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Abstract. In this work we consider the general question: for a given algebraic formal power series with coefficients in a finite field, what kind of regularity (if any) can be expected for the partial quotients of the above power series in continued fraction expansion? Such a question is natural, since by a theorem of Christol, the coefficients of an algebraic power series over a finite field form an automatic sequence. Certain algebraic continued fractions are such that the sequence of the leading coefficients of the partial quotients is automatic. Here we give a rather general family of such sequences. Moreover, inspired by these examples, we give two criteria on automatic sequences, which allow us to obtain two new families of automatic sequences in an arbitrary finite field.

1. Introduction

For a given algebraic power series over a finite field, by a theorem of Christol, it is well known that the coefficients of the power series in question form an automatic sequence (see Theorem 1 below). Then one can ask what kind of regularity can be expected for the partial quotients of the above power series in continued fraction expansion. This question was put forward at first by Mendès France, initiated by Allouche [1], Allouche et al. [2], and continued by Mkaouar [20] and Yao [21]. Until now there are only examples and counterexamples, but not any general result.

The present work is a continuation of our article [18] in which we have approached the question from another point of view: consider the leading coefficients of the partial quotients instead of the partial quotients themselves. To know more about the motivation and the history, the reader may consult the introduction of [18] and the references therein.

Let $\mathbb{F}_q$ be the finite field containing q elements, with $q = p^s$ where p is a prime number and s is an integer such that $s \geqslant 1$. We denote by $\mathbb{F}(q)$ the field of power series in $1/T$, with coefficients in $\mathbb{F}_q$, where T is a formal indeterminate. Hence, an element in $\mathbb{F}(q)$ can be written as $\alpha = \sum_{k \leqslant k_0} u(k)T^k$, with $k_0 \in \mathbb{Z}$ and $u(k) \in \mathbb{F}_q$ for all integers k such that $k \leqslant k_0$. These fields of power series are analogues of the field of real numbers. As in the real case, it is well known that the sequence of coefficients of this power series α, $(u(k))_{k \leqslant k_0}$, is ultimately periodic if and only if α is rational, i.e., $\alpha \in \mathbb{F}_q(T)$. Moreover and remarkably, due to the rigidity of the positive characteristic case, this sequence of coefficients, for all elements in $\mathbb{F}(q)$ which are algebraic over $\mathbb{F}_q(T)$, belongs to a class of particular sequences introduced

Date: October 24, 2016.
2010 *Mathematics Subject Classification.* Primary 11J70, 11T55; Secondary 11B85.
Key words and phrases. finite fields, power series over a finite field, continued fractions, finite automata, automatic sequences.

by computer scientists. The theorem below can be found in the work [8] of Christol (see also the article [9] of Christol, Kamae, Mendès France, and Rauzy).

Theorem 1 (Christol)**.** *Let α in $\mathbb{F}(q)$ with $q = p^s$. Let $\big(u(k)\big)_{k\leqslant k_0}$ be the sequence of digits of α and $v(n) = u(-n)$ for all integers $n \geqslant 0$. Then α is algebraic over $\mathbb{F}_q(T)$ if and only if the following set of subsequences of $(v(n))_{n\geqslant 0}$*

$$K(v) = \left\{ \big(v(p^i n + j)\big)_{n\geqslant 0} \mid i \geqslant 0,\, 0 \leqslant j < p^i \right\}$$

is a finite set.

The sequences having the finiteness property stated in this theorem are called p-automatic sequences. A full account on this topic and a very complete list of references can be found in the book [3] of Allouche and Shallit.

Concerning algebraic elements in $\mathbb{F}(q)$, a particular subset needs to be considered. An irrational element α in $\mathbb{F}(q)$ is called hyperquadratic, if α^{r+1}, α^r, α, and 1 are linked over $\mathbb{F}_q(T)$, where $r = p^t$, and t is an integer such that $t \geqslant 0$. The subset of all these elements, noted $\mathcal{H}(q)$, contains not only the quadratic ($r = 1$) and the cubic power series ($r = p$ and note that the vector space over $\mathbb{F}_q(T)$ generated by all the α^j's with $j \geqslant 0$ has dimension equal to 3), but also algebraic elements of arbitrarily large degree. For various reasons, $\mathcal{H}(q)$ could be regarded as the analogue of the subset of quadratic real numbers, particularly when considering the continued fraction algorithm. See [7] for more information on this notion.

An irrational element α in $\mathbb{F}(q)$ can be expanded as an infinite continued fraction $\alpha = [a_1, a_2, \ldots, a_n, \ldots]$, where the partial quotients a_n are polynomials in $\mathbb{F}_q[T]$, all of positive degree, except perhaps for the first one. The explicit description of continued fractions for algebraic power series over a finite field goes back to Baum and Sweet [5, 6], and was carried on ten years later by Mills and Robbins [19]. It happens that this continued fraction expansion can be explicitly given for various elements in $\mathcal{H}(q)$. This is certainly the case for quadratic power series, where the sequence of partial quotients is simply ultimately periodic (as it is for quadratic real numbers). It was first observed by Mills and Robbins [19] that other hyperquadratic elements have also partial quotients of bounded degrees with an explicit continued fraction expansion, as for example the famous cubic over $\mathbb{F}_2$ introduced by Baum and Sweet [5]. Some of these examples, belonging to $\mathcal{H}(p)$ with $p \geqslant 5$, are such that $a_n = \lambda_n T$, for all integers $n \geqslant 1$, with $\lambda_n \in \mathbb{F}_p^*$. Then Allouche [1] showed that for each example given in [19], with $p \geqslant 5$, the corresponding sequence of partial quotients is automatic. Another case in $\mathcal{H}(3)$ (also given in [19]), having $a_n = \lambda_n T + \mu_n$, with $\lambda_n, \mu_n \in \mathbb{F}_3^*$ for all integers $n \geqslant 1$, was treated by Allouche et al. in [2]. Recently we have investigated the existence of such hyperquadratic power series, having partial quotients of degree 1, in the largest setting with odd characteristic (see [17] and particularly the comments in the last section). However, concerning the cubic power series introduced by Baum and Sweet in [5], Mkaouar [20] showed that the sequence of partial quotients (which takes only finitely many values) is not automatic (see also [21]). Besides, we know that most of the elements in $\mathcal{H}(q)$ have partial quotients of unbounded degrees (see for example the introduction in [18]). Hence, it appears that the link between automaticity and the sequence of partial quotients is not so straight.

With each infinite continued fraction in $\mathbb{F}(q)$, we can associate a sequence in $\mathbb{F}_q^*$ as follows: if $\alpha = [a_1, a_2, \ldots, a_n, \ldots]$ with $a_n \in \mathbb{F}_q[T]$, then for all integers $n \geqslant 1$, we define $u(n)$ as the leading coefficient of the polynomial a_n. For several examples in $\mathcal{H}(q)$, we have observed that this sequence $\big(u(n)\big)_{n\geqslant 1}$ is automatic. Indeed, a first observation in this area is the results of Allouche [1] cited above. Very recently we have described in [18] three other families of hyperquadratic continued fractions and have shown that the associated sequences as indicated above are automatic. For an algebraic (even hyperquadratic) power series, the possibility of describing explicitly the continued fraction expansion and consequently the sequence $\big(u(n)\big)_{n\geqslant 1}$ is sometimes a rather difficult problem. In the next section we shall begin with such a description given recently by the first author in [15] in characteristic 2. We show that the corresponding sequences belong to a large family of automatic sequences in an arbitrary finite field, by giving the explicit algebraic equation satisfied by the generating function attached to each such sequence. In a third section, we give two criteria on automaticity, which generalize [18, Theorem 2] and [4, Theorem 2.2] respectively. In Section 4, we apply the first criterion to present other recurrent and automatic sequences with values in an arbitrary finite field, more general than the preceding ones obtained in Section 2. In a last section, we use both criteria to study other particular sequences in a finite field of odd characteristic. These sequences are also related to hyperquadratic continued fractions, and were introduced by the first author several years ago.

2. Hyperquadratic continued fractions and automaticity

The starting point of the present work is a family of sequences, defined in a finite field of characteristic 2, which are derived from an algebraic continued fraction in power series fields. The proposition stated below is a simplified version of a theorem proved recently by the first author in [15], improving an earlier result in [16, Proposition 5, p. 556]. For the effective coefficients of the algebraic equation appearing in this proposition, the reader may consult [15].

Proposition 1. *Let $q = 2^s$ and $r = 2^t$ with $s, t \geqslant 1$ integers. Let $\ell \geqslant 1$ be an integer, and $\Lambda_{\ell+2} = (\lambda_1, \lambda_2, \dots, \lambda_\ell, \varepsilon_1, \varepsilon_2) \in (\mathbb{F}_q^*)^{\ell+2}$. We define the sequence $(\lambda_n)_{n\geqslant 1}$ in $\mathbb{F}_q^*$ recursively from the ℓ-tuple $(\lambda_1, \lambda_2, \dots, \lambda_\ell)$ as follows: for $m \geqslant 0$,*

$$(1) \qquad \begin{cases} \lambda_{\ell+rm+1} &= (\varepsilon_2/\varepsilon_1)\varepsilon_2^{(-1)^{m+1}}\lambda_{m+1}^r, \\ \lambda_{\ell+rm+i} &= (\varepsilon_1/\varepsilon_2)^{(-1)^i} \quad \text{for} \quad 2 \leqslant i \leqslant r. \end{cases}$$

Then there exist $(u, v, w, z) \in (\mathbb{F}_q[T])^4$ with $uv \neq 0$, depending on $\Lambda_{\ell+2}$, such that the continued fraction $\alpha = [\lambda_1 T, \lambda_2 T, \dots, \lambda_\ell T, \dots, \lambda_n T, \dots] \in \mathbb{F}(q)$, satisfies the following algebraic equation

$$u\alpha^{r+1} + v\alpha^r + w\alpha + z = 0.$$

We shall show that all the above sequences $(\lambda_n)_{n\geqslant 1}$ are 2-automatic. Here again, this underlines the existence of a link between automaticity and certain algebraic continued fractions, mentioned in the introduction. Indeed we are going to show the automaticity, via Christol's theorem, for a larger class of sequences with values in a finite field, including in particular the above ones as special cases. Actually the following result is slightly more precise than what we want: accidently we have obtained a family of hyperquadratic power series in characteristic 2, for which the power series defined by the leading coefficients of their partial quotients are also hyperquadratic. For these new hyperquadratic power series, one can consider again the continued fraction expansion and the leading coefficients of partial quotients, and then ask whether they form also an automatic sequence.

Theorem 2. *Let $\ell \geqslant 1$ be an integer, p a prime number, $q = p^s$ and $r = p^t$ with $s, t \geqslant 1$ integers. Let $k \geqslant 1$ be an integer dividing r. Let $(\lambda_1, \lambda_2, \ldots, \lambda_\ell) \in (\mathbb{F}_q)^\ell$, $(\alpha_1, \alpha_2, \ldots, \alpha_k) \in (\mathbb{F}_q^*)^k$, and $(\beta_1, \beta_2, \ldots, \beta_{r-1}) \in (\mathbb{F}_q)^{r-1}$. We define recursively in $\mathbb{F}_q$ the sequence $(\lambda_n)_{n\geqslant 1}$ as follows: for all integers $m \geqslant 0$,*

$$\left\{ \begin{array}{rclll} \lambda_{\ell+1+r(km+i)} & = & \alpha_{i+1}\lambda^r_{km+i+1}, & \text{for} & 0 \leqslant i < k, \\ \lambda_{\ell+1+rm+j} & = & \beta_j, & \text{for} & 1 \leqslant j < r. \end{array} \right. \tag{2}$$

Set $\theta = \sum_{n\geqslant 1} \lambda_n T^{-n}$. Then there exist ρ in $\mathbb{F}(q)$, and A, B, C in $\mathbb{F}_q(T)$ with $C \neq 0$ such that $\theta = A + \rho$ and $\rho = B + C\rho^r$. Hence θ is hyperquadratic, thus algebraic over $\mathbb{F}_q(T)$. In particular, the sequence $(\lambda_n)_{n\geqslant 1}$ is p-automatic.

Remark. *Note that the sequences in* (1) *correspond, in* (2)*, to the case below:*

$$p = 2,\ k = 2,\ \alpha_1 = \varepsilon_1^{-1},\ \alpha_2 = \varepsilon_2^2\varepsilon_1^{-1},\ \text{and } \beta_j = (\varepsilon_2/\varepsilon_1)^{(-1)^j}\ (1 \leqslant j \leqslant r-1).$$

Proof. According to Christol's theorem, the sequence $(\lambda_n)_{n\geqslant 1}$ is p-automatic if and only if θ is algebraic over $\mathbb{F}_q(T)$. In the following we shall only show that θ satisfies an algebraic equation of hyperquadratic type as indicated in the statement.

Define below two subsets of positive integers:

$$\mathbf{E} = \big\{\ell + rn + 1 \mid n \geqslant 0\big\}, \quad \text{and} \quad \mathbf{F} = \big\{\ell + rn + i \mid n \geqslant 0,\ 2 \leqslant i \leqslant r\big\}.$$

Then we have the partition $\mathbb{N}^* = \big\{1, \ldots, \ell\big\} \bigcup \mathbf{F} \bigcup \mathbf{E}$. Put

$$\rho = \sum_{n\in\mathbf{E}} \lambda_n T^{-n} = \sum_{m\geqslant 0} \lambda_{\ell+1+rm} T^{-(\ell+1+rm)}, \quad \text{and} \quad \rho_1 = \sum_{n\in\mathbf{F}} \lambda_n T^{-n}.$$

Then $\theta = \sum_{m=1}^{\ell} \lambda_m T^{-m} + \rho_1 + \rho$, and by the recursive relations (2), we obtain

$$\begin{aligned} \rho_1 &= \sum_{1\leqslant j<r} \sum_{m\geqslant 0} \lambda_{\ell+1+rm+j} T^{-(\ell+1+rm+j)} = \sum_{1\leqslant j<r} \sum_{m\geqslant 0} \beta_j T^{-(\ell+1+rm+j)} \\ &= \Big(\sum_{m\geqslant 0} T^{-rm}\Big)\Big(\sum_{1\leqslant j<r} \beta_j T^{-\ell-1-j}\Big) = (1 - T^{-1})^{-r} \sum_{1\leqslant j<r} \beta_j T^{-\ell-1-j}, \end{aligned}$$

since we have $\sum_{m\geqslant 0} T^{-m} = (1 - T^{-1})^{-1}$ in $\mathbb{F}(q)$. So we can write $\theta = A + \rho$, with

$$A = \sum_{m=1}^{\ell} \lambda_m T^{-m} + (1 - T^{-1})^{-r} \sum_{1\leqslant j<r} \beta_j T^{-\ell-1-j} \in \mathbb{F}_q(T).$$

To simplify the notation, we extend the finite sequence $(\alpha_i)_{1\leqslant i\leqslant k}$ into a purely periodic sequence of period length k, denoted by $(\alpha_n)_{n\geqslant 1}$. From the recursive relations (2), by noting that $\mathbf{E} = \{\ell+1+r(km+i) \mid m \geqslant 0,\ 0 \leqslant i < k\}$, we obtain

$$\begin{aligned}\rho &= \sum_{0\leqslant i<k}\sum_{m\geqslant 0}\lambda_{\ell+1+r(km+i)}T^{-(\ell+1+r(km+i))}\\ &= \sum_{0\leqslant i<k}\sum_{m\geqslant 0}\alpha_{i+1}\lambda^r_{km+i+1}T^{-(\ell+1+r(km+i))}\\ &= \sum_{0\leqslant i<k}\sum_{m\geqslant 0}\alpha_{km+i+1}\lambda^r_{km+i+1}T^{-(\ell+1+r(km+i))}\\ &= T^{r-\ell-1}\sum_{n\geqslant 1}\alpha_n\lambda_n^rT^{-rn}.\end{aligned}$$

By using again the above partition of $\mathbb{N}^*$, we deduce

$$T^{\ell+1-r}\rho = \sum_{m=1}^{\ell}\alpha_m\lambda_m^rT^{-rm} + \sum_{n\in\mathbf{E}}\alpha_n\lambda_n^rT^{-rn} + \sum_{n\in\mathbf{F}}\alpha_n\lambda_n^rT^{-rn}. \tag{3}$$

Since k divides r, we have $\alpha_{\ell+1+rm+j} = \alpha_{\ell+1+j}$ by periodicity. By (2), we obtain

$$\begin{aligned}\sum_{n\in\mathbf{F}}\alpha_n\lambda_n^rT^{-rn} &= \sum_{1\leqslant j<r}\sum_{m\geqslant 0}\alpha_{\ell+1+rm+j}\lambda^r_{\ell+1+rm+j}T^{-r(\ell+1+rm+j)}\\ &= \sum_{1\leqslant j<r}\sum_{m\geqslant 0}\alpha_{\ell+1+j}\beta_j^rT^{-r(\ell+1+rm+j)}\\ &= \Big(\sum_{m\geqslant 0}T^{-mr^2}\Big)\Big(\sum_{1\leqslant j<r}\alpha_{\ell+1+j}\beta_j^rT^{-r(\ell+1+j)}\Big)\\ &= (1-T^{-1})^{-r^2}\sum_{1\leqslant j<r}\alpha_{\ell+1+j}\beta_j^rT^{-r(\ell+1+j)}.\end{aligned} \tag{4}$$

By periodicity again, we have $\alpha_{\ell+1+rm} = \alpha_{\ell+1}$, and thus

$$\begin{aligned}\sum_{n\in\mathbf{E}}\alpha_n\lambda_n^rT^{-rn} &= \sum_{m\geqslant 0}\alpha_{\ell+1+rm}\lambda^r_{\ell+1+rm}T^{-r(\ell+1+rm)}\\ &= \alpha_{\ell+1}\sum_{m\geqslant 0}\lambda^r_{\ell+1+rm}T^{-r(\ell+1+rm)}\\ &= \alpha_{\ell+1}\rho^r.\end{aligned} \tag{5}$$

Combining (3), (4), and (5), we obtain $\rho = B + C\rho^r$, with $C = \alpha_{\ell+1}T^{r-\ell-1}$ and

$$B = T^{r-\ell-1}\Big(\sum_{m=1}^{\ell}\alpha_m\lambda_m^rT^{-rm} + (1-T^{-1})^{-r^2}\sum_{1\leqslant j<r}\alpha_{\ell+1+j}\beta_j^rT^{-r(\ell+1+j)}\Big).$$

Thus ρ and also θ are algebraic over $\mathbb{F}_q(T)$, and the proof is complete. □

From a number-theoretic point of view, the sequences described in Proposition 1 are most important, for they are associated with an algebraic continued fraction. Such an association is not relevant for the more general sequences discussed in Theorem 2, as well as for others of a similar type, still more general, considered in Section 4. Hence coming back to the sequences $(\lambda_n)_{n\geqslant 1}$ in a finite field of characteristic 2, defined by the recursive relations (1), a natural question arises: what can be said about the algebraic degree over $\mathbb{F}_q(T)$ of the continued fraction $\alpha = [\lambda_1T, \lambda_2T, \ldots, \lambda_nT, ...]$? By Proposition 1, this degree is in the range $[2, r+1]$, since α is irrational and satisfies an algebraic equation of degree $r+1$. It is a classical fact that α is quadratic if and only if the sequence $(\lambda_n)_{n\geqslant 1}$ is ultimately periodic. Thus α is quadratic if and only if θ (the generating function of the sequence introduced in Theorem 2) is rational.

In the particular and simplest case $r=2$, we are able to give a necessary and sufficient condition to guarantee this rationality.

Proposition 2. *Let $(\lambda_n)_{n\geqslant 1}$ be the sequence defined by the recursive formulas (1), assuming that we have $r=2$. Then this sequence is periodic (and purely periodic of period length less or equal to 2) if and only if we have*

$$\lambda_m=(\varepsilon_1/\varepsilon_2)\varepsilon_2^{((-1)^\ell-(-1)^m)/2}\quad \textit{for}\quad 1\leqslant m\leqslant \ell.$$

Proof. We shall apply Theorem 2 in the particular case $p=2$, $k=2$, and $r=2$. Hence $\theta=A+\rho$ and $\rho=B+C\rho^2$. Let $V\in\mathbb{F}_q(T)$ be given. Then we have

$$V+\rho=V+B+CV^2+C(V+\rho)^2.$$

Setting $\xi=C(V+\rho)$, and multiplying this last equality by C, we obtain

$$\xi=CV+CB+(CV)^2+\xi^2=U+\xi^2, \tag{6}$$

with $U=CV+CB+(CV)^2$. Note that the sequence $(\alpha_m)_{m\geqslant 1}$ is 2-periodic, and $\alpha_m=(\varepsilon_2/\varepsilon_1)\varepsilon_2^{(-1)^m}$ for $m\geqslant 1$. Note also that $\alpha_{\ell+1}\alpha_{\ell+2}\beta_1^2=\alpha_1\alpha_2\beta_1^2=1$,

$$C=\alpha_{\ell+1}T^{1-\ell},\quad \text{and}\quad B=T^{1-\ell}\Big(\sum_{m=1}^{\ell}\alpha_m\lambda_m^2T^{-2m}+\alpha_{\ell+2}\beta_1^2(1+T)^{-4}T^{-2\ell}\Big).$$

By taking $V=\alpha_{\ell+1}^{-1}T^{1-\ell}(T+1)^{-2}$, we obtain directly

$$U=CV+CB+(CV)^2=T^{2-2\ell}\sum_{m=1}^{\ell}(\alpha_{\ell+1}\alpha_m\lambda_m^2+1)T^{-2m}.$$

Put $u_m=\alpha_{\ell+1}\alpha_m\lambda_m^2+1$ for all integers $m\,(1\leqslant m\leqslant \ell)$. Then we can write

$$U=u_1T^{-2\ell}+u_2T^{-2\ell-2}+\cdots+u_\ell T^{-4\ell+2},$$

and we obtain $\xi=\sum_{m\geqslant 0}U^{2^m}$ by the formula (6). If $U\neq 0$, there exist arbitrarily long blocks of zeros in the $(1/T)$-power series expansion of ξ, and then ξ is irrational. So $\xi\in\mathbb{F}_q(T)$ if and only if $U=0$, i.e, for all integers $m\,(1\leqslant m\leqslant \ell)$, we have

$$\lambda_m^2=(\alpha_{\ell+1}\alpha_m)^{-1}=(\varepsilon_1/\varepsilon_2)^2\varepsilon_2^{(-1)^\ell-(-1)^m},$$

which implies in turn that the sequence $(\lambda_n)_{n\geqslant 1}$ is 2-periodic: $\varepsilon_1/\varepsilon_2,\varepsilon_1/\varepsilon_2^2,\dots$ or $\varepsilon_1,\varepsilon_1/\varepsilon_2,\dots$ according to the parity of ℓ. To conclude, it suffices to note that we have $\theta=A+V+C^{-1}\xi$, with $A,V\in\mathbb{F}_q(T)$. □

Remark. *The expressions $\theta=A+V+C^{-1}\xi$ and $\xi=U+\xi^2$ had been stated without proof in* [15]*. Moreover we can also observe that θ is rational if and only if α is quadratic, and this is equivalent to say $\alpha=[\lambda_1T,\lambda_2T,\dots,\lambda_1T,\lambda_2T,\dots]$. Hence, if $(\lambda_n)_{n\geqslant 1}$ is not purely 2-periodic, then α is cubic over $\mathbb{F}_q(T)$. Furthermore, if we define $\omega(T)=[T,T,\dots,T,\dots]$ (which is the analogue in the formal case of the golden ration $(1+\sqrt{5})/2=[1,1,\dots,1,\dots]$), then α is quadratic if and only if we have $\alpha(T)=(\lambda_1/\lambda_2)^{q/2}\omega((\lambda_1\lambda_2)^{q/2}T)$.*

3. Two criteria for automatic sequences

Motivated by the form of the sequences presented in Theorem 2 above and also by [18, Theorem 2] and [4, Theorem 2.2], we shall give in this section two new criteria for automatic sequences. Here we consider sequences of the form $v = \big(v(n)\big)_{n\geqslant 1}$. Let $r \geqslant 2$ be an integer. Equivalently, the sequence v is r-automatic if its r-kernel

$$K_r(v) = \Big\{\big(v(r^i n + j)\big)_{n\geqslant 1} \mid i \geqslant 0,\, 0 \leqslant j < r^i\Big\}$$

is a finite set (see Cobham [11, p. 170, Theorem 1], see also Eilenberg [12, p. 107, Proposition 3.3]). For more details on automatic sequences, see the book [3] of Allouche and Shallit. Recall that all ultimately periodic sequences are r-automatic for all integers $r \geqslant 2$, adding or chopping off a prefix to a sequence does not change its automaticity (see [3, p. 165]), the pointwise product of two r-automatic sequences with values in a semigroup is r-automatic (see [3, Corollary 5.4.5, p. 166]), a sequence is r-automatic if and only if it is r^m-automatic for all integers $m \geqslant 1$ (see [3, Theorem 6.6.4, p. 187]), and that v is r-automatic if all the subsequences $\big(v(an+b)\big)_{n\geqslant 1}$ $(0 \leqslant b < a)$ are r-automatic, where $a \geqslant 1$ is a fixed integer (see [3, Theorem 6.8.2, p. 190]). All these results will be used later in this work.

For all integers j, n $(0 \leqslant j < r$ and $n \geqslant 1)$, define

$$(T_j v)(n) = v(rn + j).$$

Then for all integers $n, a\,(n, a \geqslant 1)$, and $b\,(0 \leqslant b < r^a)$ in r-adic expansion

$$b = \sum_{l=0}^{a-1} b_l r^l \qquad (0 \leqslant b_l < r),$$

with the help of the operators T_j $(0 \leqslant j < r)$, we have

$$v(r^a n + b) = (T_{b_{a-1}} \circ T_{b_{a-2}} \circ \cdots \circ T_{b_0} v)(n).$$

In particular, we obtain that v is r-automatic if and only if all $T_j v\,(0 \leqslant j < r)$ are r-automatic, for we have $K_r(v) = \{v\} \cup \bigcup_{j=0}^{r-1} K_r(T_j v)$.

The following criterion for automatic sequences generalizes Theorem 2 in [18].

Theorem 3. *Let $r \geqslant 2$ be an integer. Let $v = \big(v(n)\big)_{n\geqslant 1}$ be a sequence in a finite set $\mathbb{A}$, and σ a bijection on $\mathbb{A}$. Fix an integer i with $0 \leqslant i < r$. Then, for all integer $m \geqslant 0$, we have the following statement:*

(i_m) *If $(T_i v)(n+m) = \sigma\big(v(n)\big)$ for all integers $n \geqslant 1$, and $T_j v$ is r-automatic for all integers j $(0 \leqslant j < r$ and $j \neq i)$, then v is r-automatic.*

Proof. Since $\mathbb{A}$ is finite and σ is a bijection on $\mathbb{A}$, there exists an integer $l \geqslant 1$ such that $\sigma^l = \mathrm{id}_{\mathbb{A}}$, the identity mapping on $\mathbb{A}$. In the following we shall show (i_m) by induction on m. For this, we need only show that under the conditions of (i_m), the sequence $T_i v$ is r-automatic.

If $m = 0$, then under the conditions of (i_0), we have $T_i v = \sigma(v)$, and then

$$K_r(T_i v) = \{\sigma^a(T_i v) \mid 0 \leqslant a < l\} \cup \bigcup_{\substack{0\leqslant b<l \\ 0\leqslant j<r,\, j\neq i}} \sigma^b\big(K_r(T_j v)\big),$$

so $K_r(T_i v)$ is finite, as $T_j v$ is r-automatic for all integers j $(0 \leqslant j < r)$ with $j \neq i$.

If $m = 1$, under the conditions of (i_1), we have $(T_i v)(n+1) = \sigma\big(v(n)\big)$ for all integers $n \geqslant 1$, and $T_j v$ is r-automatic for all integers j $(0 \leqslant j < r)$ with $j \neq i$.

Below we distinguish two cases:

Case I: $0 \leqslant i \leqslant r-2$. Then for all integers $n \geqslant 1$, we have

$$(T_0 T_i v)(n+1) = (T_i v)(rn+r) = \sigma\big(v(rn+r-1)\big) = \sigma\big((T_{r-1}v)(n)\big),$$

hence $T_0(T_i v)$ is r-automatic, since it is obtained from $\sigma(T_{r-1}v)$ by adding a letter before, and $T_{r-1}v$ is r-automatic by hypothesis, for $r-1 \neq i$.

Let j be an integer such that $1 \leqslant j < r$. Then for all integers $n \geqslant 1$,

$$(T_j T_i v)(n) = (T_i v)(rn+j) = \sigma\big(v(rn+j-1)\big) = \sigma\big((T_{j-1}v)(n)\big).$$

Hence if $j \neq i+1$, then $T_j(T_i v)$ is r-automatic, for $j-1 \neq i$, and thus $T_{j-1}v$ is k-automatic by hypothesis. Moreover for $j = i+1$, we have $T_{i+1}(T_i v) = \sigma(T_i v)$. Note that $T_j(T_i v)$ is r-automatic for all integers j $(0 \leqslant j < r)$ with $j \neq i+1$, hence we can apply $((i+1)_0)$ with $T_i v$, and we obtain that $T_i v$ is r-automatic.

Case II: $i = r-1$. Then for all integers j, n $(1 \leqslant j < r$ and $n \geqslant 1)$, we have

$$\begin{aligned}(T_j T_{r-1} v)(n) &= (T_{r-1}v)(rn+j)\\ &= \sigma\big(v(rn+j-1)\big) = \sigma\big((T_{j-1}v)(n)\big).\end{aligned}$$

So $T_j(T_{r-1}v)$ is r-automatic, for $j-1 \neq i$, and thus $T_{j-1}v$ is r-automatic by hypothesis. Moreover for all integers $n \geqslant 1$, we have

$$\begin{aligned}(T_0 T_{r-1} v)(n+1) &= (T_{r-1}v)(rn+r)\\ &= \sigma\big(v(rn+r-1)\big) = \sigma\big((T_{r-1}v)(n)\big).\end{aligned}$$

Since $T_j(T_{r-1}v)$ is r-automatic for all integers j $(1 \leqslant j < r)$, we can apply (0_1) proved above with $T_{r-1}v$, and we obtain that $T_{r-1}v$ is r-automatic.

Now let $m \geqslant 1$ be an integer, and assume that (i_j) holds for all integers i, j with $0 \leqslant i < r$ and $0 \leqslant j \leqslant m$. We shall show that (i_{m+1}) holds for all integers i with $0 \leqslant i < r$. Namely, under the conditions that $(T_i v)(n+m+1) = \sigma\big(v(n)\big)$ for all integers $n \geqslant 1$, and $T_j v$ is r-automatic for all integers j $(0 \leqslant j < r)$ with $j \neq i$, we shall show that $T_i v$ is r-automatic. For this, we distinguish two cases below.

Write $m = r[m/r] + a$, with a an integer such that $0 \leqslant a < r$.

Case I: $0 \leqslant i < r-a-1$. Let j be an integer such that $0 \leqslant j < r$. If $j < a+1$, then for all integers $n \geqslant 1$, we have

$$\begin{aligned}(T_j T_i v)(n+[m/r]+1) &= (T_i v)(rn+r[m/r]+r+j)\\ &= (T_i v)(rn+m+r+j-a)\\ &= \sigma\big(v(rn+r+j-a-1)\big)\\ &= \sigma\big((T_{r+j-a-1}v)(n)\big),\end{aligned}$$

hence $T_j(T_i v)$ is r-automatic, since it is obtained from $\sigma(T_{r+j-a-1}v)$ by adding a prefix of length $[m/r]+1$, and the latter is r-automatic by hypothesis, for we have $j \geqslant 0 > i+a+1-r$. Assume $j \geqslant a+1$. Then for all integers $n \geqslant 1$, we have

$$\begin{aligned}(T_j T_i v)(n+[m/r]) &= (T_i v)(rn+r[m/r]+j) = (T_i v)(rn+m+j-a)\\ &= \sigma\big(v(rn+j-a-1)\big) = \sigma\big((T_{j-a-1}v)(n)\big).\end{aligned}$$

If $j \neq i+a+1$, then $T_j(T_i v)$ is r-automatic, since it is obtained from $\sigma(T_{j-a-1}v)$ by adding a prefix of length $[m/r]$, and the latter is r-automatic by hypothesis, for we have $j-a-1 \neq i$. If $j = i+a+1$, then for all integers $n \geqslant 1$, we have

$$(T_j T_i v)(n+[m/r]) = \sigma\big((T_i v)(n)\big).$$

Note here that $[m/r] \leqslant m$ and $T_j(T_i v)$ is r-automatic for all integers j $(0 \leqslant j < r)$ with $j \neq i+a+1$, hence we can apply $((i+a+1)_{[m/r]})$ with $T_i v$, and we obtain at once that $T_i v$ is r-automatic.

Case II: $r-a-1 \leqslant i < r$. Let j $(0 \leqslant j < r)$ be an integer. If $j \geqslant a+1$, then for all integers $n \geqslant 1$, we have

$$\begin{aligned}(T_j T_i v)(n+[m/r]) &= (T_i v)(rn + r[m/r] + j) = (T_i v)(rn + m + j - a) \\ &= \sigma\big(v(rn + j - a - 1)\big) = \sigma\big((T_{j-a-1}v)(n)\big),\end{aligned}$$

hence $T_j(T_i v)$ is r-automatic, since it is obtained from $\sigma(T_{j-a-1}v)$ by adding a prefix of length $[m/r]$, and the latter is r-automatic by hypothesis, for we have $i \geqslant r-a-1 > j-a-1$. If $j < a+1$, then for all integers $n \geqslant 1$, we have

$$\begin{aligned}(T_j T_i v)(n+[m/r]+1) &= (T_i v)(rn + r[m/r] + r + j) \\ &= (T_i v)(rn + m + r + j - a) \\ &= \sigma\big(v(rn + r + j - a - 1)\big) = \sigma\big((T_{r+j-a-1}v)(n)\big).\end{aligned}$$

If $j \neq i+a+1-r$, then $T_j(T_i v)$ is r-automatic, since it is obtained from $\sigma(T_{r+j-a-1}v)$ by adding a prefix of length $[m/r]+1$, and the latter is r-automatic by hypothesis, for we have $r+j-a-1 \neq i$. If $j = i+a+1-r$, then for all integers $n \geqslant 1$,

$$(T_j T_i v)(n+[m/r]+1) = \sigma\big((T_i v)(n)\big).$$

Now that $[m/r]+1 \leqslant m$ and $T_j(T_i v)$ is r-automatic for all integers j $(0 \leqslant j < r)$ with $j \neq i+a+1-r$, hence we can apply $((i+a+1-r)_{[m/r]+1})$ with $T_i v$, and we obtain that $T_i v$ is r-automatic.

Finally we obtain that (i_m) holds for all integers i, m $(0 \leqslant i < r$ and $m \geqslant 0)$. $\square$

Remarks. *1) By the same argument as above, one can show that Theorem 3 holds also for all integers $m \leqslant 0$, thus for all $m \in \mathbb{Z}$.*

2) One can ask what's the size of $K_r(v)$, which cannot be obtained from the inductive proof given above. In principle, this is not an easy task, and precisely for this reason, one cannot deduce directly from Theorem 5, in the next section, the algebraic equation of hyperquadratic type presented in Theorem 2.

Again, inspired by Theorem 2.2 obtained by Allouche and Shallit in [4], we give below another criterion for automatic sequences which generalizes and improves slightly the latter. This criterion will be needed in the last section. The proof is a reformulation of that of Theorem 2.2 cited above, but with appropriate modifications and more details.

As above, let $r \geqslant 2$ be an integer, and $v = \big(v(n)\big)_{n\geqslant 1}$ a sequence with values in a finite set $\mathbb{A}$. Fix $i, j, l \in \mathbb{Z}$ $(i \geqslant 0$ and $0 \leqslant j < r^i)$. For all $n \geqslant (1-j)r^{-i} - l$, put

$$v_{i,j}^{(l)}(n) = v\big(r^i(n+l) + j\big).$$

If $(1-j)r^{-i} - l > 1$, then we define $v_{i,j}^{(l)}(n) = a_0$ $(1 \leqslant n < (1-j)r^{-i} - l)$, with $a_0 \in \mathbb{A}$ an element fixed in advance. Let a, b be two integers such that $a, b \geqslant 0$. We denote by $\mathbb{A}_{a,b}$ the set of all tensors $(c_{i,j}^{(l)})_{|l|\leqslant a, 0\leqslant i\leqslant b, 0\leqslant j<r^i}$ with $c_{i,j}^{(l)} \in \mathbb{A}$, and by $\mathbb{M}_{a,b}$ the set of all mappings from $\mathbb{A}_{a,b}$ to $\mathbb{A}$. Both $\mathbb{A}_{a,b}$ and $\mathbb{M}_{a,b}$ are finite sets.

Theorem 4. *Let $r \geqslant 2$ be an integer, and $v = \big(v(n)\big)_{n\geqslant 1}$ be a sequence in a finite set $\mathbb{A}$. Assume that there exist integers $a, b, N_0 \geqslant 0$ and $f_m \in \mathbb{M}_{a,b}$ $(0 \leqslant m < r^{b+1})$ such that for all integers m, n $(0 \leqslant m < r^{b+1}$ and $n > N_0)$, we have*

$$v(r^{b+1}n + m) = v^{(0)}_{b+1,m}(n) = f_m\Big(\big(v^{(l)}_{i,j}(n)\big)_{|l|\leqslant a, 0\leqslant i\leqslant b, 0\leqslant j<r^i}\Big). \tag{7}$$

Then the sequence v is r-automatic.

Proof. To conclude, it suffices to construct a finite set of sequences $\mathbb{E}$ containing v such that if $u \in \mathbb{E}$, then $T_m u \in \mathbb{E}$, for all integers m $(0 \leqslant m < r)$.

Fix an integer N such that $N \geqslant N_0 + r(a+1)/(r-1)$. Let $\mathbb{E}$ be the set of all sequences $u = (u(n))_{n\geqslant 1}$ with values in $\mathbb{A}$ such that for all integers $n > N$, we have

$$u(n) = f\Big(\big(v^{(l)}_{i,j}(n)\big)_{|l|\leqslant N, 0\leqslant i\leqslant b, 0\leqslant j<r^i}\Big),$$

where $f \in \mathbb{M}_{N,b}$. Since $a < N$, we can treat $\mathbb{M}_{a,b}$ as a subset of $\mathbb{M}_{N,b}$. From the definition and the finiteness of $\mathbb{A}$ and $\mathbb{M}_{N,b}$, we know that $\mathbb{E}$ is finite. Moreover $v^{(l)}_{i,j} \in \mathbb{E}$ $(|l| \leqslant N, 0 \leqslant i \leqslant b, 0 \leqslant j < r^i)$, and by using composition of mappings, we need only to show $T_m v^{(l)}_{i,j} \in \mathbb{E}$, for all integers m $(0 \leqslant m < r)$.

For all integers $n \geqslant \max((1-j)r^{-i} - l, 1)$, we have

$$T_m v^{(l)}_{i,j}(n) = v^{(l)}_{i,j}(rn + m) = v\big(r^i(rn + m + l) + j\big).$$

By the division algorithm, we can write

$$r^i(m + l) + j = r^{i+1}x + y,$$

with $x, y \in \mathbb{Z}$ and $0 \leqslant y < r^{i+1} \leqslant r^{b+1}$. Then we obtain

$$T_m v^{(l)}_{i,j}(n) = v\big(r^{i+1}(n + x) + y\big) = v^{(x)}_{i+1,y}(n). \tag{8}$$

Note that $r^{i+1}x \leqslant r^{i+1}x + y = r^i(m + l) + j < r^i(m + l + 1)$, and

$$r^{i+1}x = r^i(m + l) + j - y > r^i(m + l) - r^{i+1}.$$

Hence $x < (m + l + 1)/r \leqslant (r + N)/r \leqslant N$, and

$$x > (m + l - r)/r \geqslant (m - N - r)/r \geqslant -(N + r)/r \geqslant -N,$$

for we have $N \geqslant N_0 + r(a + 1)/(r - 1) \geqslant r/(r - 1)$. Consequently $|x| \leqslant N$.

We distinguish two cases below.

Case I: $i < b$. Then $i + 1 \leqslant b$, and by the relation (8), we obtain $T_m v^{(l)}_{i,j} \in \mathbb{E}$.

Case II: $i = b$. By the condition (7), for all integers $n > N$, we have

$$T_m v^{(l)}_{i,j}(n) = v^{(x)}_{b+1,y}(n) = f_y\Big(\big(v^{(e+x)}_{c,d}(n)\big)_{|e|\leqslant a, 0\leqslant c\leqslant b, 0\leqslant d<r^c}\Big).$$

To conclude, it suffices to show $v^{(e+x)}_{c,d} \in \mathbb{E}$ $(|e| \leqslant a, 0 \leqslant c \leqslant b, 0 \leqslant d < r^c)$, and then apply composition of mappings. But $N \geqslant r(a + 1)/(r - 1)$, thus

$$e + x \geqslant x - a > -(N + r)/r - a \geqslant -N,$$

and $e + x \leqslant x + a < (r + N)/r + a \leqslant N$. So $|e + x| \leqslant N$, and then $v^{(e+x)}_{c,d} \in \mathbb{E}$. $\square$

4. A first family of automatic sequences

As an application of Theorem 3, we have the following result which slightly generalizes the part of Theorem 2 concerning the automaticity, leaving aside the hyperquadratic equation given there. Again, note that in general the sequences considered in this section are not related to algebraic continued fractions.

Theorem 5. *Let $\ell \geqslant 1$, $r \geqslant 2$, and $k \geqslant 1$ be integers such that k divides r. Let p be a prime number and $q = p^s$, with $s \geqslant 1$ an integer. Let $\big(u(1), u(2), \ldots, u(\ell)\big)$ be given in $\mathbb{F}_q^\ell$, and $u = \big(u(n)\big)_{n\geqslant 1}$ be the sequence in the finite field $\mathbb{F}_q$ such that for all integers m, i ($m \geqslant 0$ and $0 \leqslant i < k$), we have*

$$\text{(9)} \quad \begin{cases} u\big(\ell+1+r(km+i)\big) &= \alpha_{i+1}\big(u(km+i+1)\big)^\gamma, \\ u\big(\ell+1+r(km+i)+j\big) &= \beta_{i,j}, \qquad \textit{with } 1 \leqslant j < r, \end{cases}$$

where α_{i+1} $(0 \leqslant i < k)$ in $\mathbb{F}_q^*$, $\beta_{i,j}$ $(1 \leqslant j < r)$ in $\mathbb{F}_q$ are fixed elements, and $\gamma \geqslant 1$ is an integer coprime to $q-1$. Then the sequence u is r-automatic.

Proof. For all integers i, n ($0 \leqslant i < r$ and $n \geqslant 1$), set $u_i(n) = u(rn+i)$, and we need only show that all the u_i $(0 \leqslant i < r)$ are r-automatic.

Write $\ell+1 = ra+b$, with a, b integers such that $a \geqslant 0$, $0 \leqslant b < r$. Then $a+b \geqslant 1$. From the recursive relations (9), we deduce at once that all the u_j $(0 \leqslant j < r)$ are ultimately periodic except for $j = b$, and for all integers m, i ($m \geqslant 0$ and $0 \leqslant i < k$),

$$u_b(km+i+a) = \alpha_{i+1}\big(u(km+i+1)\big)^\gamma.$$

Since all the ultimately periodic sequences are r-automatic, it remains for us to show that the sequence u_b is r-automatic.

Extend the finite sequence $(\alpha_i)_{1\leqslant i\leqslant k}$ to be a purely periodic sequence of period length k, denoted by $(\alpha_n)_{n\geqslant 0}$. Then for all integers $n \geqslant 1$, we have

$$u_b(n-1+a) = \alpha_n\big(u(n)\big)^\gamma,$$

from which, by noting that k divides r, we obtain, for all integers $m \geqslant 1$,

$$\begin{cases} u_b(rm+a+b-1) &= \alpha_{rm+b}\big(u_b(m)\big)^\gamma = \alpha_b\big(u_b(m)\big)^\gamma, \\ u_b(rm+a+j-1) &= \alpha_{rm+j}\big(u_j(m)\big)^\gamma = \alpha_j\big(u_j(m)\big)^\gamma \ (0 \leqslant j < r,\ j \neq b). \end{cases}$$

Write $a+b-1 = rc+d$, with c, d integers such that $c \geqslant 0$ and $0 \leqslant d < r$. Then all the $T_i u_b$ $(0 \leqslant i < r)$ are ultimately periodic (thus r-automatic) except for $i = d$, for all the u_j $(0 \leqslant j < r,\ j \neq b)$ are ultimately periodic. Moreover for all $n \geqslant 1$,

$$(T_d u_b)(m+c) = u_b(rm+a+b-1) = \alpha_b\big(u_b(m)\big)^\gamma = \sigma\big(u_b(m)\big),$$

where $\sigma(x) = \alpha_b x^\gamma$, for all $x \in \mathbb{F}_q$. Now that $\alpha_b \neq 0$ and γ is coprime to $q-1$, thus the mapping σ is bijective on $\mathbb{F}_q$. To conclude, it suffices to apply Theorem 3 with u_b to obtain that u_b is also r-automatic. $\square$

Remark. *In the above theorem, if r is a power of p and $\gamma = r$, then we can argue similarly as for Theorem 2 to obtain the algebraic equation of hyperquadratic type, and the automaticity follows directly from Christol's theorem, as we have seen, so Theorem 2 is slightly more precise than Theorem 5 in this case. If r is a power of p but $\gamma \neq r$, then the generating function F of the sequence u is still algebraic by Christol's theorem, but the algebraic equation is not given and it may not be as simple (hyperquadratic type) as it is in Theorem 2. Finally if r is not a power of p, then r and p are multiplicatively independent and thus F may not be algebraic* *over $\mathbb{F}_q(T)$. Indeed if F is not rational, then u is not ultimately periodic, hence by a classical theorem of Cobham* [10]*, it cannot be p-automatic, consequently F is transcendental over $\mathbb{F}_q(T)$ by Christol's theorem.*

5. A second family of automatic sequences

In this last section, we consider families of sequences in a finite field of odd characteristic. They have been discussed by the first author in several papers, and their origin can be found in [13]. As those introduced above in Proposition 1, they are also derived from hyperquadratic continued fractions. Here, we shall consider the more general setting, as it was presented in [14], and then show by Theorem 3 and Theorem 4 that they are automatic sequences.

Below we recall briefly the origin of these sequences and the way in which they are built. For more details, the reader may consult [14, p. 252-257].

Let p be an odd prime number. Set $q = p^s$ and $r = p^t$ with $s, t \geqslant 1$ integers. Let $\ell \geqslant 1$ be an integer and fix $\Lambda_{\ell+2} = (\lambda_1, \lambda_2, \ldots, \lambda_\ell, \varepsilon_1, \varepsilon_2) \in (\mathbb{F}_q^*)^{\ell+2}$. We consider a finite set $E(r)$ of positive integers, defined as follows:

$$E(r) = \left\{mp^j + (p^j - 1)/2 \mid 1 \leqslant m \leqslant (p-1)/2,\ 0 \leqslant j \leqslant t-1\right\}.$$

Note that we have $\left\{1, \ldots, (p-1)/2\right\} \subset E(r) \subset \left\{1, \ldots, (r-1)/2\right\}$.

In the sequel, we fix an integer $k \in E(r)$. Given $\Lambda_{\ell+2}$ and k, we can build an infinite continued fraction α in $\mathbb{F}(q)$ such that

$$\alpha = [\lambda_1 T, \lambda_2 T, \ldots, \lambda_\ell T, a_{\ell+1}, \ldots, a_n, \ldots], \quad \text{with } a_n \in \mathbb{F}_q[T]\ (n \geqslant 1),$$

and there exist $u, v, w, z \in \mathbb{F}_q[T]$, explicitly depending on $\Lambda_{\ell+2}$ and k, satisfying

$$u\alpha^{r+1} + v\alpha^r + w\alpha + z = 0.$$

If $\Lambda_{\ell+2}$ and k satisfy a certain condition, then this continued fraction α is predictable, that is to say, the partial quotients can be fully described. In this case, the sequence of partial quotients is based on a particular sequence $(A_m)_{m\geqslant 1}$, depending only on k and r, of monic polynomials in $\mathbb{F}_p[T]$. Indeed, we have $a_n = \lambda_n A_{i(n)}$ for all integers $n \geqslant 1$, where $\lambda_n \in \mathbb{F}_q^*$ and $\big(i(n)\big)_{n\geqslant 1}$ is a sequence in $\mathbb{N}$, depending only on k and ℓ. Note in particular that we have $A_{i(j)} = T$, for $1 \leqslant j \leqslant \ell$.

Here, we are only interested in the sequence $(\lambda_n)_{n\geqslant 1}$, of the leading coefficients of the partial quotients of α. This sequence can be described as follows.

At first, there exists a sequence $(\delta_n)_{n\geqslant 1}$ in $\mathbb{F}_q^*$, defined by the initial values $\delta_1, \delta_2, \ldots, \delta_\ell$ (depending on $\Lambda_{\ell+2}$ and k) and by the recursive formulas

$$\delta_{(2k+1)n+\ell-2k+i} = \theta_i \varepsilon_1^{r(-1)^{n+i}} \delta_n^{r(-1)^i}, \quad \text{for } n \geqslant 1 \text{ and } 0 \leqslant i \leqslant 2k, \tag{10}$$

where $\theta_i \in \mathbb{F}_p^*$ $(0 \leqslant i \leqslant 2k)$ are constants depending only on k and i. Then the sequence $(\lambda_n)_{n\geqslant 1}$ in $\mathbb{F}_q^*$ is defined recursively from $\lambda_1, \lambda_2, \ldots, \lambda_\ell$ by the formulas

$$\begin{cases} \lambda_{(2k+1)n+\ell-2k} &= \varepsilon_1^{(-1)^n} \lambda_n^r, \\ \lambda_{(2k+1)n+\ell-2k+i} &= \theta_i' \varepsilon_1^{(-1)^{n+i}} \delta_n^{(-1)^i}, \text{ for } n \geqslant 1 \text{ and } 1 \leqslant i \leqslant 2k, \end{cases} \tag{11}$$

where $\theta_i' \in \mathbb{F}_p^*$ $(1 \leqslant i \leqslant 2k)$ are constants depending only on k and i.

Note that the case that $k = (r-1)/2$ is remarkable: in this case $A_m = T$ for all integers $m \geqslant 1$, therefore $a_n = \lambda_n T$ for all integers $n \geqslant 1$ (while if $k < (r-1)/2$, the degree of A_m tends to infinity with m). Due to this peculiarity, first examples were given in 1986 by Mills and Robbins [19], in the particular case: $r = q = p \geqslant 5$,

$k=(p-1)/2$, and $\ell=2$. Shortly afterwards, Allouche [1] showed in 1988 that the sequences of the partial quotients for these examples of Mills and Robbins are p-automatic. Equivalently, this means that the sequence $(\lambda_n)_{n\geqslant 1}$ is p-automatic.

More generally, we have the following result.

Theorem 6. *The sequences $(\lambda_n)_{n\geqslant 1}$ defined above are $(2k+1)$-automatic.*

Proof. To simplify the notation, set $K=2k+1$, and we proceed in two steps.

Step 1: we show that the sequence $(\delta_n)_{n\geqslant 1}$ is K-automatic. For all integers $n\geqslant 1$, put $v(n)=\delta_{n+\ell+1}$. Then for all integers n,i ($0\leqslant i\leqslant 2k$ and $n>\ell$),

$$v(Kn+i)=\delta_{Kn+\ell+1+i}=\theta_i\varepsilon_1^{r(-1)^{n+1+i}}\delta_{n+1}^{r(-1)^i}=\theta_i\varepsilon_1^{r(-1)^{n+1+i}}\big(v(n-\ell)\big)^{r(-1)^i}, \tag{12}$$

from which we deduce at once, for all integers $n>\ell/2$,

$$\begin{cases} v(2Kn+i) &= \theta_i\varepsilon_1^{r(-1)^{1+i}}\big(v(2n-\ell)\big)^{r(-1)^i},\\ v(2Kn+K+i) &= \theta_i\varepsilon_1^{r(-1)^{i}}\big(v(2n+1-\ell)\big)^{r(-1)^i}.\end{cases} \tag{13}$$

For all integers $n\geqslant 1$, define $v_0(n)=v(2n)$, $v_1(n)=v(2n+1)$, and

$$V(n)=\big(v_0(n),v_1(n)\big)\in(\mathbb{F}_q)^2.$$

For all integers i ($0\leqslant i<K$), with the help of the relations (13) and by distinguishing the parity of ℓ, we can find a mapping $f_i:(\mathbb{F}_q)^2\times(\mathbb{F}_q)^2\to(\mathbb{F}_q)^2$ such that for all integers $n>\ell/2+1$, we have

$$V(Kn+i)=f_i\big(V(n-[\ell/2]),V(n-[\ell/2]-1)\big).$$

Applying Theorem 4 with $b=0$, we obtain that the sequence V is K-automatic, thus both v_0 and v_1 are K-automatic, which implies in turn that v is K-automatic. Consequently the sequence $(\delta_n)_{n\geqslant 1}$ is K-automatic.

Step 2: we show that the sequence $(\lambda_n)_{n\geqslant 1}$ is K-automatic. For all integers $n\geqslant 1$, define $u(n)=\lambda_{n+1}$, and it suffices to show that the sequence $u=\big(u(n)\big)_{n\geqslant 1}$ is K-automatic. By (11), for all integers n,i ($n\geqslant 1$ and $1\leqslant i\leqslant 2k$), we have

$$u(Kn+\ell+i)=\lambda_{(2k+1)(n+1)+\ell-2k+i}=\theta'_i\varepsilon_1^{(-1)^{n+1+i}}\delta_{n+1}^{(-1)^i}, \tag{14}$$

from which we obtain directly that $\big(u(Kn+\ell+i)\big)_{n\geqslant 1}$ is K-automatic, for $(\delta_{n+1}^{(-1)^i})_{n\geqslant 1}$ is K-automatic by **Step 1**, and $(\theta'_i\varepsilon_1^{(-1)^{n+1+i}})_{n\geqslant 1}$ is periodic, thus K-automatic.

Similarly by (11), for all integers $n\geqslant 1$, we have

$$u(Kn+\ell)=\lambda_{(2k+1)(n+1)+\ell-2k}=\varepsilon_1^{(-1)^{n+1}}\lambda_{n+1}^r=\varepsilon_1^{(-1)^{n+1}}\big(u(n)\big)^r,$$

from which we deduce at once, for all integers $n\geqslant 1$,

$$\begin{cases} u(2Kn+\ell) &= \varepsilon_1^{-1}\big(u(2n)\big)^r,\\ u(2Kn+K+\ell) &= \varepsilon_1\big(u(2n+1)\big)^r.\end{cases} \tag{15}$$

For all integers $n\geqslant 1$, put $u_0(n)=u(2n)$ and $u_1(n)=u(2n+1)$. In the following we distinguish two cases.

Case 1: $\ell=2L$ with $L\geqslant 1$ an integer. For all integers $n\geqslant 1$, by (15), we have

$$\begin{cases} u_0(Kn+L) &= \varepsilon_1^{-1}\big(u_0(n)\big)^r,\\ u_1(Kn+k+L) &= \varepsilon_1\big(u_1(n)\big)^r.\end{cases} \tag{16}$$

Note also that for all integers $n \geqslant 1$, we have

$$\begin{cases} u_0(Kn+L+i) &= u(2Kn+\ell+2i), & \text{for } 1 \leqslant i \leqslant k, \\ u_1(Kn+L+i) &= u(2Kn+\ell+2i+1), & \text{for } 0 \leqslant i < k, \\ u_0(Kn+k+L+i) &= u(K(2n+1)+\ell+2i-1), & \text{for } 1 \leqslant i \leqslant k, \\ u_1(Kn+k+L+i) &= u(K(2n+1)+\ell+2i), & \text{for } 1 \leqslant i \leqslant k. \end{cases}$$

Hence all the $\big(u_0(Kn+L+i)\big)_{n\geqslant 1}$ and $\big(u_1(Kn+L+j)\big)_{n\geqslant 1}$ are K-automatic, where i, j are integers such that $1 \leqslant i \leqslant 2k$, $0 \leqslant j \leqslant 2k$, and $j \neq k$. Below we show that both u_0 and u_1 are K-automatic, which implies in turn that u is K-automatic. By the way, for all sequences $w = \big(w(n)\big)_{n\geqslant 1}$, we define $(T_i w)(n) = w(Kn+i)$, where i, n are integers such that $0 \leqslant i < K$ and $n \geqslant 1$.

(1) We show that u_0 is K-automatic. Write $L = K[L/K] + a$, with a an integer such that $0 \leqslant a < K$. Let i be an integer such that $1 \leqslant i \leqslant 2k$. Then

$$L + i \not\equiv K[L/K] + a \pmod{K}.$$

Now that all the $\big(u_0(Kn+L+i)\big)_{n\geqslant 1}$ $(1 \leqslant i \leqslant 2k)$ are K-automatic, hence all the $T_m u_0$ $(0 \leqslant m \leqslant 2k$ and $m \neq a)$ are K-automatic, for adding a prefix to a sequence does not change its automaticity. By (16), for all integers $n \geqslant 1$, we have

$$(T_a u_0)(n + [L/K]) = u_0(Kn+L) = \varepsilon_1^{-1}\big(u_0(n)\big)^r,$$

thus by Theorem 3, we obtain that u_0 is K-automatic.

(2) We show that u_1 is K-automatic. Write $k + L = K[(k+L)/K] + b$, with b an integer such that $0 \leqslant b < K$. If $0 \leqslant j \leqslant 2k$ and $j \neq k$, then we have

$$L + j \not\equiv K[(k+L)/K] + b \pmod{K}.$$

Since all the $\big(u_1(Kn+L+j)\big)_{n\geqslant 1}$ $(0 \leqslant j \leqslant 2k$ and $j \neq k)$ are K-automatic, we obtain that all the $T_i u_1$ $(0 \leqslant i \leqslant 2k$ and $i \neq b)$ are K-automatic, for the same reason as above. Again by (16), for all integers $n \geqslant 1$, we have

$$u_1\big(K(n + [(k+\mathfrak{m})/K]) + b\big) = \varepsilon_1\big(u_1(n)\big)^r,$$

so u_1 is K-automatic, by virtue of Theorem 3.

Case 2: $\ell = 2L + 1$ with $L \geqslant 0$ an integer. For all $n \geqslant 1$, by (15), we have

$$\begin{cases} u_1(Kn+L) &= \varepsilon_1^{-1}\big(u_0(n)\big)^r, \\ u_0(Kn+k+L+1) &= \varepsilon_1\big(u_1(n)\big)^r, \end{cases}$$

which implies that for all integers $n \geqslant 1$, we have

$$\begin{cases} u_1\big(K(Kn+k+L+1)+L\big) &= \varepsilon_1^{-1+r}\big(u_1(n)\big)^{r^2}, \\ u_0\big(K(Kn+L)+k+L+1\big) &= \varepsilon_1^{1-r}\big(u_0(n)\big)^{r^2}. \end{cases} \tag{17}$$

Note also that for all integers $n \geqslant 1$, we have

$$\begin{cases} u_1(Kn+L+i) &= u(2Kn+\ell+2i), & \text{for } 1 \leqslant i \leqslant k, \\ u_0(Kn+L+i+1) &= u(2Kn+\ell+2i+1), & \text{for } 0 \leqslant i < k, \\ u_1(Kn+k+L+i) &= u(K(2n+1)+\ell+2i-1), & \text{for } 1 \leqslant i \leqslant k, \\ u_0(Kn+k+L+i+1) &= u(K(2n+1)+\ell+2i), & \text{for } 1 \leqslant i \leqslant k. \end{cases}$$

Hence all the $\big(u_0(Kn+L+i)\big)_{n\geqslant 1}$ and $\big(u_1(Kn+L+j)\big)_{n\geqslant 1}$ are K-automatic, where i, j are integers such that $1 \leqslant i \leqslant 2k+1$, $1 \leqslant j \leqslant 2k$, and $i \neq k+1$.

For all sequences $w = \big(w(n)\big)_{n\geqslant 1}$, define $(T_i w)(n) = w(K^2 n + i)$, where i, n are integers such that $0 \leqslant i < K^2$ and $n \geqslant 1$. As above, we show below that both u_0 and u_1 are K-automatic.

(3) We show that u_0 is K-automatic. Write $KL + k + L + 1 = K^2 c + d$, where c, d are integers such that $0 \leqslant d < K^2$. If $1 \leqslant i \leqslant 2k + 1$ and $i \neq k + 1$, then for all integers m $(0 \leqslant m \leqslant 2k)$, we have

$$Km + L + i \not\equiv K^2 c + d \pmod{K}.$$

But now all the $\big(u_0(K(Kn + m) + L + i)\big)_{n\geqslant 1}$ $(1 \leqslant i \leqslant 2k + 1$ and $i \neq k + 1)$ are K-automatic, thus K^2-automatic, then as above, all the $T_j u_0$ $(0 \leqslant j < K^2$ and $j \neq d)$ are K^2-automatic. Note that by (17), for all integers $n \geqslant 1$, we have

$$(T_d u_0)(n + c) = u_0(K^2 n + KL + k + L + 1) = \varepsilon_1^{1-r}\big(u_0(n)\big)^{r^2},$$

hence by Theorem 3, we obtain that u_0 is K^2-automatic, and thus K-automatic.

(4) We show that u_1 is K-automatic. Write $K(k + L + 1) + L = K^2 e + f$, where e, f are integers such that $0 \leqslant f < K^2$. Let j be an integer such that $1 \leqslant j \leqslant 2k$. Then for all integers m $(0 \leqslant m \leqslant 2k)$, we have

$$Km + L + j \not\equiv K^2 e + f \pmod{K}.$$

Since all the $\big(u_1(Kn+L+j)\big)_{n\geqslant 1}$ $(1 \leqslant j \leqslant 2k)$ are K-automatic, we obtain that all the $T_i u_1$ $(0 \leqslant i < K^2$ and $i \neq f)$ are K-automatic, for the same reason as above. Again by (17), for all integers $n \geqslant 1$, we have

$$(T_f u_1)(n + e) = u_1\big(K(Kn + k + L + 1) + L\big) = \varepsilon_1^{-1+r}\big(u_1(n)\big)^{r^2},$$

so u_1 is K^2-automatic by virtue of Theorem 3, and then it is K-automatic. □

Acknowledgments. Part of the work was done while Jia-Yan Yao visited the Institut de Mathématiques de Jussieu-PRG (CNRS), and he would like to thank his colleagues, in particular Jean-Paul Allouche, for generous hospitality and interesting discussions. He would also like to thank the National Natural Science Foundation of China (Grants no. 10990012 and 11371210) for partial financial support. Finally the authors would like to thank heartily the anonymous referee for pertinent comments and valuable suggestions.

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