Mechanism Design and Auctions for Electricity Network

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Abstract We present some key aspects of wholesale electricity markets modeling and more specifically focus our attention on auctions and mechanism design. Some of the results arising from those models are the computation of an optimal allocation for the Independent System Operator, the study of the equilibria (existence and unicity in particular) and the design of mechanisms to increase the social surplus. From a more general perspective, this field of research provides clues to discuss how wholesale electricity market should be regulated. We start with a general introduction and then present some results the authors obtained recently. We also briefly expose some undergoing related work. As an illustrative example, a section is devoted to the computation of the Independent System Operator response function for a symmetric binodal setting with piece-wise linear production cost functions.

1 Introduction

Economists, engineers and mathematicians have been giving a lot of attention to electricity markets since the beginning of the liberalization era in the eighties. We present recent results and ongoing research about wholesale electricity markets in mandatory pool settings and, in particular, the optimal design of such market. Market design studies the effects of market rules on economic functioning such as oligopoly behavior, vertical integration, market power, pricing, externalities and so on. The number of recent Nobel Prize laureates with contributions in this field demonstrates its impact on the economic thinking. In this chapter we focus on recent works [1], [2], [3] and [4] as well as on some ongoing research by the authors.

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In what follows, we assume that we are in a mandatory pool market, i.e. the agents will satisfy their engagements.

The specificity of a market in term of economical, industrial and geographical setting, its dependency on the regulatory environment, the entities that compose it - producers, consumers, Independent System Operator (ISO) and network - and the time scales, and the complex physical properties of electrical networks (Kirchhoff laws for instance) explain the diversity in electrical market models.

Key modeling decisions concern the agents preferences, the uncertainties on the energy sources and demands, the information representation, the production capacities and the physics of the system. In particular one have to specify the structure used to represent the bidding strategies of the producers. Since the physics of an electrical network is a hard problem too, it is usually simplified.

The classic questions arising from any modeling attempt are the mathematical well posedness of the problem, the existence, the uniqueness and the tractability of the equilibria as well as their properties. Besides, one may ask about the existence of efficient algorithms for calculating those equilibria. We point out that models for wholesale electricity markets are often general enough to be relevant for other economical settings.

In our setting, the production allocation plan is the result of an auction. Indeed, the producers communicate their selling prices to a central agent, and then the central agent minimizes the total cost while satisfying the demand. In what follows we most of the time take into account the geography of the network (i.e. production and consumption are not co-localized at one point) as well as the losses due to the electricity transportation. Figure 1 presents a simple example of network with four nodes.

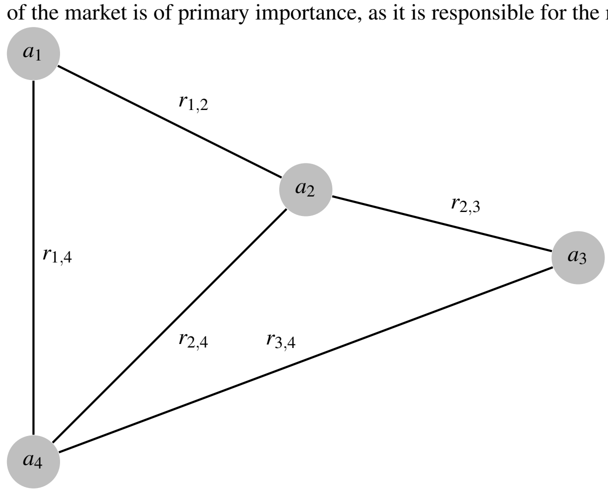
We first present the general setting of the model in Section 2. Section 3 is a short review of some recent related works in the field. We give a quantitative formulation of the problem in Section 4. We will discuss the main results in Sections 5. In Section 6 we develop the example of a two producers setting with piece-wise linear cost functions. We conclude in Section 7.

2 Setting

This section is a qualitative description of the market settings encountered in the literature. We try to be as general as possible, whereas in Section 3 we focus on the frameworks Escobar, Figueroa, Heymann and Jofré study in [1], [2], [3] and [4]. The main market model components are the agents, the demand, the network, the regulation and the structure of information (since some uncertainty is usually part of the model). Besides, different types of equilibria could be considered. An example of network is proposed in Figure 1.

The agents are divided among those who produce electricity (producers) and those who consume it (usually aggregated into an ISO). In our setting, both producers and consumer are macroscopic, but for the sack of completeness, we note

Fig. 1 An example of a wholesale electricity market network. On the demand side, at each node a_i there is a demand (load) d_i that has to be fulfilled. On the supply side, electricity can be produced at some nodes by some independent agents. The production cost function associates the quantity produced by an agent and the corresponding economical cost (it is specific to the agent). For modeling but also technical and practical reasons, those production cost functions are often approximated with functions in simple functional sets (linear, quadratic, piecewise linear...). Electricity can be sent from one node to another through the edges of the network, but there is a price for that (for instance, a loss proportional to the square of the quantity sent). The Independent System Operator (ISO) is as its name indicates a central operator that has to allocate the production so that supply meet demand while minimizing a criteria (usually the total price). To produce this allocation, the ISO needs to know the price he will have to pay for each allocation, so the producers specify a bid function that is usually in the same simple aforementioned functional set. Since the ISO has no way to know the real production cost function of the producers, it is in their interest to game the system. We point out that stochasticity may occurs on the demand. Moreover, the network aspect of the market is of primary importance, as it is responsible for the market power of the agent.



that some models use a continuum of microscopic producers. The microscopic producers models correspond to situations where no producer can have any unilateral impact on the market.

Producers incur production costs when they supply electricity to the market. This cost depends on the quantity of electricity they produce individually. This relation between the quantity the producer supply and its production cost is encoded in his production cost function. To sell electricity, producers quote a price to the market. We consider later in the chapter two structures to model the way a producer specifies a selling price: the bid function and the supply function. A bid function maps any quantity of electricity to the price a producer asks to supply such quantity. A supply function maps any price to the quantity a producer is ready to supply at such price. The objective of each producer is to maximize his individual profit (or his average profit if the model contains any form of randomness). We consider only non cooperative settings: producers are competing against each other.

We assume dispatch decisions to be centralized: a unique agent, the Independent System Operator (ISO), aggregates the demand side. We justify this aggregation by the regulatory environment and the market organization. The ISO receives bids from producers and has to supply the local electricity demand where it is needed by buying the electricity at the quoted prices (pay as bid market) The demand is most of the time (i.e. in the literature we are considering) inelastic, but it could be either deterministic or stochastic.

Usually electricity is seen as a divisible commodity. Nonetheless in our model we can also see it as a geographically differentiated product. Indeed, unless production facilities and consumers are colocalized, productions and demands are dispatched on a network. The network contains nodes and edges. Each node is a place where electricity is produced or consumed (or both) and so is characterized by its local demand and its local producers. Each edge is a place through which electricity can be sent. The ISO has the possibility to send any nodal electricity surplus where it is needed through the network cables but those cables are subject to physical limitations such as capacity constraints and online losses (due to Joule effects). The geographical differentiation comes from the fact that it the ISO who incurs those losses.

We envision different kinds of optimization problems: the standard ISO problem, the agents profit maximization problem and the mechanism design problem. The first one, and the simplest, simply consists in finding the minimal cost production plan. For this problem the ISO, (or principal, if we want to use the mechanism design terminology) receives bids from the different producers and knows the demand (deterministic or stochastic) at each node. He then has to supply this demand at each node for the cheapest total cost. This optimization is subject to the network physical limitations and is parametrized by the demand at each nodes and the producers bids. We call principal response the function that maps these parameters with the solution of the corresponding problem.

The second problem comes from the agents perspective. Knowing the principal response function, their own production costs functions and some common knowledge on their fellow producers, they optimize their bids to maximize their individual profits. This problem raises questions about best response strategies and Nash equilibria.

In mechanism design, there is an agent and a principal. In our model, we assign the role of the principal to the ISO. The mechanism design problem reverts the role of the principal and the agents: the principal builds his response function knowing that the agents will then maximize their profits. By offering the right incentives, he leads the agents. The mechanism design problem can be formulated by considering that the principal gives a new response function that is not the optimal solution of the first problem. Indeed, instead of waiting for the bids to order the production, the ISO defines in a contract a response function that he will respect in the future. This contract depends on the (future) bids of each producer and the demand at each node. Why would he do such a thing ? Because otherwise there are no incentives for the agents to tell the truth about their true production costs, so when they bid a price, they just selfishly optimize their own benefits based on the information they have.

We have shown that in some very general setting, it is possible for the principal to formulate the response function in the contract so that the producers are incentivized to reveal their true types (i.e. real production costs). Put differently, the principal can design a contract so that for each producer, it is optimal to reveal his true production costs function. To do so, the principal has to pay (virtually, through the payment function define in the contract) an information rent to the producers, but his total cost is smaller than in the previous setting.

In general the principal does not know the real production cost of the producers. This is why producers can bid higher than their production cost. The information the principal has about the producers costs is modeled by a probability distribution. The less the producer knows about the production costs, the higher the information rent.

3 Literature

Several approaches have been proposed to answer the questions raised in the previous Section. In [6] Klemperer and Meyer show that uncertainty reduces the quantity of supply function Nash equilibria. The firms bid their supply functions before demand is revealed. The existence of a Nash equilibria is shown for a symmetric oligopoly. In [7], Anderson and Philpott show how to construct optimal time dependent supply functions in electricity market setting where demand and competing generators behaviors are unknown by introducing a market distribution function. The gaming aspect of the situation is reduced by arguing that competitors do not react to the producer bids. The problem is formulated as an optimal control problem. In [8] the authors study asymmetric competition and propose a numerical solver based on GAMS to compute the optimal strategies. They compare the algorithm with the ODE method. In [9], Anderson gives a proof of existence of a pure Nash Equilibria under some technical assumptions when the network is reduced to a single point. He also gives sufficient conditions for unicity. Optimal auction design was introduced by Myerson in his 1981 seminal article [12]. Laffont and Martimort wrote in [5] an introduction to mechanism design in a general setting. The authors expose important concepts such as the revelation principle, adverse selection, participation constraints and information rent. The book does not consider interactions on a network -which is the specificity of the wholesale electricity market. Bi-level approaches with quadratic production cost functions are proposed in [10] and [11] to study the ISO response functions and the Nash equilibria.

4 Quantitative formulations

We briefly present in this section some questions of interest concerning models that fit into the general setting described in Section 2. Those questions were partially addressed in recent works by the authors.

4.1 Generality

We will generically use the notations i to refer to a producer and q_i to refer to the quantity this agent produces. The nodes are connected by edges and we denote by $h_{i,i'}$ the quantity of electricity that is sent from node i to node i'. The market network is not necessary complete. We call d_i the demand at node i. Each producer quotes a bid denoted by b_i to the principal. This bid is a function of the quantity q_i. Each producer also informs the principal of the maximum quantity $\bar{q}_i$ he can produce. In general, the allocation problem is subject to network constraints, i.e., the vector h of components $h_{i,i'}$ has to be in a set H. For example the set H could be made of all vector h such that $h_{i,i'} \leq h^{max}_{i,i'}$, which means that one cannot send an arbitrary big amount of electricity thought the network.

4.2 The standard allocation problem

The principal receives bids from the agents and allocates the production so that:

- the allocation minimizes the total cost;
- the allocation respects the network and capacity constraints;
- supply is greater than demand at any node.

The last point corresponds to the nodal constraints. The supply at a given node i is the sum of the local production q_i and the importations from neighboring nodes $\sum_{i'} h_{i',i}$. To this we need to subtract the exportations to neighboring nodes $\sum_{i'} h_{i,i'}$ and the line losses. If we send a quantity h through an edge $\{i,i'\}$, we denote by $L_{i,i'}(h)$ the corresponding loss. To get a symmetric expression we will count half of this loss at node i and the other half at node i'. We end up with the following nodal constraint:

$$q_i + \sum_{i'} h_{i',i} - h_{i,i'} - \sum_{i'} \frac{L_{i',i}(h_{i',i}) + L_{i,i'}(h_{i,i'})}{2} \geq d_i, \tag{1}$$

where the summations are performed over the nodes adjacent to i. All this being said, the generic allocation problem writes

$$\begin{aligned}
\underset{q,h}{\text{minimize}} \quad & \sum_i b_i(q_i) \\
\text{subject to} \quad & q_i + \sum_{i'} h_{i',i} - h_{i,i'} - \sum_i \frac{L_{i,i'}(h_{i',i}) + L_{i,i'}(h_{i,i'})}{2} \geq d_i \\
& q_i \leq \bar{q}_i \\
& h \in H \\
& h_{i,j} \geq 0.
\end{aligned} \tag{2}$$

We point out that if the bidding and the loss functions are convex functions and H is a convex set, then the problem is convex. For instance, one can take the bid functions linear, the loss functions quadratic and H is $\mathbb{R}^+$. Note that the bid functions b_i and the demand vector d can be seen as parameters of the optimization problem. We could make the solution of this problem stochastic by adding a dependency of d to a random event ω. This would not change the solution of the problem from the operator perspective, but it would change the market setting for the agents. **What is the solution of this problem? What are the analytical properties of this solution? How can we compute it?**

4.3 *The agent problem*

The objective of each producer is to maximize his profit. Note that by solving the principal allocation problem, we have the response function of the ISO to the agents bids. It is stochastic if the demand is stochastic. We can map each bidding profile of the agents with the expected profit of each agent. So by competing against each other, the agents are playing a game. In addition, producer i does not know the production cost functions of his fellow agents. So we are in an imperfect information setting. We assume that for each agent i there is a probability distribution f_i over the production functions that represents the information the other agents have about agent i. We assume those probability distributions to be independent. The profit of an agent of type (i.e. production cost function) c_i that bids b_i is given by

$$\pi_i(c,x) = \int_{C_{-i}} [b_i(q_i(b_i, b(c_{-i})) - c(q_i(b_i, b(c_{-i}))))] f_{-i}(c_{-i}) dc_{-i}, \tag{3}$$

where the integral is performed over the types of the other agents. Then the maximized profit is

$$\overline{\pi}_i(c) = \max_{b_i} \int_{C_{-i}} [(b_i(q_i(b_i, b(c_{-i})) - c(q_i(b_i, b(c_{-i}))] f_{-i}(c_{-i}) dc_{-i} \tag{4}$$

and an optimal strategy $\bar{b}_i(c)$ must satisfy:

$$\bar{b}_i(c) \in arg\max_{b_i} \int_{C_{-i}} [(b_i(q_i(b_i, b(c_{-i})) - c(q_i(b_i, b(c_{-i}))]f_{-i}(c_{-i})dc_{-i}. \tag{5}$$

So for each agent the best response strategy to the other agents is the solution of an optimization problem on the set of maps from the types (i.e. production cost functions) c to the bids b. Usually, the production cost functions will be characterized by a vector of $\mathbb{R}^n$. In this case this is an optimization over the functions from $\mathbb{R}^n$ to $\mathbb{R}^n$. Of course, it is natural to ask about the Nash Equilibria of the game. We point out that when $r = 0$, and there are no network and capacity constraints (and of course, the network is connected), the problem corresponds to the classic setting of first best auction theory (see Figure 2). **What can we say about this game? Is there an equilibrium ? Is it unique ? Can the agents 'game' the system** ?

4.4 The optimal mechanism design problem

In this part we assume everybody knows the demand. In order to decrease the market power of the agents and increase social welfare, we reverse the role of the principal and the agents, i.e. the principal "bids" a contract to the agent. The contract should associate each bid profile $(b_i)_i$ with two vectors q and x, where q_i is the quantity of electricity agent i has to produce and x_i is amount of money he will receive. Of course, this contract has to be incentive compatible, i.e. the payments described by the principal need to be high enough to make the agent willing to stay in the market. In this situation, the problem we are solving is the design of the optimal contract:

$$\begin{aligned}
\underset{q_j, h_{i,k}, x_j}{\text{minimize}} \quad & \sum_j \mathbb{E}x_j(c) \\
\text{subject to} \quad & q_j(c) + \sum_i h_{i,j} - h_{j,i} - \sum_i \frac{L_{i,j}(h_{i,j}) + L_{j,i}(h_{j,i})}{2} \geq d_j \\
& \mathbb{E}x_j(c) - C_j(q_j(c), c) \geq \mathbb{E}x_j(b) - C_j(q_j(b), c) \\
& \mathbb{E}x_j(c) - C_j(q_j(c), c)) \geq 0 \\
& h_{i,j}, x_j \geq 0,
\end{aligned} \tag{6}$$

where $\mathbb{E}$ denotes the mean operator with respect to the f_i's, c denote the vector of production cost functions and the constraints should be verified for all c. We refer to [4] for a justification of the formulation. We point out that this is an optimization problem over a functional set (so infinite dimensional) with an infinite number of constraints. We display some results in Figures 3 and 4. **How do we build such problem? How can we solve it? How better is the social surplus with an optimal design?**

4.5 A differential equation

We introduce a fictitious play like game for the binodal symmetric setting:

- 2 agents;
- $L_{i,i'}(h_{i,i'}) = rh_{i,i'}$;
- $H = \mathbb{R}^2_+$;
- $\bar{q}_i = +\infty$;
- the cost functions and the bid functions are linear;
- $d_1 = d_2$: the demand is equal at each node.
- $f_1 = f_2 = f$

We look for a symmetric equilibrium. If the agents iteratively change their bid functions proportionally to the corresponding increase in profit this will produce, the bid functions dynamics should be described by this formal differential equation.

$$\partial_t b(c,t) = \partial_b \pi_b(c, b(c,t))) \tag{7}$$

with

$$\pi_b(c,s) = \int_{C_{-i}} (s-c)(q_i(s, b(c_{-i}))f(c_{-i})dc_{-i}. \tag{8}$$

Is this dynamics well posed? What conclusions can we draw from its study? Can we build such dynamics for more general settings?

5 Important results

Escobar and Jofré show in [3] that in a random environment a Walrasian and a non-cooperative equilibria exist (for the non-cooperative equilibrium the distribution need to be atom-less) in this setting, the demand is elastic and the ISO maximize the sum of the utility functions. Utility functions and cost functions are general. Escobar and Jofré demonstrate in [1], the existence of non-cooperative and Walrasian equilibrium when the ISO solve the standard ISO problem and demand is uncertain. The paper finishes with a welfare theorem for wholesale electricity auction. Escobar and Jofré give in [2] a lower bound on the market power exercised by each producers. The existence of a mixed strategy Nash equilibrium is given.The authors also give some regularity property on the ISO response function (condition to be a singleton, continuity and Lipschitzianity). The cost functions are general. Figueroa, Jofré and Heymann study in [4], a bi-nodal symmetric market with linear production cost functions and quadratic losses (as shown in Figure 2). The principal minimal cost production plan problem was already solved in [2] and an explicit solution given. If we define

$$F(x,y) = d + \frac{1}{2r}\left(\frac{x-y}{x+y}\right)^2 - \frac{1}{r}\left(\frac{x-y}{x+y}\right) \quad \text{and} \quad \overline{q} = 2\left[\frac{1-\sqrt{1-2dr}}{r}\right] \tag{9}$$

the solution to this problem can be written as

$$q_i(c_i,c_{-i}) = \begin{cases} F(c_i,c_{-i}) & \text{if } F(c_i,c_{-i}) \geq 0 \text{ and } F(c_{-i},c_i) \geq 0 \\ \overline{q} & \text{if } F(c_{-i},c_i) < 0 \text{ and } F(c_i,c_{-i}) \geq 0 \\ 0 & \text{if } F(c_i,c_{-i}) < 0 \text{ and } F(c_{-i},c_i) \geq 0 \end{cases} \tag{10}$$

This solution is used to compute an explicit solution of the mechanism design problem. The mechanism design solution is then compared to the standard setting for which numerical simulations are performed. The authors assume that the function $J_i : c_i \longrightarrow c_i + \frac{F_i(c_i)}{f_i(c_i)}$ is increasing in c_i, where f_i is the distribution of the marginal cost of producer i and F_i is its integral. Then the main result is

Proposition 1. *Define*

$$\tilde{q} = 2\left[\frac{1-\sqrt{1-2dr}}{r}\right]$$

Then an optimal mechanism is given by

$$\hat{q}_i(c) = \begin{cases} F(J_i(c_i),J_{-i}(c_{-i})) & \text{if } F(J_i(c_i),J_{-i}(c_{-i})) \geq 0 \\ & \text{and } F(J_{-i}(c_{-i}),J_i(c_i)) \geq 0 \\ 0 & \text{if } F(J_i(c_i),J_{-i}(c_{-i})) \leq 0 \\ \tilde{q} & \text{if } F(J_{-i}(c_{-i}),J_i(c_i)) \leq 0 \end{cases}$$

$$\hat{h}_i(c) = \begin{cases} \frac{1}{r}\left[\frac{J_{-i}(c_{-i})-J_i(c_i)}{J_{-i}(c_{-i})+J_i(c_i)}\right] & \text{if } J_i(c_i) \leq J_{-i}(c_{-i}) \text{ and } F(J_{-i}(c_{-i}),J_i(c_i)) \geq 0 \\ \tilde{q}-1 & \text{if } J_i(c_i) \leq J_{-i}(c_{-i}) \text{ and } F(J_{-i}(c_{-i}),J_i(c_i)) \leq 0 \\ 0 & \text{if not} \end{cases}$$

$$\hat{x}_i(c) = c_i\hat{q}_i(c) + \int_{c_i}^{\overline{c}_i} q_i(s,c_{-i})ds$$

We point out that the mechanism design is built with the standard ISO response function, we just replace c_i by $J_i(c_i)$.

6 Example: computing the ISO response function for a symmetric binodal setting with piecewise linear production cost functions

6.1 Introduction

In this section we derive an explicit expression for a specific example of ISO allocation problem. We study the bi-nodal market with quadratic externalities displayed in Figure 2. The production cost functions of both agents are made of two linear pieces, with a slope change when the production level is equal to $\bar{q}$. We denote by c_1

(resp. c_2) producer 1 (resp. 2) marginal cost when his production level is below $\bar{q}$, and by $\bar{c}_1$ (resp. $\bar{c}_2$) when it is above. The production cost functions are convex i.e. $c_i < \bar{c}_i$ and the demand d is the same at both nodes. We end-up with the following formulation for the ISO allocation problem:

$$\begin{aligned}
\underset{q_i,\bar{q}_i,h}{\text{minimize}} \quad & c_1 q_1 + \bar{c}_1 \bar{q}_1 + c_2 q_2 + \bar{c}_2 \bar{q}_2 \\
\text{subject to} \quad & q_i + \bar{q}_i + (-1)^i h \geq \frac{r}{2}(h^2) + d \qquad (\lambda_i) \;\; \text{for i} = 1,2 \\
& q_i, \bar{q}_i \geq 0 \qquad (\mu_i) \;\; \text{for i} = 1,2 \\
& q_i \leq \bar{q} \qquad (\gamma_i) \;\; \text{for i} = 1,2.
\end{aligned} \tag{11}$$

In this formulation, q_i is the quantity produced by agent i at marginal cost c_i, and $\bar{q}_i$ is the quantity produced by i at marginal cost $\bar{c}_i$. These quantities are subject to positivity constraints with multipliers μ_i and $\bar{\mu}_i$. We also introduce λ_i the multipliers of the nodal constraints, and γ_i the multipliers of the constraints $q_i \leq \bar{q}$. We denote

$$F(\lambda_1,\lambda_2) = d + \frac{1}{r}\frac{\lambda_2 - \lambda_1}{\lambda_1 + \lambda_2} + \frac{1}{2r}\left(\frac{\lambda_2 - \lambda_1}{\lambda_1 + \lambda_2}\right)^2, \tag{12}$$

$$P(h) = h + \frac{rh^2}{2} + d, \tag{13}$$

$$k(\lambda_1,\lambda_2) = P\left(\frac{\lambda_2 - \lambda_1}{r(\lambda_1 + \lambda_2)}\right), \tag{14}$$

and

$$q_i^{tot} = q_i + \bar{q}_i. \tag{15}$$

We assume without loss of generality that $\bar{q} < 2d$ and $c_1 < c_2$. It is clear that if $q_1 < \bar{q}$, then $\bar{q}_1 = 0$. To solve this problem, we check whether $d < \bar{q}$ or $d \geq \bar{q}$.

6.2 *If* $d < \bar{q}$

By hypothesis $c_1 < c_2$. This implies that $q_1 \geq q_2$. So $\bar{q}_2 > 0$ implies that $q_1 = q_2 = \bar{q} > d$, which is not optimal. Therefore we can set $\bar{q}_2 = 0$. We can also relax the constraint $q_2 < \bar{q}$ because it won't be binding for the optimal solution. So we rewrite the problem

$$\begin{aligned}
&\underset{q_i,\bar{q}_1,h}{\text{minimize}} \quad c_1 q_1 + \bar{c}_1 \bar{q}_1 + c_2 q_2 \\
&\text{subject to} \\
&q_1 + \bar{q}_1 - h \geq \frac{r}{2}(h^2) + d \qquad (\lambda_1) \\
&q_2 + h \geq \frac{r}{2}(h^2) + d \qquad (\lambda_2) \\
&q_1, q_2, \bar{q}_1 \geq 0 \qquad (\mu_i) \\
&q_1 \leq \tilde{q} \qquad (\gamma_1)
\end{aligned}$$

The first order conditions give

$$c_1 - \lambda_1 - \mu_1 + \gamma_1 = 0 \tag{16}$$

$$c_2 - \lambda_2 - \mu_2 = 0 \tag{17}$$

$$\bar{c}_1 - \lambda_1 - \bar{\mu}_1 = 0 \tag{18}$$

$$h = \frac{\lambda_2 - \lambda_1}{r(\lambda_1 + \lambda_2)} \tag{19}$$

There are four possible cases.

6.2.1 *Case 1*: $P(\frac{c_2 - c_1}{r(c_1 + c_2)}) \leq \bar{q}$

We consider a relaxation of the problem by removing the constraint $q_1 \leq \bar{q}$. In this relaxed problem, any optimal solution should verify $\bar{q}_1 = 0$ so the relaxed problem is equivalent to the linear cost functions allocation problem with costs c_i, for which we have an explicit formula of the solution. **We then notice that the optimal solution of the relaxed problem is admissible, so it is also the solution of** (11)**.**

6.2.2 *Case 2*: $P(\frac{c_2 - c_1}{r(c_1 + c_2)}) > \bar{q}$ and $P(\frac{c_2 - \bar{c}_1}{r(\bar{c}_1 + c_2)}) \leq \bar{q}$

We show that $\bar{q}_1 = 0$ **and** $q_1 = \bar{q}$**.**
If $\bar{q}_1 > 0$, then by complementarity of the multiplier $\bar{\mu}_1 = 0$, so with (18) $\lambda_1 = \bar{c}_1$. So by (19) we have $h = \frac{\lambda_2 - \bar{c}_1}{r(\bar{c}_1 + \lambda_2)}$. Then by hypothesis and the fact that P is increasing and $\lambda_2 \leq c_2$ we have that $P(h) \leq P(\frac{c_2 - \bar{c}_1}{r(\bar{c}_1 + c_2)}) \leq \bar{q}$. So $q_1 + \bar{q}_1 = q^{tot} \leq \bar{q}$. Then using the fact that $q_1 < \bar{q} \Rightarrow \bar{q}_1 = 0$, we deduce that $\bar{q}_1$ is null, which is not the hypothesis. We conclude that $\bar{q}_1 = 0$.
By hypothesis, $\bar{q} < 2d$ and less than $\bar{q}$ is produced at node 1. Summing the two nodal constraints we see that $q_2 > 0$, so that $\lambda_2 = c_2$.
Now if $q_1 < \bar{q}$, then by complementarity of the multiplier $\gamma_1 = 0$, then with (16) $c_1 = \lambda_1$ and with (19), $h = \frac{c_2 - c_1}{r(c_1 + c_2)}$. Therefore we get $q_1 + \bar{q}_1 = q_1^{tot} \geq P(\frac{c_2 - c_1}{r(c_1 + c_2)})$ (by the first nodal constraint), so by hypothesis $q_1^{tot} \geq \bar{q}$ and so $\bar{q}_1$ which we know as false. So $q_1 = \bar{q}$.

6.2.3 *Case 3*: $P(\frac{c_2-c_1}{r(c_1+c_2)}) > \bar{q}$ and $P(\frac{c_2-\bar{c}_1}{r(\bar{c}_1+c_2)}) > \bar{q}$ and $P(\frac{\bar{c}_1-c_2}{r(\bar{c}_1+c_2)}) > 0$

We show that $q_2 > 0$**,** $q_1 = \bar{q}$**,** $\bar{q}_1 > 0$**, and,** $q_2 = P(\frac{\bar{c}_1-c_2}{r(\bar{c}_1+c_2)})$**.**

First we show that $q_2 > 0$. If $q_2 = 0$, then by the second nodal constraint $h \geq \frac{rh^2}{2} + d$, which means that $P(-h) \leq 0$. Moreover, $\bar{q}_1 > 0$ because $2d > \bar{q}$. So by (18) $\lambda_1 = \bar{c}_1$. With (17) and (19) we have $P(\frac{\bar{c}_1-c_2}{r(\bar{c}_1+c_2)}) \leq P(\frac{\bar{c}_1-\lambda_2}{r(\bar{c}_1+\lambda_2)}) = P(-h) \leq 0$, which is false by hypothesis. So $q_2 > 0$. We deduce from this and (17) that $\lambda_2 = c_2$.
If $q_1 < \bar{q}$ then by complementarity of the multiplier $\gamma_1 = 0$, so by (16), $\lambda_1 = c_1$ and by (19), $h = \frac{c_2-c_1}{r(c_1+c_2)}$. Then we get $q_1^{tot} \geq P(\frac{c_2-c_1}{r(c_1+c_2)})$ (by the first nodal constraint), so by hypothesis $q_1^{tot} \geq \bar{q}$, which implies $q_1 = \bar{q}$, which is absurd since we assumed $q_1 < \bar{q}$. So $q_1 = \bar{q}$.
If $\bar{q}_1 = 0$ then with (18), $\lambda_1 \leq \bar{c}_1$ and so with (19), $h \geq \frac{c_2-\bar{c}_1}{r(\bar{c}_1+c_2)}$. We then deduce by nodal constraint 1 and the hypothesis that $q_1^{tot} \geq P(h) \geq \bar{q}$, which implies that $\bar{q}_1 > 0$, which is absurd. So $\bar{q} > 0$.
We know that $\bar{q}_2 = 0$. Using the second nodal constraint, we get $q_2 = P(\frac{\bar{c}_1-c_2}{r(\bar{c}_1+c_2)})$. With the first nodal constraint, we have $\bar{q}_1 = P(\frac{c_2-\bar{c}_1}{r(\bar{c}_1+c_2)}) - \bar{q}$.

6.2.4 *Case 4*: $P(\frac{c_2-c_1}{r(c_1+c_2)}) > \bar{q}$ and $P(\frac{c_2-\bar{c}_1}{r(\bar{c}_1+c_2)}) > \bar{q}$ and $P(\frac{\bar{c}_1-c_2}{r(\bar{c}_1+c_2)}) \leq 0$

We show that $q_2 = 0$**.**
Indeed, if $q_2 > 0$, then $\lambda_2 = c_2$. Using the same reasoning as the one used in the third case, we would show that $q_1 = \bar{q}$ and $\bar{q}_1 > 0$. So that $h = -\frac{\bar{c}_1-c_2}{r(\bar{c}_1+c_2)}$. So using $P(\frac{\bar{c}_1-c_2}{r(\bar{c}_1+c_2)}) \leq 0$, we see that nodal constraint 2 is satisfied with $q_2 = 0$, so the solution is not optimal, which is absurd. So $q_2 = 0$.

6.2.5 We conclude

Theorem 1. *Assuming $d < \bar{q} < 2d$, then:*

$$q_1^{tot} = k(c_1, c_2) \text{ and } q_2^{tot} = k(c_2, c_1) \text{ if } k(c_1, c_2) \leq \bar{q}$$
$$q_1^{tot} = \bar{q} \text{ and } q_2^{tot} = \bar{q} - 2\frac{-1 + \sqrt{1 + 2r(\bar{q} - d)}}{r} \text{ if } k(c_1, c_2) > \bar{q} \text{ and } k(\bar{c}_1, c_2) \leq \bar{q}$$
$$q_1^{tot} = k(\bar{c}_1, c_2) \text{ and } q_2^{tot} = k(c_2, \bar{c}_1) \text{ if } k(c_1, c_2) > \bar{q}, k(\bar{c}_1, c_2) > \bar{q} \text{ and } k(c_2, \bar{c}_1) > 0$$
$$q_1^{tot} = 2\frac{1 - \sqrt{1 - 2dr}}{r} \text{ and } q_2^{tot} = 0 \text{ if } k(c_1, c_2) > \bar{q},\ k(\bar{c}_1, c_2) > \bar{q} \text{ and } k(c_2, \bar{c}_1) \leq 0$$

6.3 case $d \geq \bar{q}$

Since we consider that c_1 is smaller than c_2, there are two possibilities. Either the $\bar{c}_i$ are all bigger than the c_i, or $\bar{c}_1$ is smaller than c_2.

6.3.1 If the $\bar{c}_i$ are all bigger than the c_i

In this case, we first show that $q_1 = q_2 = \bar{q}$. The problem then writes

$$
\begin{aligned}
\underset{q_i,\bar{q}_i,h}{\text{minimize}} \quad & \bar{c}_1\bar{q}_1 + \bar{c}_2\bar{q}_2 \\
\text{subject to} \quad & \bar{q}_1 + -h \geq \frac{r}{2}(h^2) + d - \bar{q} & (\lambda_1) \\
& \bar{q}_2 + h \geq \frac{r}{2}(h^2) + d - \bar{q} & (\lambda_2) \\
& \bar{q}_i \geq 0 \qquad (\mu_i) \ \text{ for i} = 1,2
\end{aligned}
$$

which corresponds to the linear case problem with a demand of $d - \bar{q}$ and costs of $\bar{c}_i$.

6.3.2 If $\bar{c}_1$ is smaller than c_2

We point out that replacing c_1 by $\bar{c}_1$ does not change the solution.
If $F(c_2, \bar{c}_1) \leq \bar{q}$, we show that $\bar{q}_2 = 0$ the problem can be reduced to the linear production cost problem with demand d and marginal costs $\bar{c}_1$ and c_2.
If $F(c_2, \bar{c}_1) > \bar{q}$, we show that we can reduce the linear production cost problem with demand $d - q$ and marginal costs $\bar{c}_1$ and $\bar{c}_2$.

Theorem 2. *If $d \geq \bar{q}$, then*

- *If $c_i \leq \bar{c}_j$ for all i, j, then we get the result by solving the linear problem with demand $d - \bar{q}$ and costs $\bar{c}_i$ and adding $\bar{q}$ to the quantity we get.*
- *If $0 \leq F(c_2, \bar{c}_1) \leq \bar{q}$ we reduce to the linear allocation problem with demand d and marginal cost $\bar{c}_1$ and c_2.*
- *If $F(c_2, \bar{c}_1) > \bar{q}$, we reduce the problem to the linear allocation problem with demand $d - q$ and marginal costs $\bar{c}_1$ and $\bar{c}_2$ and add $\bar{q}$ to the q_is we get.*

7 On going work

We are currently working on generalization of those results. In particular, we have shown that for a market with n-pieces piecewise linear production cost functions and any number of producers, there is a mechanism design with an explicit formulation.

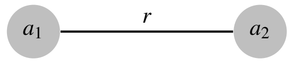


Fig. 2 In [4] and [2], the authors consider a binodal market with quadratic line losses. The demand is the same at both nodes. The production cost functions are linear. There are no network and capacity constraints. A very intuitive justification of the market power induced by the line losses is given in [2]. Indeed, in a symmetric perfect information setting with linear production cost function of slope c, the equilibrium strategy of both producers is to bid $\frac{c}{1-2dr} > c$.

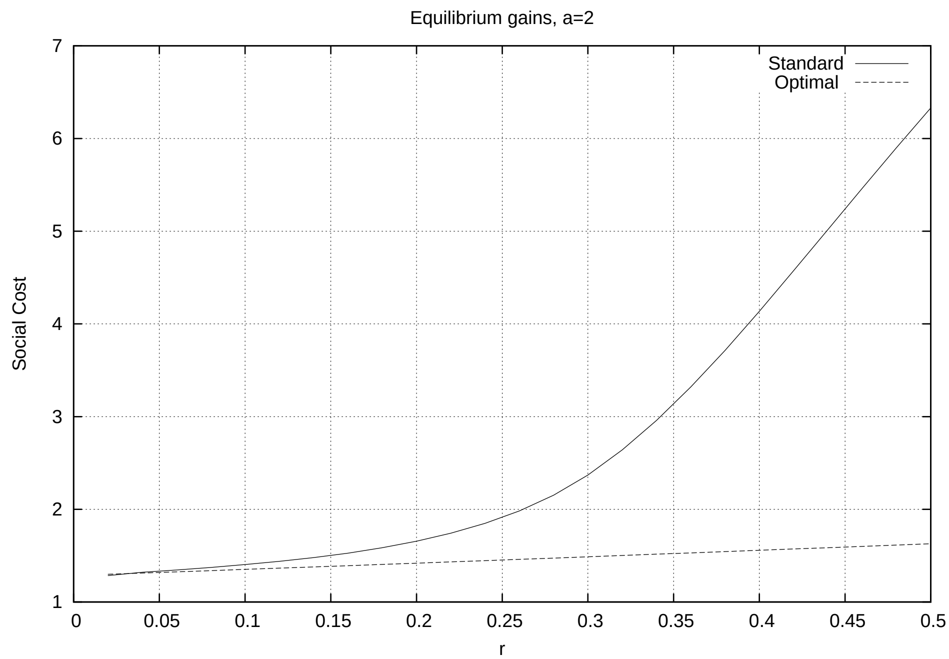


Fig. 3 The average total cost for the ISO (in the market described in Figure 2) as a function of the loss coefficient r for the standard mechanism and the optimal mechanism. We take $f_1(c) = f_2(c) = 2c + (1 - \frac{2}{4})1_{c \le \frac{1}{2}} + -2c + (1 + \frac{3}{2})1_{c \ge \frac{1}{2}}$. Note how r influences the social cost in the standard mechanism. The agents market power increase with r. When r goes to zero, the two mechanisms lead to the same social cost. When $r = 0$ we recover a classic result on first and second best auctions.

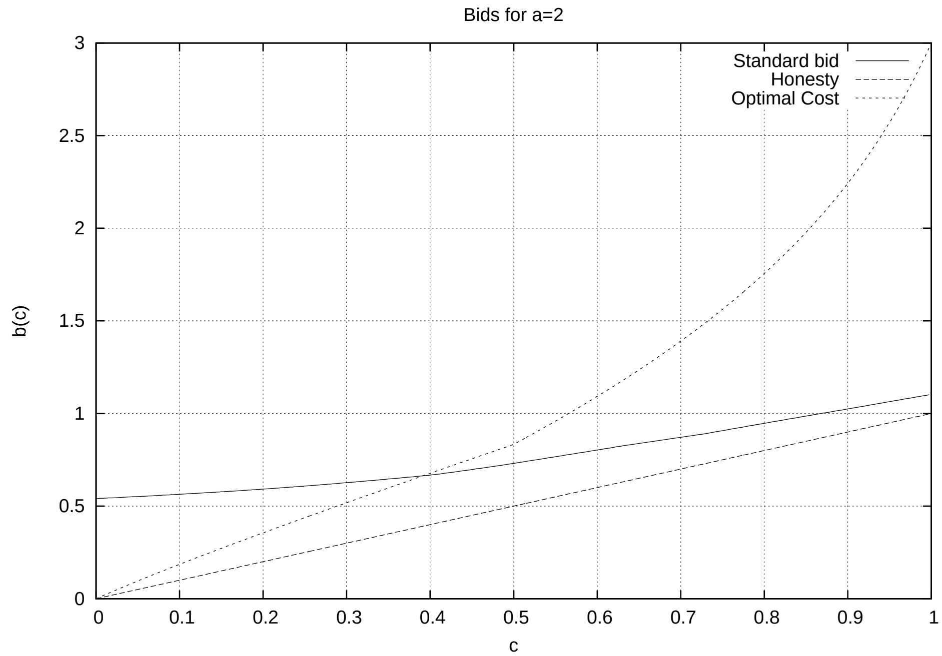


Fig. 4 Comparison of bidding strategy and information rent for the market described in Figure 2. The standard bid strategy corresponds to the equilibrium strategy of the Bayesian Game. The Honesty strategy correspond to a producer telling the truth. The optimal cost correspond to the sum of the truth-telling strategy and the information rent.

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