we specify in a second time the value $n = 4$. Remember that the force system is set up by $\Pi = (p_1, \dots, p_n)$ (see Fig. 1) and the local coordinate system is $\theta = (\theta_1, \dots, \theta_n)$. The configuration manifold is then $M = (S^1)^n$.

Straightforward computations give the following expression for the 1-forme Q_Π (all bars have the same length ℓ):

$$
\begin{aligned}
Q_\Pi(\theta) &= -\ell \sum_{k=1}^{n} p_k \sum_{j=1}^{k} (\cos\theta_j \sin\theta_k - \cos\theta_k \sin\theta_j) d\theta_j, \\
&= -\ell \sum_{k=1}^{n} p_k \sum_{j=1}^{k} \sin(\theta_j - \theta_k) d\theta_j, \\
&= -\ell \sum_{j=1}^{n} \sum_{k>j} p_k \sin(\theta_j - \theta_k) d\theta_j, \\
&= \sum_{j=1}^{n} Q_{\Pi,j}(\theta) d\theta_j,
\end{aligned} \tag{15}
$$

with

$$
Q_{\Pi,j}(\theta) = \ell \sum_{k>j} p_k \sin(\theta_k - \theta_j). \tag{16}
$$

Calculations then give (for sake of simplicity $\ell = 1$):

$$
dQ_\Pi(\theta) = \sum_{i<j} p_j \cos(\theta_j - \theta_i) d\theta_i \wedge d\theta_j.
$$

Suppose now that $n = 4$. Then

$$
\begin{aligned}
dQ_\Pi =\ & p_2 \cos(\theta_2 - \theta_1) d\theta_1 \wedge d\theta_2 + p_3 \cos(\theta_3 - \theta_1) d\theta_1 \wedge d\theta_3 + p_4 \cos(\theta_4 - \theta_1) d\theta_1 \wedge d\theta_4, \\
&+ p_3 \cos(\theta_3 - \theta_2) d\theta_2 \wedge d\theta_3 + p_4 \cos(\theta_4 - \theta_2) d\theta_2 \wedge d\theta_4 + p_4 \cos(\theta_4 - \theta_3) d\theta_3 \wedge d\theta_4,
\end{aligned} \tag{17}
$$

and the square of dQ_Π reads:

$$
\begin{aligned}
dQ_\Pi^2(\theta) =\ & dQ_\Pi(\theta) \wedge dQ_\Pi(\theta) = (p_2 p_4 \cos(\theta_1 - \theta_2) \cos(\theta_3 - \theta_4), \\
&- p_3 p_4 \cos(\theta_1 - \theta_3) \cos(\theta_2 - \theta_4) + p_3 p_4 \cos(\theta_1 - \theta_4) \cos(\theta_2 - \theta_3)) d\theta_1 \wedge d\theta_2 \wedge d\theta_3 \wedge d\theta_4,
\end{aligned}
$$

and obviously because $\dim M = 4$, $dQ_\Pi^3 = 0$.

Remark also that when $p_1 = p_2 = p_3 = 0$, $dQ_\Pi^2 = 0$ and the class of dQ_Π is 1 meaning that when the follower load is acting only at the extremity of the system (only $p_4 \neq 0$), the (non-linear) degree of nonconservativity is 1. In this case, the unique constraint then reads $f^1(\theta) = a\theta_4 + b$ with a, b any constant. Indeed, in this case:

$$
dQ_\Pi = p_4 (\cos(\theta_4 - \theta_1) d\theta_1 \wedge d\theta_4 + \cos(\theta_4 - \theta_3) d\theta_3 \wedge d\theta_4) = 0,
$$

on N defined by $f^1 = 0$ because then $d\theta_4 = 0$. We then find again that, for usual Ziegler system, when the action of the force p_4 is constrained to keep a fixed direction ($\theta_4 = cst$), the system becomes conservative. In a neighborhood of 0, that reads $\theta_4 = 0$ which has been found in [12] in the linear framework.

Except on the closed hypersurface S of M whose equation reads

$$
p_2 p_4 \cos(\theta_1 - \theta_2) \cos(\theta_3 - \theta_4) - p_3 p_4 \cos(\theta_1 - \theta_3) \cos(\theta_2 - \theta_4) + p_3 p_4 \cos(\theta_1 - \theta_4) \cos(\theta_2 - \theta_3) = 0, \tag{18}
$$

$dQ_\Pi^2 \neq 0$ which proves that the geometric degree of nonconservativity is generically 2 meaning on $M \backslash S$. On S, the geometric degree of nonconservativity is again 1. According to (18), the constraint reads, supposing $p_4 \neq 0$:

$$
g(\theta) = p_2 \cos(\theta_1 - \theta_2) \cos(\theta_3 - \theta_4) - p_3 \cos(\theta_1 - \theta_3) \cos(\theta_2 - \theta_4) + p_3 \cos(\theta_1 - \theta_4) \cos(\theta_2 - \theta_3) = 0.
$$

We now tackle the generic issue supposing that the point $m \in M \backslash S$. We are then looking for the families of two constraints $f^1 = 0, f^2 = 0$ defining the family of submanifolds N. That is equivalent to find a coordinate system $(y^i)_{i=1,\dots,4}$ of M such that

$$
dQ_\Pi = dy^1 \wedge dy^3 + dy^2 \wedge dy^4, \tag{19}
$$

whose existence is ensured by Darboux's theorem.

For finding the constraints, we follow the approach of S. Lang in [24]. First, since the calculations are local, we choose $\theta_0 = 0$ and we investigate the non-linear problem in a neighborhood U of $\theta_0 = 0$. Then

$$dQ_\Pi(0) = \sum_{i<j} p_j d\theta_i \wedge d\theta_j.$$

Let us then put, for $\theta \in U$, $\omega_t(\theta) = dQ_\Pi(0) + t(dQ_\Pi(\theta) - dQ_\Pi(0))$. Straight forward calculations give:

$$\omega_t(\theta) = \sum_{i<j} p_j((1-t) + t\cos(\theta_i - \theta_j))d\theta_i \wedge d\theta_j = \sum_{i<j} p_j((1 + t(\cos(\theta_i - \theta_j) - 1))d\theta_i \wedge d\theta_j.$$

We then are led to define the time-dependent vector field ξ_t such that for all $0 \le t \le 1$:

$$\omega_t \circ \xi_t = Q_\Pi(\theta) - Q_\Pi(0) = Q_\Pi(\theta), \tag{20}$$

because $Q_\Pi(0) = 0$ and where the 1-form $\omega_t \circ \xi_t$ is defined in U by:

$$\omega_t \circ \xi_t(X) = \omega_t(\xi_t, X),$$

for all vector field X on U. In local coordinates θ on U, (20) reads:

$$\sum_{i<j} p_j((1 + t(\cos(\theta_i - \theta_j) - 1))d\theta_i \wedge d\theta_j(\xi_t, X) = \sum_{j=1}^{n} Q_{\Pi,j}(\theta)d\theta_j(X), \tag{21}$$

for all vector field X on U and using the basis $(\frac{\partial}{\partial\theta_i})_{i=1,\dots,n}$, (21) becomes:

$$\sum_{j<i} p_j((1 + t(\cos(\theta_i - \theta_j) - 1))\xi_t^j - \sum_{j>i} p_j((1 + t(\cos(\theta_i - \theta_j) - 1))\xi_t^j = Q_{\Pi,j}(\theta) \ \ \forall i = 1, \dots, n,$$

where $\xi_t = \sum_{i=1}^n \xi_t^i \frac{\partial}{\partial\theta_i}$ and then according to (16)

$$\sum_{j<i} p_j((1 + t(\cos(\theta_i - \theta_j) - 1))\xi_t^j - \sum_{j>i} p_j((1 + t(\cos(\theta_i - \theta_j) - 1))\xi_t^j = \sum_{j>i} p_j \sin(\theta_j - \theta_i), \tag{22}$$

for $i = 1, \dots, n$.

From now on we restrict the calculations to the case $n = 4$.

The vector field ξ_t is then the solution of the linear system

$$\begin{cases} -p_2\psi_{12}(t,\theta)\xi_t^2 - p_3\psi_{13}(t,\theta)\xi_t^3 - p_4\psi_{14}(t,\theta)\xi_t^4 = Q_{\Pi,1}(\theta), \\ p_1\psi_{12}(t,\theta)\xi_t^1 - p_3\psi_{23}(t,\theta)\xi_t^3 - p_4\psi_{24}(t,\theta)\xi_t^4 = Q_{\Pi,2}(\theta), \\ p_1\psi_{13}(t,\theta)\xi_t^1 + p_2\psi_{23}(t,\theta)\xi_t^2 - p_4\psi_{34}(t,\theta)\xi_t^4 = Q_{\Pi,3}(\theta), \\ p_1\psi_{14}(t,\theta)\xi_t^1 + p_2\psi_{24}(t,\theta)\xi_t^2 + p_3\psi_{34}(t,\theta)\xi_t^3 = Q_{\Pi,4}(\theta), \end{cases} \tag{23}$$

where $\psi_{ij}(t,\theta) = \psi_{ji}(t,\theta) = 1 + t(\cos(\theta_i - \theta_j) - 1)$ for all $i \neq j$ and $Q_{\Pi,j}$ is given by (16). Solving (23) gives the functions $(t,\theta) \mapsto \xi_t^j(\theta)$ for all $j = 1,2,3,4$ whose analytic expressions (found with MAPLE) are complicated but may always be found because of the linearity of the system with unknowns ξ_t^j, $j = 1,2,3,4$ (see the annex for the solutions of (23)).

Let now α_t be the flow of ξ_t. This is the first step that may be not analytical effective because we cannot always find an analytic explicit expression of α_t.

It may be proved (see [24] pp. 152–153) from (20) that

$$\frac{d(\alpha_t^*\omega_t)}{dt} = 0,$$

namely $\alpha_t^*\omega_t$ is constant and thus

$$\alpha_1^*\omega_1 = \alpha_1^* dQ_\Pi = dQ_\Pi(0).$$

(Here, as usually, if ϕ: $M \to N$ is a differential map between two manifolds M and N, $\phi^*\alpha$ is the pull-back of any differential p-form α on N: $\phi^*\alpha$ is then a differential p-form on M.)

But the constant closed 2-form $dQ_\Pi(0)$ is of class 2. There is then a new system of coordinates $u = (u_1, u_2, u_3, u_4)$ in a neighborhood of 0 such that

$$dQ_\Pi(0) = du_1 \wedge du_3 + du_2 \wedge du_4,$$

and this system of coordinates may be found by usual algebraic methods. Here, we find for example:

$$u_1 = \theta_1, u_2 = \theta_1 + \theta_2 + \theta_3, u_3 = p_2\theta_2, u_4 = p_3\theta_3 + p_4\theta_4, \tag{24}$$

and conversely

$$\theta_1 = u_1, \theta_2 = \frac{1}{p_2}u_3, \theta_3 = -u_1 + u_2 - \frac{1}{p_2}u_3, \theta_4 = \frac{p_3}{p_4}u_1 - \frac{p_3}{p_4}u_2 + \frac{p_3}{p_2 p_4}u_3 + \frac{1}{p_4}u_4, \tag{25}$$

(24) is obviously (5) because, as mentioned in (2.2), $dQ_\Pi(0) = \phi$.

Finally, if the diffeomorphism $\alpha_1\colon V \to U$ reads, after having used (25) to pass from variables θ to variables u:

$$\alpha_1(u_1, u_2, u_3, u_4) = \alpha_1(u) = (h_1(u), h_2(u), h_3(u), h_4(u)),$$

then the coordinates $y = (y_1, y_2, y_3, y_4)$ leading to the wanted form (19) are given by the relations $u_i = h_i(y)$ or $y_i = h_i^{-1}(u) = g_i(\theta)$ (thanks to (24)) for all $i = 1, \dots, 4$. Inverting the diffeomorphism h is the second non-analytical effective step of the calculation of the non-linear constraints.

This example highlights the two steps that are the obstructions to get an analytic explicit expression of the non-linear kinematic constraints.

The other way using the proof by induction leads to other issues of same level of difficulty. Building the nonlinear kinematic constraints even for this four d.o.f. example is a real challenge. A possible alternative should be to ask whether there are conditions for an analytic expansion of the canonical coordinates in Darboux's theorem and, if so, to build at least this expansion up to a given order. If not, the obstacles might however be overcome thanks to a numerical approach.

Conclusion

In this paper, we focus on the so-called geometric degree or index of nonconservativity of a discrete mechanical system that has been introduced and investigated in [12] in a linear framework. It is here tackled in a linear and nonlinear context as well. By definition, this index is the minimal number of kinematic constraints in order that the constrained system becomes conservative or Lagrangian. The paper deals with positional non-Lagrangian mechanical systems like circulatory elastic systems and viscosity or friction in the system are not directly taken into account. In the linear framework (namely on the tangent space of the configuration space at a configuration q_e), the geometric degree of nonconservativity is defined and investigated through the skew symmetric part $K_a(q_e)$ of the stiffness matrix $K(q_e)$ of the mechanical system Σ whereas, in the non-linear framework, it is studied in the language of analytic mechanics. Firstly, we use the language of exterior p-forms in order to get an effective solution of the building of the constraints. This method leads to a more effective solution than the one proposed in [12]. Secondly, the building of the complete set of all the solutions is provided thanks to the calculations of the Lagrangian planes of an appropriate symplectic space. Thirdly, the extension to the nonlinear case is done via the language of differential forms. The degree of nonconservativity is then the class of a closed 2-form and the corresponding constraints are linked with the canonical form of this 2-form whose existence is due to well-known Darboux's theorem. An 4 degree of freedom example illustrates all the results. This example shows how building the constraints in the nonlinear framework but also highlights the steps of the used method that are non-analytically effective. An interesting open issue could then be first to find conditions for an possible analytic expansion of the canonical coordinates and second to perform an algorithm providing such an expansion at least up to a fixed order.

Annex

We now give the expanded expression of the first component $\xi_t^1 = \xi_t^1(\theta) = \xi_1(t,\theta)$ of the vector field $\xi_t = \sum_{i=1}^4 \xi_t^i \frac{\partial}{\partial \theta_i}$ for the 4 degree of freedom example. Calculations have been done by Maple. The other components are of the same vein. Maple fails to calculate analytically the flow α_t of ξ_t and then to find by this way an analytic expression of the non-linear constraints.

$$\begin{aligned}
\xi_t^1(\theta) = & -[p_4 t\sin(\theta_3-\theta_4) + p_4 t\cos(\theta_3-\theta_4)\sin(\theta_2-\theta_4) - p_4 t\cos(\theta_2-\theta_4)\sin(\theta_3-\theta_4),\\
& - \ p_3 t\sin(\theta_2-\theta_3 + p_3 t\cos(\theta_3-\theta_4)\sin(\theta_2-\theta_2) + p_4\sin(\theta_2-\theta_4)\\
& + \ p_3\sin(\theta_2-\theta_3) - p_4\sin(\theta_3-\theta_4)]/p_1[-t\cos(\theta_2-\theta_4) - t\cos(\theta_1-\theta_3),\\
& - t^2\cos(\theta_1-\theta_3)\cos(\theta_2-\theta_4) + t^2\cos(\theta_1-\theta_3) + t^2\cos(\theta_2-\theta_4) + t\cos(\theta_1-\theta_4) + t\cos(\theta_2-\theta_3),\\
& + t^2\cos(\theta_2-\theta_3)\cos(\theta_1-\theta_4) - t^2\cos(\theta_2-\theta_3) - t^2\cos(\theta_1-\theta_4) + 1 + t\cos(\theta_3-\theta_4),\\
& - 2t + t\cos(\theta_1-\theta_2) + t^2\cos(\theta_1-\theta_2)\cos(\theta_3-\theta_4) - t^2\cos(\theta_1-\theta_2) - t^2\cos(\theta_3-\theta_4) + t^2].
\end{aligned}$$

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