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ON A NONLINEAR CRITICAL ELLIPTIC EQUATION WITH A PERTURBED HARDY POTENTIAL

HUSSEIN CHEIKH ALI

ABSTRACT. In this work, we use a minimisation argument to show the existence of a nontrivial solution to

$$
\begin{cases}
-\Delta u - \frac{a(x)}{|x|^2} u = u^{2^*-1} & \text{in } \Omega \\
u \geq 0 & \text{a.e. in } \Omega \\
u = 0 & \text{in } \partial \Omega
\end{cases}
$$

where $\Omega \subset \mathbb{R}^n$, $n \geq 3$, is a smooth domain.

1. Introduction

Let $\Omega$ be a smooth bounded domain of $\mathbb{R}^n$, $n \geq 3$, such that $0 \in \partial \Omega$. We let $a \in C^{0,\theta}(\bar{\Omega})$ for some $\theta \in (0,1]$ be a Hölder continuous function. In this note, we investigate the existence of weak solutions $u \in H^1_0(\Omega)$, $u \not\equiv 0$, to the problem

$$
(H) \begin{cases}
-\Delta u - \frac{a(x)}{|x|^2} u = u^{2^*-1} & \text{in } \Omega \\
u \geq 0 & \text{a.e. in } \Omega \\
u = 0 & \text{in } \partial \Omega
\end{cases}
$$

where $2^* := \frac{2n}{n-2}$ and $H^1_0(\Omega)$ is the completion of $C_0^\infty(\Omega)$ for the norm $u \mapsto \|\nabla u\|_2$.

The exponent $2^*$ is critical from the viewpoint of Sobolev embeddings: $H^1_0(\Omega)$ is continuously embedded in $L^q(\Omega)$ for all $1 \leq q \leq 2^*$, and the embedding is compact if $1 \leq q < 2^*$. Such type of equations has been investigated by many authors since the pioneer work of Brezis-Nirenberg [1] (which is the case $a(x) := \lambda|x|^2$ for some $\lambda \in \mathbb{R}$). For the case of nontrivial Hardy potential, without claiming any exhaustivity, we refer to Jannelli [4], Ruiz-Willem [5] (both for the case $0 \in \Omega$), the recent survey by Ghoussoub-Robert [3] and the references therein.

Concerning terminology, we say that $u \in H^1_0(\Omega)$ is a weak solution to $(H)$ if

$$
\int_\Omega \nabla u \nabla v dx - \int_\Omega \frac{a(x)}{|x|^2} uv dx = \int_\Omega v^{2^*-1} v dx \quad \text{for all } v \in H^1_0(\Omega).
$$

This definition makes sense because of the Sobolev embedding $H^1_0(\Omega) \hookrightarrow L^{2^*}(\Omega)$ and the Hardy inequality

$$
\frac{(n-2)^2}{4} \int_\Omega \frac{v^2}{|x|^2} dx \leq \int_\Omega |
abla v|^2 dx
$$

for all $v \in H^1_0(\Omega)$. Since $0 \in \partial \Omega$, this classical inequality is improved and one has

$$
(1) \quad \gamma_H(\Omega) \int_\Omega \frac{v^2}{|x|^2} dx \leq \int_\Omega |
abla v|^2 dx
$$

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for all \( v \in H_0^1(\Omega) \), where

\[
\frac{(n-2)^2}{4} < \gamma_H(\Omega) \leq \frac{n^2}{4}
\]

is the best possible constant such that (1) holds. Note that when \( \Omega \) is convex, or more generally when \( \Omega \subset \mathbb{R}^n_+ := \{ x \in \mathbb{R}^n / x_1 > 0 \} \), then \( \gamma_H(\Omega) \) achieves the maximal value \( \frac{n^2}{4} \). We refer to Ghoussoub-Robert [2] for discussions and references about the value of \( \gamma_H(\Omega) \).

In the sequel, we assume that the operator \(-\Delta - \frac{a(x)}{|x|^2}\) is coercive, that is there exists \( c > 0 \) such that

\[
\int_{\Omega} (|\nabla v|^2 - \frac{a(x)v^2}{|x|^2}) \ dx \geq c \int_{\Omega} |v|^2 \ dx \text{ for all } v \in H_0^1(\Omega).
\]

We investigate weak solutions to (H) as minima of the functional defined for \( u \in H_0^1(\Omega) \setminus \{0\} \) by

\[
J_a^\Omega(u) := \frac{\int_{\Omega} |\nabla u|^2 \ dx - \int_{\Omega} \frac{a(x)u^2}{|x|^2} \ dx}{(\int_{\Omega} |u|^2 \ dx)^{\frac{2}{n}}}. \tag{2}
\]

We define

\[
\mu_a(\Omega) := \inf \{ J_a^\Omega(u) | u \in H_0^1(\Omega) \setminus \{0\} \}.
\]

In the following \( D^{1,2}(\mathbb{R}^n_+) \) is the completion of \( C_0^\infty(\mathbb{R}^n_+) \) for the norm \( u \mapsto \| \nabla u \|_2 \).

We prove the following result.

**Theorem 1.1.** Let \( \Omega \) be a smooth bounded domain of \( \mathbb{R}^n \), \( n \geq 3 \). Let \( a \in C^{0,\theta}(\overline{\Omega}) \) for some \( \theta \in (0,1) \). We assume that \(-\Delta - \frac{a(x)}{|x|^2}\) is coercive and that \( \gamma := a(0) < \frac{n^2}{4} \).

We assume that

\[
\mu_a(\Omega) < \frac{1}{K(n,\gamma)}
\]

where

\[
\frac{1}{K(n,\gamma)} := \inf_{u \in D^{1,2}(\mathbb{R}^n_+) \setminus \{0\}} \frac{\int_{\mathbb{R}^n_+} (|\nabla v|^2 - \frac{\gamma v^2}{|x|^2}) \ dx}{(\int_{\mathbb{R}^n_+} |v|^2 \ dx)^{\frac{2}{n}}}. \tag{3}
\]

Then there exists \( u \in H_0^1(\Omega) \setminus \{0\} \) that is a weak solution of (H). Moreover, \( u \) can be achieved as a minimizer for \( \mu_a(\Omega) \).

**Remark 1.** For example, if \( a(x) = \gamma \) and \( \gamma < \frac{(n-2)^2}{4} \), we have that \(-\Delta - \frac{a(x)}{|x|^2}\) is coercive in \( \mathbb{R}^n \). More generally, it is necessary and sufficient that \( \gamma < \gamma_H(\Omega) \) or \( \gamma_H(\Omega) \) is the Hardy-Sobolev constant.

This note is devoted to the proof of Theorem 1.1.

2. **Proof of Theorem 1.1**

Let \((u_k)_{k \in \mathbb{N}} \in H_0^1(\Omega) \setminus \{0\}\) be a minimizing sequence for \( \mu_a(\Omega) \), that is \( J_a^\Omega(u_k) = \mu_a(\Omega) + o(1) \) as \( k \to +\infty \). Without loss of generality, we can assume that

\[
\| u_k \|_{L^2}^2 = 1 \text{ for all } k \in \mathbb{N},
\]

and then

\[
\int_{\Omega} |\nabla u_k|^2 - \frac{a(x)u_k^2}{|x|^2} \ dx = \mu_a(\Omega) + o(1) \text{ as } k \to +\infty. \tag{2}
\]

**Step 1:** We claim that \((u_k)_{k \in \mathbb{N}}\) is bounded in \( H_0^1(\Omega) \).
Proof. Since the operator is coercive, there exists $C > 0$ such that
\begin{equation}
\int_{\Omega} \left( |\nabla u_k|^2 - \frac{a(x)u_k^2}{|x|^2} \right) \, dx \geq C \int_{\Omega} |\nabla u_k|^2 \, dx
\end{equation}
for all $k \in \mathbb{N}$. We combine (2) and (3) to get
\[ C \int_{\Omega} |\nabla u_k|^2 \, dx \leq \mu a(\Omega) + o(1) \]
as $k \to +\infty$. Therefore $(u_k)_{k \in \mathbb{N}}$ is bounded in $H^1_0(\Omega)$. This proves the claim.

As a consequence, up to the extraction of a subsequence, there exists $u \in H^1_0(\Omega)$ such that
\begin{align*}
u_k &\rightharpoonup u \quad \text{converges weakly in } H^1_0(\Omega) \\
u_k &\to u \quad \text{converges strongly in } L^2(\Omega) \\
u_k &\to u \quad \text{converges a.e in } \Omega.
\end{align*}
We define $v_k = u_k - u$, we use equation (2) and we get
\begin{equation}
\int_{\Omega} |\nabla (v_k + u)|^2 \, dx - \int_{\Omega} \frac{a(x)(v_k + u)^2}{|x|^2} \, dx = \mu a(\Omega) + o(1).
\end{equation}
Therefore
\begin{align*}
\int_{\Omega} |\nabla v_k|^2 \, dx + \int_{\Omega} |\nabla u|^2 \, dx + 2 \int_{\Omega} \langle \nabla v_k, \nabla u \rangle \, dx \\
- \int_{\Omega} \frac{a(x)v_k^2}{|x|^2} \, dx - \int_{\Omega} \frac{a(x)u^2}{|x|^2} \, dx - 2 \int_{\Omega} \frac{a(x)v_k u}{|x|^2} \, dx &= \mu a(\Omega) + o(1).
\end{align*}
Since $v_k \to 0$ converges weakly in $H^1_0(\Omega)$ as $k \to +\infty$, we get that
\begin{equation}
\int_{\Omega} |\nabla v_k|^2 \, dx + \int_{\Omega} |\nabla u|^2 \, dx - \int_{\Omega} \frac{a(x)v_k^2}{|x|^2} \, dx - \int_{\Omega} \frac{a(x)u^2}{|x|^2} \, dx = \mu a(\Omega) + o(1)
\end{equation}
as $k \to +\infty$.

Step 2: We claim that
\begin{equation}
\lim_{k \to +\infty} \left[ \int_{\Omega} \frac{a(x)v_k^2}{|x|^2} \, dx - \int_{\Omega} \frac{a(0)v_k^2}{|x|^2} \, dx \right] = 0
\end{equation}
where $a(0) = \gamma$.

Proof. Since $a \in C^{0, \theta}(\Omega)$ then there exists $c > 0$ such that
\[ |a(x) - a(0)| \leq c|x|^\theta \]
for all $x \in \Omega$.
This gives
\begin{align*}
\left| \int_{\Omega} \frac{a(x)v_k^2}{|x|^2} \, dx - \int_{\Omega} \frac{a(0)v_k^2}{|x|^2} \, dx \right| &= \int_{\Omega} \frac{|a(x) - a(0)|v_k^2}{|x|^2} \, dx \\
&\leq \int_{\Omega} \frac{|a(x) - a(0)|v_k^2}{|x|^2} \, dx \\
&\leq c \int_{\Omega} \frac{|x|^\theta v_k^2}{|x|^2} \, dx.
\end{align*}
We let $\delta > 0$ to be fixed later. We have that
\[
\left| \int_{\Omega} \frac{u(x)v_k^2}{|x|^2} dx - \int_{\Omega} \frac{a(0)v_k^2}{|x|^2} dx \right| \leq c \int_{B(0,\delta)} \frac{|x|^0v_k^2}{|x|^2} dx + c \int_{\Omega \setminus B(0,\delta)} \frac{|x|^0v_k^2}{|x|^2} dx
= I_1(k) + I_2(k)
\]
as $k \to +\infty$. Concerning $I_1(k)$, we use the Hardy inequality (1) to get
\[
I_1(k) = c \int_{B(0,\delta)} \frac{|x|^0v_k^2}{|x|^2} dx \leq \delta^0 \int_{B(0,\delta)} \frac{v_k^2}{|x|^2} dx \leq \frac{c}{\gamma_H(\Omega)} \delta^0 \int_{\Omega} |\nabla v_k|^2 dx
\]
for all $k \in \mathbb{N}$. Therefore, since $(v_k)_k$ is bounded in $H^1_0(\Omega)$, we get that
\[
(6) \quad I_1(k) \leq c_1 \delta^0
\]
for all $k \in \mathbb{N}$. Concerning $I_2(k)$, we have that
\[
I_2(k) = c \int_{\Omega \setminus B(0,\delta)} \frac{|x|^0v_k^2}{|x|^2} dx \leq c_2 \delta^{q-2} \int_{\Omega \setminus B(0,\delta)} v_k^2 dx
\]
for all $k \in \mathbb{N}$. Since $v_k \to 0$ strongly in $L^2$, we get that
\[
(7) \quad \lim_{k \to \infty} I_2(k) = 0.
\]
Let $\epsilon > 0$ be a positive number. Since $\lim_{k \to 0} c_1 \delta^0 = 0$, there exists $\delta > 0$ such that $c_1 \delta^0 < \frac{\epsilon}{2}$, and then (6) yields $I_1(k) < \frac{\epsilon}{2}$ for all $k \in \mathbb{N}$. Moreover, it follows from (7) that there exists $k_0$ such that $I_2(k) < \frac{\epsilon}{2}$ for all $k > k_0$. Then,
\[
\left| \int_{\Omega} \frac{a(x)v_k^2}{|x|^2} dx - \int_{\Omega} \frac{a(0)v_k^2}{|x|^2} dx \right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\]
for all $k > k_0$. Therefore,
\[
\lim_{k \to \infty} \left[ \int_{\Omega} \frac{a(x)v_k^2}{|x|^2} dx - \int_{\Omega} \frac{a(0)v_k^2}{|x|^2} dx \right] = 0.
\]
This proves the claim. \qed

**Step 3:** We claim that $\|u\|_{L^2}^2 = 1$.

**Proof.** We have
\[
(8) \quad \|u\|_{L^2}^2 \leq \int_{\Omega} |\nabla u|^2 dx - \int_{\Omega} \frac{a(x)u^2}{|x|^2} dx.
\]
It follows from Ghoussoub-Robert [2], Proposition 4.3 that for any $\epsilon > 0$, there exists $C_\epsilon > 0$ such that
\[
\|v\|_{L^2}^2 \leq (K(n, \gamma) + \epsilon) \int_{\Omega} \left( |\nabla v|^2 - \frac{\gamma v^2}{|x|^2} \right) dx + C_\epsilon \int_{\Omega} v^2 dx
\]
for all $v \in H^1_0(\Omega)$. When $k \to +\infty$, $v_k \to 0$ in $L^2(\Omega)$, we then get that
\[
(9) \quad \|v_k\|_{L^2}^2 \leq (K(n, \gamma) + \epsilon) \int_{\Omega} \left( |\nabla v_k|^2 - \frac{\gamma v_k^2}{|x|^2} \right) dx + o(1).
\]
We combine (4), (5), (8) and (9) and to get
\[
\int_{\Omega} |\nabla u|^2 dx + \int_{\Omega} |\nabla u|^2 dx - \int_{\Omega} \frac{a(x)v_k^2}{|x|^2} dx - \int_{\Omega} \frac{a(x)u^2}{|x|^2} dx = \mu_3(\Omega) + o(1)
\]
\[
\int_{\Omega} |\nabla u|^2 \, dx - \int_{\Omega} \frac{a(x)u^2}{|x|^2} \, dx = \mu_a(\Omega) - \int_{\Omega} |\nabla v_k|^2 \, dx + \int_{\Omega} \frac{a(x)v_k^2}{|x|^2} \, dx + o(1)
\]

\[
\|u\|_{2^*}^2 \mu_a(\Omega) \leq \mu_a(\Omega) - \int_{\Omega} |\nabla v_k|^2 \, dx + \int_{\Omega} \frac{a(x)v_k^2}{|x|^2} \, dx + o(1)
\]

\[
(\|u\|_{2^*}^2 - 1) \mu_a(\Omega) \leq - \int_{\Omega} |\nabla v_k|^2 \, dx + \int_{\Omega} \frac{a(x)v_k^2}{|x|^2} \, dx + o(1)
\]

\[
(1 - \|u\|_{2^*}^2) \mu_a(\Omega) \geq \int_{\Omega} |\nabla v_k|^2 \, dx - \int_{\Omega} \frac{a(x)v_k^2}{|x|^2} \, dx + o(1)
\]

\[
\geq \int_{\Omega} |\nabla v_k|^2 \, dx - \int_{\Omega} \frac{a(0)v_k^2}{|x|^2} \, dx + o(1)
\]

\[
(1 - \|u\|_{2^*}^2) \mu_a(\Omega) \geq \frac{1}{(K(n, \gamma) + \epsilon)} \|v_k\|_{2^*}^2 + o(1)
\]

as \( k \to +\infty \), since \( a(0) = \gamma \). It follows from Brezis-Lieb that

\[
\|u_k\|_{2^*}^2 = \|v_k\|_{2^*}^2 + \|u\|_{2^*}^2 + o(1)
\]

as \( k \to +\infty \). Since \( \|u_k\|_{2^*}^2 = 1 \), we get

\[
1 = \|v_k\|_{2^*}^2 + \|u\|_{2^*}^2 + o(1).
\]

Inequality (10) and (11) yield

\[
(1 - \|u\|_{2^*}^2) \mu_a(\Omega) \geq \frac{1}{(K(n, \gamma) + \epsilon)} (1 - \|u\|_{2^*}^2)
\]

\[
(1 - \|u\|_{2^*}^2) \mu_a(\Omega) - \frac{1}{(K(n, \gamma) + \epsilon)} \geq 0.
\]

Since \( \mu_a(\Omega) < \frac{1}{(K(n, \gamma) + \epsilon)} \), then there exists \( \epsilon > 0 \) such that \( \mu_a(\Omega) - \frac{1}{(K(n, \gamma) + \epsilon)} < 0 \), and therefore

\[
\|u\|_{2^*}^2 \geq 1.
\]

Since \( u_k \rightharpoonup u \) converges weakly in \( H_0^1(\Omega) \) as \( k \to +\infty \), we have that \( \|u\|_{2^*}^2 \leq \liminf \|u_k\|_{2^*}^2 = 1 \). Therefore, we get that

\[
\|u\|_{2^*}^2 = 1.
\]

This proves the claim. \( \square \)

**Step 4:** We claim that \( u \) is a minimizer for \( \mu_a(\Omega) \).

*Proof.* Equation (4) rewrites

\[
(12) \int_{\Omega} |\nabla u|^2 \, dx - \int_{\Omega} \frac{a(x)u^2}{|x|^2} \, dx = \mu_a(\Omega) - \int_{\Omega} |\nabla v_k|^2 \, dx + \int_{\Omega} \frac{a(x)v_k^2}{|x|^2} \, dx + o(1).
\]

Equation (5) gives

\[
\int_{\Omega} |\nabla v_k|^2 \, dx - \int_{\Omega} \frac{a(x)v_k^2}{|x|^2} \, dx = \int_{\Omega} |\nabla v_k|^2 \, dx - \int_{\Omega} \frac{a(0)v_k^2}{|x|^2} \, dx + o(1)
\]

as \( k \to +\infty \). Then, combining (9) with (12), we get that

\[
\int_{\Omega} |\nabla u|^2 \, dx - \int_{\Omega} \frac{a(x)u^2}{|x|^2} \, dx \leq \mu_a(\Omega) - \frac{1}{(K(n, \gamma) + \epsilon)} \|v_k\|_{2^*}^2 + o(1)
\]

\[
\leq \mu_a(\Omega) + o(1)
\]
as $k \to +\infty$. Therefore, since $\|u\|_2^2 = 1$, we get

$$\int_\Omega |\nabla u|^2 \, dx - \int_\Omega \frac{a(x)u^2}{|x|^2} \, dx \leq \mu_a(\Omega).$$

Combining (13) with the definition of $\mu_a(\Omega)$, we get that

$$\mu_a(\Omega) = \int_\Omega |\nabla u|^2 \, dx - \int_\Omega \frac{a(x)u^2}{|x|^2} \, dx \|u\|_2^2.$$ 

Therefore, $u$ is a minimizer of $\mu_a(\Omega)$ and the claim is proved. \qed

**Step 5:** We claim that, up to multiplying by a positive constant, $u$ is a weak solution to (H).

**Proof.** Indeed, since $u$ is a minimum for $\mu_a(\Omega)$, it is a critical point of $J_\Omega a$ and then, using that $\|u\|_2^2 = 1$, we may as well assume that $u \geq 0$ on $\Omega$ (otherwise we replace $u$ by $|u|$ in the minimization process) and we get that

$$\int_\Omega \nabla u \nabla v \, dx - \int_\Omega \frac{a(x)u^2}{|x|^2} v \, dx = \mu_a(\Omega) \int_\Omega |u|^{2^*-2} v \, dx$$

for all $v \in H^1_0(\Omega)$, and $u$ is a weak solution to equation

$$-\Delta u - \frac{a(x)}{|x|^2} u = \mu_a(\Omega) u^{2^*-1} \text{ in } H^1_0(\Omega).$$

We define $u = k\tilde{u}$ where $k := \mu_a(\Omega)^{\frac{1}{2^*-1}}$. As one checks, $\tilde{u}$ is still a minimizer of $J_a^\Omega$ and $\tilde{u}$ is a weak solution to (H). This proves the claim and Step 5. \qed

**Remark 2.** It follows from the above analysis that $u_k \to u$ strongly in $H^1_0(\Omega)$. Indeed, we use equation (4), the coercivity of the operator and that $u$ is minimum to conclude that

$$\int_\Omega |\nabla v_k|^2 \, dx - \int_\Omega \frac{a(x)v_k^2}{|x|^2} \, dx = o(1)$$

and then

$$\int_\Omega |\nabla v_k|^2 \, dx \leq o(1)$$

as $k \to +\infty$. Since $v_k \to 0$ strongly in $L^2(\Omega)$, we get that $v_k \to 0$ strongly in $H^1_0(\Omega)$. Therefore $u_k \to u$ strongly in $H^1_0(\Omega)$.

**References**


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