Free vibrations of non-uniform and axially functionally graded beams using Haar wavelets

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Vibrations of non-uniform and functionally graded (FG) beams with various boundary conditions and varying cross-sections are investigated using the Euler–Bernoulli theory and Haar matrices. It is assumed that the cross-section and material properties vary along the beam in the axial direction. The system of the governing equations is transformed with the aid of a set of simplest wavelets. To validate the present results, the non-homogeneity of the beams is discussed in detail and the calculated frequencies are compared with those of the existing literature. The results show that the Haar wavelet approach is capable of calculating frequencies for the beams with different shapes, rigidity, mass density, small or large translational and rotational boundary coefficients. The advantage of the novel approach consists in its simplicity, accuracy and swiftness.

1. Introduction

In contemporary engineering conditions, the requirements for structural materials and their properties are becoming more stringent. This is particularly true for the materials which are used in constructional elements or assembly units and are utilized in extremely severe environment or adverse exploitation. Nevertheless, traditional means for improving characteristics and performance of natural materials are depleted. Therefore, an increasing interest in composite materials and the materials with gradients in composition is evident. The tendency is also provoked by economical aspects: extraction and processing of natural resources is limited and expensive.

FG materials withstand high temperatures and resist corrosion. On account of comparatively good fracture toughness, FG materials are less exposed to delamination or cracking in comparison to uniform or homogeneous beams; therefore, FG materials have been under important consideration among engineers in recent decades. A detailed overview of the advanced materials, their development, elemental composition, microstructure, properties, design and application are described in [1] by Byrd. A more comprehensive research on thermoelastic behaviour of FG structures was first conducted by Chakraborty et al. in 2003 yet. Static, free vibrations and wave propagation were investigated by the beam element approach which required an exact solution of the static part of the governing differential equations [2]. Aydogdu and Taskin studied free vibrations of simply supported FG beams with the aid of classical beam theory, parabolic and exponential shear deformation beam theories. The governing equations were found by the Navier type solution [3]. A unified approach for analysing both static and dynamic behaviour of FG beams was proposed by Li extending the Timoshenko beam theory [4]. A fundamental frequency analysis using different higher-order beam theories was carried out by Simsek [5]. In 2010, Alshorbagy et al. suggested FEM for calculating dynamic characteristics of FG beams with material graduation in axially or transversally through the thickness based on the power law [6]. The same method was applied by Shahna et al. for stability analysis of FG tapered Timoshenko beam [7]. Xiang and Yang studied forced vibrations of a three-layer laminated FG Timoshenko beam with arbitrary end supports and varying thickness due to the applied heat [8]. Recently Simsek and Kocatürk studied dynamic behaviour of FG simply-supported beams under a concentrated moving harmonic load. The approach was based on Lagrange's equations [9]. Bending and vibration of cylindrical beams with arbitrary radial non-homogeneity were investigated by Huang and Li [10]. A dynamic system with a moving mass was broadly studied by Simsek in [11,12], and Khalili et al. [13]. A new approach for calculating free vibration of FG beams with non-uniform cross-section area and varying physical properties along its longitude was proposed by Huang and Li [14] last year. The approach was based on the Fredholm integration equation.

The analytical method for studying free vibrations of FG beams was provided by Sina et al. [15] and Mahi et al. [16] only a few years ago. The equation of deflection was derived applying Hamilton's principle. The Galerkin method was employed to analyse free vibration of sandwich beams with FG core in [17].

Despite the variety of methods and approaches for analytical and computational analysis of non-uniform and FG beams, no simple and fast solutions applicable for both free and forced vibrations in such beams with different boundary conditions and varying cross-section area were proposed. Only few solutions are found for the studies on the axially FG beams. Hereof, the purpose of the present work is to introduce the Haar wavelet approach for calculating natural frequencies in non-uniform and FG beams. The paper is organized in five sections. Section 2 describes integration of Haar wavelets. In Section 3, the problem and the solution are stated. Various numerical examples can be found in Section 4. The main conclusions are drawn in Section 5.

2. Integration of Haar wavelets

The Haar wavelet is one of the simplest wavelets which is discontinuous and resembles a step function. In other words, the Haar wavelets belong to the special class of discrete orthonormal wavelets. The other wavelets generated from the same mother wavelet form a basis whose elements are orthonormal to each other and are normalized to unit length. This property allows each wavelet coefficient to be computed independently of other wavelets. The Haar wavelet family for $\xi \in [0,1]$ is defined as follows:

$$h_i(\xi) = \begin{cases} 1 & \text{for } \xi \in [\xi^{(1)}, \xi^{(2)}], \\ -1 & \text{for } \xi \in [\xi^{(2)}, \xi^{(3)}], \\ 0 & \text{elsewhere.} \end{cases} \tag{1}$$

In (1), notations

$$\xi^{(1)} = \frac{k}{m}, \qquad \xi^{(2)} = \frac{k+0.5}{m}, \qquad \xi^{(3)} = \frac{k+1}{m} \tag{2}$$

are introduced. Integer $m = 2^j$ $(j = 0, 1, \ldots, J)$ is the factor of scale; $k = 0, 1, \ldots, m-1$ is the factor of delay. Integer J determines the maximal level of resolution. Index i in (1) is calculated as $i = m + k + 1$; the minimal value for i is one (if $j = 0$, then $m = 1$, $k = 0$); the maximal value of i is $2M$, which is 2^{J+1}. If index i is equal to one, the corresponding scaling function is $h_1(\xi) = 1$ if $\xi \in [0, 1]$, and $h_1(\xi) = 0$ elsewhere.

In [18], the Haar coefficient matrix $H_{(2M\times 2M)}(i, l) = h_i(\xi_l)$ is introduced; the collocation points are defined as:

$$\xi_l = \frac{l-0.5}{2M}, \quad l = 1, 2, \ldots, 2M. \tag{3}$$

For further studies, the integrals of the wavelets

$$p_{\alpha,i}(\xi) = \int_0^{\xi} p_{\alpha_{i-1,i}(\xi)} \mathrm{d}\xi \tag{4}$$

are required. In (4), $p_{0,i}(\xi) = h_i(\xi)$. These integrals are calculated analytically [19]. In case $i = 1$, the integral of the wavelet is $p_{\alpha,1}(\xi) = \xi^{\alpha}/\alpha!$, and in case $i > 1$ is

$$p_{\alpha,i}(\xi) = \begin{cases} 0 \quad \text{for } \xi < \xi^{(1)}, \\ \dfrac{1}{\alpha!}\left(\xi - \dfrac{k}{m}\right)^{\alpha} \quad \text{for } \xi \in [\xi^{(1)}, \xi^{(2)}], \\ \dfrac{1}{\alpha!}\left[\left(\xi - \dfrac{k}{m}\right)^{\alpha} - 2(\xi - \xi^{(2)})^{\alpha}\right] \\ \quad \text{for } \xi \in [\xi^{(2)}, \xi^{(3)}], \\ \dfrac{1}{\alpha!}\left[\left(\xi - \dfrac{k}{m}\right)^{\alpha} - 2(\xi - \xi^{(2)})^{\alpha} + (\xi - \xi^{(3)})^{\alpha}\right] \\ \quad \text{for } \xi > \xi^{(3)}. \end{cases} \tag{5}$$

Values $p_{\alpha,i}(0)$ and $p_{\alpha,i}(1)$ should be calculated in order to satisfy the boundary conditions. Evaluating integrals (5) in the collocation points, the following form could be obtained

$$P^{(\alpha)}(i, l) = p_{\alpha,i}(\xi_l), \tag{6}$$

where $P^{(\alpha)}$ is a $2M \times 2M$ matrix. It should be noted that calculations of matrices $H(i, l)$ and $P^{(\alpha)}(i, l)$ must be carried out only once.

3. Problem statement and method of solution

Consider an axially graded Euler–Bernoulli beam with a variable cross-section of length L. In the present study, it is assumed that the material properties and cross-section of the beam vary continuously along the length. Introducing the quantities:

$$\xi = \frac{x}{L}, \qquad k^4 = \frac{\rho_0 A_0 \omega^2 L^4}{E_0 I_0}, \tag{7}$$

the equation of motion for transverse vibrations is given by

$$\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\left[E(\xi)I(\xi)\frac{\mathrm{d}^2 W(\xi)}{\mathrm{d}\xi^2}\right] - k^4 m(\xi) W(\xi) = 0, \quad \xi \in [0, 1], \tag{8}$$

where $W(\xi)$ is the transverse deflection, $m(\xi) = \rho(\xi)A(\xi)$ is the mass at position ξ, $E(\xi)I(\xi) = D(\xi)$ is the bending stiffness; $\rho(\xi)$ is the mass density of the beam material, $E(\xi)$ is the Young's modulus, $A(\xi)$ is the cross-section area and $I(\xi)$ is the moment of inertia at ξ. In (7), k is the dimensionless natural frequency, and ρ_0, A_0, E_0, I_0 denote the values of ρ, A, E, I at $\xi = 0$, respectively. In the present study, it is assumed that functions $E(\xi)$ and $I(\xi)$ have derivatives up to the second order. From (8), it yields that

$$\begin{aligned} &\frac{\mathrm{d}^4 W(\xi)}{\mathrm{d}\xi^4}E(\xi)I(\xi) + 2\frac{\mathrm{d}^3 W(\xi)}{\mathrm{d}\xi^3}\left[\frac{\mathrm{d}E(\xi)}{\mathrm{d}\xi}I(\xi) + \frac{\mathrm{d}I(\xi)}{\mathrm{d}\xi}E(\xi)\right] \\ &\quad + \frac{\mathrm{d}^2 W(\xi)}{\mathrm{d}\xi^2}\left[\frac{\mathrm{d}^2 E(\xi)}{\mathrm{d}\xi^2}I(\xi) + 2\frac{\mathrm{d}E(\xi)}{\mathrm{d}\xi}\frac{\mathrm{d}I(\xi)}{\mathrm{d}\xi} + \frac{\mathrm{d}^2 I(\xi)}{\mathrm{d}\xi^2}E(\xi)\right] \\ &\quad - k^4 W(\xi)\rho(\xi)A(\xi) = 0, \quad \xi \in [0, 1]. \end{aligned} \tag{9}$$

For a general case, the solution of (9) is not available. According to [18,19], a highest-order derivative is expanded into the Haar series instead of solving the differential equation. Therefore, it is assumed that the fourth derivative of the solution (9) is sought in the following form:

$$W^{IV}(\xi) = \sum_{i=1}^{2M} a_i h_i(\xi), \tag{10}$$

where a_i are unknown wavelet coefficients. Integrating (10) four times and taking into account (4) and (5), we obtain

$$\begin{aligned} W'''(\xi) &= \sum_{i=1}^{2M} a_i p_{1,i}(\xi) + W'''(0), \\ W''(\xi) &= \sum_{i=1}^{2M} a_i p_{2,i}(\xi) + W'''(0)\xi + W''(0), \\ W'(\xi) &= \sum_{i=1}^{2M} a_i p_{3,i}(\xi) + \frac{1}{2}W'''(0)\xi^2 + W''(0)\xi + W'(0), \\ W(\xi) &= \sum_{i=1}^{2M} a_i p_{4,i}(\xi) + \frac{1}{6}W'''(0)\xi^3 \\ &\quad + \frac{1}{2}W''(0)\xi^2 + W'(0)\xi + W(0). \end{aligned} \tag{11}$$

In (11), quantities $W(0), W'(0), W''(0), W'''(0)$ can be evaluated from the boundary conditions. In the present work, the following boundary conditions are considered:

(i) Cantilever beams (CF)

In this case, one end $\xi = 0$ is clamped, while the other end $\xi = 1$ is free. The boundary conditions for the beam are $W(0) =$

$W'(0) = 0, W''(1) = W'''(1) = 0$ and the system is derived from (11) as follows:

$$\sum_{i=1}^{2M} a_i p_{1,i}(1) + W'''(0) = 0,$$
$$\sum_{i=1}^{2M} a_i p_{2,i}(1) + W'''(0) + W''(0) = 0. \tag{12}$$

Evaluating the system, $W''(0)$ and $W'''(0)$ are obtained

$$W''(0) = \sum_{i=1}^{2M} a_i q_{1,i},$$
$$W'''(0) = -\sum_{i=1}^{2M} a_i p_{1,i}(1), \tag{13}$$

where $q_{1,i} = p_{1,i}(1) - p_{2,i}(1)$. In this case, the mode shape W is described by the following equation:

$$W(\xi) = \sum_{i=1}^{2M} a_i \left[p_{4,i}(\xi) - \frac{1}{6} p_{1,i}(1)\xi^3 + \frac{1}{2} q_{1,i}\xi^2 \right]. \tag{14}$$

(ii) Simply supported beams (SS)

In the case of simply supported beam, the boundary conditions are $W(0) = W''(0) = 0, W(1) = W''(1) = 0$ and the system is

$$\sum_{i=1}^{2M} a_i p_{2,i}(1) + W'''(0) = 0,$$
$$\sum_{i=1}^{2M} a_i p_{4,i}(1) + \frac{1}{6} W'''(0) + W'(0) = 0. \tag{15}$$

From (15) we get

$$W'''(0) = -\sum_{i=1}^{2M} a_i p_{2,i}(1),$$
$$W'(0) = \frac{1}{6} \sum_{i=1}^{2M} a_i q_{2,i}, \tag{16}$$
$$W(\xi) = \sum_{i=1}^{2M} a_i \left[p_{4,i}(\xi) - \frac{1}{6} p_{2,i}(1)\xi^3 + \frac{1}{6} q_{2,i}\xi \right],$$

where $q_{2,i} = p_{2,i}(1) - 6p_{4,i}(1)$.

(iii) Clamped beams (CC)

Satisfying the boundary conditions, $W(0) = W'(0) = 0, W(1) = W'(1) = 0$ for the clamped–clamped beams one could get

$$W'''(0) = 6\sum_{i=1}^{2M} a_i q_{3,i},$$
$$W''(0) = \sum_{i=1}^{2M} a_i q_{4,i}, \tag{17}$$
$$W(\xi) = \sum_{i=1}^{2M} a_i \left[p_{4,i}(\xi) + q_{3,i}\xi^3 + \frac{1}{2} q_{4,i}\xi^2 \right],$$

where $q_{3,i} = 2p_{4,i}(1) - p_{3,i}(1)$ and $q_{4,i} = 2p_{3,i}(1) - 6p_{4,i}(1)$.

(iiii) Clamped–pinned (CP) beams

As a representative, it is assumed that one end $\xi = 0$ is clamped or fixed and the other end $\xi = 1$ is pinned or simply supported. Thus, the corresponding boundary conditions are $W(0) = W'(0) = 0, W(1) = W''(1) = 0$. In this case from (11) one could get

$$W'''(0) = \sum_{i=1}^{2M} a_i q_{5,i},$$
$$W''(0) = \sum_{i=1}^{2M} a_i q_{6,i}, \tag{18}$$
$$W(\xi) = \sum_{i=1}^{2M} a_i \left[p_{4,i}(\xi) + \frac{1}{6} q_{5,i}\xi^3 + \frac{1}{2} q_{6,i}\xi^2 \right],$$

where $q_{5,i} = 3p_{4,i}(1) - \frac{3}{2} p_{2,i}(1)$ and $q_{6,i} = -q_{5,i} - p_{2,i}(1)$.

For the illustration of the method, consider a case of the cantilever beam. Next the notation is introduced as follows:

$$a(:)H(:, l) = \sum_{i=1}^{2M} a_i h_i(\xi_l). \tag{19}$$

Substituting (10), (11) and (13) into (9), taking into account (19) and discretizing the results by taking $\xi \to \xi_l$, the governing system is derived

$$\begin{aligned} a(:)\Big\{ & H(:, l)U_1(l) + 2\left[P^{(1)}(:, l) - P^{(1)}(:, 1)E(l)\right] U_2(l) \\ & + \left[P^{(2)}(:, l) - P^{(1)}(:, 1)\xi_l E(l) + q_1(:)\xi_l^2 E(l)\right] U_3(l) \\ & - k^4 \left[P^{(4)}(:, l) - \frac{1}{6} P^{(1)}(:, 1)\xi_l^3 E(l) + \frac{1}{2} q_1(:)\xi_l^2 E(l)\right] U_4(l) \Big\}, \\ & l = 1, \dots, 2M, \end{aligned} \tag{20}$$

where

$$\begin{aligned} U_1(l) &= E(\xi_l)I(\xi_l), \\ U_2(l) &= E'(\xi_l)I(\xi_l) + E(\xi_l)I'(\xi_l), \\ U_3(l) &= E''(\xi_l)I(\xi_l) + 2E'(\xi_l)I'(\xi_l) + E(\xi_l)I''(\xi_l), \\ U_4(l) &= \rho(\xi_l)A(\xi_l) \end{aligned} \tag{21}$$

and $E(l)$ is a unit row vector. The system (20) is linear and homogeneous with regard to a_i and contains frequency parameter ω. For deriving a nontrivial solution, the determinant of system (20) must be zero. According to this requirement, the values for ω are evaluated.

4. Numerical examples

4.1. A comparison of the results for uniform homogeneous beams

Consider a homogeneous cantilever with a uniform cross-section. In this case, $D = EI$ and $m = \rho A$ are constants. The comparisons of the exact [20] and calculated dimensionless natural frequencies (DNF) k for different levels of resolution J are presented in Table 1. It is seen in the table that high accuracy is obtained using a small number of grid points.

4.2. Beams with non-uniform cross-section and elastic end constraints

The suggested approach of the Haar wavelets was applied to the wedge beam with a rectangular cross-section and clamped-free ends. Due to the shape, the breadth and the height of the beam were described by the formulae:

$$h(\xi) = h_0\left[1 + (\alpha_h - 1)\xi\right], \qquad b(\xi) = b_0\left[1 + (\alpha_b - 1)\xi\right], \tag{22}$$

where α_b stands for the ratio between the breadths at the beginning and at the end of the beam; α_h is the ratio between the heights respectively. In Table 2, the first dimensionless natural

Table 1
The first four DNF k_n for uniform cantilever beams.

n	Exact [20]	$J=2$	$J=3$	$J=4$	$J=5$	$J=6$
1	1.8751	1.8783	1.8759	1.8753	1.8752	1.8751
2	4.6941	4.7342	4.7040	4.6966	4.6948	4.6943
3	7.8548	7.9991	7.8899	7.8635	7.8570	7.8553
4	10.9956	11.3505	11.0804	11.0165	11.0008	10.9969

Table 2
The first DNF k^2 for the wedge and cone cantilever.

α	Wedge beam		Cone beam	
	[21]	Present	[21]	Present
0.1	4.63074	4.6305	7.20500	7.2055
0.4	3.93428	3.9343	5.00906	5.0088
0.6	3.73708	3.7373	4.31879	4.3189
0.7	3.66675	3.6670	4.06694	4.0671
0.9	3.55870	3.5589	3.67371	3.6739

Table 3
The first two DNF k^2 for a beam with translational constraints and $k_{rr}=k_{rl}=0$.

$k_{tl}=k_{tr}$	$n=1$		$n=2$	
	[21]	Present	[21]	Present
0.001	0.21656	0.2166	0.31795	0.3180
0.01	0.38510	0.3851	0.5639	0.5654
0.1	0.68462	0.6846	1.00528	1.0053
1	1.21404	1.2140	1.78509	1.7851
10	2.10096	2.1009	3.13023	3.1303
100	3.07241	3.0723	5.06670	5.0668
1000	3.37553	3.3754	6.56963	6.5697

frequencies k^2 for the wedge beam with $\alpha_b=1, \alpha_h=\alpha$ and the cone beam with $\alpha_b=\alpha_h=\alpha$ are presented. The results calculated by Hsu et al. [21] are given in columns two and four. The level of resolution was taken as $J=5$. According to the results, the Haar wavelets approach works accurately with insignificant errors in the cases of simple wedge beam models.

Next, consider a non-uniform beam with elastic end constants. The boundary conditions in the presence of translational and rotational spring constants at $\xi=0$ are presented as [21]:

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\left[I(\xi)\frac{\mathrm{d}^2 w(\xi)}{\mathrm{d}\xi^2}\right]+k_{tl}w(\xi)=0, \tag{23}$$

$$I(\xi)\frac{\mathrm{d}^2 w(\xi)}{\mathrm{d}\xi^2}-k_{rl}\frac{\mathrm{d}w(\xi)}{\mathrm{d}\xi}=0. \tag{24}$$

In (23) and (24), the non-dimensional translational and rotational spring coefficients k_t and k_r are

$$k_{tl}=\frac{K_{TL}L^3}{EI_0},\qquad k_{rl}=\frac{K_{RL}L}{EI_0}. \tag{25}$$

The boundary conditions on the right end $\xi=1$ are defined analogically. The first two natural frequencies with $\alpha_b=\alpha_h=1.4$, fixed rotational spring constraints $k_{rr}=k_{rl}=0$ and varying translational spring constraints are presented in Table 3. The results are compared with Hsu's [21] calculations.

It is important to note that the provided method of Haar wavelets is capable of calculating different frequencies for both small and large translational coefficients. The results are precise, however, an attempt to improve the accuracy was done on account of resolution. The model and conditions were remained the same as in the previous example, but the resolution was increased from five to seven.

As it can be seen from Table 4, the accuracy of calculations grows with the growth of resolution. However, according to Lepik [19], it is so only to a certain extend of the resolution.

Table 4
The third DNF k^2 for the beam with translational constraints and $k_{rr}=k_{rl}=0$ with different resolutions.

$k_{tl}=k_{tr}$	[21]	Present $J=5$	Present $J=6$	Present $J=7$
0.001	5.19178	5.1927	5.1920	5.1918
0.01	5.19196	5.1929	5.1922	5.1920
0.1	5.19381	5.1948	5.1940	5.1939
1	5.21223	5.2132	5.2125	5.2123
10	5.39376	5.3948	5.3940	5.3938
100	6.71152	6.7125	6.7118	6.7116
1000	9.28876	9.2894	9.2889	9.2888

Table 5
The first DNF k^2 for the beam with varying translational and fixed rotational end constraints ($J=5$).

$k_{tl}=k_{tr}$	$k_{rr}=0.5, k_{rl}=1$	$k_{rr}=1, k_{rl}=0.5$
0.001	0.2115	0.2115
0.01	0.3760	0.3760
0.1	0.6682	0.6684
1	1.1803	1.1828
10	2.0210	2.0345
100	2.9628	2.9852

Table 6
The first three DNF k^2 for the beam with parabolic-taper width and linear-taper height ($J=5$).

α	$n=1$		$n=2$		$n=3$	
	Present	[22]	Present	[22]	Present	[22]
0.1	5.8383	5.8382	16.6901	16.696	34.8190	34.854
0.3	4.7574	4.7577	17.5988	17.600	41.6669	41.660
0.5	4.2101	4.2100	18.9250	18.922	47.9306	47.907
0.8	3.7303	3.7301	20.8437	20.838	56.4426	56.453

Table 7
The first DNF of the cantilever with parabolic thickness (rectangular and circular cross-section) versus the dimensionless coordinate of the fixed end ($J=5$).

ξ_1	Cross-section			
	Rectangular		Circular	
	[23]	Present	[23]	Present
−0.7	1.050	1.0474	1.011	1.0028
−0.5	1.936	1.9363	2.238	2.2357
−0.3	3.070	3.0704	3.940	3.9394
0.0	5.576	5.5774	7.886	7.8857
0.3	10.17	10.1784	15.31	15.3134
0.5	16.27	16.2805	25.26	25.2880
0.7	30.47	30.4827	48.49	48.4945

The Haar wavelet approach was also applied to the beams with varying translational and fixed rotational end constraints ($k_{rr}=0.5, k_{rl}=1$) with $\alpha_b=2$ and $\alpha_h=1$ (Table 5).

The study on the frequencies proves that there is slight dependence between the rotational constraints and the cross-section area. The influence can be observed with the growth of the translational constraint value.

Now consider a truncated at α tapered beam [22] with parabolic-taper width $b(\xi)=b_0\sqrt{\xi}$ and linear-taper height $h(\xi)=h_0\xi$. The first three DNF for boundary conditions (on the left end $k_{rl}=k_{tl}=0$ and on the right end $k_{rr}\to\infty, k_{tr}\to\infty$) with $J=5$ are presented in Table 6. The calculated results correspond well to the previous works. Finally, examine the Haar wavelet approach on the cantilever with parabolic thickness versus the dimensionless coordinate of the fixed end ($h=h_0(1-\xi^2)$) and the cantilever with circular cross-section and parabolic thickness. The results of the calculations and comparison are provided in Table 7. The calculated results correspond to the previous research.

Table 8
The first DNF of axially graded beam versus the gradient parameter ($J = 5$).

β	S–S		C–C		C–P	
	Present	[14]	Present	[14]	Present	[14]
−10	11.4481	11.4532	24.0269	24.0576	16.3837	16.4775
−3	11.2422	11.2443	23.9384	23.9456	16.0307	16.0219
0	10.8660	10.8663	24.3749	24.3752	15.8729	15.8734
3	10.3670	10.3669	24.9371	24.9375	15.7171	15.7171
10	9.9366	9.9358	24.8080	24.7949	15.4930	15.4956

Table 9
The first DNF of the axially graded tapered cantilever with variable Young's modulus and mass density ($J = 5$).

γ	β				
	−0.2	−0.1	0.0	0.1	0.2
−0.2	2.6468	2.6051	2.5691	2.5378	2.5102
−0.1	3.0553	3.0103	2.9716	2.9378	2.9082
0.0	3.6085	3.5589	3.5162	3.4789	3.4460
0.1	4.4492	4.3922	4.3430	4.2999	4.2618
0.2	6.0214	5.9518	5.8909	5.8372	5.7893

4.3. *Uniform beams with variable flexural rigidity and mass density*

Consider a uniform beam with axial non-homogeneity. Assume that bending stiffness and mass density vary according to the following equation:

$$Q(\xi) = \begin{cases} Q_0\left(1 - \dfrac{e^{\beta\xi} - 1}{e^{\beta} - 1}\right) + Q_1\dfrac{e^{\beta\xi} - 1}{e^{\beta} - 1}, & \beta \neq 0, \\ Q_0(1-\xi) + Q_1\xi, & \beta = 0. \end{cases} \tag{26}$$

In (26), Q_0 and Q_1 stand for the corresponding material properties at the ends $\xi = 0, \xi = 1$ respectively, and β is the gradient. The material is chosen to consist of aluminium and zirconia as follows [14]

$$\begin{aligned} &\text{Al: } E_0 = 70\ \text{GPa}, \qquad \rho_0 = 2702\ \text{kg/m}^3, \\ &\text{ZrO}_2\text{: } E_1 = 200\ \text{GPa}, \qquad \rho_1 = 5700\ \text{kg/m}^3. \end{aligned} \tag{27}$$

The first natural frequencies for the functionally graded beams with different boundary conditions are calculated and presented in Table 8.

4.4. *Beams with non-uniform cross-section, variable flexural rigidity and mass density*

Next, consider a functionally graded beam with non-uniform cross-section. Assume that the cross-section of the beam has constant width and linearly varying height, i.e. $A/A_0 = 1 + \beta\xi, I/I_0 = (1+\beta\xi)^3$. The Young's modulus and the mass density are considered as trigonometric functions:

$$E(\xi) = E_0\left[1 + \gamma\cos(\pi\xi)\right], \qquad \rho(\xi) = \rho_0\left[1 + \delta\cos(\pi\xi)\right], \tag{28}$$

where $|\gamma| < 1$ and $|\delta| < 1$ are parameters. In Table 9, the calculated first natural frequencies for $\delta = 4\gamma$ and different values of β are presented. The natural frequencies of the tapered cantilever are sensitive to parameter γ.

4.5. *Non-homogeneous beams with variable flexural rigidity, mass density and elastic end constraints*

Here we consider the cases of axially FG beams with unusual boundary conditions. Let the flexural rigidity and mass density of the beam vary in the following form:

$$D(\xi) = D_0\left[1 + \alpha\cos(\pi\xi)\right], \qquad \rho(\xi) = \rho_0\left[1 + \beta\cos(\pi\xi)\right], \tag{29}$$

Table 10
The first DNF for beams the with variable flexural rigidity, mass density and elastic end constraints ($J = 5, k_{rr} = k_{tr} = 1$).

$\alpha = \beta$	$k_{rl} = k_{tl}$				
	1.0	2.0	3.0	4.0	5.0
−0.2	1.4408	2.0732	2.5756	3.0206	3.4420
−0.15	1.4120	2.0309	2.5191	2.9473	3.3462
0.15	1.2950	1.8632	2.3030	2.6794	3.0176
0.2	1.2818	1.8446	2.2795	2.6510	2.9842

Table 11
The first DNF for beams with variable flexural rigidity, elastic end constraints and intermediate rigid support ($J = 5, k_{rr} = k_{tr} = k_{rl} = k_{tl} = 1$).

$\alpha = \beta$	γ				
	0.1	0.3	0.5	0.7	0.9
−0.2	1.8902	2.8150	4.0124	3.9385	3.2176
−0.15	1.9120	2.8620	4.0277	3.8784	3.1657
0.15	2.0578	3.1553	4.0527	3.5405	2.8874
0.2	2.0851	3.2077	4.0475	3.4879	2.8455

where $|\alpha| < 1$ and $|\beta| < 1$ are parameters. The conditions insure that $D(\xi)$ and $\rho(\xi)$ are positive. The ends of the beam are fixed by elastic spring supports. The boundary conditions in the presence of rotational and translational spring constants are given by Eqs. (23)–(25). In Table 10, the first natural frequencies are presented for the case of fixed translational and rotational spring constants at the right end ($k_{rr} = k_{tr} = 1$) and variable constants at the left end.

4.6. *Non-homogeneous beams with variable flexural rigidity, mass density, elastic end constraints and intermediate rigid support*

Finally, the present method can also be applied for beams with additional intermediate constraints. Let us consider an axially FG beam with variable flexural rigidity and mass density varying according to the Eq. (29). The beam has rotational and translational flexible ends and an additional rigid support at $\xi = \gamma$. In the presence of the rigid support, the condition is

$$W(\gamma) = 0. \tag{30}$$

According to (18), the Eq. (30) takes the form:

$$W(\gamma) = \sum_{i=1}^{2M} a_i\left[p_{4,i}(\gamma) + \frac{1}{6}q_{5,i}\gamma^3 + \frac{1}{2}q_{6,i}\gamma^2\right]. \tag{31}$$

In Table 11, the calculated first natural frequencies for the fixed translational and rotational spring constants $k_{rr} = k_{tr} = k_{rl} = k_{tl} = 1$, varying $\alpha = \beta$ and different values of γ are presented. It can be seen in Table 11 that in the case of symmetric boundary conditions, the natural frequencies at the right end and left end are not the same. This is explained by non-symmetric FG material distribution in the beam.

5. Conclusions

The Haar wavelet approach was presented to solve free vibrations of non-uniform Euler–Bernoulli beams with continuously varying flexural rigidity and mass density. A numerical method was developed for general nonlinear functions. The obtained results were compared with those given in [14,21–23]. The benefits of the Haar wavelet approach are its simplicity and sparse matrices of presentation. The computational time is therefore comparatively small. High accuracy is obtained even with a small number of grid points. The method is easily applicable for systems with discontinuities and works effectively in the case of non-linearity.

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