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Analytic approach for reflected Brownian motion in the quadrant

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Abstract. Random walks in the quarter plane are an important object both of combinatorics and probability theory. Of particular interest for their study, there is an analytic approach initiated by Fayolle, Iasnogorodski and Malýšev, and further developed by the last two authors of this note. The outcomes of this method are explicit expressions for the generating functions of interest, asymptotic analysis of their coefficients, etc. Although there is an important literature on reflected Brownian motion in the quarter plane (the continuous counterpart of quadrant random walks), an analogue of the analytic approach has not been fully developed to that context. The aim of this note is twofold: it is first an extended abstract of two recent articles of the authors of this paper, which propose such an approach; we further compare various aspects of the discrete and continuous analytic approaches.

Keywords: Reflected Brownian motion in the quadrant; (Random) Walks in the quarter plane; Stationary distribution; Laplace transform; Generating function; Boundary value problem; Asymptotic analysis

1 Introduction

1.1 *Random walks in the quarter plane*

Since the seventies and the pioneered papers Malýšev (1972); Fayolle and Iasnogorodski (1979), random walks in the quarter plane (cf. Figure 1) are extensively studied. They are indeed an important object of probability theory and have been studied for their recurrence/transience, for their links with queueing systems (Fayolle and Iasnogorodski (1979)), representation theory (Biane (1992)), potential theory. Moreover, the state space $\mathbb{N}^2 = \{0, 1, 2, \dots\}^2$ offers a natural framework for studying any two-dimensional population; accordingly, quadrant walks appear as models in biology and in finance (Cont and de Larard (2013)). Another interest of random walks in the quarter plane is that in the large class of random processes in cones, they form a family for which remarkable exact formulas exist. Moreover, quadrant walks are popular in combinatorics, see Bousquet-Mélou and Mishna (2010); Bostan and Kauers (2010); Kurkova and Raschel (2012). Indeed, many models of walks are in bijection with other combinatorial objects: maps, permutations, trees, Young tableaux, etc. In combinatorics again, famous models have emerged from quadrant walks, as Kreweras' or Gessel's ones, see Bousquet-Mélou and Mishna (2010); Bostan and Kauers (2010). Finally, walks in the quarter plane are interesting for the numerous tools used

for their analysis: combinatorial (Bousquet-Mélou and Mishna (2010)), from complex analysis (Malyšev (1972); Fayolle and Iasnogorodski (1979); Fayolle et al. (1999); Kurkova and Raschel (2011, 2012); Bernardi et al. (2015)), computer algebra (Bostan and Kauers (2010)), for instance.

1.2 *Issues and technicalities of the analytic approach*

In the literature (see, e.g., Malyšev (1972); Fayolle and Iasnogorodski (1979); Fayolle et al. (1999); Kurkova and Raschel (2011, 2012)), the analytic approach relies on six key steps:

- (i) Finding a functional equation between the generating functions of interest;
- (ii) Reducing the functional equation to boundary value problems (BVP);
- (iii) Solving the BVP;
- (iv) Introducing the group of the walk;
- (v) Defining the Riemann surface naturally associated with the model, continuing meromorphically the generating functions and finding the conformal gluing function;
- (vi) Deriving the asymptotics of the (multivariate) coefficients.

Before commenting these different steps, let us note that altogether, they allow for studying the following three main problems:

- (P1) Explicit expression for the generating functions of interest (needs (i), (ii), (iii) and (v));
- (P2) Algebraic nature of these functions (needs (iv) and (v));
- (P3) Asymptotics of their coefficients in various regimes (needs (iii), (v) and (vi)).

The point (i) reflects the inherent properties of the model and is easily obtained. Point (ii), first shown in Fayolle and Iasnogorodski (1979), is now standard (see Fayolle et al. (1999)) and follows from algebraic manipulations of the functional equation of (i). Item (iii) uses specific literature devoted to BVP (our main reference for BVP is the book of Litvinchuk (2000)). The idea of introducing the group of the model (iv) was proposed in Malyšev (1972), and brought up to light in the combinatorial context in Bousquet-Mélou and Mishna (2010). Point (v) is the most technical a priori; it is however absolutely crucial, as it allows to access key quantities (as a certain conformal gluing function which appears in the exact formulation of (iii)). Finally, (vi) uses a double refinement of the classical saddle point method: the uniform steepest descent method.

1.3 *Reflected Brownian motion in the quarter plane*

There is a large literature on reflected Brownian motion in quadrants (and also in orthants, generalization to higher dimension of the quadrant), to be rigorously introduced in Section 3. First, it serves as an approximation of large queuing networks (see Foddy (1984); Baccelli and Fayolle (1987)); this was the initial motivation for its study. In the same vein, it is the continuous counterpart of (random) walks in the quarter plane. In other directions, it is studied for its Lyapunov functions in Dupuis and Williams (1994), cone points of Brownian motion in Le Gall (1987), intertwining relations and crossing probabilities in Dubédat (2004), and of particular interest for us, for its recurrence/transience in Hobson and

Rogers (1993). The asymptotic behavior of the stationary distribution (when it exists) is now well known, see Harrison and Hasenbein (2009); Dai and Miyazawa (2011); Franceschi and Kurkova (2016). There exist, however, very few results giving exact expressions for the stationary distribution. Let us mention Foddy (1984) (dealing with the particular case of a Brownian motion with identity covariance matrix), Baccelli and Fayolle (1987) (on a diffusion having a quite special behavior on the boundary), Harrison and Williams (1987b); Dieker and Moriarty (2009) (on the special case when stationary densities are exponential) and Franceschi and Raschel (2016) (on the particular case of an orthogonal reflection). We also refer to Burdzy et al. (2015) for the analysis of reflected Brownian motion in bounded planar domains by complex analysis techniques.

1.4 Main results and plan

This note is an extended abstract of the papers Franceschi and Kurkova (2016); Franceschi and Raschel (2016), whose main contributions are precisely to export the analytic method for reflected Brownian motion in the quarter plane. Our study constitutes one of the first attempts to apply these techniques to the continuous setting, after Foddy (1984); Baccelli and Fayolle (1987). In addition of reporting about the works Franceschi and Kurkova (2016); Franceschi and Raschel (2016), we also propose a comparative study of the discrete/continuous cases.

Our paper is organized as follows: Section 2 concerns random walks and Section 3 Brownian motion. For clarity of exposition we have given the same structure to Sections 2 and 3: in Section 2.1/3.1 we first state the key functional equation (a kernel equation), which is the starting point of our entire analysis. We study the kernel (a second degree polynomial in two variables). In Section 2.2/3.2 we state and solve the BVP satisfied by the generating functions. We then move to asymptotic results (Section 2.3/3.3). In Section 2.4/3.4 we introduce the Riemann surface of the model and some important related facts.

2 Random walks in the quarter plane

This section is devoted to the discrete case and is based mainly on Fayolle et al. (1999).

2.1 Functional equation

One considers a piecewise homogeneous random walk with sample paths in $\mathbb{N}^2$. There are four domains of spatial homogeneity (the interior of $\mathbb{N}^2$, the horizontal and vertical axes, the origin), inside of which the transition probabilities (of unit size) are denoted by $p_{i,j}$, $p'_{i,j}$, $p''_{i,j}$ and $p^0_{i,j}$, respectively. See Figure 1. The inventory polynomial of the inner domain is called the kernel and equals

$$K(x, y) = xy \{ \sum_{-1 \leq i, j \leq 1} p_{i,j} x^i y^j - 1 \}. \quad (1)$$

The inventory polynomials associated to the other homogeneity domains are

$$k(x, y) = x \{ \sum p'_{i,j} x^i y^j - 1 \}, \quad \tilde{k}(x, y) = y \{ \sum p''_{i,j} x^i y^j - 1 \}, \quad k_0(x, y) = \{ \sum p^0_{i,j} x^i y^j - 1 \}.$$

Assuming the random walk ergodic (we refer to (Fayolle et al., 1999, Theorem 1.2.1) for necessary and sufficient conditions), we denote the invariant measure by $\{\pi_{i,j}\}_{i,j \geq 0}$ and introduce the generating functions

$$\pi(x, y) = \sum_{i,j \geq 1} \pi_{i,j} x^{i-1} y^{j-1}, \quad \pi(x) = \sum_{i \geq 1} \pi_{i,0} x^{i-1}, \quad \tilde{\pi}(y) = \sum_{j \geq 1} \pi_{0,j} y^{j-1}.$$

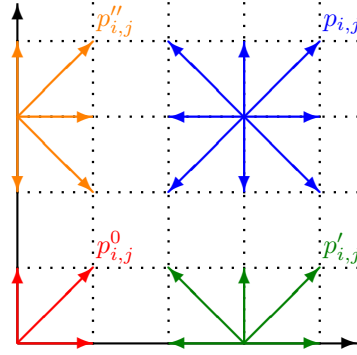


Fig. 1: Transition probabilities of the reflected random walk in the quarter plane, with four domains of spatial homogeneity

Writing the balance equations at the generating function level, we have (see (Fayolle et al., 1999, Equation (1.3.6)) for the original statement):

Lemma 1 *The fundamental functional equation holds*

$$-K(x, y)\pi(x, y) = k(x, y)\pi(x) + \tilde{k}(x, y)\tilde{\pi}(y) + k_0(x, y)\pi_{0,0}. \quad (2)$$

Equation (2) holds a priori in the region $\{(x, y) \in \mathbb{C}^2 : |x| \leq 1, |y| \leq 1\}$. Indeed, the $\pi_{i,j}$ sum up to 1, so that the generating functions $\pi(x, y)$, $\pi(x)$ and $\tilde{\pi}(y)$ are well defined on the (bi)disc. The identity (2) is a kernel equation, and a crucial role will be played by the kernel (1). This polynomial K is of second order in both x and y ; its roots $X(y)$ and $Y(x)$ defined by

$$K(X(y), y) = K(x, Y(x)) = 0 \quad (3)$$

are thus algebraic of degree 2. Writing the kernel as $K(x, y) = a(y)x^2 + b(y)x + c(y)$ and defining its discriminant $d(y) = b(y)^2 - 4a(y)c(y)$, one has obviously

$$X(y) = \frac{-b(y) \pm \sqrt{d(y)}}{2a(y)}.$$

The polynomial d has three or four roots, and exactly two of them are located in the unit disc, see (Fayolle et al., 1999, Lemma 2.3.8). They are named y_1, y_2 , cf. Figure 2. On (y_1, y_2) one has $d(y) < 0$, so that the two values (or branches) of $X(y)$ (that we shall call $X_0(y)$ and $X_1(y)$) are complex conjugate of one another. In particular, the set

$$\mathcal{M} = X([y_1, y_2]) = \{x \in \mathbb{C} : K(x, y) = 0 \text{ and } y \in [y_1, y_2]\}$$

is symmetrical w.r.t. the real axis (Figure 2). This curve will be used to set a boundary condition for the unknown function π (Lemma 2).

2.2 Statement and resolution of the BVP

The analytic approach of Malyšev (1972); Fayolle and Iasnogorodski (1979); Fayolle et al. (1999) proposes a way for solving the functional equation (2), by reduction to a BVP. Generally speaking, a BVP consists of a regularity condition and a boundary condition.

Lemma 2 *The function π satisfies the following BVP:*

- π is meromorphic in the bounded domain delimited by $\mathcal{M}$ and has there identified poles;
- for any $x \in \mathcal{M}$,

$$\frac{k(x, Y_0(x))}{\tilde{k}(x, Y_0(x))} \pi(x) - \frac{k(\bar{x}, Y_0(\bar{x}))}{\tilde{k}(\bar{x}, Y_0(\bar{x}))} \pi(\bar{x}) = \frac{k_0(\bar{x}, Y_0(\bar{x}))}{\tilde{k}(\bar{x}, Y_0(\bar{x}))} \pi_{0,0} - \frac{k_0(x, Y_0(x))}{\tilde{k}(x, Y_0(x))} \pi_{0,0}.$$

Proof: The regularity condition follows from Theorem 5, which provides a (maximal) meromorphic continuation of the function π . We now turn to the boundary condition. For $i \in \{0, 1\}$, we evaluate the functional equation (2) at $(X_i(y), y)$ and divide by $\tilde{k}(X_i(y), y)$. We then make the difference of the identities corresponding to $i = 0$ and $i = 1$. Finally, we substitute $X_0(y) = x$ and $X_1(y) = \bar{x}$, noting that when $y \in [y_1, y_2]$, $x \in \mathcal{M}$ by construction. Notice that we have chosen the segment $[y_1, y_2]$ connecting the points inside of the unit disc (Figure 2), in which we know that the generating function $\tilde{\pi}$ is well defined. $\square$

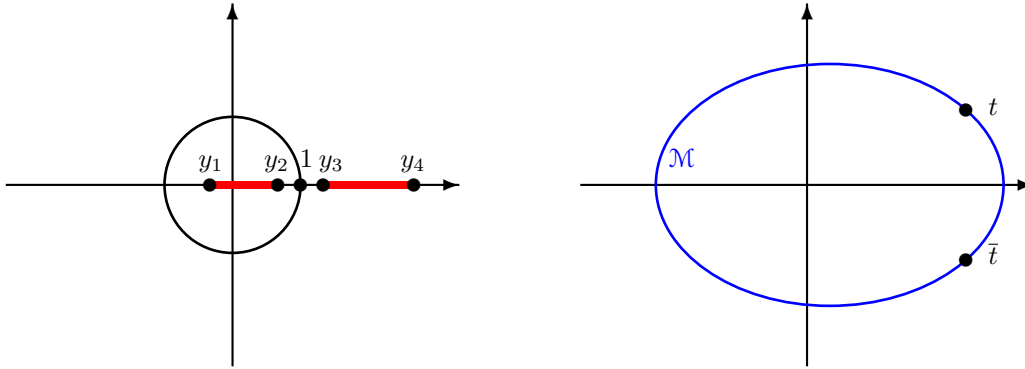


Fig. 2: Left: the polynomial d has three or four roots, denoted by y_1, y_2, y_3, y_4 ; exactly two of them are inside the unit disc. Right: the curve $\mathcal{M} = X([y_1, y_2])$ is symmetrical w.r.t. the horizontal axis

Although Lemma 2 could be written more precisely (by giving the number and the location of the poles of π), we shall prefer the above version, since we focus in this note on the main ideas of the analytic approach.

Lemma 2 happens to characterize the generating functions, as it eventually leads to an explicit expression for π , see Theorem 3. Before stating this central result (borrowed from (Fayolle et al., 1999, Theorem 5.4.3)), we need to introduce a function w called a conformal gluing function. By definition it satisfies $w(x) = w(\bar{x})$ for $x \in \mathcal{M}$ and is one-to-one inside of $\mathcal{M}$. This function will be constructed in Theorem 6 of Section 2.4.

Theorem 3 *There exist two functions f and g , constructed from k , $\tilde{k}$, k_0 and w , such that the following integral formulation for π holds:*

$$\pi(x) = f(x) \int_{\mathcal{M}} g(u) \frac{w'(u)}{w(u) - w(x)} du.$$

A similar contour integral representation exists for $\tilde{\pi}$, and eventually the functional equation (2) provides us with an explicit expression for the bivariate function $\pi(x, y)$.

2.3 Asymptotics of the stationary probabilities

The asymptotics of coefficients $\pi_{i,j}$ of unknown generating functions satisfying the functional equation (2) has been obtained by Malyšev (1973) via analytic arguments. He computed the asymptotics of the stationary probabilities $\pi_{i,j}$ as $i, j \rightarrow \infty$ and $j/i = \tan \alpha$, for any given $\alpha \in (0, \pi/2)$. Let us briefly present these results. It is assumed in Malyšev (1973) that the random walk is simple, meaning that

$$p_{-1,1} = p_{1,1} = p_{-1,-1} = p_{1,-1} = 0. \quad (4)$$

It is also assumed that both coordinates of the interior drift vector are negative (as in Figure 3). For $\alpha \in (0, \pi/2)$, we define the point $(x(\alpha), y(\alpha))$ as follows. Introducing as in Kurkova and Raschel (2011) the function $P(u, v) = \sum_{i,j} p_{i,j} e^{iu} e^{jv}$ on $\mathbb{R}^2$, the mapping

$$(u, v) \mapsto \frac{\nabla P(u, v)}{|P(u, v)|}$$

is a homeomorphism between $\{(u, v) \in \mathbb{R}^2 : P(u, v) = 1\}$ and the unit circle. The point $(u(\alpha), v(\alpha))$ is the unique solution to $\frac{\nabla P(u, v)}{|P(u, v)|} = (\cos \alpha, \sin \alpha)$. Finally, $(x(\alpha), y(\alpha)) = (e^{u(\alpha)}, e^{v(\alpha)})$.

Following Malyšev (1973), we introduce the sets of parameters

$$\begin{aligned} \mathcal{P}_{--} &= \{(\{p_{i,j}\}, \alpha) : k(\psi(x(\alpha), y(\alpha))) \leq 0 \text{ and } \tilde{k}(\phi(x(\alpha), y(\alpha))) \leq 0\}, \\ \mathcal{P}_{+-} &= \{(\{p_{i,j}\}, \alpha) : k(\psi(x(\alpha), y(\alpha))) > 0 \text{ and } \tilde{k}(\phi(x(\alpha), y(\alpha))) \leq 0\}, \end{aligned}$$

and $\mathcal{P}_{-+}$ and $\mathcal{P}_{++}$ accordingly. The automorphisms ψ and ϕ are defined in Section 2.4 by (8). The following theorem is proven in Malyšev (1973).

Theorem 4 *Let $(i, j) = (r \cos \alpha, r \sin \alpha)$ with $\alpha \in (0, \pi/2)$. Then as $r \rightarrow \infty$ we have*

$$\pi_{i,j} = (1 + o(1)) \cdot \begin{cases} \frac{C_0(\alpha)}{\sqrt{r}} x^{-i}(\alpha) y^{-j}(\alpha) & \text{in } \mathcal{P}_{--}, \\ C_1 p_1^{-i} q_1^{-j} & \text{in } \mathcal{P}_{-+}, \\ C_2 p_2^{-i} q_2^{-j} & \text{in } \mathcal{P}_{+-}, \\ C_1 p_1^{-i} q_1^{-j} + C_2 p_2^{-i} q_2^{-j} & \text{in } \mathcal{P}_{++}, \end{cases} \quad (5)$$

where C_0 , C_1 and C_2 are constants that can be expressed in terms of the functions π and $\tilde{\pi}$. The point (p_1, q_1) is a solution of the system $\{K(x, y) = 0, k(\psi(x, y)) = 0\}$ and similarly, (p_2, q_2) is a solution of $\{K(x, y) = 0, \tilde{k}(\phi(x, y)) = 0\}$.

Proof: The stationary probabilities $\pi_{i,j}$ are first written as two-dimensional Cauchy integrals, then reduced via the residue theorem to one-dimensional integrals along some contours. The asymptotics of these integrals is characterized either by the saddle point $(x(\alpha), y(\alpha))$ in the case of the set of parameters $\mathcal{P}_{--}$ or by a pole (p_1, q_1) or (p_2, q_2) that is encountered when moving the integration contour to the saddle point; this happens for the sets of parameters $\mathcal{P}_{-+}$, $\mathcal{P}_{+-}$ and $\mathcal{P}_{++}$. $\square$

This approach has been applied for the analysis of the join the shortest queue problem in Kurkova and Suhov (2003), and for computing the asymptotics of Green functions for transient random walks in the quarter plane reflected at the axes (see Kurkova and Malyshev (1998)) or killed at the axes (cf. Kurkova and Raschel (2011)). Moreover, as illustrated in Kurkova and Raschel (2011); Kurkova and Suhov (2003), the assumption (4) is not crucial for the applicability of the method. The limiting cases $\alpha = 0$ and $\alpha = \pi/2$ can also be treated via this approach, with some additional technical details (the saddle point then coincides with a branch point of the integrand), it is done in Kurkova and Raschel (2011).

2.4 Riemann surface and related facts

In Section 2.1 the set

$$\mathcal{K} = \{(x, y) \in \mathbb{C}^2 : K(x, y) = 0\} = \{(x, y) \in \mathbb{C}^2 : \sum p_{i,j} x^i y^j = 1\}$$

has appeared very naturally, since in order to state the BVP (our Lemma 2), we introduced the functions $X(y)$ and $Y(x)$, which by construction cancel the kernel, see (3).

In this section the central idea is to consider the (global) complex structure of $\mathcal{K}$. The set $\mathcal{K}$ turns out to be a Riemann surface of genus 1, i.e., a torus. This simply comes from the reformulation of the identity $K(x, y) = 0$ as

$$\{2a(y)x + b(y)\}^2 = d(y).$$

Moreover, the Riemann surface of the square root of a polynomial of degree 3 or 4 is classically a torus (with this terminology, the roots of the discriminant are branch points).

This new point of view on $\mathcal{K}$ brings powerful tools. Of particular interest is a parametrization of $\mathcal{K}$ in terms of Weierstrass elliptic functions:

$$\mathcal{K} = \{(x(\omega), y(\omega)) : \omega \in \mathbb{C}/(\omega_1\mathbb{Z} + \omega_2\mathbb{Z})\}. \quad (6)$$

This parametrization is totally explicit: $x(\omega)$ and $y(\omega)$ are rational functions in the $\wp$ -Weierstrass function and its derivative $\wp'$ (see (Fayolle et al., 1999, Lemma 3.3.1)); the periods ω_1 and ω_2 admit expressions as elliptic integrals in terms of $\{p_{i,j}\}$ (cf. (Fayolle et al., 1999, Lemma 3.3.2)), etc. Moreover, as any functions of x and/or y , the functions $\pi(x)$ and $\tilde{\pi}(y)$ can be lifted on $\mathcal{K}$ by setting

$$\Pi(\omega) = \pi(x(\omega)), \quad \tilde{\Pi}(\omega) = \tilde{\pi}(y(\omega)). \quad (7)$$

Group of the walk

Introduced in Malyšev (1972) in a probabilistic context and further used in Fayolle et al. (1999); Bousquet-Mélou and Mishna (2010), the group of the walk is a dihedral group generated by

$$\zeta(x, y) = \left(x, \frac{\sum_i p_{i,-1} x^i}{\sum_i p_{i,+1} x^i} \frac{1}{y} \right), \quad \eta(x, y) = \left(\frac{\sum_j p_{-1,j} y^j}{\sum_j p_{+1,j} y^j} \frac{1}{x}, y \right). \quad (8)$$

(One easily verifies that these generators are idempotent: $\zeta^2 = \eta^2 = 1$.) The group $\langle \zeta, \eta \rangle$ can be finite or infinite, according to the order of the element $\zeta \circ \eta$. The generator ζ (resp. η) exchanges the roots in y (resp. in x) of $K(x, y) = 0$. Viewed as a group of birational transformations in Bousquet-Mélou and Mishna (2010), we shall rather see it as a group of automorphisms of the Riemann surface $\mathcal{K}$.

This group has many applications. First, it allows for a continuation of the functions $\pi(x)$ and $\tilde{\pi}(y)$ (Theorem 5 below). It further connects the algebraic nature of the generating functions to the (in)finiteness of the group (Theorem 7). Finally, in the finite group case, elementary algebraic manipulations of the functional equations can be performed (typically, via the computation of certain orbit-sums) so as to eventually obtain D-finite expressions for the unknowns, see (Fayolle et al., 1999, Chapter 4) and Bousquet-Mélou and Mishna (2010).

Using the structure of the automorphisms of a torus, the lifted versions of ζ and η admit simple expressions (Fayolle et al., 1999, Section 3.1.2):

$$\zeta(\omega) = -\omega + \omega_1 + \omega_2, \quad \eta(\omega) = -\omega + \omega_1 + \omega_2 + \omega_3, \quad (9)$$

where, as the periods ω_1 and $\omega_2, \omega_3 \in (0, \omega_2)$ is an elliptic integral (Fayolle et al., 1999, Lemma 3.3.3). Accordingly, the group is finite if and only if $\omega_2/\omega_3 \in \mathbb{Q}$, which provides a nice criterion in terms of elliptic integrals.

Continuation

While the generating function $\pi(x)$ is defined through its power series in the unit disc, it is a priori unclear how to continue it to a larger domain. This is however crucial, since the curve $\mathcal{M}$ on which it satisfies a BVP (Lemma 2) is not included in the unit disc in general.

Theorem 5 *The function π can be continued as a meromorphic function to $\mathbb{C} \setminus [x_3, x_4]$.*

We notice that $\mathcal{M}$ does not intersect $[x_3, x_4]$ by (Fayolle et al., 1999, Theorem 5.3.3), so that Theorem 5 indeed provides a continuation of the generating function in the domain delimited by $\mathcal{M}$.

Proof: This result, stated as Theorem 3.2.3 in Fayolle et al. (1999), is a consequence of a continuation of the lifted generating functions (7) on the Riemann surface (or, better, on its universal covering — but we shall not go into these details here). The continuation on $\mathcal{K}$ uses the (lifted) functional equation (2) and the group of the walk $\langle \zeta, \eta \rangle$. $\square$

Conformal mapping

In the integral expression of Theorem 3, the conformal gluing function w is all-present, as it appears in the integrand and in f and g as well. The introduction of the Riemann surface $\mathcal{K}$ allows to derive an expression for this function (this is another major interest of introducing $\mathcal{K}$). Let us recall that ω_1 and ω_2 are the periods of the elliptic functions of the parametrization (6), while ω_3 comes out in the lifted expression (9) of the automorphisms.

Theorem 6 *The conformal gluing function w admits the expression:*

$$w(x) = \wp(\wp^{-1}(x; \omega_1, \omega_2); \omega_1, \omega_3),$$

where for $i \in \{2, 3\}$, $\wp(\cdot; \omega_1, \omega_i)$ is the $\wp$ -Weierstrass elliptic function with periods ω_1 and ω_i .

Proof: While on the complex plane, $\mathcal{M}$ is a quartic curve, it becomes on $\mathcal{K}$ much simpler (typically, a segment). This remark (which again illustrates all the benefit of having introduced the Riemann surface) is used in (Fayolle et al., 1999, Section 5.5.2) so as to obtain the above expression for w . $\square$

Algebraic nature of the generating functions

Recall that a function of one variable is D-finite if it satisfies a linear differential equation with polynomial coefficients.

Theorem 7 *If the group is finite, the generating functions $\pi(x)$ and $\tilde{\pi}(y)$ are D-finite.*

Proof: This follows from manipulations on the Riemann surface, see (Fayolle et al., 1999, Chapter 4). The D-finiteness is proved on $\mathbb{R}$; refined results (in the combinatorial context of the enumeration of paths) can be found in Bousquet-Mélou and Mishna (2010), where the D-finiteness is proved on $\mathbb{Q}$. $\square$

The converse of Theorem 7 is not shown in full generality. It is true in combinatorics, see Kurkova and Raschel (2012).

3 Reflected Brownian motion in the quadrant

Defining reflected Brownian motion in the quadrant

The object of study here is the reflected Brownian motion with drift in the quarter plane $\mathbb{R}_+^2$

$$Z(t) = Z_0 + W(t) + \mu t + RL(t), \quad \forall t \geq 0, \quad (10)$$

associated to the triplet (Σ, μ, R) , composed of a non-singular covariance matrix, a drift and a reflection matrix, see Figure 3:

$$\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{21} \\ \sigma_{12} & \sigma_{22} \end{pmatrix}, \quad \mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \quad R = (R^1, R^2) = \begin{pmatrix} r_{11} & r_{21} \\ r_{12} & r_{22} \end{pmatrix}.$$

In Equation (10), Z_0 is any initial point in $\mathbb{R}_+^2$, the process $(W(t))_{t \geq 0}$ is an unconstrained planar Brownian motion starting from the origin, and for $i \in \{1, 2\}$, $L^i(t)$ is a continuous non-decreasing process, that increases only at time t such that $Z^i(t) = 0$, namely $\int_0^t \mathbb{1}_{\{Z^i(s) \neq 0\}} dL^i(s) = 0$. The columns R_1 and R_2 represent the directions in which the Brownian motion is pushed when the axes are reached.

The reflected Brownian motion $(Z(t))_{t \geq 0}$ associated with (Σ, μ, R) is well defined, see for instance Williams (1995). Its stationary distribution exists and is unique if and only if the following (geometric flavored) conditions are satisfied (see, e.g., Harrison and Williams (1987a); Hobson and Rogers (1993))

$$r_{11} > 0, \quad r_{22} > 0, \quad r_{11}r_{22} - r_{12}r_{21} > 0, \quad r_{22}\mu_1 - r_{12}\mu_2 < 0, \quad r_{11}\mu_2 - r_{21}\mu_1 < 0. \quad (11)$$

More that the Brownian motion in the quadrant, all results presented below concern the Brownian motion in two-dimensional cones (by a simple linear transformation of the cones). This is a major difference and interest of the continuous case, which also illustrates that the analytic approach is very well suited to that context.

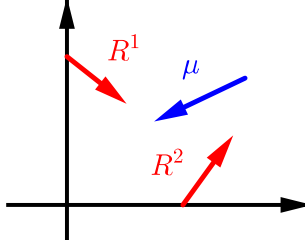


Fig. 3: Drift μ and reflection vectors R^1 and R^2

3.1 Functional equation

Laplace transform of the stationary distribution

The continuous analogues of the generating functions are the Laplace transforms. As their discrete counterparts, they characterize the stationary distribution. Under assumption (11), that we shall do throughout the manuscript, the stationary distribution is absolutely continuous w.r.t. the Lebesgue measure, see Harrison and Williams (1987a); Dai (1990). We denote its density by $\pi(x) = \pi(x_1, x_2)$. Let the Laplace transform of π be defined by

$$\varphi(\theta) = \mathbb{E}_\pi[e^{\langle \theta | Z \rangle}] = \iint_{\mathbb{R}_+^2} e^{\langle \theta | x \rangle} \pi(x) dx.$$

We further define two finite boundary measures ν_1 and ν_2 with support on the axes, by mean of the formula

$$\nu_i(B) = \mathbb{E}_\pi \left[\int_0^1 \mathbb{1}_{\{Z(t) \in B\}} dL^i(t) \right].$$

The measures ν_i are continuous w.r.t. the Lebesgue measure by Harrison and Williams (1987a), and may be viewed as boundary invariant measures. We define their moment Laplace transform by

$$\varphi_2(\theta_1) = \int_{\mathbb{R}_+} e^{\theta_1 x_1} \nu_2(x_1) dx_1, \quad \varphi_1(\theta_2) = \int_{\mathbb{R}_+} e^{\theta_2 x_2} \nu_1(x_2) dx_2.$$

Functional equation

There is a functional equation between the Laplace transforms φ , φ_1 and φ_2 , see (12), which is reminiscent of the discrete functional equation (2).

Lemma 8 *The following key functional equation between the Laplace transforms holds*

$$-\gamma(\theta)\varphi(\theta) = \gamma_1(\theta)\varphi_1(\theta_2) + \gamma_2(\theta)\varphi_2(\theta_1), \quad (12)$$

where

$$\begin{cases} \gamma(\theta) = \frac{1}{2} \langle \theta | \sigma \theta \rangle + \langle \theta | \mu \rangle = \frac{1}{2} (\sigma_{11} \theta_1^2 + \sigma_{22} \theta_2^2 + 2\sigma_{12} \theta_1 \theta_2) + \mu_1 \theta_1 + \mu_2 \theta_2, \\ \gamma_1(\theta) = \langle R^1 | \theta \rangle = r_{11} \theta_1 + r_{21} \theta_2, \\ \gamma_2(\theta) = \langle R^2 | \theta \rangle = r_{12} \theta_1 + r_{22} \theta_2. \end{cases}$$

By definition of the Laplace transforms, this equation holds at least for any $\theta = (\theta_1, \theta_2)$ with $\Re \theta_1 \leq 0$ and $\Re \theta_2 \leq 0$. The polynomial γ in (12) is the kernel and is the continuous analogue of the kernel (1) in the discrete case. Polynomials γ_1 and γ_2 are the counterparts of k and $\tilde{k}$.

Proof: To show (12), the main idea is to use an identity called a basic adjoint relationship (first proved in Harrison and Williams (1987a) in some particular cases, then extended in Dai and Harrison (1992)), which characterizes the stationary distribution. (It is the continuous analogue of the well-known equation $\pi Q = 0$, where π is the stationary distribution of a recurrent continuous-time Markov chain with infinitesimal generator Q .) This basic adjoint relationship connects the stationary distribution π and the corresponding boundary measures ν_1 and ν_2 . We refer to Foddy (1984); Dai and Miyazawa (2011) for the details. $\square$

Elementary properties of the kernel

The kernel γ in (12) can be alternatively written as

$$\gamma(\theta_1, \theta_2) = \tilde{a}(\theta_2)\theta_1^2 + \tilde{b}(\theta_2)\theta_1 + \tilde{c}(\theta_2) = a(\theta_1)\theta_2^2 + b(\theta_1)\theta_2 + c(\theta_1). \quad (13)$$

The equation $\gamma(\theta_1, \theta_2) = 0$ defines a two-valued algebraic function $\Theta_1(\theta_2)$ by $\gamma(\Theta_1(\theta_2), \theta_2) = 0$, and similarly $\Theta_2(\theta_1)$ such that $\gamma(\theta_1, \Theta_2(\theta_1)) = 0$. Expressions of their branches are given by

$$\Theta_2^\pm(\theta_1) = \frac{-b(\theta_1) \pm \sqrt{d(\theta_1)}}{2a(\theta_1)},$$

where $d(\theta_1) = b^2(\theta_1) - 4a(\theta_1)c(\theta_1)$ is the discriminant. The polynomial d has two zeros, real and of opposite signs; they are denoted by $\theta_1^\pm$ and are branch points of the algebraic function Θ_2 . In the same way we define $\Theta_1^\pm$ and its branch points $\theta_2^\pm$.

Finally, notice that d is negative on $\mathbb{R} \setminus [\theta_1^-, \theta_1^+]$. Accordingly, the branches $\Theta_2^\pm$ take complex conjugate values on this set.

3.2 Statement and resolution of the BVP

An important hyperbola

For further use, we need to introduce the curve

$$\mathcal{R} = \{\theta_2 \in \mathbb{C} : \gamma(\theta_1, \theta_2) = 0 \text{ and } \theta_1 \in (-\infty, \theta_1^-)\} = \Theta_2^\pm((-\infty, \theta_1^-)). \quad (14)$$

It is the analogue of the curve $\mathcal{M}$ in Section 2.1. The curve $\mathcal{R}$ is symmetrical w.r.t. the real axis, see Figure 4 (this is a consequence of d being negative on $(-\infty, \theta_1^-)$, see above). Furthermore, it is a (branch of a) hyperbola by Baccelli and Fayolle (1987). We shall denote by $\mathcal{G}_{\mathcal{R}}$ the open domain of $\mathbb{C}$ bounded by $\mathcal{R}$ and containing 0, see Figure 4. Obviously $\overline{\mathcal{G}_{\mathcal{R}}}$, the closure of $\mathcal{G}_{\mathcal{R}}$, is equal to $\mathcal{G}_{\mathcal{R}} \cup \mathcal{R}$.

BVP for orthogonal reflections

In the case of orthogonal reflections (see Figure 3), R is the identity matrix in (10), and we have $\gamma_1(\theta_1, \theta_2) = \theta_1$ and $\gamma_2(\theta_1, \theta_2) = \theta_2$. We set

$$\psi_1(\theta_2) = \frac{1}{\theta_2} \varphi_1(\theta_2), \quad \psi_2(\theta_1) = \frac{1}{\theta_1} \varphi_2(\theta_1). \quad (15)$$

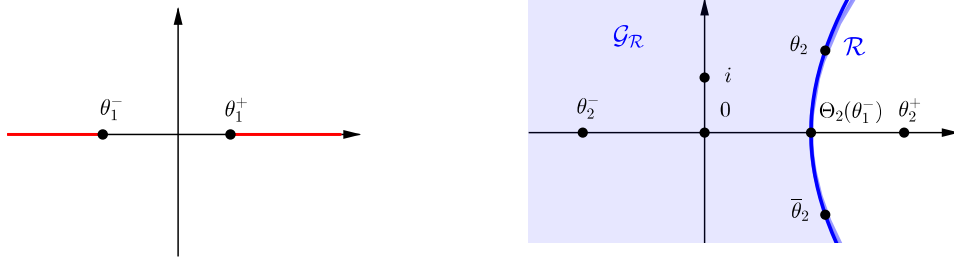


Fig. 4: Left: the discriminant d has two roots θ_1^- and θ_1^+ of opposite signs. Right: the curve $\mathcal{R}$ in (14) is symmetric w.r.t. the horizontal axis and $\mathcal{G}_{\mathcal{R}}$ is the domain in blue

Lemma 9 *The function ψ_1 in (15) satisfies the following BVP:*

- (i) ψ_1 is meromorphic on $\mathcal{G}_{\mathcal{R}}$ with a single pole at 0, of order 1 and residue $\varphi_1(0)$, and vanishes at infinity;
- (ii) ψ_1 is continuous on $\overline{\mathcal{G}_{\mathcal{R}}} \setminus \{0\}$ and

$$\psi_1(\overline{\theta_2}) = \psi_1(\theta_2), \quad \forall \theta_2 \in \mathcal{R}. \quad (16)$$

Proof: The regularity condition of point (i) follows from Theorem 15, which provides a (maximal) meromorphic continuation of the function. Let us now consider (ii). Evaluating the (continued) functional equation (12) at $(\theta_1, \Theta_2^\pm(\theta_1))$, we obtain $\psi_1(\Theta_2^\pm(\theta_1)) + \psi_2(\theta_1) = 0$, which immediately implies that

$$\psi_1(\Theta_2^+(\theta_1)) = \psi_1(\Theta_2^-(\theta_1)). \quad (17)$$

Choosing $\theta_1 \in (-\infty, \theta_1^-)$, the two quantities $\Theta_2^+(\theta_1)$ and $\Theta_2^-(\theta_1)$ are complex conjugate the one of the other, see Section 3.1. Equation (17) can then be reformulated as (16), using the definition (14) of the curve $\mathcal{R}$. $\square$

The BVP stated in Lemma 9 is called a homogeneous BVP with shift (the shift stands here for the complex conjugation, but the theory applies to more general shifts, see Litvinchuk (2000)). It has a simpler form than the BVP in Lemma 2 for the discrete case, because there is no inhomogeneous term (as $\pi_{0,0}$) and also because in the coefficients in front of the unknowns there is no algebraic function (as Y_0) involved. Due to its particularly simple form, we can solve it in an explicit way, using the two following steps:

- Using a certain conformal mapping w (to be introduced below), we can construct a particular solution to the BVP of Lemma 9.
- The solution to the BVP of Lemma 9 is unique (see the invariant Lemma 2 in (Litvinchuk, 2000, Section 10.2)). In other words, two different solutions must coincide, and the explicit solution constructed above must be the function ψ_1 .

In Franceschi and Raschel (2016) it is explained that the above method may be viewed as a variation of Tutte's invariant approach, first introduced by Tutte for solving a functional equation arising in the enumeration of properly colored triangulations, see Tutte (1995).

The function w glues together the upper and lower parts of the hyperbola $\mathcal{R}$. There are at least two ways to find such a w . First, it turns out that in the literature there exist expressions for conformal gluing functions for relatively simple curves as hyperbolas, see (Baccelli and Fayolle, 1987, Equation (4.6)). Here (based on Franceschi and Raschel (2016)), we use instead the Riemann sphere $\mathbb{S}$, as we will see in Section 3.4. Indeed, many technical aspects (and in particular finding the conformal mapping) happen to be quite simpler on that surface.

We will deduce from Section 3.4 that function w can be expressed in terms of the generalized Chebyshev polynomial

$$T_a(x) = \cos(a \arccos(x)) = \frac{1}{2} \left\{ (x + \sqrt{x^2 - 1})^a + (x - \sqrt{x^2 - 1})^a \right\}$$

as follows:

$$w(\theta_2) = T_{\frac{\pi}{\beta}} \left(-\frac{2\theta_2 - (\theta_2^+ + \theta_2^-)}{\theta_2^+ - \theta_2^-} \right), \quad (18)$$

where we have noted

$$\beta = \arccos -\frac{\sigma_{12}}{\sqrt{\sigma_{11}\sigma_{22}}}. \quad (19)$$

In the case of orthogonal reflection, this method leads to the main result of Franceschi and Raschel (2016), which is:

Theorem 10 *Let R be the identity matrix in (10). The Laplace transform φ_1 is equal to*

$$\varphi_1(\theta_2) = \frac{-\mu_1 w'(0)}{w(\theta_2) - w(0)} \theta_2.$$

Statement of the BVP in the general case

We would like to close Section 3.2 by stating the BVP in the case of arbitrary reflections (non-necessarily orthogonal). Let us define for $\theta_2 \in \mathcal{R}$

$$G(\theta_2) = \frac{\gamma_1}{\gamma_2} (\Theta_1^-(\theta_2), \theta_2) \frac{\gamma_2}{\gamma_1} (\Theta_1^-(\theta_2), \overline{\theta_2}).$$

Similarly to Lemma 9, there is the following result:

Lemma 11 *The function φ_1 satisfies the following BVP:*

- (i) φ_1 is meromorphic on $\mathcal{G}_{\mathcal{R}}$ with at most one pole p of order 1 and is bounded at infinity;
- (ii) φ_1 is continuous on $\overline{\mathcal{G}_{\mathcal{R}}} \setminus \{p\}$ and

$$\varphi_1(\overline{\theta_2}) = G(\theta_2) \varphi_1(\theta_2), \quad \forall \theta_2 \in \mathcal{R}. \quad (20)$$

Due to the presence of the function $G \neq 1$ in (20), this BVP (still homogeneous with shift) is more complicated than the one encountered in Lemma 9 and cannot be solved thanks to an invariant lemma. Instead, the resolution is less combinatorial and far more technical, and the solution should be expressed in terms of both Cauchy integrals and the conformal mapping w of Theorem 10. This will be achieved in a future work.

3.3 Asymptotics of the stationary probabilities

Overview

Let Π be a random vector that has the stationary distribution of the reflected Brownian motion. Dai and Miyazawa (2011) obtain the following asymptotic result: for a given directional vector $c \in \mathbb{R}_+^2$ they find (up to a multiplicative constant) a function $f_c(x)$ such that

$$\lim_{x \rightarrow \infty} \frac{\mathbb{P}[\langle c | \Pi \rangle \geq x]}{f_c(x)} = 1.$$

In Franceschi and Kurkova (2016) we solve a harder problem arisen in (Dai and Miyazawa, 2011, §8), namely computing the asymptotics of $\mathbb{P}[\Pi \in xc + B]$ as $x \rightarrow \infty$, where $c \in \mathbb{R}_+^2$ is any directional vector and $B \subset \mathbb{R}_+^2$ any compact subset. Furthermore, we are able to find the full asymptotic expansion of the density $\pi(x_1, x_2)$ of Π as $x_1, x_2 \rightarrow \infty$ and $x_2/x_1 \rightarrow \tan(\alpha)$, for any given angle $\alpha \in (0, \pi/2)$.

Main results

First we need to introduce some notations. The equation $\gamma(\theta) = 0$ determines an ellipse $\mathcal{E}$ on $\mathbb{R}^2$ passing through the origin, see Figure 5. Here we restrict ourselves to the case $\mu_1 < 0$ and $\mu_2 < 0$, although our methods can be applied without additional difficulty to other cases. For a given angle $\alpha \in [0, \pi/2]$, let us

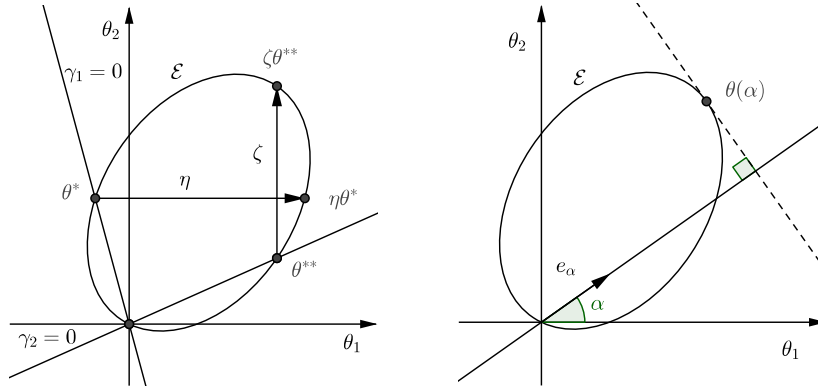


Fig. 5: Left: representation of the ellipse $\mathcal{E}$, straight lines $\{\gamma_1(\theta) = 0\}$, $\{\gamma_2(\theta) = 0\}$, and points θ^* , θ^{**} , $\eta\theta^*$ and $\zeta\theta^{**}$. Right: geometric interpretation of the point $\theta(\alpha)$ in (21) on $\mathcal{E}$

define the point $\theta(\alpha)$ on the ellipse $\mathcal{E}$ by

$$\theta(\alpha) = \operatorname{argmax}_{\theta \in \mathcal{E}} \langle \theta | e_\alpha \rangle, \quad \text{where } e_\alpha = (\cos \alpha, \sin \alpha). \quad (21)$$

The coordinates of $\theta(\alpha)$ can be given explicitly. One can also construct $\theta(\alpha)$ geometrically as on Figure 5.

Secondly, consider the straight lines $\{\gamma_1(\theta) = 0\}$ and $\{\gamma_2(\theta) = 0\}$, depending on the reflection matrix R only. They cross the ellipse $\mathcal{E}$ at the origin. The line $\{\gamma_1(\theta) = 0\}$ (resp. $\{\gamma_2(\theta) = 0\}$) intersects the

ellipse at a second point called θ^* (resp. θ^{**}). To present our results, we need to define the images on $\mathcal{E}$ of these points via the so-called Galois automorphisms ζ and η , to be introduced in Section 3.4. Namely, for the point $\theta^* = (\theta_1^*, \theta_2^*) \in \mathcal{E}$ there exists a unique point $\eta\theta^* = (\eta\theta_1^*, \theta_2^*) \in \mathcal{E}$ with the same second coordinate. Likewise, there exists a unique point $\zeta\theta^{**} = (\theta_1^{**}, \zeta\theta_2^{**}) \in \mathcal{E}$ with the same first coordinate as $\theta^{**} = (\theta_1^{**}, \theta_2^{**}) \in \mathcal{E}$. Points θ^* , θ^{**} , $\eta\theta^*$ and $\zeta\theta^{**}$ are pictured on Figure 5. Their coordinates can be made explicit.

Similarly to the discrete case, we introduce the set of parameters

$$\mathcal{Q}_{--} = \{((\Sigma, \mu, R), \alpha) : \gamma_1(\eta\theta(\alpha)) < 0 \text{ and } \gamma_2(\zeta\theta(\alpha)) < 0\}$$

and $\mathcal{Q}_{+-}$, $\mathcal{Q}_{-+}$ and $\mathcal{Q}_{++}$ accordingly. The following theorem provides the main term in the asymptotic expansion of $\pi(r \cos \alpha, r \sin \alpha)$.

Theorem 12 *Let $(x, y) = (r \cos \alpha, r \sin \alpha)$ with $\alpha \in (0, \pi/2)$. We assume that $\theta(\alpha) \in \mathbb{R}_+^2$. Then as $r \rightarrow \infty$ we have*

$$\pi(re_\alpha) = (1 + o(1)) \cdot \begin{cases} \frac{C_0}{\sqrt{r}} e^{-r\langle e_\alpha | \theta(\alpha) \rangle} & \text{in } \mathcal{Q}_{--}, \\ C_1 e^{-r\langle e_\alpha | \eta\theta^* \rangle} & \text{in } \mathcal{Q}_{+-}, \\ C_2 e^{-r\langle e_\alpha | \zeta\theta^{**} \rangle} & \text{in } \mathcal{Q}_{-+}, \\ C_1 e^{-r\langle e_\alpha | \eta\theta^* \rangle} + C_2 e^{-r\langle e_\alpha | \zeta\theta^{**} \rangle} & \text{in } \mathcal{Q}_{++}, \end{cases} \quad (22)$$

where C_0 , C_1 and C_2 are constants that can be expressed in terms of functions φ_1 and φ_2 and the parameters.

In Franceschi and Kurkova (2016) the constants mentioned in Theorem 12 are specified in terms of functions φ_1 and φ_2 . But these functions are for now unknown. As we explained in Section 3.2, in a next work we are going to obtain φ_1 and φ_2 as solutions of BVP, thereby determining the constants in Theorem 12.

Proof of the key step of Theorem 12: Theorem 12 is proven in Franceschi and Kurkova (2016). The first step consists in continuing meromorphically the functions φ_1 and φ_2 on $\mathbb{C} \setminus [\theta_2^+, \infty)$ or on the Riemann surface $\mathcal{S}$, see Section 3.4. Then by the functional equation (12) and the inversion formula of Laplace transform (we refer to Doetsch (1974) and Brychkov et al. (1992)), the density $\pi(x_1, x_2)$ can be represented as a double integral. Using standard computations from complex analysis, we are able to reduce it to a sum of single integrals. We obtain the following (with the notation (13)):

$$\begin{aligned} \pi(x_1, x_2) &= \frac{-1}{(2\pi i)^2} \int_{-i\infty}^{i\infty} \int_{-i\infty}^{i\infty} e^{-x_1\theta_1 - x_2\theta_2} \frac{\gamma_1(\theta)\varphi_1(\theta_2) + \gamma_2(\theta)\varphi_2(\theta_1)}{\gamma(\theta)} d\theta_1 d\theta_2 \\ &= \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \varphi_2(\theta_1) \gamma_2(\theta_1, \Theta_2^+(\theta_1)) e^{-x_1\theta_1 - x_2\Theta_2^+(\theta_1)} \frac{d\theta_1}{\sqrt{d(\theta_1)}} \\ &\quad + \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \varphi_1(\theta_2) \gamma_1(\Theta_1^+(\theta_2), \theta_2) e^{-x_1\Theta_1^+(\theta_2) - x_2\theta_2} \frac{d\theta_2}{\sqrt{\tilde{d}(\theta_2)}}. \end{aligned}$$

These integrals are typical to apply the saddle point method, see Fedoryuk (1986). The coordinates of the saddle point are the critical points of the functions

$$\cos(\alpha)\theta_1 + \sin(\alpha)\Theta_2^+(\theta_1) \quad \text{and} \quad \cos(\alpha)\Theta_1^+(\theta_2) + \sin(\alpha)\theta_2.$$

It is the point $\theta(\alpha)$. Then we have to shift the integration contour up to new contours which coincide with the steepest-descent contour near the saddle point. When we shift the contours we have to take into account the poles of the integrands and their residues. The asymptotics will be determined by the pole if we cross a pole when we shift the contour and by the saddle point otherwise. $\square$

3.4 Riemann surface and related facts

Riemann surface

The Riemann surface

$$\mathcal{S} = \{(\theta_1, \theta_2) \in \mathbb{C}^2 : \gamma(\theta_1, \theta_2) = 0\}$$

may be viewed as the set of zeros of the kernel (equivalently, it is the Riemann surface of the algebraic functions Θ_2 and Θ_1). Due to the degree of γ , the surface $\mathcal{S}$ has genus 0 and is a Riemann sphere, i.e., homeomorphic to $\mathbb{C} \cup \{\infty\}$, see Franceschi and Kurkova (2016). It admits a very useful rational parametrization, given by

$$\theta_1(s) = \frac{\theta_1^+ + \theta_1^-}{2} + \frac{\theta_1^+ - \theta_1^-}{4} \left(s + \frac{1}{s} \right), \quad \theta_2(s) = \frac{\theta_2^+ + \theta_2^-}{2} + \frac{\theta_2^+ - \theta_2^-}{4} \left(\frac{s}{e^{i\beta}} + \frac{e^{i\beta}}{s} \right), \quad (23)$$

with β as in (19). The equation $\gamma(\theta_1(s), \theta_2(s)) = 0$ holds and $\mathcal{S} = \{(\theta_1(s), \theta_2(s)) : s \in \mathbb{C} \cup \{\infty\}\}$.

Group of the process

We finally introduce the notion of group of the model, similar to the notion of group of the walk in the discrete setting (see Malyšev (1972); Fayolle et al. (1999); Bousquet-Mélou and Mishna (2010)). This group $\langle \zeta, \eta \rangle$ is generated by ζ and η , given by (with the notation (13))

$$\zeta(\theta_1, \theta_2) = \left(\theta_1, \frac{c(\theta_1)}{a(\theta_1)} \frac{1}{\theta_2} \right), \quad \eta(\theta_1, \theta_2) = \left(\frac{\tilde{c}(\theta_2)}{\tilde{a}(\theta_2)} \frac{1}{\theta_1}, \theta_2 \right).$$

By construction, the generators satisfy $\gamma(\zeta(\theta_1, \theta_2)) = \gamma(\eta(\theta_1, \theta_2)) = 0$ as soon as $\gamma(\theta_1, \theta_2) = 0$. In other words, there are (covering) automorphisms of the surface $\mathcal{S}$. Since $\zeta^2 = \eta^2 = 1$, the group $\langle \zeta, \eta \rangle$ is a dihedral group, which is finite if and only if the element $\zeta \circ \eta$ (or equivalently $\eta \circ \zeta$) has finite order.

Algebraic nature of the Laplace transforms

With the above definition, it is not clear how to see if the group is finite, nor to see if its finiteness would have any implication on the problem. In fact, we have, with β defined in (19):

Lemma 13 *The group $\langle \zeta, \eta \rangle$ is finite if and only if $\pi/\beta \in \mathbb{Q}$.*

The proof of Lemma 13 is simple, once the elements ζ and η have been lifted and reformulated on the sphere $\mathcal{S}$:

$$\zeta(s) = \frac{1}{s}, \quad \eta(s) = \frac{e^{2i\beta}}{s}.$$

These transformations leave invariant $\theta_1(s)$ and $\theta_2(s)$, respectively, see (23). In particular, we have the following result (see Franceschi and Raschel (2016)), which connects the nature of the solution of the BVP to the finiteness of the group. Such a result holds for discrete walks, see our Theorem 7 and Bousquet-Mélou and Mishna (2010); Bernardi et al. (2015).

Theorem 14 *The solution φ_1 given in Theorem 10 and the conformal gluing function w in (18) are algebraic if and only if the group $\langle \zeta, \eta \rangle$ is finite.*

Conformal mapping

The conformal gluing function w introduced in Section 3.2 may be lifted on $\mathcal{S}$. In fact its expression is even simpler using the parametrization of $\mathcal{S}$. We show in Franceschi and Raschel (2016) that

$$w(\theta_2(s)) = -\frac{i}{2} \{ (-s)^{\frac{\pi}{\beta}} + (-s)^{-\frac{\pi}{\beta}} \} = -\frac{i}{2} \{ e^{\frac{\pi}{\beta} \log(-s)} + e^{-\frac{\pi}{\beta} \log(-s)} \}, \quad (24)$$

where we make use of the principal determination of the logarithm.

Continuation of the Laplace transforms

To establish the BVP, we have stated a boundary condition for the functions φ_1 and φ_2 , on curves which lie outside their natural domains of definition (the half-plane with negative real-part), see Figure 4. In the same way, in the asymptotic study we use the steepest descent method on some curves outside of the initial domain of definition. We therefore need to extend the domain of definition of the Laplace transforms.

Theorem 15 *The function φ_1 can be continued meromorphically on the cut plane $\mathbb{C} \setminus [\theta_2^+, \infty)$.*

Proof: The first step is to continue meromorphically $\varphi_1(\theta_2)$ to the open and simply connected set $\{\theta_2 \in \mathbb{C} : \Re \theta_2 \leq 0 \text{ or } \Re \Theta_1^-(\theta_2) < 0\}$, by setting

$$\varphi_1(\theta_2) = \frac{\gamma_2}{\gamma_1} (\Theta_1^-(\theta_2), \theta_2) \varphi_2(\Theta_1^-(\theta_2)).$$

This is immediate (see Franceschi and Kurkova (2016) for the details). It is then possible to pursue the extension to the whole $\mathcal{S}$ using the invariance properties by the automorphisms ζ and η satisfied by the lifted Laplace transforms on $\mathcal{S}$. $\square$

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