Density regularity for multidimensional jump diffusions with position dependent jump rate

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Résumé

On considère une diffusion $X = (X_t)_t$, avec des sauts, correspondant au générateur infinitésimal suivant :

$$L\psi(x) = \frac{1}{2}\sum_{1\leq i,j\leq d} a_{ij}(x)\frac{\partial^2\psi(x)}{\partial x_i\partial x_j} + g(x)\nabla\psi(x) + \int_{\mathbb{R}^d}(\psi(x+c(z,x)) - \psi(x))\gamma(z,x)\mu(\mathrm{d}z)$$

où μ est de masse totale infinie. En notant $(X_t(x))$ un tel processus partant du point x, et en utilisant une approche basé sur un Calcul de Malliavin fini-dimensionnel, nous étudions la régularité jointe de celui-ci dans le sens suivant : on fixe $b \geq 1$ et $p > 1$, K un ensemble compact de $\mathbb{R}^d$, et nous donnons des conditions suffisantes pour avoir $P(X_t(x) \in dy) = p_t(x,y)dy$ avec $(x,y) \mapsto p_t(x,y)$ appartenant à $W^{bp}(K \times \mathbb{R}^d)$.

Abstract

We consider a jump type diffusion $X = (X_t)_t$ with infinitesimal generator given by

$$L\psi(x) = \frac{1}{2}\sum_{1\leq i,j\leq d} a_{ij}(x)\frac{\partial^2\psi(x)}{\partial x_i\partial x_j} + g(x)\nabla\psi(x) + \int_{\mathbb{R}^d}(\psi(x+c(z,x)) - \psi(x))\gamma(z,x)\mu(\mathrm{d}z)$$

where μ is of infinite total mass. Denoting $(X_t(x))$ such process starting from point x, and using an finite-dimensional Malliavin Calculus based approach, we study the joint regularity in the following sense : let $b \geq 1$, $p > 1$ and K be a compact set in $\mathbb{R}^d$, then we give sufficient conditions in order to have $P(X_t(x) \in dy) = p_t(x,y)dy$ with $(x,y) \mapsto p_t(x,y)$ in $W^{bp}(K \times \mathbb{R}^d)$.

Key words : Diffusions with jumps, Malliavin Calculus, Regularity, Density, Finite dimensional Malliavin Calculus.

MSC 2000 : 60 J 55, 60 J 35, 60 F 10, 62 M 05

Contents

1 Differential calculus and Integration by part

1.1 Introduction

The purpose of this work is to study the regularity of the density of the following stochastic equation

$$\begin{aligned} X_t = & x + \sum_{l=1}^{m} \int_0^t \sigma_l(X_s)\, \mathrm{d}W_s^l + \int_0^t b(X_s)\, \mathrm{d}s \\ & + \int_0^t \int_{E\times\mathbb{R}_+} c(X_{s-}, z) \mathbb{1}_{\{u\le\gamma(X_{s-},z\}} N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) \end{aligned}$$

where the existence and uniqueness is proved for example in [10] or Graham [6] (1992).

The way to do so used here is based on a two step strategy. First, we construct an approximation (F_M) of the process X_t (basically given a non-decreasing sequence of subsets $\big(B_M\big)_{M\in\mathbb{N}^*}$, with $\mu(B_M) < \infty$, recovering E, the approximation F_M will be constructed (for each M) from a restriction of the processes X_t based on the restriction of the random measure N on the subset B_M) verifying an integration by part formula :

$$\mathrm{E}\left[\varphi'(F_M)\right] = \mathrm{E}\left[\varphi(F_M) H_M\right].$$

This integration by part is obtained within a general framework developed in [3] by V. Bally and E. Clément, whose main results used in the sequel are presented in this section.

The second step consists in proving the density regularity itself. The idea is to use a certain balance between the error $\mathrm{E}\left[|F_M - X_t|\right]$ (which tends to 0) and the weight $\mathrm{E}\left[|H_M|\right]$ (which tends to ∞). This was the strategy used in [3] as well. But here the estimates of $\mathrm{E}\left[|H_M|\right]$ will appear to be more delicate than the corresponding one in [3] because of the additional Brownian part $\sigma\, \mathrm{d}W$. Moreover, the balance used in [3] was based on a Fourier transform method while here we use the new method developed by V. Bally and L. Caramellino in [2].

This new method allowed us also to extend the result to the regularity of the density considering additionally the variation of the starting point of the process, which was fixed in [3] ; the part of [2] used for our purpose is presented, in this section, subsection 1.6.

1.2 Notations, tools of differential calculus

1.2.1 Notations and differentials operators

We consider a sequence $(V_i)_{i\in\mathbb{N}^*}$ of random variables on a probability space $(\Omega, \mathcal{F}, \mathrm{P})$, a σ-algebra $\mathcal{G} \subset \mathcal{F}$ and a $\mathcal{G}$-measurable random variable J, with values in $\mathbb{N}$. We assume that the variables (V_i) and J satisfy the following integrability condition :

$$\forall p \ge 1, \qquad \mathrm{E}\left[J^p\right] + \mathrm{E}\left[\left(\sum\nolimits_{i=1}^{J} V_i^2\right)^p\right] < \infty.$$

Following Bally and Clément, we will define a differential calculus based on the variables (V_i), conditionally on $\mathcal{G}$.

First we will define the following set

Definition 1.1 *Let $\mathcal{M}$ be the class of functions $f : \Omega \times \mathbb{R}^{\mathbb{N}^*} \to \mathbb{R}$ such that :*

- *f can be written as*

$$f(\omega, v) = \sum_{j=1}^{\infty} f^j(\omega, v_1, \ldots, v_j) \mathbb{1}_{\{J(\omega)=j\}}$$

  where $f^j : \Omega \times \mathbb{R}^j \to \mathbb{R}$ are $\mathcal{G} \times \mathcal{B}(\mathbb{R}^j)$-measurable functions ;

- *there exists a random variable $C \in \bigcap_{q\ge1} \mathrm{L}^q(\Omega, \mathcal{F}, \mathrm{P})$ and $p \in \mathbb{N}^*$ such that*

$$|f(\omega, v)| \le C(\omega)\Big(1 + \left(\sum\nolimits_{i=1}^{J(\omega)} v_i^2\right)^p\Big)$$

  (in other words, conditionally on $\mathcal{G}$, the functions of $\mathcal{M}$ have polynomial growth with respect to the variables (v_i)).

We will define also

- $\mathcal{G}_i$ the σ-algebra generated by $\mathcal{G} \cup \sigma(V_j, 1 \le j \le J, j \neq i)$,
- $(a_i(\omega))$ and $(b_i(\omega))$ two sequences of $\mathcal{G}_i$-measurable random variables satisfying

$$-\infty \le a_i(\omega) < b_i(\omega) \le +\infty, \qquad \forall i \in \mathbb{N}^*$$

- O_i the open set of $\mathbb{R}^{\mathbb{N}^*}$ defined by $O_i = P_i^{-1}(]a_i, b_i[)$, where P_i is the coordinate map $\mathbb{R}^{\mathbb{N}^*}$ (*ie.* $P_i(v) = v_i$).

We localize the differential calculus on the sets (O_i) by introducing some weights (π_i), satisfying the following hypothesis.

Hypothesis 1.1 *For all $n \in \mathbb{N}^*$, $\pi_i \in \mathcal{M}$ and*

$$\{\pi_i > 0\} \subset O_i.$$

Moreover for all $j \ge 1$, π_i^j is infinitely differentiable with bounded derivatives with respect to the variables $(v_1, \ldots, v_j)$.

At last, we associate to these weights π_i, the spaces $C_\pi^k \subset \mathcal{M}$, $k \in \mathbb{N}^*$, defined recursively as follows.

- For $k = 1$, C_π^1 denotes the space of functions $f \in \mathcal{M}$ such that for each $i \in \mathbb{N}^*$, f admits a partial derivative with respect to the variable v_i on the open set O_i. We then define

$$\partial_i^\pi f(\omega, v) \stackrel{\text{def}}{=} \pi(\omega, v) \frac{\partial}{\partial v_i} f(\omega, v)$$

  and we assume that $\partial_i^\pi f \in \mathcal{M}$.

- Suppose now that C_π^k is already defined. For a multi-index $\alpha = (\alpha_1, \ldots, \alpha_k) \in {\mathbb{N}^*}^k$, we define recursively $\partial_\alpha^\pi = \partial_{\alpha_1}^\pi \cdots \partial_{\alpha_k}^\pi$ and C_π^{k+1} is the space of functions $f \in C_\pi^k$ such that for every multi-index $\alpha = (\alpha_1, \ldots, \alpha_k) \in {\mathbb{N}^*}^k$ we have $\partial_\alpha^\pi f \in C_\pi^1$. Notice that if $\partial_\alpha^\pi f \in \mathcal{M}$ for each α with $|\alpha| \le k$.

- Finally we define

$$C_\pi^\infty \stackrel{\text{def}}{=} \bigcap_{k \in \mathbb{N}^*} C_\pi^k.$$

Definition 1.2 (Simple functionals) *A random variable F is called a simple functional if there exist $f \in C_\pi^\infty$ such that $F = f(\omega, V)$, where $V = (V_i)$. We denote by $\mathcal{S}$ the space of the simple functionals (it is an algebra) ; moreover, it is worth to notice that, conditionally on $\mathcal{G}$, $F = f^J(V_1, \ldots, V_J)$.*

Definition 1.3 (Simple processes.) *A simple process is a sequence of random variables $U = (U_i)_{i \in \mathbb{N}^*}$ such that for each $i \in \mathbb{N}^*$, $U_i \in \mathcal{S}$. Consequently, conditionally on $\mathcal{G}$, we have $U_i = u_i^J(V_1, \ldots, V_J)$. We denote by $\mathcal{P}$ the space of the simple processes and we define the scalar product*

$$\langle U, V \rangle_J = \sum_{i=1}^J U_i V_i \qquad (\in \mathcal{S}).$$

We can now define the derivative operator and state the integration by parts formula.

Definition 1.4 (The derivative operator.) *We define $\mathrm{D} : \mathcal{S} \to \mathcal{P}$ by*

$$\mathrm{D}\, F \stackrel{def}{=} (\mathrm{D}_i\, F) \in \mathcal{P} \quad \textit{where} \quad \mathrm{D}_i\, F \stackrel{def}{=} \partial_i^\pi f(\omega, v).$$

Notice that $\mathrm{D}_i\, F = 0$, for $i > J$.

Definition 1.5 (Malliavin covariance matrix) *For $F = (F^1, \ldots, F^d) \in \mathcal{S}^d$, the Malliavin covariance matrix is defined by*

$$\sigma^{k,k'}(F) = \langle \mathrm{D}\, F^k, \mathrm{D}\, F^{k'} \rangle_J = \sum_{i=1}^J \mathrm{D}_i\, F^k\, \mathrm{D}_i\, F^{k'}$$

We denote

$$\Lambda(F) = \{\det \sigma(F) \neq 0\} \qquad \text{and} \qquad \gamma(F)(\omega) = \sigma^{-1}(F)(\omega), \quad \omega \in \Lambda(F)$$

In order to derive an integration by parts formula, we need some additional assumptions on the random variables (V_i). The main hypothesis is that conditionally on $\mathcal{G}$, the law of the vector $(V_1, \ldots, V_J)$, admits a locally smooth density with respect to the Lebesgue measure on $\mathbb{R}^J$.

Hypothesis 1.2 *1. Conditionally on $\mathcal{G}$, the vector $(V_1, \ldots, V_J)$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^J$ and we denote by p_J the conditional density.*

2. The set $\{p_J > 0\}$ is open in $\mathbb{R}^J$ and on $\{p_J > 0\}$, $\ln p_J \in C_\pi^\infty$.

3. For all $q > 1$, there exists a constant C_q such that

$$(1 + |v|^q) p_J \leq C_q$$

where $|v|$ stands for the euclidean norm of the vector $(v_1, \ldots, v_J)$.

Assumption 3) implies in particular that conditionally on $\mathcal{G}$, the functions of $\mathcal{M}$ are integrable with respect to p_J and that for $f \in \mathcal{M}$:

$$\mathrm{E}_{\mathcal{G}}[f(\omega, V)] = \int_{\mathbb{R}^J} f^J \times p_J(\omega, v_1, \ldots, v_J) \, \mathrm{d}v_1, \ldots, \mathrm{d}v_J.$$

Definition 1.6 (The divergence operator) *Let $U = (U_i)_{i \in \mathbb{N}^*} \in \mathcal{P}$ with $U \in \mathcal{S}$. We define $\delta : \mathcal{P} \to \mathcal{S}$ by*

$$\delta_i(U) \stackrel{def}{=} -(\partial_{v_i}(\pi_i U_i) + U_i \mathbb{1}_{\{p_J > 0\}} \partial_i^\pi \ln p_J) \tag{1.1}$$

$$\delta(U) = \sum_{i=1}^{J} \delta_i(U) \tag{1.2}$$

For $F \in \mathcal{S}$, we then define

$$\mathrm{L}(F) \stackrel{\text{def}}{=} \delta(\mathrm{D}\, F) \tag{1.3}$$

1.3 Duality and integration by parts formulae

1.4 IPP

The duality between δ and D is given by the following proposition.

Proposition 1.1 *Assuming the two preceding hypothesis, then for all $F \in \mathcal{S}$ and for all $U \in \mathcal{P}$ we have*

$$\mathrm{E}_{\mathcal{G}}[\langle \mathrm{D}\, F, U \rangle_J] = \mathrm{E}_{\mathcal{G}}[F \delta(U)].$$

Lemma 1.2 *Let $\phi : \mathbb{R}^d \to \mathbb{R}$ be a smooth function and $F = (F^1, \ldots, F^d) \in \mathcal{S}^d$. Then $\phi(F) \in \mathcal{S}$ and*

$$\mathrm{D}\, \phi(F) = \sum_{r=1}^{d} \partial_r \phi(F) \, \mathrm{D}\, F^r. \tag{1.4}$$

If $F \in \mathcal{S}$ and $U \in \mathcal{P}$, then

$$\delta(FU) = F\delta(U) - \langle \mathrm{D}\, F, U \rangle_J.$$

Moreover, for $F = (F^1, \ldots, F^d) \in \mathcal{S}^d$, we have

$$\mathrm{L}\phi(F) = \sum_{r=1}^{d} \partial_r \phi(F) \mathrm{L} F^r - \sum_{r,r'=1}^{d} \partial_{r,r'} \phi(F) \langle \mathrm{D}\, F^r, \mathrm{D}\, F^{r'} \rangle_J.$$

We can now state the main results of this subsection.

Theorem 1.3 *Assuming the two preceding hypothesis, let $F = (F^1, \ldots, F^d) \in \mathcal{S}^d$, $G \in \mathcal{S}$ and $\phi : \mathbb{R}^d \to \mathbb{R}$ be a smooth bounded function with bounded derivatives. Let $\Lambda \in \mathcal{G}$, $\Lambda \subset \Lambda(F)$ such that*

$$\mathrm{E}\left[|\det \gamma(F)|^p \mathbb{1}_\Lambda \right] < \infty, \qquad \forall p \geq 1.$$

Then,

1. for every $r = 1, \ldots, d$,

$$\mathrm{E}_{\mathcal{G}}[\partial_r \phi(F) G] \mathbb{1}_\Lambda = \mathrm{E}_{\mathcal{G}}[\phi(F) H_r(F,G)] \mathbb{1}_\Lambda$$

with

$$H_r(F,G) = \sum_{r'=1}^{d} \delta(G \gamma^{r',r}(F) \, \mathrm{D} F^{r'}) = \sum_{r'=1}^{d} \Big(G\delta(\gamma^{r',r}(F) \, \mathrm{D} F^{r'}) - \gamma^{r',r} \langle \mathrm{D} F^{r'}, \mathrm{D} G \rangle_J \Big) ; \tag{1.5}$$

2. for every multi-index $\beta = (\beta_1, \ldots, \beta_q) \in \{1, \ldots, d\}^q$

$$\mathrm{E}_{\mathcal{G}}[\partial_\beta \phi(F) G] \mathbb{1}_\Lambda = \mathrm{E}_{\mathcal{G}}[\phi(F) H^q_\beta(F,G)] \mathbb{1}_\Lambda \tag{1.6}$$

where the weights H^q are defined recursively by (1.5) *and*

$$H^q_\beta(F,G) = H_{\beta_1}\Big(F, H^{q-1}_{(\beta_2,\ldots,\beta_q)}(F,G)\Big). \tag{1.7}$$

1.5 Estimations of H^q

In order to estimate the weights H^q appearing in the integration by parts formulae of the previous subsection, we first need to define iterations of the derivative operator. Let $\alpha = (\alpha_1, \ldots, \alpha_k)$ be a multi-index, with $\alpha_i \in \{1, \ldots, J\}$, for $i = 1, \ldots, k$ and $|\alpha| = k$.

For $F \in \mathcal{S}$ we define recursively

$$\mathrm{D}^k_{(\alpha_1,\ldots,\alpha_k)} F \stackrel{\text{def}}{=} \mathrm{D}_{\alpha_k}\left(\mathrm{D}^{k-1}_{(\alpha_1,\ldots,\alpha_{k-1})} F \right) \qquad \text{and} \qquad \mathrm{D}^k F \stackrel{\text{def}}{=} \Big(\mathrm{D}^k_{(\alpha_1,\ldots,\alpha_k)} F \Big)_{\alpha_i \in \{1,\ldots,J\}}.$$

Notice that $\mathrm{D}^k F \in \mathbb{R}^{J \otimes k}$, and consequently we define the norm of $\mathrm{D}^k F$ as

$$\big| \mathrm{D}^k F \big| \stackrel{\text{def}}{=} \sqrt{\sum_{\alpha_1,\ldots,\alpha_k=1}^{J} \big| \mathrm{D}^k_{(\alpha_1,\ldots,\alpha_k)} F \big|^2}. \tag{1.8}$$

Moreover we introduce the following norms, for $F \in \mathcal{S}$:

$$|F|_{1,l} \stackrel{\text{def}}{=} \sum_{k=1}^{l} \big| \mathrm{D}^k F \big| \qquad \text{and} \qquad |F|_l \stackrel{\text{def}}{=} |F| + |F|_{1,l} = \sum_{k=0}^{l} \big| \mathrm{D}^k F \big|. \tag{1.9}$$

For $F = (F_1, \ldots, F_d) \in \mathcal{S}^d$:

$$|F|_{1,l} \stackrel{\text{def}}{=} \sum_{r=1}^{d} \big| F^r \big|_{1,l} \qquad \text{and} \qquad |F|_l \stackrel{\text{def}}{=} \sum_{r=1}^{d} \big| F^r \big|_l,$$

and, similarly, for $F = \big(F^{r,r'} \big)_{r,r'=1,\ldots,d}$

$$|F|_{1,l} \stackrel{\text{def}}{=} \sum_{r,r'=1}^{d} \big| F^{r,r'} \big|_{1,l} \qquad \text{and} \qquad |F|_l \stackrel{\text{def}}{=} \sum_{r,r'=1}^{d} \big| F^{r,r'} \big|_l.$$

Notation 1.4
- *In the sequel, we will generally denote simply D^k_α by D_α (where α is a multi-index of length k).*
- *We will also use the following generalisation for $F \in \mathcal{S}^d$ and $G \in \mathcal{S}^{d \times k}$: we will simply set*

$$\mathrm{D}_\alpha F \stackrel{\text{def}}{=} \Big(\mathrm{D}_\alpha F_i \Big)_{1 \le i \le d} \qquad \textit{and} \qquad \mathrm{D}^k G \stackrel{\text{def}}{=} \Big(\mathrm{D}^k G_{i,j} \Big)_{\substack{1 \le i \le d \\ 1 \le j \le k}}.$$

1.5.1 Differentiability lemmas

In this subsection we will use directly the notations from Section 2 defined in 2.9 and 2.10, where we will apply the previous general differential framework.

In order to express the form of the different multi-derivatives we will use in the next section, let us set the following notations :

- if $F \in \mathcal{S}^d$, we will denote the n-th derivative

$$\mathrm{D}_{(k_n,r_n)}\Big(\mathrm{D}_{(k_{n-1},r_{n-1})}\Big(\cdots\Big(\mathrm{D}_{(k_1,r_1)}(F)\Big)\Big)$$

  by

$$\mathrm{D}_\alpha(F)$$

  with $\alpha = (\alpha_1, \dots, \alpha_n)$ and, for all $i \in \{1, \dots, n\}$, $\alpha_i \overset{\text{def}}{=} (k_i, r_i)$;

- for $1 \le l \le n$, we denote by

$$\mathcal{M}_n(l) = \Big\{M = (M_1, \dots, M_l),\ \bigcup_{i \in [\![1,l]\!]} M_i = \{1, \dots, n\} \text{ and } M_i \cap M_j = \varnothing,\ \text{for } i \neq j\Big\}\ ;$$

  the set of the partitions of length l of $\{1, \dots, n\}$.

Remark 1.5 *The multi-derivatives defined above are not commutative : in general*

$$\mathrm{D}_{(k,r)}\Big(\mathrm{D}_{(m,n)}(F)\Big) \neq \mathrm{D}_{(m,n)}\Big(\mathrm{D}_{(k,r)}(F)\Big).$$

We can now state :

Lemma 1.6 *Let $A, B \in \mathcal{S}$, $\phi : \mathbb{R}^d \to \mathbb{R}$ and $c : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be smooth functions and $F = (F^1, \dots, F^d)$, $G = (G^1, \dots, G^d) \in \mathcal{S}^d$. Then*

1. *for every $(k, r) \in [\![1, J]\!] \times [\![1, d]\!]$,*

$$\mathrm{D}_{k,r}(AB) = \mathrm{D}_{k,r}(A)B + A\,\mathrm{D}_{k,r}(B)\ ; \tag{1.10}$$

   and for every $\alpha = (\alpha_1, \dots, \alpha_n)$, with $\alpha_i \overset{def}{=} (k_i, r_i) \in [\![1, J]\!] \times [\![1, d]\!]$,

$$\mathrm{D}_\alpha(AB) = \sum_{\substack{\alpha_i \oplus \alpha_j = \alpha \\ \alpha_i, \alpha_j \ \textit{ordered}}} \mathrm{D}_{\alpha_i} A\, \mathrm{D}_{\alpha_j} B\ ; \tag{1.11}$$

   (by "ordered" we mean that if $\alpha_i = (\alpha_{i_1}, \dots, \alpha_{i_k})$, then $i_1 < \cdots < i_k$)

2. *for every $\alpha = (\alpha_1, \dots, \alpha_n)$, with $\alpha_i \overset{def}{=} (k_i, r_i) \in [\![1, J]\!] \times [\![1, d]\!]$,*

$$\begin{aligned} \mathrm{D}_\alpha\, \phi(F) &= \sum_{l=1}^{n} \sum_{\substack{\beta=(\beta_1,\dots,\beta_l) \\ \beta_i \in [\![1,d]\!]}} \sum_{M \in \mathcal{M}_n(l)} \partial_\beta \phi(F)\, \mathrm{D}_{M_1(\alpha)} F_{\beta_1} \cdots \mathrm{D}_{M_l(\alpha)} F_{\beta_l} && (1.12) \\ &= T_\alpha\big(\phi\big)(F) + \nabla\phi(F)\, \mathrm{D}_\alpha F, && (1.13) \end{aligned}$$

   where

   - *for $M = (M_1, \dots, M_l) \in \mathcal{M}_n(l)$, if $M_j = (i_1, \dots, i_r) \subseteq \{1, \dots, n\}$,*

$$M_j(\alpha) \overset{def}{=} (\alpha_{i_1}, \dots, \alpha_{i_r}),$$

   - *and*

$$T_\alpha\big(\phi\big)(F) \overset{def}{=} \sum_{l=2}^{n} \sum_{\substack{\beta=(\beta_1,\dots,\beta_l) \\ \beta_i \in [\![1,d]\!]}} \sum_{M \in \mathcal{M}_n(l)} \partial_\beta \phi(F)\, \mathrm{D}_{M_1(\alpha)} F_{\beta_1} \cdots \mathrm{D}_{M_l(\alpha)} F_{\beta_l} \tag{1.14}$$

3. for every $\alpha = (\alpha_1, \ldots, \alpha_n)$, with $\alpha_i \overset{def}{=} (k_i, r_i) \in [\![1, J]\!] \times [\![1, d]\!]$, and using the same notations,

$$\begin{aligned} \mathrm{D}_\alpha\, c(F,G) =& \sum_{l=1}^{n} \sum_{\substack{\beta=(\beta_1,\ldots,\beta_r,\beta'_{r+1},\ldots,\beta'_l)\\ \beta_i\in[\![1,d]\!],\ \beta'_j\in[\![d+1,2d]\!]}} \sum_{M\in\mathcal{M}_n(l)} \partial_\beta c(F,G)\, \mathrm{D}_{M_1(\alpha)} F_{\beta_1} \cdots \mathrm{D}_{M_r(\alpha)} F_{\beta_r} \\ & \qquad\qquad\qquad\qquad\qquad\qquad \times \mathrm{D}_{M_{r+1}(\alpha)} G_{\beta'_{r+1}} \cdots \mathrm{D}_{M_l(\alpha)} G_{\beta'_l} \\ =& U_\alpha(c)(F,G) + \nabla_f c(F,G)\, \mathrm{D}_\alpha F + \nabla_g c(F,G)\, \mathrm{D}_\alpha G. \end{aligned}$$

Remark 1.7 *The non-symmetric form (1.13) is used in the sequel in recurrence's purpose : all the elements $M_i(\alpha)$ from T_α are such that $|M_i(\alpha)| < \alpha$ so the degree of derivation of $\mathrm{D}_{M_i(\alpha)}$ is strictly inferior to the one of D_α itself.*

With the same notations :

Lemma 1.8 *Let $\phi : \mathbb{R}^d \to \mathbb{R}$ a smooth function and $F = (F^1, \ldots, F^d) \in \mathcal{S}^d$, α a multi-index and $n \overset{def}{=} |\alpha|$. Then there exists $C_{n,p,d} > 0$ such that*

$$|\,\mathrm{D}_\alpha\, \phi(F)|^{2p} \le C_{n,p,d}\big(|\phi|_n(F)\big)^{2p} \sum_{l=0}^{n} \sum_{M\in\mathcal{M}_n(l)} \Big(|\,\mathrm{D}_{M_1(\alpha)} F|^{2pd} + \cdots + |\,\mathrm{D}_{M_l(\alpha)} F|^{2pd}\Big) \tag{1.15}$$

with $|\phi|_n(F) \overset{def}{=} \sup_{|\beta|\le n} |\partial_\beta \phi(F)|$.

Proof :

From Lemma 1.6 we have

$$\mathrm{D}_\alpha\, \phi(F) = \sum_{l=1}^{n} \sum_{\substack{\beta=(\beta_1,\ldots,\beta_l)\\ \beta_i\in[\![1,d]\!]}} \sum_{M\in\mathcal{M}_n(l)} \partial_\beta \phi(F)\, \mathrm{D}_{M_1(\alpha)} F_{\beta_1} \cdots \mathrm{D}_{M_l(\alpha)} F_{\beta_l}.$$

It follows

$$|\,\mathrm{D}_\alpha\, \phi(F)| \le C_n |\phi|_n(F) \sum_{l=1}^{n} \sum_{\substack{\beta=(\beta_1,\ldots,\beta_l)\\ \beta_i\in[\![1,d]\!]}} \sum_{M\in\mathcal{M}_n(l)} |\,\mathrm{D}_{M_1(\alpha)} F_{\beta_1}| \cdots |\,\mathrm{D}_{M_d(\alpha)} F_{\beta_l}|$$

$$|\,\mathrm{D}_\alpha\, \phi(F)|^{2p} \le C_{n,p}\big(|\phi|_n(F)\big)^{2p} \sum_{l=1}^{n} \sum_{\substack{\beta=(\beta_1,\ldots,\beta_l)\\ \beta_i\in[\![1,d]\!]}} \sum_{M\in\mathcal{M}_n(l)} |\,\mathrm{D}_{M_1(\alpha)} F_{\beta_1}|^{2p} \cdots |\,\mathrm{D}_{M_d(\alpha)} F_{\beta_l}|^{2p}$$

Now[1]

$$|\,\mathrm{D}_{M_1(\alpha)} F|^{2p} \cdots |\,\mathrm{D}_{M_l(\alpha)} F|^{2p} \le \frac{1}{d}\Big(|\,\mathrm{D}_{M_1(\alpha)} F|^{2pd} + \cdots + |\,\mathrm{D}_{M_d(\alpha)} F|^{2pd}\Big),$$

so

$$|\,\mathrm{D}_\alpha\, \phi(F)|^{2p} \le C_{n,p,d}\big(|\phi|_n(F)\big)^{2p} \sum_{l=1}^{n} \sum_{M\in\mathcal{M}_n(l)} \Big(|\,\mathrm{D}_{M_1(\alpha)} F|^{2pd} + \cdots + |\,\mathrm{D}_{M_d(\alpha)} F|^{2pd}\Big).$$

•

We will also need an extended version of the first item of Lemma 1.6 :

[1]Since, if $a_1, \ldots, a_n \in \mathbb{R}_+^*$, it is well-known that $\sqrt[n]{\prod_{i=1}^n a_i} \le \frac{1}{n}\sum_{i=1}^n a_i$,

$$\prod_{i=1}^{n} a_i \le \frac{1}{n^n}\Big(\sum_{i=1}^{n} a_i\Big)^n \le \frac{1}{n^n} n^{n-1} \sum_{i=1}^{n} a_i^n,$$

so

$$\prod_{i=1}^{n} a_i \le \frac{1}{n} \sum_{i=1}^{n} a_i^n.$$

Lemma 1.9 *Let $A, B \in \mathcal{S}^{d\times d}$. Then for every $(k,r) \in [\![1,J]\!] \times [\![1,d]\!]$,*

$$\mathrm{D}_{k,r}(AB) = \mathrm{D}_{k,r}(A)B + A\,\mathrm{D}_{k,r}(B)\,; \tag{1.16}$$

and for every $\alpha = (\alpha_1, \ldots, \alpha_n)$, with $\alpha_i \stackrel{def}{=} (k_i, r_i) \in [\![1,J]\!] \times [\![1,d]\!]$,

$$\mathrm{D}_\alpha(AB) = \sum_{\substack{\alpha_i \oplus \alpha_j = \alpha \\ \alpha_i, \alpha_j \ ordered}} \mathrm{D}_{\alpha_i} A\, \mathrm{D}_{\alpha_j} B\,; \tag{1.17}$$

(by "ordered" we mean that if $\alpha_i = (\alpha_{i_1}, \ldots, \alpha_{i_k})$, then $i_1 < \cdots < i_k$).

Proof : Let $A = (A_{i,j})_{1\le i,j\le d}$ and $B = (B_{i,j})_{1\le i,j\le d}$ with $A_{i,j},\, B_{i,j} \in \mathcal{S}$. Then,

$$\begin{aligned}
\mathrm{D}_{k,r}(AB) &= \mathrm{D}_{k,r}\Big(\big(\sum_{m=1}^{d} A_{i,m}B_{m,j}\big)_{1\le i,j\le d}\Big) \\
&= \Big(\big(\sum_{m=1}^{d} \mathrm{D}_{k,r}\left(A_{i,m}B_{m,j}\right)\big)_{1\le i,j\le d}\Big) \\
&= \Big(\big(\sum_{m=1}^{d} \mathrm{D}_{k,r}\left(A_{i,m}\right)B_{m,j} + A_{i,m}\,\mathrm{D}_{k,r}\left(B_{m,j}\right)\big)_{1\le i,j\le d}\Big) \qquad \text{(using (1.10))} \\
&= \mathrm{D}_{k,r}(A)B + A\,\mathrm{D}_{k,r}(B).
\end{aligned}$$

But the proof of (1.17) only requires an induction over the formal relation (1.16) (and does not need any commutativity in the product of A by B). •

Corollary 1.10 *Let $A = (A_{i,j})_{1\le i,j\le d}$, $B = (B_{i,j})_{1\le i,j\le d}$ with $A_{i,j},\, B_{i,j} \in \mathcal{S}$ and $l \in \mathbb{N}^*$. Then there exists $C_l > 0$ such that*

$$|AB|_l \le C_l |A|_l |B|_l. \tag{1.18}$$

We also have the following result proven in [3] (Lemma 8) :

Lemma 1.11 *Let $\phi : \mathbb{R}^d \to \mathbb{R}$ be a $\mathcal{C}^\infty$ function and $F \in \mathcal{S}^d$ then for all $l \ge 1$ we have*

$$|\phi(F)|_{1,l} \le |\nabla\phi(F)||F|_{1,l} + C_l \sup_{2\le\beta\le l} |\partial_\beta \phi(F)||F|_{1,l-1}^{l}.$$

Result that we will essentially use in this work through this corollary :

Corollary 1.12 *Let $\phi : \mathbb{R}^d \to \mathbb{R}$ be a $\mathcal{C}^\infty$ bounded function with bounded derivatives of any order and $F \in \mathcal{S}^d$ then for all $l \ge 1$ there exists $C_{\phi,l} > 0$ such that*

$$|\phi(F)|_l \le C_{\phi,l}\big(1 + |F|_l + |F|_{l-1}^{l}\big). \tag{1.19}$$

1.5.2 Some bounds on H^q

The further theorem, proven in [3], gives some estimates for the weights H^q in terms of the derivatives of G, F, $\mathrm{L}F$ and $\gamma(F)$.

Theorem 1.13 *For $F \in \mathcal{S}^d$, $G \in \mathcal{S}$ and for all $q \in N^*$ there exists a universal constant $C_{q,d}$ such that for every multi-index $\beta = (\beta_1, \ldots, \beta_q)$*

$$\Big|H_\beta^q(F,G)\Big| \le \frac{C_{q,d}|G|_q(1+|F|_{q+1})^{(6d+1)q}}{|\det\sigma(F)|^{3q-1}}\big(1 + |\mathrm{L}F|_{q-1}^{q}\big).$$

Remark 1.14 *In the sequel, we will simply denote $H_\beta^q(F,1)$ by $H_\beta^q(F)$.*

1.6 Interpolation method : notations and theoretical result

All this subsection is directly taken from the article of Bally and Caramelino [2].

1.6.1 Notations and definitions

Let us define for ξ and γ some multi-indexes

$$x^\gamma \stackrel{\text{def}}{=} \prod_{i=1}^{d} x_i^{\gamma_i} \tag{1.20}$$

$$f_{\xi,\gamma}(x) \stackrel{\text{def}}{=} x^\gamma \partial_\xi f(x) \tag{1.21}$$

$$\|f\|_{k,l,p} = \sum_{0\leq|\gamma|\leq l} \sum_{0\leq|\xi|\leq k} \|f_{\xi,\gamma}\|_p \tag{1.22}$$

For all γ such that $|\gamma| \leq l$,

$$|x^\gamma| \stackrel{\text{def}}{=} \prod_{i=1}^{d} |x_i|^{\gamma_i} \leq \prod_{i=1}^{d} |x|^{\gamma_i} \leq |x|^{\sum \gamma_i} \leq (1+|x|)^l$$

so

$$\|f\|_{k,l,p} \leq C_d \sum_{0\leq|\xi|\leq k} \|(1+|x|)^l \partial_\xi f(x)\|_p. \tag{1.23}$$

Since in the sequel we will have to bound the quantity $\|f_M\|_{2m+q,2m,p}$, let us notice that we have directly

$$\|f_M\|_{2m+q,2m,p} \leq C_d \sum_{0\leq|\xi|\leq 2m+q} \Big(\int_{\mathbb{R}^d\times\mathbb{R}^d} \big[(1+|(x,y)|^{2m})\partial_\xi\big(f_M(x,y)\big)\big]^p \,\mathrm{d}x\,\mathrm{d}y \Big)^{\frac{1}{p}}. \tag{1.24}$$

Moreover, we will define a distance between two measures μ and ν in the following way :

$$d_k(\mu,\nu) \stackrel{\text{def}}{=} \sup\Big\{ \Big| \int \phi \,\mathrm{d}\mu - \int \phi \,\mathrm{d}\nu \Big| : \phi \in \mathcal{C}^\infty(\mathbb{R}^d), \sum_{0\leq|\xi|\leq k} \|\partial_\xi \phi\|_\infty \leq 1 \Big| \Big\}. \tag{1.25}$$

Remark 1.15 *Here, we will only use the case $k=1$, which is called the bounded variation distance (or, also, the Fortet-Mourier distance).*

1.6.2 Main result

Theorem 1.16 *Let $q,k \in \mathbb{N}$, $m \in \mathbb{N}^*$, $p>1$ and set*

$$\eta > \frac{q+k+d/p^*}{2m}. \tag{1.26}$$

We consider a non negative finite measure μ and a family of finite non negative measures

$$\mu_\delta(\mathrm{d}x) = f_\delta(x)\,\mathrm{d}x, \qquad \delta > 0.$$

We assume that there exist C, $r>0$ such that

$$\lambda_{q,m}(\delta) \stackrel{def}{=} \sup_{\delta\leq\delta'\leq 1} \|f_{\delta'}\|_{2m+q,2m,p} \leq C\delta^{-r}$$

and moreover, with η given in (1.26),

$$\lambda_{q,m}(\delta)^\eta d_k(\mu,\mu_\delta) \leq C. \tag{1.27}$$

Then $\mu(\mathrm{d}x) = f(x)\,\mathrm{d}x$ with $f \in W^{q,p}$.

2 Regularity of the Density

2.1 Introduction

As we briefly mentioned in the introduction of the last section, the main purpose of this second part is to study the regularity of the law of the random variable X_t solution of the following stochastic equation with jumps :

$$X_t = x + \int_0^t \sigma(X_s)\,\mathrm{d}W_s + \int_0^{t+}\int_{E\times\mathbb{R}_+} c(z, X_{s^-})\mathbb{1}_{\{u\le\gamma(z,X_{s^-})\}} N(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u) + \int_0^t g(X_s)\,\mathrm{d}s \tag{2.28}$$

(again, for the existence and the uniqueness of such stochastic equation see [6]).

Our global aim is to give sufficient conditions in order to prove that the law of X_t is absolutely continuous with respect to the Lebesgue measure and has a smooth density. That was the point of the study made in [3] as well, with an equation of this type but without the Brownian part.

But, here, we will not only consider the existence and the regularity of the density $y \mapsto p_{X_t}(y)$ (defined by $\mathrm{P}_{X_t}(\mathrm{d}y) = p_{X_t}(y)\,\mathrm{d}y$) with a given starting point $x \in \mathbb{R}^d$: we will consider instead the behaviour of $(x, y) \mapsto p_{X_t^x}(y)$ (with $\mathrm{P}_{X_t^x}(\mathrm{d}y) = p_{X_t^x}(y)\,\mathrm{d}y$ where X_t^x stands for the solution of (2.28) starting at x).

This joint density regularity property, in addition to being obviously a stronger result, will allow us (and it was, at first, one of the motivations to extend the result obtained for the regularity of $y \mapsto p_{X_t^x}(y)$) to obtain an interesting application concerning the Harris-recurrence of the process (section 5).

2.2 Hypothesis and notations

Let us recall that the associated intensity measure of the counting measure N is given by

$$\hat{N}(\mathrm{d}t,\mathrm{d}z,\mathrm{d}u) = \mathrm{d}t \times \mu(\mathrm{d}z) \times \mathbb{1}_{\{0,\infty\}}(u)\,\mathrm{d}u$$

where $(z, u) \in X = \mathbb{R}^d \times \mathbb{R}_+$ and $\mu(\mathrm{d}z) = h(z)\,\mathrm{d}z$.

In this subsection we make the following hypothesis on the functions γ, g, h and c.

Hypothesis 2.1 *We assume that γ, g, h and c are infinitely differentiable functions in both variables z and x. Moreover we assume that*

- *g and its derivatives are bounded ;*
- *$\ln h$ has bounded derivatives ;*
- *both γ and $\ln\gamma$ have bounded derivatives.*

Hypothesis 2.2 *We assume that there exist two functions $\overline{\gamma}, \underline{\gamma} : \mathbb{R}^d \to \mathbb{R}_+$ such that*

$$\overline{C} \ge \overline{\gamma}(z) \ge \gamma(z, x) \ge \underline{\gamma}(z) \ge 0, \qquad \forall x \in \mathbb{R}^d$$

where $\overline{C}$ is a constant.

Hypothesis 2.3 *1. We assume that there exists a non negative and bounded function $\overline{c} : \mathbb{R}^d \to \mathbb{R}_+$ such that $\int_{\mathbb{R}^d} \overline{c}(z)\mu(\mathrm{d}z) < \infty$ and*

$$|c(z, x)| + |\partial_z^\beta \partial_x^\alpha c(z, x)| \le \overline{c}(z), \qquad \forall z, x \in \mathbb{R}^d.$$

We need this hypothesis in order to estimate the Sobolev norms.

2. There exists a measurable function $\hat{c} : \mathbb{R}^d \to \mathbb{R}_+$ such that $\int_{\mathbb{R}^d} \hat{c}(z)\mu(\mathrm{d}z) < \infty$ and

$$\|\nabla_x c \times (\mathrm{Id} + \nabla_x c)^{-1}(z, x)\| \le \hat{c}(z), \qquad \forall z, x \in \mathbb{R}^d.$$

In order to simplify the notations we assume that $\hat{c}(z) = \overline{c}(z)$.

3. There exists a non negative function $\underline{c} : \mathbb{R}^d \to \mathbb{R}_+$ such that, for all $z \in \mathbb{R}^d$,

$$\sum_{r=1}^d \langle \partial_{z^r} c(z, x), \xi\rangle^2 \ge \underline{c}^2(z)|\xi|^2, \qquad \forall \xi \in \mathbb{R}^d$$

and we assume that there exists $\theta \in \overline{\mathbb{R}}_+^*$ such that

$$\liminf_{a\to\infty} \frac{1}{\ln a} \int_{\{\underline{c}^2 \ge \frac{1}{a}\}} \underline{\gamma}(z)\mu(\mathrm{d}z) = \theta. \tag{2.29}$$

Remark : assumptions 2) and 3) give sufficient conditions to prove the non degeneracy of the Malliavin covariance matrix as defined in the previous section.

2.3 Main result

We are now able to state the density property of X_t^x and the joint regularity (in x and y) of it : we fix $q \geq 1$ and $p > 1$, K a compact set of $\mathbb{R}^d$, and we will give sufficient conditions in order to have $\mathrm{P}_{X_t^x}(\mathrm{d}y) = p_{X_t^x}\,\mathrm{d}y$ with $(x,y) \mapsto p_{X_t^x} \in W^{q,p}(K \times \mathbb{R}^d)$.

Theorem 2.1 *Let q, $p \geq 1$. We assume that hypotheses 2.1, 2.2 and 2.3 hold.*
Let $\left(B_M\right)_{M\in\mathbb{N}^*}$ such that $\bigcup_{M\in\mathbb{N}^*} B_M = E$ and, for all $i \in \mathbb{N}^*$

$$B_i \subset B_{i+1} \qquad \textit{and} \qquad \mu(B_i) < +\infty.$$

Let K a compact set of $\mathbb{R}^d$ and (with p^* such that $\frac{1}{p} + \frac{1}{p^*} = 1$)

$$\eta > \frac{q+1+d/p^*}{2}. \tag{2.30}$$

If there exists C, $r > 0$ such that

$$\mu(B_M)^{6(d+q+3)^3} \leq CM^r \tag{2.31}$$

and if

$$\limsup_M \left(\mu(B_M)^{6(d+q+3)^3\eta} \Big(\int_{B_M^c} \overline{c}(z)\overline{\gamma}(z)\,\mathrm{d}\mu(z) + \sqrt{\int_{B_{M+1}^c} \underline{c}^2(z)\underline{\gamma}(z)\,\mathrm{d}\mu(z)} \Big) \right) < +\infty, \tag{2.32}$$

then, for $t > \frac{4d(3q'-1)}{\theta}$, with $q' = d+q+2$, and for every $x \in K$, the law of X_t^x is absolutely continuous with respect to the Lebesgue measure, ie. $\mathrm{P}_{X_t^x}(\mathrm{d}y) = p_{X_t^x}(y)\,\mathrm{d}y$, and the function $(x,y) \mapsto p_{X_t^x}(y)$ belongs to $W^{q,p}(K \times \mathbb{R}^d)$.

Remark 2.2 *The quantity η and the related condition 2.30, come directly from the main theorem of the interpolation method (Theorem 1.16), in the particular case $k = 1$, as we stated in Remark 1.15, and with $m = 1$ (this last choice is discussed in Remark 2.12).*

Remark 2.3 *If $\theta = \infty$, then, for all $t > 0$, the law of X_t^x is absolutely continuous with respect to the Lebesgue measure and the density $p_{X_t^x}$ belongs to $W^{q,p}(K \times \mathbb{R}^d)$.*

Remark 2.4 *Recalling that (cf. Brézis [5], p.168, Corollaire IX.13, for example), with $k \overset{def}{=} \left[q - \frac{d}{p}\right]$, we have (with $O \in \mathbb{R}^n$ an open ball), $W^{q,p}(O) \subset \mathcal{C}^k(O)$, in the sense that each element of $W^{q,p}(O)$ has a $\mathcal{C}^k$ representative, this theorem can also be used to characterize the $\mathcal{C}^k$ behaviour of the function $(x,y) \mapsto p_{X_t^x}(y)$ (as we will briefly see in the examples at 2.3.1).*

Notation 2.5 *In the sequel, since x will belong to a fixed compact set, we will often write simply Y_t instead of Y_t^x for any process Y starting at x, if this precision is not strictly needed (that is why the starting point will never explicitly appear within the Section 3 but will be always used in Section 4).*

Before starting the proof itself, we will first try to give a sketch of the strategy that we will use. The global idea is articulated in two steps :

1. to obtain an integration by part formula on an appropriate approximation of the process X_t ;
2. to use this last result to prove the regularity of the density.

The terms from the condition (2.32) are a direct consequence of this pattern.

For the first step, and first of all, given a non-decreasing sequence of subsets $\left(B_M\right)_{M\in\mathbb{N}^*}$, with $\mu(B_M) < \infty$, recovering E, we construct (for each M) an approximation X_t^M of the process X_t based on the restriction N_M of the random measure N on the subset B_M.

Using a similar result as the Lemma **??**, given in the first part of this work, we can then say that the L^1-distance between these two processes is bounded as follows :

$$\forall t \leq T, \qquad \mathrm{E}\left[\left|X_t - X_t^M\right|\right] \leq C_T \int_{B_M^c} \overline{c}(z)\overline{\gamma}(z)\mu(\mathrm{d}z),$$

which explains the presence of the term $\int_{B_M^c} \overline{c}(z)\overline{\gamma}(z)\,\mathrm{d}\mu(z)$ in the condition (2.32).

Since $\mu(B_M) < +\infty$, the random measure N_M may be represented as a compound Poisson process (where the jump times will be denoted by T_k^M, $k \in \mathbb{N}$) and the Poisson part of process X_t^M could be expressed as a sum ; nevertheless, because of the indicator function from the original equation, the coefficients of the equation verified by X_t^M are still (for the Poisson part) discontinuous and therefore, we cannot use directly the differential calculus presented earlier. Instead we prove that X_t^M has the same law as the process $\overline{X}_t^M$ which verifies an equation with smooth coefficients.

At this point, one would like to obtain an integration by part formula for $\overline{X}_t^M$, but there remains one last difficulty : it is clear that, for $t < T_1^M$ (the first jump of N_M), the random measure N_M produces no noise, and consequently there is no chance to use it for an integration by part (the Malliavin covariance matrix being, of course, degenerated).

That is why one last process will be introduced :

$$F_M \stackrel{\text{def}}{=} \overline{X}_t^M + \sqrt{U_M(t)} \times \Delta,$$

where Δ Gaussian and where $U_M(t)$ is defined by $U_M(t) = t\int_{B_{M+1}^c} \underline{c}^2(z)\underline{\gamma}(z)\,\mathrm{d}\mu(z)$.

The L^1-distance between F_M and $\overline{X}_t^M$ is then bounded, for $t \leq T$, by

$$K_T\sqrt{\int_{B_{M+1}^c} \underline{c}^2(z)\underline{\gamma}(z)\,\mathrm{d}\mu(z)},$$

which gives a natural interpretation for the last term of the condition (2.32).

We are now able to obtain an integration by part formula for the process F_M :

$$\mathrm{E}\left[\varphi'(F_M)\right] = \mathrm{E}\left[\varphi(F_M)H_M\right]. \tag{I_M}$$

The second step consists in proving the density regularity. The idea is to use a certain balance between the error $\mathrm{E}\left[|F_M - X_t|\right]$ (which tends to 0) and the weight $\mathrm{E}\left[|H_M|\right]$ (which tends to ∞). This was the strategy used in [3] as well. But here the estimates of $\mathrm{E}\left[|H_M|\right]$ have been more delicate then the corresponding one in [3] because of the additional brownian part $\sigma\,\mathrm{d}W$. Moreover, the balance used in [3] was based on a Fourier transform method while here we use the new method developed in [2].

This new method allows us also to extend the result to the regularity of the density considering additionally the variation of the starting point of the process, which was fixed in [3] ; finally, we give an application of this improvement since we can then consider a regenerative scheme to obtain an interesting result concerning the Harris-recurrence of the process (section 5).

2.3.1 Examples

In this example we assume that $h = 1$ so $\mu(\mathrm{d}z) = \mathrm{d}z$ and $\underline{\gamma}(z)$ is equal to a constant $\underline{\gamma} > 0$. We then have

$$\mu(B_M) = r_d M^d$$

where r_d is the volume of the unit ball in $\mathbb{R}^d$. We will also assume that x is in some compact set $K \stackrel{\text{def}}{=} \overline{B(0,R)}$, $R > 0$.

We will consider two types of behaviour for c.

i) Exponential decay : we assume that $\overline{c}(z) = e^{-a|z|^c}$ for some constants $0 < b \leq a$ and $c > 0$. We then have

$$\int_{\{\overline{c}^2 > \frac{1}{u}\}} \underline{\gamma}(z)\,\mathrm{d}\mu(\mathrm{d}z) = \underline{\gamma}\frac{r_d}{(2a)^{\frac{d}{c}}} \times (\ln u)^{\frac{d}{c}}.$$

we then deduce for the constant θ (definied in (2.29))

$$\begin{aligned}
&\theta = 0 && \text{if} \quad c > d,\\
&\theta = \infty && \text{if} \quad 0 < c < d,\\
&\theta = \frac{\gamma r_d}{2a} && \text{if} \quad c = d.
\end{aligned} \tag{2.33}$$

If $c > d$, hypothesis 2.3.3 fails, which is coherent with the result of Bichteler, Gravereaux and Jacod in [4]. Now observe that

$$\int_{B_M^c} \overline{c}(z)\overline{\gamma}(z)\,\mathrm{d}\mu(z) + \sqrt{\int_{B_{M+1}^c} \underline{c}^2(z)\underline{\gamma}(z)\,\mathrm{d}\mu(z)} \leq Ke^{-\xi M^c}$$

for some $\xi > 0$, so the condition (2.32) is always well verified.

When $0 < c < d$, since $\theta = \infty$, for every $t > 0$, $(x, y) \mapsto p_{X_t^x}(y)$ belongs to $W^{\infty,p}(K \times \mathbb{R}^d)$ $(\forall p \geq 1)$, which implies, according to the Remark 2.4, that $(x, y) \mapsto p_{X_t^x}(y)$ can be considered as an element of $\mathcal{C}^\infty(K \times \mathbb{R}^d)$.

If $c = d$, then appears a more particular behaviour and it is interesting to compare the result obtained here with the example from [3] (recalling that, in this last case, X_t does not possess a Brownian part, though), assuming here, for that sake, that x is fixed, $k \in \mathbb{N}$ and $p > 1$:

	[3]	Present work
Domain	$t > \frac{8da}{\gamma r_d}(3d+2)$	$t > \frac{8da}{\gamma r_d}\Big(3d\Big(1+\frac{1}{p}\Big)+3k+8\Big)$
Regularity of p_{X_t}	$\mathcal{C}^k$ with $k \leq \frac{1}{3}\Big(1+\frac{\gamma r_d}{8da}t\Big) - d$	$\mathcal{C}^k$ with $k \leq \frac{1}{3}\Big(1+\frac{\gamma r_d}{8da}t\Big) - d\Big(1+\frac{1}{p}\Big) - 2$

Remark 2.6 *In fact, in this work, $(x, y) \mapsto p_{X_t^x}(y)$ belongs to $W^{q,p}(K \times \mathbb{R}^d)$ $(\forall p \geq 1)$ with*

$$q \leq \frac{1}{3}\Big(1 + \frac{\gamma r_d}{8da}t\Big) - d - 2,$$

so, using again the Remark 2.4, $(x, y) \mapsto p_{X_t^x}(y)$ can be considered as an element of $\mathcal{C}^k(K \times \mathbb{R}^d)$, with

$$k \overset{def}{=} \Big[q - \frac{d}{p}\Big] \leq q - \frac{d}{p} \leq \frac{1}{3}\Big(1 + \frac{\gamma r_d}{8da}t\Big) - d\Big(1 + \frac{1}{p}\Big) - 2.$$

In particular it requires, at least, $q \geq \frac{d}{p}$ to obtain some regularity.

ii) Polynomial decay : We assume now that $\overline{c}(z) = \frac{b}{1+|z|^v}$ and $\underline{c}(z) = \frac{a}{1+|z|^v}$ for some constants $0 < a \leq b$ and $v > d$. We have here

$$\int_{\{\overline{c}^2 > \frac{1}{u}\}} \underline{\gamma}(z)\, \mathrm{d}\mu(\mathrm{d}z) = \underline{\gamma} r_d \times (a\sqrt{u} - 1)^{\frac{d}{v}},$$

so $\theta = \limsup_{u\to\infty} \frac{1}{\ln u}\underline{\gamma} r_d (a\sqrt{u} - 1)^{\frac{d}{v}} = \infty$ and then, in this case, the regularity result stands for every $t > 0$.

A simple computation gives us the following bounds :

$$\int_{B_M^c} \overline{c}(z)\overline{\gamma}(z)\, \mathrm{d}\mu(z) \leq \frac{C}{M^{v-d}} \qquad \text{and} \qquad \int_{B_{M+1}^c} \underline{c}^2(z)\underline{\gamma}(z)\, \mathrm{d}\mu(z) \leq \frac{C}{M^{2v-d}}.$$

So with C and $r > 0$ such that $\mu(B_M)^{6(d+q+3)^3} \leq CM^r$ (condition (2.31)), we have

$$\begin{aligned}\mu(B_M)^{6(d+q+3)^3\eta}\Big(\int_{B_M^c} \overline{c}(z)\overline{\gamma}(z)\, \mathrm{d}\mu(z) + \sqrt{\int_{B_{M+1}^c} \underline{c}^2(z)\underline{\gamma}(z)\, \mathrm{d}\mu(z)}\Big) \\ \leq C' M^{r\eta}\Big(\frac{1}{2M^{v-d}} + \frac{1}{2M^{2v-d}}\Big) \\ \leq C' M^{r\eta - v + d}.\end{aligned}$$

Hence, the condition (2.32) is true if

$$\eta \leq \frac{v-d}{r}$$

and, since here $\mu(B_M) = r_d M^d$, (2.31) gives $r = 6d(d+q+3)^3$ and, with (2.30), we have the following condition :

$$\frac{q + 1 + d/p^*}{2} \leq \frac{v-d}{r}.$$

Finally, with v such that

$$v > 6d(d+q+3)^3 \frac{q+1+d}{2}, \tag{2.34}$$

$(x, y) \mapsto p_{X_t^x}(y)$ belongs to $W^{q,p}(K \times \mathbb{R}^d)$, for all $p \geq 1$.

2.4 Approximation of X_t

In order to prove that the process X_t, solution of (2.28), has a smooth density, we will apply the differential calculus and the integration by parts formula from Section 1. But since the random variable X_t cannot be viewed as a simple functional, the first step consists in approximating it. We describe in this subsection our approximation procedure. We consider a non-negative and smooth function φ such that $\varphi(z) = 0$ for $|z| > 1$ and $\int_{\mathbb{R}^d} \varphi(z)\,\mathrm{d}z = 1$. And for $M \in \mathbb{N}$, we denote

$$\Phi_M(z) = \varphi * \mathbb{1}_{B_M}$$

with $B_M = \{z \in \mathbb{R}^d \;:\; |z| < M\}$. Then $\Phi_M \in C_b^\infty$ and we have $\mathbb{1}_{B_{M-1}} < \Phi_M < \mathbb{1}_{B_{M+1}}$. We denote by X_t^M the solution of the equation

$$X_t^M = x + \int_0^t \sigma(X_s^M)\,\mathrm{d}W_s + \int_0^t \int_E c_M(z, X_{s-}^M)\mathbb{1}_{\{u \leq \gamma(z, X_{s-}^M)\}} N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) + \int_0^t g(X_s^M)\,\mathrm{d}s. \tag{2.35}$$

where

$$c_M(z,x) \stackrel{\text{def}}{=} c(z,x)\Phi_M(z).$$

If we set

$$N_M(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) \stackrel{\text{def}}{=} \mathbb{1}_{B_{M+1}}(z) \times \mathbb{1}_{[0,2\overline{C}]}(u) N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u),$$

since $\{u < \gamma(z, X_{s-}^M)\} \subset \{u < 2\bar{C}\}$ and $\Phi_M(z) = 0$ if $|z| > M+1$,we may replace N by N_M in the above equation and consequently X_t^M is solution of the equation

$$X_t^M = x + \int_0^t \sigma(X_s^M)\,\mathrm{d}W_s + \int_0^t \int_E c_M(z, X_{s-}^M)\mathbb{1}_{\{u \leq \gamma(z, X_{s-}^M)\}}\,\mathrm{d}N_M(s,z,u) + \int_0^t g(X_s^M)\,\mathrm{d}s.$$

Since the intensity measure $\hat{N}_M$ is finite, we may represent the random measure N_M by a compound Poisson process. Let

$$\lambda_M \stackrel{\text{def}}{=} 2\bar{C} \times \mu(B_{M+1}) = t^{-1}\,\mathrm{E}\,[N_M(t,E)]$$

and let J_t^M a Poisson process of parameter λ_M. We denote by T_k^M, $k \in \mathbb{N}$, the jump times of J_t^M. We also consider two sequences of independent random variables $(Z_k^M)_{k \in \mathbb{N}}$ and $(U_k)_{k\in\mathbb{N}}$, respectively in $\mathbb{R}^d$ and $\mathbb{R}_+$, which are independent of J_t^M and such that

$$Z_k^M \sim \frac{1}{\mu(B_{M+1})}\mathbb{1}_{B_{M+1}}(z)\mu(\mathrm{d}z), \qquad \text{and} \qquad U_k \sim \frac{1}{2\bar{C}}\mathbb{1}_{[0,2\bar{C}]}(u)\,\mathrm{d}u.$$

Then, the last equation may be written as

$$X_t^M = x + \int_0^t \sigma(X_s^M)\,\mathrm{d}W_s + \sum_{k=1}^{J_t^M} c_M(Z_k^M, X_{T_k^M-}^M)\mathbb{1}_{(U_k,\infty)}(\gamma(Z_k^M, X_{T_k^M-}^M)) + \int_0^t g(X_s^M)\,\mathrm{d}s. \tag{2.36}$$

The random variable X_t^M solution of (2.36) is a function of $(Z_1, \ldots, Z_{J_t^M})$ but it is not a simple functional, as defined in Section 1, because the coefficient $c_M(z,x)\mathbb{1}_{\{u \leq \gamma(z,x)\}}$ is not differentiable with respect to z. In order to avoid this difficulty we use the following alternative representation. Let $z_M^* \in \mathbb{R}^d$ such that $|z_M^*| = M + 3$. We define

$$q_M(z,x) \stackrel{\text{def}}{=} \varphi(z - z_M^*)\theta_{M,\gamma}(x) + \frac{1}{2\overline{C}\mu(B_{M+1})}\mathbb{1}_{B_{M+1}}(z)\gamma(z,x)h(z) \tag{2.37}$$

$$\theta_{M,\gamma}(x) \stackrel{\text{def}}{=} \frac{1}{\mu(B_{M+1})}\int_{\{|z| \leq M+1\}} \left(1 - \frac{1}{2\overline{C}}\gamma(z,x)\right)\mu(\mathrm{d}z). \tag{2.38}$$

We recall that φ is the function defined at the beginning of this subsubsection : a non-negative and smooth function with $\int \varphi = 1$ and which is null outside the unit ball. Moreover from Hypothesis 2.2, $0 \leq \gamma(z,x) \leq \overline{C}$ and then

$$1 \geq \theta_{M,\gamma}(x) \geq \frac{1}{2}. \tag{2.39}$$

From this last inequality it is easy to deduce the following result :

Lemma 2.7 *Let q_M defined as in* (2.37)*. Then $\ln q_M$ has bounded derivatives of any order.*

By construction the function q_M satisfies $\int q_M(z,x)\,\mathrm{d}z = 1$. Hence we can check that (*cf.* appendix A.4 for a proof)

$$\mathrm{E}\left[f(X^M_{T^M_k-}) \mid X^M_{T^M_k-} = x\right] = \int_{\mathbb{R}^d} f(x + c_M(z,x)) q_M(z,x)\,\mathrm{d}z. \tag{2.40}$$

From the relation (2.40) we construct a process $(\overline{X}^M_t)$ equal in law to (X^M_t) in the following way.

Let $0 \le u \le v$ and $y \in \mathbb{R}^d$, we denote by $\Psi_{u,v}(y)$ the solution of

$$\Psi_{u,v}(x) = y + \int_u^v \sigma(\Psi_{u,s}(y))\,\mathrm{d}W_s + \int_u^v g(\Psi_{u,s}(y))\,\mathrm{d}s.$$

We assume that the times T_k, $k \in \mathbb{N}$ are fixed and we consider a sequence $(z_k)_{k\in\mathbb{N}}$ with $z_k \in \mathbb{R}^d$. Then we define x_t, $t \ge 0$ by $x_0 = x$ and, if x_{T_k} is given, then

$$\begin{aligned} x_t &= \Psi_{T_k,t}(x_{T_k}), \qquad T_k \le t < T_{k+1}, \\ x_{T_{k+1}} &= x_{T^-_{k+1}} + c_M\big(z_{k+1}, x_{T^-_{k+1}}\big). \end{aligned} \tag{2.41}$$

We note that for $T_k \le t < T_{k+1}$, x_t is a function of x, z_1, ..., z_k. Notice also that x_t solves the equation

$$x_t = x + \int_0^t \sigma(x_s)\,\mathrm{d}W_s + \sum_{k=1}^{J^M_t} c_M(z_k, x_{T^-_k}) + \int_0^t g(x_s)\,\mathrm{d}s.$$

We consider now a sequence of random variables $(\overline{Z}_k)$, $k \in \mathbb{N}^*$, independent of the Brownian motion W_t, and we denote $\mathcal{G}_k = \sigma(T_p, p \in \mathbb{N}) \vee \sigma(\overline{Z}_p, p \le k)$ and

$$\overline{X}^M_t = x_t(\overline{Z}_1, \dots \overline{Z}_{J^M_t}). \tag{2.42}$$

We assume that the law of $\overline{Z}_{k+1}$ conditionally on $\mathcal{G}_k$ is given by

$$\mathrm{P}(\overline{Z}_{k+1} \in \mathrm{d}z \mid \mathcal{G}_k) = q_M(z, x_{T^-_{k+1}}(\overline{Z}_1, \dots \overline{Z}_k))\,\mathrm{d}z = q_M(z, \overline{X}^M_{T^-_{k+1}})\,\mathrm{d}z. \tag{2.43}$$

Then $\overline{X}^M_t$ satisfies the equation

$$\overline{X}^M_t = x + \int_0^t \sigma(\overline{X}^M_s)\,\mathrm{d}W_s + \sum_{k=1}^{J^M_t} c_M(\overline{Z}_k, \overline{X}^M_{T^-_k}) + \int_0^t g(\overline{X}^M_s)\,\mathrm{d}s \tag{2.44}$$

and $\overline{X}^M_t$ has the same law as X^M_t. Moreover we can prove a bit more.

Lemma 2.8 *For a locally bounded and measurable function $\psi : \mathbb{R}^d \to \mathbb{R}$ let*

$$\overline{S}_t(\psi) = \sum_{k=1}^{J^M_t} (\Phi_M \psi)(\overline{Z}_k), \qquad S_t(\psi) = \sum_{k=1}^{J^M_t} (\Phi_M \psi)(Z_k) \mathbb{1}_{\{\gamma(Z_k, X^M_{T^-_k}) > U_k\}},$$

then $(\overline{X}^M_t, \overline{S}_t(\psi))_{t\ge 0}$ has the same law as $(X^M_t, S_t(\psi))_{t\ge0}$.

Proof : Observing that $(\overline{X}^M_t, \overline{S}_t(\psi))_{t\ge0}$ solves a system of equations similar to (2.44), but in dimension $d+1$, it suffices to prove that $(\overline{X}^M_t)_{t\ge0}$ has the same law as $(X^M_t)_{t\ge0}$, which is done in detail in the appendix A.5. •

2.5 The integration by part formula

The random variable $\overline{X}^M_t$ constructed previously is a simple functional but unfortunately its Malliavin covariance matrix is degenerated. To avoid this problem we use a classical regularization procedure. Instead of the variable $\overline{X}^M_t$, we consider the regularized one F_M defined by

$$F_M \stackrel{\text{def}}{=} \overline{X}^M_t + \sqrt{U_M(t)} \times \Delta, \tag{2.45}$$

where Δ is a d-dimensional standard Gaussian variable independent of the variables $(\overline{Z}_k)_{k\ge1}$ and $(T_k)_{k\ge1}$ and $U_M(t)$ is defined by

$$U_M(t) = t \int_{B^c_{M+1}} \underline{c}^2(z) \underline{\gamma}(z)\,\mathrm{d}\mu(z). \tag{2.46}$$

Notation 2.9 *We observe that* $F_M \in \mathcal{S}^d$ *where* $\mathcal{S}$ *is the space of simple functionals for the differential calculus based on the variables* $(\overline{Z}_k)_{k\geq 1}$ *with* $\overline{Z}_0 = (\Delta^r)_{1\leq r\leq d}$ *and* $\overline{Z}_k = (\overline{Z}^r_k)_{1\leq r\leq d}$ *and we are now in the framework of the previous section (subsection 1.2) by taking* $\mathcal{G} \overset{def}{=} \sigma(T_k, k\in\mathbb{N})$ *and defining the weights* (π_k) *by setting* $\pi^r_0 = 1$ *and*

$$\pi^r_k \overset{def}{=} \phi_M(\bar{Z}_k) \tag{2.47}$$

for $1 \leq r \leq d$.

Conditionally on $\mathcal{G}$, the density of the law of $(\overline{Z}_1, \ldots, \overline{Z}_{J^M_t})$ is given by

$$p_M(\omega, z_1, \ldots, z_{J^M_t}) = \prod_{j=1}^{J^M_t} q_M\big(z_j, \Psi_{T_{j-1},T_j}(\overline{X}^M_{T_{j-1}})\big) \tag{2.48}$$

where $\overline{X}^M_{T_{j-1}}$ is a function of z_i, $1\leq i\leq j-1$ (moreover, we can notice that $\Psi_{T_{j-1},T_j}(\overline{X}^M_{T_{j-1}}) = \overline{X}^M_{T_j-})$; we can check that p_M satisfies the Hypothesis 1.2 of Section 1.

Notation 2.10 *To clarify the notation, the derivative operator can be written in this framework for* $F\in\mathcal{S}$ *by* $\mathrm{D}\,F = (\mathrm{D}_{k,r}\,F)$ *where* $\mathrm{D}_{k,r} = \pi^r_k \partial_{\overline{Z}^r_k}$ *for* $k\geq 0$ *and* $1\leq r\leq d$. *Consequently we deduce that* $\mathrm{D}_{k,r}\,F^{r'}_M = \mathrm{D}_{k,r}\,\overline{X}^{M,r'}_t$, *for* $k\geq 1$ *and* $\mathrm{D}_{0,r}\,F^{r'}_M = \sqrt{U_M(t)}\delta_{r,r'}$ *with* $\delta_{r,r'} = 0$ *if* $r\neq r'$, $\delta_{r,r'} = 1$ *otherwise.*

The Malliavin covariance matrix of $\overline{X}^M_t$ is equal to

$$\sigma\big(\overline{X}^M_t\big)_{i,j} = \sum_{k=1}^{J^M_t}\sum_{r=1}^{d} \mathrm{D}_{k,r}\,\mathrm{D}_{k,r}\,\overline{X}^{M,i}_t \overline{X}^{M,j}_t$$

for $1\leq i,j\leq d$ and finally the Malliavin covariance matrix of F_M is given by

$$\sigma\big(F_M\big) = \sigma\big(\overline{X}^M_t\big) + U_M(t)\times \mathrm{Id}\,.$$

Using the results of Section 1, we can state an integration by part formula and give a bound for the weight $H^q(F_M,1)$ in terms of the Sobolev norms of F_M, the divergence $\mathrm{L}\,F_M$ and the determinant of the inverse of the Malliavin covariance matrix $\det\sigma\big(F_M\big)$.

The control of these last three quantities is rather technical and is studied in detail in the next section.

Since we are looking here also for the regularity with respect to the starting point x, and in order to use the Interpolation method (*cf.* 1.6) we will have to look a little bit further. It is clear, from its definition, that the law $P_{F^x_M}$ of F^x_M possesses a smooth density : $P_{F^x_M}(\mathrm{d}y) = p_{F^x_M}(y)\,\mathrm{d}y$. We will then define

$$f_M(x,y) \overset{\mathrm{def}}{=} \Psi_K(x) p_{F^x_M}(y) \tag{2.49}$$

where Ψ_K is a smooth version with bounded derivatives of any order of the indicator function $\mathbb{1}_K$, and study its behaviour with respect to the norm defined by (1.22). More precisely, we will admit for the moment the following result (for the proof, see subsection 4.3) :

Lemma 2.11 *Let* $q\in\mathbb{N}$, $m\in\mathbb{N}^*$, $p>1$. *Then*

$$\|f_M\|_{2m+q,2m,p} \leq C\mu(B_M)^{6(d+2m+q+1)^3} \tag{2.50}$$

where C *does not depend on* M.

2.5.1 Proof of the main result

To do so, as we said earlier, we will use a more powerful method then the usual "balance" that can be made, with some reasonable conditions, when an integration by part formula is available for a convergent sequence of processes (for a more detailed explanation, see [2], from which this new tool is taken) : here, we will use the Theorem 1.16, taken directly from this last cited article.

Proof : Let $t > \frac{4d(3q'-1)}{\theta}$, with $q' = d+2+q$, and let us define the measure μ_X defined by (where $P_{X^x_t}$ is the law of X^x_t)

$$\mu_X(\mathrm{d}x,\mathrm{d}y) \overset{\mathrm{def}}{=} \Psi_K(x) P_{X^x_t}(\mathrm{d}y)\,\mathrm{d}x \tag{2.51}$$

where Ψ_K is a smooth version with bounded derivatives of any order of the indicator function $\mathbb{1}_K$. A natural approximation of $\mu_X(\mathrm{d}x,\mathrm{d}y)$ would then be $\Psi_K(x)p_{X^M_t}(x,y)\,\mathrm{d}x\,\mathrm{d}y$. But in order to use the Malliavin calculus developed in this work, it is more convenient to use, instead of X^M_t, the approximation (in law) F_M of it. Let us recall that

- $F_M \stackrel{\text{def}}{=} \overline{X}_t^M + \sqrt{U_M(t)} \times \Delta$, where Δ is Gaussian and where $U_M(t)$ is defined by $U_M(t) = t\int_{B_{M+1}^c} \underline{c}^2(z)\underline{\gamma}(z)\,\mathrm{d}\mu(z)$
- $f_M \stackrel{\text{def}}{=} \Psi_K(x)p_{F_t^M}(x,y)$.

We will then use the Theorem 1.16, with $\delta \stackrel{\text{def}}{=} M^{-1}$ (to be rigorous with the notations, we should define and work with $\tilde{f}_\delta \stackrel{\text{def}}{=} f_M$, but we will simply use f_M).

On one hand, using Lemma 2.11 with $m=1$, we find that

$$\|f_M\|_{2+q,2,p} \leq C\mu(B_M)^{6(d+q+3)^3} \tag{2.52}$$

where C does not depend on M.

On the other hand, using the definition (1.25) of the distance d_k in the case $k=1$:

$$\begin{aligned} d_1(\mu_X, f_M) &\stackrel{\text{def}}{=} \sup_{\substack{g\in\mathcal{C}^\infty(\mathbb{R}^d\times\mathbb{R}^d)\\ \|g\|_\infty,\|\nabla g\|_\infty\leq 1}} \Big|\iint_{\mathbb{R}^d\times\mathbb{R}^d} g(x,y)\mu_X(\mathrm{d}x,\mathrm{d}y) - \iint_{\mathbb{R}^d\times\mathbb{R}^d} g(x,y)f_M(x,y)\,\mathrm{d}x\,\mathrm{d}y\Big| \\ &= \int_{\mathbb{R}^d} \Psi_K(x)\Big(\mathrm{E}\left[g(x,X_t(x)\right] - \mathrm{E}\left[g(x,F_t^M(x)\right]\Big)\,\mathrm{d}x \\ &\leq \int_{\mathbb{R}^d} \Psi_K(x)\,\mathrm{E}\left[\left|X_t(x) - F_t^M(x)\right|\right]\mathrm{d}x \end{aligned}$$

so

$$d_1(\mu_X, f_M) \leq C_K\Big(\int_{B_M^c} \overline{c}(z)\overline{\gamma}(z)\,\mathrm{d}\mu(z) + \sqrt{\int_{B_{M+1}^c} \underline{c}^2(z)\underline{\gamma}(z)\,\mathrm{d}\mu(z)}\Big). \tag{2.53}$$

It follows that, with the conditions (2.31) and (2.32), the conditions of the Theorem 1.16 are well verified and we can directly conclude. •

Remark 2.12 *Let us give a quick explanation why we only considered the case where the parameter m is equal to 1. This parameter was made to loosen up the lower bound condition for η in Theorem 1.16 (roughly speaking this lower bound is a $O\left(\frac{1}{m}\right)$) which could help to obtain a better condition from (1.27). But, the upper bound obtained for $\|f_M\|_{2m+q,2m,p}$ with respect to m, is a $O(m^3)$, so we lose here completely the possible advantage of taking $m>1$.*

3 Bounding of the weights $H_\beta^q(F_M)$

3.1 Introduction

In this section we consider that the starting point x is fixed.

The final result of that part is to bound the quantity $H_\beta^q(F_M)$; to do so, we will use the bounding given by the Theorem 1.13, which implies :

$$\left|H_\beta^q(F_M)\right| \leq C_{q,d}\frac{1}{|\det\sigma(F_M)|^{3q-1}}\big(1+|F_M|_{q+1}\big)^{(6d+1)q}\big(1+|\mathrm{L}F_M|_{q-1}^q\big),$$

and so will be brought to bound, in particular, on one hand $\|\frac{1}{|\det\,\sigma(F_M)|}\|_p$, which will be done at Lemma 3.14 and, on the other hand $\||F_M|_n\|_p$ and $\||\mathrm{L}F_M|_n\|_p$ (where $\|.\|_p$ is the L^p-norm). To do this last thing, because of the similar structure of the linear equations verified by the different processes involved here, we will develop in the first place a way to bound this type of processes, in a recursive way (which is natural, since we want, in particular, to bound successive derivatives of our process). Moreover, this theoretical result will be helpful in the Section 4, when we will study further the density continuity of the process X_t.

The upper bound of this quantity allows us to prove, under some similar conditions to (2.32), the existence of a regular density for X_t : (with $q\geq 1$ and $p>1$ fixed) we have $\mathrm{P}_{X_t}(\mathrm{d}y) = p_{X_t}\,\mathrm{d}y$ with $p_{X_t}\in W^{q,p}(\mathbb{R}^d)$ (using a Fourier transform method as in [3], or some weaker version of the interpolation method (*cf.* [2]) quoted here). In this sense, this section is "self-contained" ; that is one of the reasons why we give the Lemma 3.15. The other reason (and it is globally true for the whole section) is to show a pattern of the proof, in a simpler case, which will be used again in the more general Lemma 2.11 (proved in the subsection 4.3).

Even though, to conclude in the general case (joint regularity), we need some further results, made in the next section, the main part of the needed techniques is presented in this one, with less heavier notations, since the starting point x is momentarily put aside.

Notations

- In all the sequel we will denote by E_W the expectation with respect to the Brownian motion ; *i.e.* conditionally with respect to the Poisson measure.
- As we have already pointed out in 2.5, in all this section, since x will belong to a fixed compact set, we will always write Y_t instead of Y_t^x for any process Y starting at x.

3.2 An upper bound lemma for a family of linear SDE's

In this subsection we give L^p bounds for the solution of a family of linear equations which represent the general framework in which the Malliavin derivatives fit.

Hypothesis

We fix a finite set I and we consider the multi-indexes of the type $\alpha \stackrel{\text{def}}{=} (\alpha_1, \ldots, \alpha_n)$, with $\alpha_i \in I$. We define and denote the length of α by $|\alpha| \stackrel{\text{def}}{=} n$. We also consider the void multi-index $\alpha = \emptyset$ and in this case we put $|\alpha| \stackrel{\text{def}}{=} 0$.

Then we denote

$$A_n \stackrel{\text{def}}{=} \{\alpha \stackrel{\text{def}}{=} (\alpha_1, \ldots, \alpha_n) \ : \ \alpha_i \in I\}$$

the set of multi-indexes of length n and define then

$$A \stackrel{\text{def}}{=} \bigcup_{n \in \mathbb{N}} A_n$$

and

$$n_k \stackrel{\text{def}}{=} \#\{\alpha \in A, \ |\alpha| \leq k\}$$

(since I is finite, n_k is a defined finite number).

We define a family of process $(\overline{V}_t^\alpha)_{t \geq 0}$, $\alpha \in A$ in the following way.

- If $|\alpha| = 0$ we put $\overline{V}_t^0 \stackrel{\text{def}}{=} \overline{X}_t^M$ with $\overline{X}_t^M$ solution of the equation (2.44) :

$$\overline{X}_t^M = x + \int_0^t \sigma(\overline{X}_s^M) \, \mathrm{d}W_s + \sum_{k=1}^{J_t^M} c_M(\overline{Z}_k, \overline{X}_{T_k^-}^M) + \int_0^t g(\overline{X}_s^M) \, \mathrm{d}s.$$

- Suppose now that we have already defined $\overline{V}^\alpha$ for $|\alpha| < n - 1$. We denote by

$$\overline{V}_{(k-1)}(t) \stackrel{\text{def}}{=} \left(\overline{V}_t^\beta\right)_{|\beta| \leq k-1}$$

(so $\overline{V}_{(0)}(t) = (\overline{V}_t^0) = (\overline{X}_t^M)$, a family of d-dimensional one element). Then let $\overline{V}^\alpha$ be, for $|\alpha| = n$, the solution of

$$\begin{aligned} \overline{V}_t^\alpha = \overline{V}_0^\alpha &+ \int_0^t G^\alpha\big(\overline{V}_{(k-1)}(s)\big) \, \mathrm{d}W_s + \sum_{j=1}^{J_t^M} d_j^\alpha\big(\overline{Z}_j, \overline{V}_{(k-1)}(T_j^-)\big) + \int_0^t g^\alpha\big(\overline{V}_{(k-1)}(s)\big) \, \mathrm{d}s \\ &+ \sum_{l=1}^m \int_0^t \rho_l^\alpha(\overline{V}_s^0) \overline{V}_s^\alpha \, \mathrm{d}W_s^l + \sum_{j=1}^{J_t^M} \beta^\alpha\big(\overline{Z}_j, \overline{V}_{T_j^-}^0\big) \overline{V}_{T_j^-}^\alpha + \int_0^t b^\alpha(\overline{V}_s^0) \overline{V}_s^\alpha \, \mathrm{d}s, \end{aligned} \tag{3.54}$$

with the functions $G^\alpha : \mathbb{R}^{d \times n_{k-1}} \to \mathbb{R}^{d \times m}$, $d_j^\alpha : \mathbb{R}^d \times \mathbb{R}^{d \times n_{k-1}} \to \mathbb{R}^d$, $g^\alpha : \mathbb{R}^{d \times n_{k-1}} \to \mathbb{R}^d$, $\rho_l^\alpha : \mathbb{R}^d \to \mathbb{R}^{d \times d}$, $\beta^\alpha : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^{d \times d}$, $b^\alpha : \mathbb{R}^d \to \mathbb{R}^{d \times d}$ and the following Hypothesis :

Hypothesis 3.1 *1. There exist $w \in \mathbb{N}$ and $K \in \mathbb{R}_+$ such that, for all $v \in \mathbb{R}^{n_{k-1}}$ and $z \in \mathbb{R}^d$,*

$$|G^\alpha(v)| \leq K(1 + |v|)^w, \qquad |g^\alpha(v)| \leq K(1 + |v|)^w, \tag{3.55}$$

and

$$|\beta^\alpha(z, v)| \leq \overline{\beta}(z), \qquad |d_j^\alpha(z, v)| \leq K\overline{c}(z)(1 + |v|)^w \quad (\forall j \in \mathbb{N}), \tag{3.56}$$

with, for all $n \in \mathbb{N}$,

- $\int_{\mathbb{R}^d} \overline{c}(z)^n \mu(\mathrm{d}z) < \infty$;
- $\int_{\mathbb{R}^d} \overline{\beta}(z)^n \mu(\mathrm{d}z) < \infty$

2. *bounding conditions :*

- $\overline{\rho} \overset{def}{=} \sup_{s\in\mathbb{R}^d} \sup_{\alpha\in A} \sup_{1\le l\le m} |\rho_l^\alpha(s)| < \infty$;
- $\overline{b} \overset{def}{=} \sup_{s\in\mathbb{R}^d} \sup_{\alpha\in A} |b^\alpha(s)| < \infty$.

Lemma 3.1 *Let $p \in \mathbb{N}^*$. We assume that the Hypothesis 3.1 holds and we set*

$$\Theta_{p,k}(t) = \sup_{\substack{\alpha\in A\\ |\alpha|\le k}} \mathrm{E}_W\left[|\overline{V}_t^\alpha|^{2p}\right].$$

For all $T > 0$, there exists a constant $C_{T,p,k}$ (which does not *depend on M* nor *on the set A, and, in particular, does not depend on the size of I) such that*

$$\mathrm{E}\left[\Theta_{p,k}(t)\right] \le C_{T,p,k}. \tag{3.57}$$

Proof : In order to use stochastic calculus, we will come back to the process X_t^M, so, with the same notations as before, and

$$V_{(k)}(t) \overset{\text{def}}{=} \left(V_t^\beta\right)_{|\beta|\le k}$$

with the convention $V_t^0 \overset{\text{def}}{=} X_t^M$ (so $V_{(0)}(t) = X_t^M$), V_t^α is then defined as a solution of the following SDE (where $k = |\alpha|$):

$$\begin{aligned} V_t^\alpha = V_0^\alpha &+ \int_0^t G^\alpha\big(V_{(k-1)}(s)\big)\,\mathrm{d}W_s + \sum_{j=0}^{J_t^M} d_j^\alpha\big(Z_j, V_{(k-1)}(T_j^-)\big)\mathbb{1}_{\{U_j\le\gamma(Z_j,X^M_{T_j-})\}} + \int_0^t g^\alpha\big(V_{(k-1)}(s)\big)\,\mathrm{d}s \\ &+ \sum_{l=1}^m \int_0^t \rho_l^\alpha(V_s^0)V_s^\alpha\,\mathrm{d}W_s^l + \sum_{j=0}^{J_t^M} \beta^\alpha\big(Z_j, V^0_{T_j^-}\big)\mathbb{1}_{\{U_j\le\gamma(Z_j,X^M_{T_j-})\}} V^\alpha_{T_j^-} + \int_0^t b^\alpha(V_s^0)V_s^\alpha\,\mathrm{d}s. \end{aligned} \tag{3.58}$$

In order to express the first compound Poisson process with an integral with respect to the Poisson measure, we put (with the convention $T_0 = 0$)

$$e_s^\alpha(z, v) \overset{\text{def}}{=} \sum_{j=1}^\infty \mathbb{1}_{]T_{j-1},T_j]}(s)d_j^\alpha(z, v)$$

from (3.56) it is clear that,

$$|e_s^\alpha(z, v)| \le K\overline{c}(z)(1 + |v|)^w. \tag{3.59}$$

Therefore

$$\begin{aligned} V_t^\alpha = V_0^\alpha &+ \int_0^t G^\alpha\big(V_{(k-1)}(s)\big)\,\mathrm{d}W_s + \sum_{j=0}^{J_t^M} e^\alpha_{T_j^-}\big(Z_j, V_{(k-1)}(T_j^-)\big)\mathbb{1}_{\{U_j\le\gamma(Z_j,X^M_{T_j-})\}} + \int_0^t g^\alpha\big(V_{(k-1)}(s)\big)\,\mathrm{d}s \\ &+ \sum_{l=1}^m \int_0^t \rho_l^\alpha(V_s^0)V_s^\alpha\,\mathrm{d}W_s^l + \sum_{j=0}^{J_t^M} \beta^\alpha\big(Z_j, V^0_{T_j^-}\big)\mathbb{1}_{\{U_j\le\gamma(Z_j,X^M_{T_j-})\}} V^\alpha_{T_j^-} + \int_0^t b^\alpha(V_s^0)V_s^\alpha\,\mathrm{d}s \\ = V_0^\alpha &+ \int_0^t G^\alpha\big(V_{(k-1)}(s)\big)\,\mathrm{d}W_s + \int_0^t\int_E e^\alpha_{s^-}\big(z, V_{(k-1)}(s^-)\big)\mathbb{1}_{\{u\le\gamma(z,X^M_{s-})\}}N(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u) + \int_0^t g^\alpha\big(V_{(k-1)}(s)\big)\,\mathrm{d}s \\ &+ \sum_{l=1}^m \int_0^t \rho_l^\alpha(V_s^0)V_s^\alpha\,\mathrm{d}W_s^l + \int_0^t\int_E \beta^\alpha\big(z, V^0_{s-}\big)\mathbb{1}_{\{u\le\gamma(z,X^M_{s-})\}}V^\alpha_{s-}N(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u) + \int_0^t b^\alpha(V_s^0)V_s^\alpha\,\mathrm{d}s. \end{aligned}$$

From Lemma 2.8, V_t^α and $\overline{V}_t^\alpha$ are sharing the same law, so we will prove that, for all $k \ge 0$,

$$\mathrm{E}\Big[\sup_{\substack{\alpha\in A\\ |\alpha|\le k}} \mathrm{E}_W\left[|V_t^\alpha|^{2p}\right]\Big] \le C_{T,p}, \tag{3.60}$$

which will be shown by recurrence on k.

- For $k = 0$, $V_t^0 = X_t^M$, it is the Proposition B.1.
- For $k \geq 1$, we first simplify the notations by writing :

$$\begin{aligned} G^\alpha(s) &\stackrel{\text{def}}{=} G^\alpha\big(V_{(k-1)}(s)\big), & g^\alpha(s) &\stackrel{\text{def}}{=} g^\alpha\big(V_{(k-1)}(s)\big), \\ \rho_l^\alpha(s) &\stackrel{\text{def}}{=} \rho_l^\alpha(V_s^0), & b^\alpha(s) &\stackrel{\text{def}}{=} b^\alpha(V_s^0), \\ h^\alpha(s^-, z, u) &\stackrel{\text{def}}{=} e_{s^-}^\alpha\big(z, V_{(k-1)}(s^-)\big)\mathbb{1}_{\{u \leq \gamma(z, X_{s^-}^M)\}}, & \beta^\alpha(s^-, z, u) &\stackrel{\text{def}}{=} \beta^\alpha\big(z, V_{s^-}^0\big)\mathbb{1}_{\{u \leq \gamma(z, X_{s^-}^M)\}}, \end{aligned}$$

which gives

$$\begin{aligned} V_t^\alpha = V_0^\alpha &+ \int_0^t G^\alpha(s)\,\mathrm{d}W_s + \int_0^t \int_E h^\alpha(s^-, z, u) N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) + \int_0^t g^\alpha(s)\,\mathrm{d}s \\ &+ \sum_{l=1}^m \int_0^t \rho_l^\alpha(s) V_s^\alpha\,\mathrm{d}W_s^l + \int_0^t \int_E \beta^\alpha(s^-, z, u) V_{s^-}^\alpha N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) + \int_0^t b^\alpha(s) V_s^\alpha\,\mathrm{d}s. \end{aligned}$$

In order to use the recurrence hypothesis, we bound the coefficients of this last equation in the following way : with

$$\overline{h}_k(s) \stackrel{\text{def}}{=} K(1 + |V_{(k-1)}(s)|)^w$$

and

$$\overline{\xi}(z, u) \stackrel{\text{def}}{=} (\overline{c}(z) + \overline{\beta}(z))\mathbb{1}_{\{u \leq \overline{\gamma}(z)\}}$$

we have (according to the hypothesis 3.1)

$$\sup_{\substack{\alpha \in A \\ |\alpha| \leq k}} G^\alpha(s) \leq \overline{h}_k(s) \qquad \text{and} \qquad \overline{h}_k(s) \stackrel{\text{def}}{=} \sup_{\substack{\alpha \in A \\ |\alpha| \leq k}} g^\alpha(s) \leq \overline{h}_k(s)$$

and (using (3.59) and (3.56))

$$\forall s, z, u, \qquad |h^\alpha(s^-, z, u)| \leq \overline{h}_k(s^-)\overline{\xi}(z, u) \qquad \text{and} \qquad |\beta^\alpha(s^-, z, u)| \leq \overline{\xi}(z, u). \tag{3.61}$$

At last, we have to notice, using the recurrence hypothesis for $k - 1$, that, for all $n \in \mathbb{N}^*$,

$$\sup_{0 \leq s \leq T} \mathrm{E}\left[|\overline{h}_k(s)|^n\right] < \infty; \tag{3.62}$$

this last result comes directly from the following bounding :

$$\begin{aligned} \mathrm{E}\left[|V_{(k-1)}(s)|^{2m}\right] = \mathrm{E}\left[\mathrm{E}_W\left[|V_{(k-1)}(s)|^{2m}\right]\right] &\leq K\,\mathrm{E}\left[\sum_{\beta \in A_{k-1}} \mathrm{E}_W\left[|V_s^\beta|^{2m}\right]\right] \\ &\leq K' \sqrt{\mathrm{E}\left[\Big(\sup_{\beta \in A_{k-1}} \mathrm{E}_W\left[|V_t^\beta|^{2m}\right]\Big)^2\right]} \\ &\leq K' \sqrt{C_{T,m,k-1}} < \infty. \end{aligned}$$

Step 1

In order to use the Itô's formula, we will first have to localize our problem by using the sequence $(\tau_K^M)_{K \in \mathbb{N}^*}$ of stopping times defined by

$$\tau_K^M(k) \stackrel{\text{def}}{=} \inf\{t > 0 \ : \ \sup_{s \leq t} \sum_{|\alpha| \leq k} |V_s^\alpha| \geq K\}. \tag{3.63}$$

Let us prove that a.s. $\lim_{K \to \infty} \tau_K^M = \infty$.
From the hypothesis made on the coefficients of V_t^α, it is clear that, for all $t \geq 0$,

$$\sum_{|\alpha| \leq k} \mathrm{E}\left[\sup_{s \leq t} |V_s^\alpha|\right] \leq \infty. \tag{3.64}$$

We have, for $t \geq 0$

$$\lim_{K\to\infty} \mathrm{P}(\tau_K^M < t) = \lim_{K\to\infty} \mathrm{P}(\sup_{s\leq t} \sum_{|\alpha|\leq k} |V_s^\alpha| > K)$$
$$\leq \lim_{K\to\infty} \frac{1}{K}\,\mathrm{E}\big[\sup_{s\leq t} \sum_{|\alpha|\leq k} |V_s^\alpha|\big] = 0.$$

$(\tau_K^M)_{K\in\mathbb{N}^*}$ tends to ∞ in probability and so, there exists a subsequence (that we will continue to denote by $(\tau_K^M)_{K\in\mathbb{N}^*}$) which tends to ∞ a.s.

In this case we have

$$|V_t^\alpha|^{2p}\mathbb{1}_{\tau_K^M>t} \uparrow |V_t^\alpha|^{2p} \qquad \text{a.s.}$$

so

$$\mathrm{E}_W\left[|V_t^\alpha|^{2p}\mathbb{1}_{\tau_K^M>t}\right] \uparrow \mathrm{E}_W\left[|V_t^\alpha|^{2p}\right] \qquad \text{a.s.}$$

and

$$\sup_{\substack{\alpha\in A\\|\alpha|\leq k}} \mathrm{E}_W\left[|V_t^\alpha|^{2p}\mathbb{1}_{\tau_K^M>t}\right] \uparrow \sup_{\substack{\alpha\in A\\|\alpha|\leq k}} \mathrm{E}_W\left[|V_t^\alpha|^{2p}\right] \qquad \text{a.s.}$$

If we admit for the moment that there exists a constant $C_{p,T,k}$ which does not depend on K and M and such that, for all $0 \leq t \leq T$,

$$\mathrm{E}\Big[\sup_{\substack{\alpha\in A\\|\alpha|\leq k}} \mathrm{E}_W\left[|V_{t\wedge\tau_K^M}^\alpha|^{2p}\right]\Big] \leq C_{T,p,k} \tag{3.65}$$

The monotone convergence theorem implies then

$$\mathrm{E}\big[\sup_{\substack{\alpha\in A\\|\alpha|\leq k}} \mathrm{E}_W\left[|V_t^\alpha|^{2p}\right]\big] = \sup_K \mathrm{E}\big[\sup_{\substack{\alpha\in A\\|\alpha|\leq k}} \mathrm{E}_W\left[|V_t^\alpha|^{2p}\mathbb{1}_{\tau_K^M>t}\right] = \sup_K \mathrm{E}\big[\sup_{\substack{\alpha\in A\\|\alpha|\leq k}} \mathrm{E}_W\left[|V_{t\wedge\tau_K^M}^\alpha|^{2p}\right]\big] \leq C_{T,p,k}.$$

Step 2

We have to establish now (3.65).

For a single component we have (omitting for a moment the parameter α in order to simplify the notations)

$$\begin{aligned} V_t^i = V_0^i &+ \sum_{l=1}^m \int_0^t G_{il}(s)\,\mathrm{d}W_s^l + \int_0^t \int_E h^i(s^-,z,u)N(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u) + \int_0^t g^i(s)\,\mathrm{d}s \\ &+ \sum_{l=1}^m \sum_{h=1}^d \int_0^t \rho_{ih}^l(s)V_s^h\,\mathrm{d}W_s^l + \sum_{h=1}^d \int_0^t \int_E \beta_{ih}(s^-,z,u)V_{s^-}^h N(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u) + \sum_{h=1}^d \int_0^t b_{ih}(s)V_s^h\,\mathrm{d}s \end{aligned}$$

Then, applying Itô's formula with $f(x) = x^{2p}$

$$\begin{aligned} (V_{t\wedge\tau_K^M}^i)^{2p} = (V_0^i)^{2p} &+ \sum_{l=1}^m \int_0^{t\wedge\tau_K^M} 2p(V_s^i)^{2p-1}\Big(G_{il}(s) + \sum_{h=1}^d \rho_{ih}^l(s)V_s^h\Big)\,\mathrm{d}W_s^l \\ &+ 2p\int_0^{t\wedge\tau_K^M} (V_s^i)^{2p-1}\Big(g_i(s) + \sum_{h=1}^d b_{ih}(s)V_s^h\Big)\,\mathrm{d}s \\ &+ p(2p-1)\sum_{l=1}^m \int_0^{t\wedge\tau_K^M} (V_s^i)^{2p-2}\Big(G_{il}(s) + \sum_{h=1}^d \rho_{ih}^l(s)V_s^h\Big)^2\,\mathrm{d}s \\ &+ \int_0^{t\wedge\tau_K^M} \int_E \Big(V_{s^-}^i + h_i(s^-,z,u) + \sum_{h=1}^d \beta_{ih}(s^-,z,u)V_{s^-}^h\Big)^{2p} - (V_{s^-}^i)^{2p} N(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u) \end{aligned}$$

We take now the expectation with respect to the Brownian motion (*i.e.* conditionally with respect to all the others random quantities) :

$$\begin{aligned}
\mathrm{E}_W\left[(V^i_{t\wedge\tau_K^M})^{2p}\right] =&\, \mathrm{E}_W\left[(V^i_0)^{2p}\right] + 2p\int_0^{t\wedge\tau_K^M}\mathrm{E}_W\left[(V^i_s)^{2p-1}\Big(g_i(s)+\sum_{h=1}^d b_{ih}(s)V^h_s\Big)\right]\mathrm{d}s\\
&+p(2p-1)\sum_{l=1}^m\int_0^{t\wedge\tau_K^M}\mathrm{E}_W\left[(V^i_s)^{2p-2}\Big(G_{il}(s)+\sum_{h=1}^d\rho^l_{ih}(s)V^h_s\Big)^2\right]\mathrm{d}s\\
&+\int_0^{t\wedge\tau_K^M}\int_E\mathrm{E}_W\left[\Big(V^i_{s^-}+h_i(s^-,z,u)+\sum_{h=1}^d\beta_{ih}(s^-,z,u)V^h_{s^-}\Big)^{2p}-(V^i_{s^-})^{2p}\right]N(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u).
\end{aligned}$$

Since $s\le t\wedge\tau_K^M$, we have $X_s = X_{s\wedge\tau_K^M}$, and obviously $t\le t\wedge\tau_K^M$, so we have

$$\begin{aligned}
\mathrm{E}_W\left[|V^i_{t\wedge\tau_K^M}|^{2p}\right] =&\, \mathrm{E}_W\left[|V^i_0|^{2p}\right] + 2p\int_0^{t}\mathrm{E}_W\left[|V^i_{s\wedge\tau_K^M}|^{2p-1}\Big(|g_i(s)|+\sum_{h=1}^d |b_{ih}(s)||V^h_{s\wedge\tau_K^M}|\Big)\right]\mathrm{d}s\\
&+p(2p-1)\sum_{l=1}^m\int_0^{t}\mathrm{E}_W\left[|V^i_{s\wedge\tau_K^M}|^{2p-2}\Big(|G_{il}(s)|+\sum_{h=1}^d|\rho^l_{ih}(s)||V^h_{s\wedge\tau_K^M}|\Big)^2\right]\mathrm{d}s\\
&+\int_0^{t\wedge\tau_K^M}\int_E\mathrm{E}_W\left[\Big|\Big(V^i_{s^-}+h_i(s^-,z,u)+\sum_{h=1}^d\beta_{ih}(s^-,z,u)V^h_{s^-}\Big)^{2p}-(V^i_{s^-})^{2p}\Big|\right]N(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u).
\end{aligned}$$

Since

$$\begin{aligned}
&\Big|\Big(V^i_{s^-}+h_i(s^-,z,u)+\sum_{h=1}^d\beta_{ih}(s^-,z,u)V^h_{s^-}\Big)^{2p}-(V^i_{s^-})^{2p}\Big|\\
&\qquad=\Big|\sum_{k=1}^{2p}\binom{2p}{k}\Big(h_i(s^-,z,u)+\sum_{h=1}^d\beta_{ih}(s^-,z,u)V^h_{s^-}\Big)^k {V^i_{s^-}}^{2p-k}\Big|\\
&\qquad\le\sum_{k=1}^{2p}\binom{2p}{k}(|h|(s^-,z,u)+|\beta|(s^-,z,u)|V|_{s^-})^k|V|_{s^-})^{2p-k}\\
&\qquad=(|V|_{s^-}+|h|(s^-,z,u)+|\beta|(s^-,z,u)|V|_{s^-})^{2p}-|V|^{2p}_{s^-},
\end{aligned}$$

it follows (with $\rho\stackrel{\text{def}}{=}\sup_l|\rho_l|$)

$$\begin{aligned}
\mathrm{E}_W[(V^i_{t\wedge\tau_K^M})^{2p}]\le&\,\mathrm{E}_W[|V_0|^{2p}]+2p\int_0^{t\wedge\tau_K^M}\mathrm{E}_W[|V_s|^{2p-1}(|g|(s)+|b|(s)|V_s|)]\,\mathrm{d}s\\
&+p(2p-1)\int_0^{t\wedge\tau_K^M}\mathrm{E}_W[|V_s|^{2p-2}(|G|(s)+\rho(s)|V_s|)^2]\,\mathrm{d}s\\
&+\int_0^{t\wedge\tau_K^M}\int_E\mathrm{E}_W[(|V_{s^-}|+|h|(s^-,z,u)+|\beta|(s^-,z,u)|V_{s^-}|)^{2p}-|V_{s^-}|^{2p}]N(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u).
\end{aligned}$$

Then (writing again from now on the parameter α, in order to see clearly which components depend of it or not), we have (using, among others things, the inequality (3.61))

$$\begin{aligned}
\mathrm{E}_W[|V^\alpha_{t\wedge\tau_K^M}|^{2p}]\le&\,\mathrm{E}_W[|V^\alpha_0|^{2p}]+2p\int_0^{t\wedge\tau_K^M}\mathrm{E}_W[|V^\alpha_s|^{2p-1}(\overline{h}_k(s)+\overline{b}|V^\alpha_s|)]\,\mathrm{d}s\\
&+p(2p-1)\int_0^{t\wedge\tau_K^M}\mathrm{E}_W[|V^\alpha_s|^{2p-2}(\overline{h}_k(s)+\overline{\rho}|V^\alpha_s|)^2]\,\mathrm{d}s\\
&+\int_0^{t\wedge\tau_K^M}\int_E\mathrm{E}_W[(|V^\alpha_{s^-}|+\overline{h}_k(s^-)\overline{\xi}(z,u)+\overline{\xi}(z,u)|V^\alpha_{s^-}|)^{2p}-|V^\alpha_{s^-}|^{2p}]N(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u).
\end{aligned}$$

(Notice that, since V^α_t is an adapted process, $|V^\alpha_0|$ is a constant, so $\mathrm{E}_W[|V^\alpha_0|^{2p}]=|V^\alpha_0|^{2p}$.)

To bound the first integral, using the elementary inequality

$$\forall x,y\ge0,\ \forall u,v>0\qquad x^uy^v\le x^{u+v}+y^{u+v}, \tag{3.66}$$

we notice that :

$$\begin{aligned} \big||V_s^\alpha|^{2p-1}\overline{h}_k(s)+\bar{b}|V_s^\alpha|^{2p}\big| &\le \overline{h}_k(s)^{2p}+|V_s^\alpha|^{2p}+\bar{b}|V_s^\alpha|^{2p} \\ &= \overline{h}_k(s)^{2p}+(1+\bar{b})|V_s^\alpha|^{2p}, \end{aligned}$$

and, similarly, for the second one,

$$\begin{aligned} |V_s^\alpha|^{2p-2}(\overline{h}_k(s)+\bar{\rho}|V_s^\alpha|)^2 &\le 2\overline{h}_k(s)^{2p}+2|V_s^\alpha|^{2p}+2\bar{\rho}^2|V_s^\alpha|^{2p} \\ &= 2\overline{h}_k(s)^{2p}+2(1+\bar{\rho}^2)|V_s^\alpha|^{2p}. \end{aligned}$$

It follows that,

$$\begin{aligned} 2p\int_0^{t\wedge\tau_K^M} \mathrm{E}_W[|V_s^\alpha|^{2p-1}(\overline{h}_k(s)+\bar{b}|V_s^\alpha|)]\,\mathrm{d}s + p(2p-1)\int_0^{t\wedge\tau_K^M} \mathrm{E}_W[|V_s^\alpha|^{2p-2}(\overline{h}_k(s)+\bar{\rho}|V_s|)^2]\,\mathrm{d}s \\ \le C_p\int_0^{t\wedge\tau_K^M} \mathrm{E}_W[\overline{h}_k(s)^{2p}]\,\mathrm{d}s + C_p'\int_0^{t\wedge\tau_K^M} \mathrm{E}_W[|V_s^\alpha|^{2p}]\,\mathrm{d}s. \end{aligned}$$

For the third integral, we will see that

$$(|V_{s-}^\alpha|+\overline{h}_k(s^-)\bar{\xi}(z,u)+\bar{\xi}(z,u)|V_{s-}^\alpha|)^{2p}-|V_{s-}^\alpha|^{2p} \le \bar{\xi}(z,u)P(\bar{\xi}(z,u))\big(|V_{s-}^\alpha|^{2p}+(\overline{h}_k(s^-))^{2p}\big) \tag{3.67}$$

where P is a polynomial function.

Let us prove now (3.67) : if $u,v\ge 0$,

$$\big|u^{2p}-v^{2p}\big| \le |u-v|(u+v)^{2p-1}, \tag{3.68}$$

so, it follows that, for a, b, $c\ge 0$:

$$\begin{aligned} (a+c(b+a))^{2p}-a^{2p} &\le c(b+a)(a(2+c)+cb)^{2p-1} \\ &\le 2^{2p-1}c(b+a)(a^{2p-1}(2+c)^{2p-1}+(cb)^{2p-1}) \\ &\le 2^{2p-1}c(a^{2p}(2+c)^{2p-1}+a(cb)^{2p-1}+ba^{2p-1}(2+c)^{2p-1}+c^{2p-1}b^{2p}) \end{aligned}$$

using (3.66), we have

$$a(cb)^{2p-1}\le a^{2p}+(cb)^{2p} \qquad \text{and} \qquad ba^{2p-1}(2+c)^{2p-1}\le a^{2p}+b^{2p}(2+c)^{2p(2p-1)}$$

which brings to

$$(a+c(b+a))^{2p}-a^{2p} \le 2^{2p-1}c\big[a^{2p}(2+(2+c)^{2p-1})+b^{2p}(c^{2p-1}+c^{2p}+(2+c)^{2p(2p-1)})\big],$$

or, more generally, to

$$(a+c(b+a))^{2p}-a^{2p} \le cP(c)\big[a^{2p}+b^{2p})\big], \tag{3.69}$$

where $P\in\mathbb{R}[X]$, which proves (3.67).

Gathering all those results,

$$\begin{aligned} \mathrm{E}_W[|V_{t\wedge\tau_K^M}^\alpha|^{2p}] \le{}& \mathrm{E}_W[|V_0^\alpha|^{2p}] + C_p\int_0^{t\wedge\tau_K^M} \mathrm{E}_W[\overline{h}_k(s)^{2p}]\,\mathrm{d}s + C_p'\int_0^{t\wedge\tau_K^M} \mathrm{E}_W[|V_s^\alpha|^{2p}]\,\mathrm{d}s \\ &+ \int_0^{t\wedge\tau_K^M}\int_E \bar{\xi}P(\bar{\xi})\,\mathrm{E}_W[\bar{h}_k^{2p}(s^-)]N(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u) + \int_0^{t\wedge\tau_K^M}\int_E \bar{\xi}P(\bar{\xi})\,\mathrm{E}_W[|V_{s-}^\alpha|^{2p}]N(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u). \end{aligned}$$

We then have, directly, with $\Theta_{p,k}(t\wedge\tau_K^M) \overset{\text{def}}{=} \sup_{\substack{\alpha\in A\\ |\alpha|\le k}} \mathrm{E}_W\left[|V_{t\wedge\tau_K^M}^\alpha|^{2p}\right]$,

$$\begin{aligned} \Theta_{p,k}(t\wedge\tau_K^M) \le{}& \Theta_{p,k}(0) + C_p\int_0^{t\wedge\tau_K^M} \mathrm{E}_W[\overline{h}_k(s)^{2p}]\,\mathrm{d}s + C_p'\int_0^{t\wedge\tau_K^M} \Theta_{p,k}(s)\,\mathrm{d}s \\ &+ \int_0^{t\wedge\tau_K^M}\int_E \bar{\xi}P(\bar{\xi})\,\mathrm{E}_W[\bar{h}_k^{2p}(s^-)]N(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u) + \int_0^{t\wedge\tau_K^M}\int_E \bar{\xi}P(\bar{\xi})\Theta_{p,k}(s-)N(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u), \end{aligned} \tag{3.70}$$

Since $s \leq t \wedge \tau_K^M$, we have, for any process Y, $Y_s = Y_{s\wedge\tau_K^M}$, and obviously $t \leq t \wedge \tau_K^M$, so we have

$$\begin{aligned}\Theta_{p,k}(t\wedge\tau_K^M) \leq &\Theta_{p,k}(0) + C_p\int_0^t \mathrm{E}_W[\bar{h}_k(s\wedge\tau_K^M)^{2p}]\,\mathrm{d}s + C'_p\int_0^t \Theta_{p,k}(s\wedge\tau_K^M)\,\mathrm{d}s \\ &+ \int_0^t\int_E \bar{\xi}P(\bar{\xi})\,\mathrm{E}_W[\bar{h}_k^{2p}((s\wedge\tau_K^M)^-)]N(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u) + \int_0^t\int_E \bar{\xi}P(\bar{\xi})\Theta_{p,k}((s\wedge\tau_K^M)-)N(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u),\end{aligned} \tag{3.71}$$

Step 2

The last step is to bound

$$\mathrm{E}\left[\Theta_{p,k}(t)\right].$$

By setting

$$R_1 \overset{\text{def}}{=} \int \bar{\xi}P(\bar{\xi})\,\mathrm{d}u\mu(\mathrm{d}z)$$

we have (using the isometry in F_p^1, *cf.* (**??**))

$$\begin{aligned}&\mathrm{E}\left[\Theta_{p,k}(t\wedge\tau_K^M)\right] \\ &\quad\leq \mathrm{E}\left[\Theta_{p,k}(0)\right] + C_p\,\mathrm{E}\left[\int_0^t \mathrm{E}_W[\bar{h}_k(s\wedge\tau_K^M)^{2p}]\,\mathrm{d}s\right] + C'_p\,\mathrm{E}\left[\int_0^t \Theta_{p,k}(s\wedge\tau_K^M)\,\mathrm{d}s\right] \\ &\quad\quad + \mathrm{E}\left[\int_0^t\int_E \bar{\xi}P(\bar{\xi})\,\mathrm{E}_W[\bar{h}_k^{2p}((s\wedge\tau_K^M)^-))]N(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u)\right] + \mathrm{E}\left[\int_0^t\int_E \bar{\xi}P(\bar{\xi})\Theta_{p,k}((s\wedge\tau_K^M)^-))N(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u)\right] \\ &\quad= \mathrm{E}\left[\Theta_{p,k}(0)\right] + C_p\int_0^t \mathrm{E}[\bar{h}_k(s\wedge\tau_K^M)^{2p}]\,\mathrm{d}s + C'_p\int_0^t \mathrm{E}[\Theta_{p,k}(s\wedge\tau_K^M)]\,\mathrm{d}s \\ &\quad\quad + R_1\int_0^t \mathrm{E}[\bar{h}_k^{2p}((s\wedge\tau_K^M)^-))]\,\mathrm{d}s + \int_0^t \mathrm{E}[\Theta_{p,k}((s\wedge\tau_K^M)^-))]\,\mathrm{d}s \\ &\quad= \mathrm{E}\left[\Theta_{p,k}(0)\right] + (C_p+R_1)\int_0^t \mathrm{E}[\bar{h}_k(s\wedge\tau_K^M)^{2p}]\,\mathrm{d}s + (C'_p+R_1)\int_0^t \mathrm{E}[\Theta_{p,k}(s\wedge\tau_K^M)]\,\mathrm{d}s\end{aligned}$$

With $A_p(T) \overset{\text{def}}{=} \mathrm{E}\left[\Theta_{p,k}(0)\right] + (C_p+R_1)\int_0^T \mathrm{E}[\bar{h}_k(s)^{2p}]\,\mathrm{d}s$, which, by virtue of (3.62), is a finite quantity, the Gronwall's lemma gives here :

$$\mathrm{E}\left[\Theta_{p,k}(t\wedge\tau_K^M)\right] \leq A_p(T)\exp[(C'_p+R_1)t]$$

which proves the assertion (3.65).

•

To bound (in L^p, $p \geq 1$) the Sobolev norm $\left|\bar{X}_t^M\right|_l$, we will proceed by recurrence on $l \in \mathbb{N}^*$, and we will show in detail the case corresponding to the first order norm, since in this particular case, the structure of the general method already appears with lesser notations than used in the general case.

In all the following we will set

$$A_k(L) \overset{\text{def}}{=} ([\![1,L]\!]\times[\![1,d]\!])^k$$

and

$$A(L) \overset{\text{def}}{=} \bigcup_{k\in\mathbb{N}^*} A_k(L).$$

We will also need the following lemma's :

Lemma 3.2 *Let $n,p \geq 1$ and $F \in \mathcal{S}^d$. Then*

$$\mathrm{E}\left[|D^nF|^{2p}\right] \leq d^{np}\sqrt{\mathrm{E}\left[(J_t^M)^{2np}\right]}\sup_L\sqrt{\mathrm{E}\left[\max_{\alpha\in A_n(L)}\mathrm{E}_W\left[|\,\mathrm{D}_\alpha F|^{4p}\right]\right]}.$$

Proof : By definition

$$\mathrm{E}\left[|D^nF|^{2p}\right] = \mathrm{E}\left[\Big(\sum_{\alpha\in([\![1,J_t^M]\!]\times[\![1,d]\!])^n}|\mathrm{D}_\alpha F|^2\Big)^p\right];$$

on the other hand

$$\mathrm{E}\left[|D^nF|^{2p}\right] = \sup_L \mathrm{E}\left[|D^nF|^{2p}\,\mathbb{1}_{J_t^M\le L}\right]$$

and

$$\begin{aligned}
\mathrm{E}\left[|D^nF|^{2p}\,\mathbb{1}_{J_t^M\le L}\right] &= \mathrm{E}\Big[\Big(\sum_{\alpha\in([\![1,J_t^M]\!]\times[\![1,d]\!])^n} |\mathrm{D}_\alpha F|^2\Big)^p \mathbb{1}_{J_t^M\le L}\Big]\\
&\le \mathrm{E}\Big[(dJ_t)^{n(p-1)} \sum_{\alpha=((k_n,r_n),\ldots,(k_1,r_1))\in A_n(L)} |\mathrm{D}_\alpha F|^{2p} \prod_{i=1}^n \mathbb{1}_{k_i\le J_t^M\le L}\Big]\\
&\le \mathrm{E}\Big[(dJ_t)^{n(p-1)} \sum_{\alpha\in A_n(L)} \mathrm{E}_W\left[\,|\mathrm{D}_\alpha F|^{2p}\,\right]\Big]\\
&\le \mathrm{E}\Big[d^{np}J_t^{np} \max_{\alpha\in A_n(L)} \mathrm{E}_W\left[\,|\mathrm{D}_\alpha F|^{2p}\,\right]\Big]\\
&\le d^{np}\sqrt{\mathrm{E}\left[(J_t^M)^{2np}\right]}\sqrt{\mathrm{E}\big[\max_{\alpha\in A_n(L)} \mathrm{E}_W\left[|\,\mathrm{D}_\alpha F|^{4p}\right]\big]}.
\end{aligned}$$

•

Lemma 3.3 *Let $j\ge 1$. Then there exists $C_{|\alpha|}>0$ such that*

$$|\,\mathrm{D}_\alpha \bar Z_j| \le C_{|\alpha|}.$$

Proof : For $|\alpha|=1$, $\alpha=(k,r)$ and (recalling that $\pi_k^r=\phi_M(\bar Z_k)$)

$$\mathrm{D}_{k,r}\bar Z_j = \begin{pmatrix} \pi_k^r\partial_{\bar Z_k^r}\bar Z_j^1\\ \vdots\\ \pi_k^r\partial_{\bar Z_k^r}\bar Z_j^r\\ \vdots\\ \pi_k^r\partial_{\bar Z_k^r}\bar Z_j^d\end{pmatrix} = \begin{pmatrix}0\\ \vdots\\ \phi_M(\bar Z_k)\delta_{k,j}\\ \vdots\\ 0\end{pmatrix} \tag{3.72}$$

so $|\,\mathrm{D}_{k,r}\bar Z_j|\le\|\phi\|_\infty=1$.

Since ϕ has bounded derivatives of any order, the recursive differentiation of (3.72) gives the general bounding property ; although this recursive differentiation is rather clear, we show the case $|\alpha|=2$, to highlight the mechanism of it :

let $\alpha=((m,n)(k,r))$, then (using (3.72)),

$$\mathrm{D}_\alpha\bar Z_j = \mathrm{D}_{m,n}\mathrm{D}_{k,r}\bar Z_j = \begin{pmatrix}0\\ \vdots\\ \pi_m^n\partial_{\bar Z_m^n}\big(\phi_M(\bar Z_k)\big)\delta_{k,j}\\ \vdots\\ 0\end{pmatrix}$$

with

$$\pi_m^n\partial_{\bar Z_m^n}\big(\phi_M(\bar Z_k)\big)\delta_{k,j} = \phi_M(\bar Z_m)\partial_n\phi_M(\bar Z_k)\delta_{m,k}\delta_{k,j},$$

and a derivative of higher order will be of the following form :

$$\mathrm{D}_\alpha\bar Z_j = \begin{pmatrix}0\\ \vdots\\ \sum c\prod\partial_\beta\phi_M(\bar Z_l))\\ \vdots\\ 0\end{pmatrix}.$$

so (since the sum and the product are finite),

$$|\,\mathrm{D}_\alpha\bar Z_j| \le \sum c\prod\|\partial_\beta\phi\|_\infty \overset{\text{def}}{=} C_{|\alpha|}.$$

$\bullet$

Recalling the following notation (*cf.* subsection 1.5.1) : for $1 \le l \le n$,

$$\mathcal{M}_n(l) \overset{\text{def}}{=} \left\{ M = (M_1, \dots, M_l),\ \bigcup_{i \in [\![1,l]\!]} M_i = \{1, \dots, n\} \text{ and } M_i \cap M_j = \varnothing,\ \text{for } i \neq j \right\},$$

we have, in fact, a more precise result :

Lemma 3.4 *Let $k \ge 1$ and $\alpha = ((k_n, r_n), \dots, (k_1, r_1))$.*

$$\mathrm{D}_\alpha \bar{Z}_k^r = \delta_{k_n,k} \cdots \delta_{k_1,k} \delta_{r_1,r} f_\alpha(\bar{Z}_k) \tag{3.73}$$

where (with $r \overset{def}{=} (r_n, \dots, r_2)$)

$$f_\alpha(\bar{Z}_k) \overset{def}{=} \sum_{\beta = (\beta_1, \cdots, \beta_n) \in \mathcal{M}_{n-1}(n)} c_\beta \prod_{i=1}^n \partial_{\beta_i(r)} \phi_M(\bar{Z}_k) \tag{3.74}$$

with

- $c_\beta \in \mathbb{N}$;
- *we denote again $\mathcal{M}_{n-1}(n)$, but, here, we allow β_i to be empty.*

Proof : By induction over the length k of α. $\bullet$

3.3 First order norm

Proposition 3.5 *If $p \ge 1$, for all $T > 0$, exists a constant $C_{T,p} > 0$ such that*

$$\forall t \in [0, T], \qquad \|D^1 \bar{X}_t^M\|_{2p} \le C_{T,p} \sqrt{\|J_T^M\|_{2p}}. \tag{3.75}$$

Proof :
First, from Lemma 3.2, we have

$$\mathrm{E}\left[\left|D^1 \bar{X}_t^M\right|^{2p}\right] \le d^p \sqrt{\mathrm{E}\left[(J_T^M)^{2p}\right]} \sup_L \sqrt{\mathrm{E}\left[\max_{\alpha \in A_1(L)} \mathrm{E}_W\left[|\mathrm{D}_\alpha \bar{X}_t^M|^{4p}\right]\right]}$$

Hence, to conclude, it remains to bound, independently of L, the quantity $\mathrm{E}\left[\max_{\alpha \in A_1(L)} \mathrm{E}_W\left[|\mathrm{D}_\alpha \bar{X}_t^M|^{4p}\right]\right]$. Recalling that $\bar{X}_t^M$ solves the following diffusion equation :

$$\bar{X}_t^M = x + \int_0^t \sigma(\bar{X}_s^M)\,\mathrm{d}W_s + \sum_{j=1}^{J_t^M} c_M(\bar{Z}_j, \bar{X}_{T_j-}^M) + \int_0^t g(\bar{X}_s^M)\,\mathrm{d}s,$$

we have

$$\begin{aligned}
\mathrm{D}_{k,r} \bar{X}_t = \nabla_z c_M(\bar{Z}_k, \bar{X}_{T_k-}^M)\,\mathrm{D}_{k,r} \bar{Z}_k &+ \sum_{l=1}^m \int_{T_k}^t \nabla \sigma_l(\bar{X}_s^M)\,\mathrm{D}_{k,r} \bar{X}_s^M\,\mathrm{d}W_s^l \\
&+ \sum_{j=k+1}^{J_t^M} \nabla_x c_M(\bar{Z}_j, \bar{X}_{T_j-}^M)\,\mathrm{D}_{k,r} \bar{X}_{T_j-}^M \\
&+ \int_{T_k}^t \nabla_x g(\bar{X}_s^M)\,\mathrm{D}_{k,r} \bar{X}_s^M\,\mathrm{d}s.
\end{aligned}$$

We can then apply the bounding Lemma 3.1 with $\alpha = (k, r)$ and $\overline{V}_t^\alpha \overset{\text{def}}{=} \mathrm{D}_{k,r} \bar{X}_t$, since the Hypothesis 3.1 are well verified, for we have ($\alpha = (k, r)$)

$$G^\alpha = 0, \quad d_j^\alpha(\overline{Z}_j, \overline{V}_{(0)}(T_j^-)) = \partial_{z_r} c_M(\bar{Z}_j, \bar{X}_{T_j-}^M) \Phi_M(\bar{Z}_j) \delta_{k,j}, \quad g^\alpha = 0$$

$$\rho_l^\alpha(\overline{V}_s^0) \overline{V}_s^\alpha = \nabla \sigma_l(\bar{X}_s^M), \quad \beta^\alpha(\overline{Z}_j, \overline{V}_{T_j^-}^0) \overline{V}_{T_j^-}^\alpha = \nabla_x c_M(\bar{Z}_j, \bar{X}_{T_j-}^M)\,\mathrm{D}_{k,r} \bar{X}_{T_j-}^M,$$

$$b^\alpha(\overline{V}^0_s)\overline{V}^\alpha_s = \nabla_x g(\bar{X}^M_s)\,\mathrm{D}_{k,r}\,\bar{X}^M_s$$

with (using Lemma 3.3)

$$|\partial_{z_r} c_M(\bar{Z}_j, \bar{X}^M_{T_j-})\Phi_M(\bar{Z}_j)\delta_{k,j}| \le C_1\bar{c}(\bar{Z}_j) \qquad \text{and} \qquad |\nabla_x c_M(\bar{Z}_j, \bar{X}^M_{T_j-})| \le \bar{c}(\bar{Z}_j),$$

which completes the proof.

•

3.4 Norm of higher order

Following the very same path as before, we find, recalling that $\lambda_M \stackrel{\text{def}}{=} 2\bar{c}\mu(E_M)$ (where E_1 is chosen in order to have $\mu(E_1) > 0$)

Proposition 3.6 *If $p \ge 1$ and $l \in \mathbb{N}^*$, there exists a constant $C_{p,l,T} > 0$ such that*

$$\forall t \in [0,T], \qquad \big\| \, \big|\bar{X}^M_t\big|_l \, \big\|_{2p} \le C_{p,l,T}(1 + \sqrt{\|(J^M_T)^l\|_{2p}}), \tag{3.76}$$

and, consequently, there exists a constant $C'_{p,l,T} > 0$ such that[2]

$$\forall t \in [0,T], \qquad \big\| \, \big|\bar{X}^M_t\big|_l \, \big\|_{2p} \le C'_{p,l,T}\sqrt{(\lambda_M)^l}. \tag{3.77}$$

Proof : We have

$$\mathrm{E}\left[\big|\bar{X}^M_t\big|_l^{2p}\right] \le C_p \sum_{k=0}^{l} \mathrm{E}\left[\big|D^k\bar{X}^M_t\big|^{2p}\right]$$

and, using Lemma 3.2,

$$C_p \sum_{k=0}^{l} \mathrm{E}\left[\big|D^k\bar{X}^M_t\big|^{2p}\right] \le C_p \sum_{k=0}^{l} d^{kp}\sqrt{\mathrm{E}\left[(J^M_T)^{2kp}\right]}\,\sup_L \sqrt{\mathrm{E}\left[\max_{\alpha\in A_k(L)} \mathrm{E}_W\left[|\,\mathrm{D}_\alpha\,\bar{X}^M_t|^{4p}\right]\right]}.$$

So, if we admit for a moment that, for $k \in [\![0,l]\!]$, $\sup_L \sqrt{\mathrm{E}\left[\max_{\alpha\in A_k(L)} \mathrm{E}_W\left[|\,\mathrm{D}_\alpha\,\bar{X}^M_t|^{4p}\right]\right]} \le C_{k,p}$, then

$$\mathrm{E}\left[\big|\bar{X}^M_t\big|_l^{2p}\right] \le C_p \sum_{k=0}^{l} d^{kp}\sqrt{\mathrm{E}\left[(J^M_T)^{2kp}\right]}C_{k,p} \le C_{p,l,T}(1 + \sqrt{\mathrm{E}\left[(J^M_T)^{2lp}\right]}) \tag{3.78}$$

which proves the proposition.

Hence, to conclude, it remains to bound, independently of L, the quantity $\mathrm{E}\left[\max_{\alpha\in A_k(L)} \mathrm{E}_W\left[|\,\mathrm{D}_\alpha\,\bar{X}^M_t|^{4p}\right]\right]$, which will be done by recurrence on $|\alpha|$ in the next lemma.

•

Lemma 3.7 *Let $p \ge 1$, and $n \in \mathbb{N}$ and ; there exists $C_{n,p}$ such that*

$$\sup_L \mathrm{E}\left[\max_{\substack{\alpha\in A(L)\\ |\alpha|\le n}} \mathrm{E}_W\left[|\,\mathrm{D}_\alpha\,\bar{X}^M_t|^{2p}\right]\right] \le C_{n,p}. \tag{3.79}$$

Proof :

The case $|\alpha| = 1$ is corresponding to the first order norm case.

Else, starting again from

$$\bar{X}^M_t = x + \int_0^t \sigma(\bar{X}^M_s)\,\mathrm{d}W_s + \sum_{j=1}^{J^M_t} c_M(\bar{Z}_j, \bar{X}^M_{T_j-}) + \int_0^t g(\bar{X}^M_s)\,\mathrm{d}s,$$

[2] We note, indeed, that $J^M_t \sim \mathcal{P}(t\lambda_M)$ which implies $\mathrm{E}\left[{J^M_t}^n\right] = P(t\lambda_M)$, where P is polynomial of degree n : when M is growing, (for $t \le T$), we have

$$\mathrm{E}\left[{J^M_t}^n\right] = O(\lambda_M^n).$$

we have (using Lemma 1.6, with

$$\sum_{(1)} \stackrel{\text{def}}{=} \sum_{l=2}^{k} \sum_{\substack{\beta=(\beta_1,\dots,\beta_l)\\ \beta_i\in[\![1,d]\!]}} \sum_{M\in\mathcal{M}_k(l)} \qquad \text{and} \qquad \sum_{(2)} \stackrel{\text{def}}{=} \sum_{l=1}^{k} \sum_{\substack{\beta=(\beta_1,\dots,\beta_r,\beta'_{r+1},\dots,\beta'_l)\\ \beta_i\in[\![1,d]\!],\ \beta'_j\in[\![d+1,2d]\!]}} \sum_{M\in\mathcal{M}_k(l)},$$

where $k \stackrel{\text{def}}{=} |\alpha|$, in order to shorten the equation)

$$\begin{aligned}
\mathrm{D}_\alpha \bar{X}_t^M &= \sum_{l=1}^{m} \int_0^t \mathrm{D}_\alpha(\sigma_l(\bar{X}_s^M))\,\mathrm{d}W_s^l + \sum_{j=1}^{J_t^M} \mathrm{D}_\alpha(c_M(\bar{Z}_j, \bar{X}_{T_j-}^M)) + \int_0^t \mathrm{D}_\alpha(g(\bar{X}_s^M))\,\mathrm{d}s \\
&= \sum_{l=1}^{m} \int_0^t \sum_{(1)} \partial_\beta\sigma_l(\bar{X}_s^M)\,\mathrm{D}_{M_1(\alpha)}(\bar{X}_s^M)_{\beta_1}\cdots\mathrm{D}_{M_l(\alpha)}(\bar{X}_s^M)_{\beta_l} + \nabla\sigma_l(\bar{X}_s^M)\,\mathrm{D}_\alpha(\bar{X}_s^M)\,\mathrm{d}W_s^l \\
&\quad + \sum_{j=1}^{J_t^M}\sum_{(2)} \partial_\beta c_M(\bar{Z}_j, \bar{X}_{T_j-}^M)\,\mathrm{D}_{M_1(\alpha)}\,\bar{Z}_j^{\beta_1}\cdots\mathrm{D}_{M_r(\alpha)}\,\bar{Z}_j^{\beta_r}\times\mathrm{D}_{M_{r+1}(\alpha)}(\bar{X}_{T_j-}^M)_{d-\beta'_{r+1}}\cdots\mathrm{D}_{M_l(\alpha)}(\bar{X}_{T_j-}^M)_{d-\beta'_l} \\
&\qquad\qquad + \nabla_z c_M(\bar{Z}_j, \bar{X}_{T_j-}^M)\,\mathrm{D}_\alpha(\bar{Z}_j) + \nabla_x c_M(\bar{Z}_j, \bar{X}_{T_j-}^M)\,\mathrm{D}_\alpha(\bar{X}_{T_j-}^M) \\
&\quad + \int_0^t \sum_{(1)} \partial_\beta(g(\bar{X}_s^M))\,\mathrm{D}_{M_1(\alpha)}(\bar{X}_s^M)_{\beta_1}\cdots\mathrm{D}_{M_l(\alpha)}(\bar{X}_s^M)_{\beta_l} + \nabla g(\bar{X}_s^M)\,\mathrm{D}_\alpha(\bar{X}_s^M)\,\mathrm{d}s
\end{aligned} \tag{3.80}$$

Then we apply the upper bound Lemma 3.1 with $\overline{V}_t^\alpha \stackrel{\text{def}}{=} \mathrm{D}_\alpha \bar{X}_t$ (and consequently $\overline{V}_{(k-1)}(t) = \left(\mathrm{D}_\beta \bar{X}_t\right)_{|\beta|<|\alpha|}$). Using Lemma 3.3, it follows that the Hypothesis 3.1 are well verified ; for example :

$$\begin{aligned}
G_l^\alpha\big(\overline{V}_{(k-1)}(s)\big) &= G_l^\alpha\Big(\left(\mathrm{D}_\beta \bar{X}_t\right)_{|\beta|<|\alpha|}\Big) \\
&\stackrel{\text{def}}{=} \sum_{(1)} \partial_\beta(\sigma_l(\bar{X}_s^M))\,\mathrm{D}_{M_1(\alpha)}(\bar{X}_s^M)_{\beta_1}\cdots\mathrm{D}_{M_l(\alpha)}(\bar{X}_s^M)_{\beta_l}
\end{aligned}$$

so there exists $w \in \mathbb{N}$ such that

$$|G^\alpha(v)| \le K(1+|v|)^w;$$

and

$$\begin{aligned}
& d_j^\alpha\big(\overline{Z}_j, \overline{V}_{(k-1)}(T_j^-)\big) \\
&= d_j^\alpha\big(\overline{Z}_j, \left(\mathrm{D}_\beta \bar{X}_{T_j^-}\right)_{|\beta|<|\alpha|}\big) \\
&\stackrel{\text{def}}{=} \sum_{(2)} \partial_\beta c_M(\bar{Z}_j, \bar{X}_{T_j-}^M)\,\mathrm{D}_{M_1(\alpha)}\,\bar{Z}_j^{\beta_1}\cdots\mathrm{D}_{M_r(\alpha)}\,\bar{Z}_j^{\beta_r}\times\mathrm{D}_{M_{r+1}(\alpha)}(\bar{X}_{T_j-}^M)_{d-\beta'_{r+1}}\cdots\mathrm{D}_{M_l(\alpha)}(\bar{X}_{T_j-}^M)_{d-\beta'_l}.
\end{aligned}$$

Notice that it is legitimate to consider each $\mathrm{D}_{M_u(\alpha)}\,\bar{Z}_j^{\beta_u}$ as a function of j and $\bar{Z}_j$ since, using directly the Lemma 3.4

$$\mathrm{D}_{M_u(\alpha)}\,\bar{Z}_j^{\beta_u} = \delta_{(M_u(\alpha))_n^1, j}\cdots\delta_{(M_u(\alpha))_1^1, j}\delta_{(M_u(\alpha))_1^2, \beta_u} f_\alpha(\bar{Z}_j)$$

where f_α is defined in (3.74).

Since, from the Lemma 3.3, every $|\,\mathrm{D}_{M_i(\alpha)}\,\bar{Z}_j^{\gamma_i}|$ is bounded, there comes an inequality of the form

$$|d_j^\alpha(\overline{Z}_j, \overline{V}_{(k-1)}(T_j^-))| \le K\overline{c}(\overline{Z}_j)(1+|\overline{V}_{(k-1)}(T_j^-)|)^w,$$

and likewise to the other quantities.

•

3.5 Operator L

Proposition 3.8 *For $p \le 1$, and all $l \in \mathbb{N}^*$, it exists $N_{T,l,p} > 0$ such that*

$$\big\|\big|\,\mathrm{L}\,\bar{X}_t^M\big|_l\big\|_{2p} \le N_{T,l,p}\big(\lambda_M\big)^{(l+2)^2}, \tag{3.81}$$

with $\lambda_M = \mu(E_M)$.

Proof :

In the following C_p, C_l, $C_{p,l}$ are "flying constants" which may change during the calculation ; we have

$$\mathrm{L}(F) = -\sum_{k=1}^{J_t^M}\sum_{r=1}^{d} \partial_{k,r}(\pi_{k,r})\,\mathrm{D}_{k,r} F + \mathrm{D}_{k,r}(\mathrm{D}_{k,r} F) + \mathrm{D}_{k,r}\ln p_J\,\mathrm{D}_{k,r} F, \tag{3.82}$$

hence, using the fact that $|.|_l$ is a norm (and with $|AB|_l \leq C_l|A|_l|B|_l$: *cf.* 1.10)

$$\begin{aligned}
|\,\mathrm{L}(F)|_l &\leq \sum_{k=1}^{J_t^M}\sum_{r=1}^{d} |\partial_{k,r}(\pi_{k,r})\,\mathrm{D}_{k,r} F|_l + |\,\mathrm{D}_{k,r}(\mathrm{D}_{k,r} F)|_l + \sum_{k=1}^{J_t^M}\sum_{r=1}^{d} |\,\mathrm{D}_{k,r}\ln p_J\,\mathrm{D}_{k,r} F|_l \\
&\leq C_l\sum_{k=1}^{J_t^M}\sum_{r=1}^{d} |\partial_{k,r}(\pi_{k,r})|_l|\,\mathrm{D}_{k,r} F|_l + \sum_{k=1}^{J_t^M}\sum_{r=1}^{d} |\,\mathrm{D}_{k,r}(\mathrm{D}_{k,r} F)|_l + C_l\sum_{k=1}^{J_t^M}\sum_{r=1}^{d} |\,\mathrm{D}_{k,r}\ln p_J||\,\mathrm{D}_{k,r} F|_l \\
&\leq C_l\sum_{k=1}^{J_t^M}\sum_{r=1}^{d} |\partial_{k,r}(\pi_{k,r})|_l^2 + |\,\mathrm{D}_{k,r} F|_l^2 + \sum_{k=1}^{J_t^M}\sum_{r=1}^{d} |\,\mathrm{D}_{k,r}(\mathrm{D}_{k,r} F)|_l + C_l\sum_{k=1}^{J_t^M}\sum_{r=1}^{d} |\,\mathrm{D}_{k,r}\ln p_J|_l^2 + |\,\mathrm{D}_{k,r} F|_l^2 \\
&\leq \sum_{k=1}^{J_t^M}\sum_{r=1}^{d} |\,\mathrm{D}_{k,r}(\mathrm{D}_{k,r} F)|_l + C_l(|\pi_{k,r}|_{l+1}^2 + |\ln p_J|_{l+1}^2 + 2|F|_{l+1}^2)
\end{aligned}$$

(the last inequality follows from the fact that $\sum_{k=1}^{J_t^M}\sum_{r=1}^{d} |\,\mathrm{D}_{k,r} G|_l^2 = \sum_{k=1}^{J_t^M}\sum_{r=1}^{d}\sum_{|\alpha|\leq l} |\,\mathrm{D}_\alpha\,\mathrm{D}_{k,r} G|^2 \leq \sum_{|\alpha|\leq l+1} |\,\mathrm{D}_\alpha G|^2 = |G|_{l+1}^2$), which implies :

$$\begin{aligned}
|\,\mathrm{L}(F)|_l^{2p} &\leq C_p, l\Big(\Big(\sum_{k=1}^{J_t^M}\sum_{r=1}^{d} |\,\mathrm{D}_{k,r}(\mathrm{D}_{k,r} F)|_l\Big)^{2p} + |\pi_{k,r}|_{l+1}^{4p} + |F|_{l+1}^{4p} + |\ln p_J|_{l+1}^{4p}\Big) \\
&\leq C_p, l\Big(\Big(J_t^M\sum_{k=1}^{J_t^M}\sum_{r=1}^{d} |\,\mathrm{D}_{k,r}(\mathrm{D}_{k,r} F)|_l^2\Big)^{p} + |\pi_{k,r}|_{l+1}^{4p} + |F|_{l+1}^{4p} + |\ln p_J|_{l+1}^{4p}\Big) \\
&\leq C_p, l\Big(\big(J_t^M\big)^p|F|_{l+2}^{2p} + |\pi_{k,r}|_{l+1}^{4p} + |F|_{l+1}^{4p} + |\ln p_J|_{l+1}^{4p}\Big).
\end{aligned}$$

And, finally (noticing that, as a consequence of Lemma 3.3, $|\pi_{k,r}|_{l+1}^{4p} \leq C_{l,p}$),

$$\mathrm{E}\Big[|\,\mathrm{L}(F)|_l^{2p}\Big] \leq C_p, l\Big(1 + \sqrt{\mathrm{E}\big[\big(J_t^M\big)^{2p}\big]}\sqrt{\mathrm{E}\big[|F|_{l+2}^{4p}\big]} + \mathrm{E}\big[|F|_{l+1}^{4p}\big] + \mathrm{E}\big[|\ln p_J|_{l+1}^{4p}\big]\Big) \tag{3.83}$$

We put $F = \bar{X}_t^M$. Let us recall that, from (3.78)

$$\mathrm{E}\Big[\big|\bar{X}_t^M\big|_{l+1}^{4n}\Big] \leq C_{p,l,T}(1 + \sqrt{\mathrm{E}\big[(J_T^M)^{4n(l+1)}\big]}) \leq C'_{p,l,T}\sqrt{\lambda_M)^{4n(l+1)}}.$$

Then, if we admit for the moment Lemma 3.9, we obtain the following upper bound :

$$\mathrm{E}\Big[|\,\mathrm{L}(\bar{X}_t^M)|_l^{2p}\Big] \leq N_{T,l,p}\big(\lambda_M\big)^{2p(l+2)^2},$$

which ends the proof. •

Hence, all that remains is to prove the following result :

Lemma 3.9 *Let $l, n \in \mathbb{N}$.*

$$\mathrm{E}\Big[|\ln p_J|_{l+1}^{2n}\Big] \leq C_{q_M,l,n,T}\big(\mu(E_M)\big)^{n(l+2)^2}.$$

Proof :

We recall that

$$\ln p_J = \sum_{j=1}^{J_T^M} \ln q_M(\bar{Z}_j, \bar{X}_{T_j-}^M),$$

and recalling that $\ln q_M$ has bounded derivatives of any order, using Lemma 2.7, which implies, by corollary 1.12, the existence of $C_{q_M,l} > 0$ such that

$$|q_M(F)|_{l+1} \le C_{q_M,l}\big(1 + |F|_{l+1} + |F|_l^{l+1}\big)$$

we have,

$$\begin{aligned}
|\ln p_J|_{l+1} &\le \sum_{j=1}^{J_T^M} |\ln q_M(\bar Z_j, \bar X^M_{T_j-})|_{l+1} \\
&\le C_{q_M,l} \sum_{j=1}^{J_T^M} 1 + |(\bar Z_j, \bar X^M_{T_j-})|_{l+1} + |(\bar Z_j, \bar X^M_{T_j-})|_l^{l+1} \\
&\le C_{q_M,l}\Big(J_T^M + \sum_{j=1}^{J_T^M} |\bar Z_j|_{l+1} + |\bar X^M_{T_j-}|_{l+1} + 2^l(|\bar Z_j|_l^{l+1} + |\bar X^M_{T_j-}|_l^{l+1})\Big) \\
&\le C'_{q_M,l}\Big(J_T^M + \sum_{j=1}^{J_T^M} |\bar X^M_{T_j-}|_{l+1} + |\bar X^M_{T_j-}|_l^{l+1}\Big)
\end{aligned}$$

it directly follows

$$\begin{aligned}
\mathrm{E}\Big[|\ln p_J|_{l+1}^{2n}\Big] &\le C_{q_M,l,n}\Big(\mathrm{E}\left[(J_T^M)^{2n}\right] + \mathrm{E}\Big[\Big(J_T^M \sum_{j=1}^{J_T^M} |\bar X^M_{T_j-}|_{l+1}^2 + (|\bar X^M_{T_j-}|_l^{l+1})^2\Big)^n\Big]\Big) \\
&\le C_{q_M,l,n}\Big(\mathrm{E}\left[(J_T^M)^{2n}\right] + \mathrm{E}\Big[\Big(J_T^M\Big)^n\Big(\sum_{j=1}^{J_T^M} |\bar X^M_{T_j-}|_{l+1}^2 + |\bar X^M_{T_j-}|_l^{2(l+1)}\Big)^n\Big]\Big) \\
&\le C_{q_M,l,n}\Big(\mathrm{E}\left[(J_T^M)^{2n}\right] + \sqrt{\mathrm{E}\Big[\Big(J_T^M\Big)^{2n}\Big]}\sqrt{\mathrm{E}\Big[\Big(\sum_{j=1}^{J_T^M} |\bar X^M_{T_j-}|_{l+1}^2 + |\bar X^M_{T_j-}|_l^{2(l+1)}\Big)^{2n}\Big]}\Big)
\end{aligned}$$

Our aim is then to bound the second term of the rhs of this last inequality.
We have

$$|\bar X^M_{T_j-}|_{l+1}^2 = \sum_{|\alpha|\le l+1} |\,\mathrm{D}_\alpha \bar X^M_{T_j-}|^2$$

and from (3.80) we know that we can put $\overline{V}_t^\alpha \overset{\text{def}}{=} \mathrm{D}_\alpha \bar X_t$ with $\overline{V}_t^\alpha$ defined in (3.54) ; then, there exists a process V_t^α with the same law and verifying (3.58). So :

$$\begin{aligned}
\mathrm{E}\Big[\Big(\sum_{j=1}^{J_T^M} |\bar X^M_{T_j-}|_{l+1}^2\Big)^{2n}\Big] &= \mathrm{E}\Big[\Big(\sum_{j=1}^{J_T^M}\sum_{|\alpha|\le l+1} |\,\mathrm{D}_\alpha \bar X^M_{T_j-}|^2\Big)^{2n}\Big] \\
&= \mathrm{E}\Big[\Big(\sum_{j=1}^{J_T^M}\sum_{|\alpha|\le l+1} |\,\mathrm{D}_\alpha \bar X^M_{T_j-}|^2\Big)^{2n}\Big] \\
&= \mathrm{E}\Big[\Big(\sum_{j=1}^{J_T^M}\sum_{|\alpha|\le l+1} |\overline{V}^\alpha_{T_j-}|^2\Big)^{2n}\Big] \\
&= \mathrm{E}\Big[\Big(\sum_{j=1}^{J_T^M}\sum_{|\alpha|\le l+1} |V^\alpha_{T_j-}|^2\Big)^{2n}\Big] \\
&\le C_n\sqrt{\mathrm{E}\Big[\big(J_T^M\big)^{4n-2}\Big]}\sqrt{\mathrm{E}\Big[\sum_{j=1}^{J_T^M}\Big(\sum_{|\alpha|\le l+1} |V^\alpha_{T_j-}|^2\Big)^{2n}\Big]}.
\end{aligned}$$

In the same way,

$$\mathrm{E}\Big[\Big(\sum_{j=1}^{J_T^M}|\bar{X}^M_{T_j-}|_l^{2(l+1)}\Big)^{2n}\Big] \le C'_n\sqrt{\mathrm{E}\Big[\big(J_T^M\big)^{4n-2}\Big]}\sqrt{\mathrm{E}\Big[\sum_{j=1}^{J_T^M}\Big(\sum_{|\alpha|\le l}|V^\alpha_{T_j-}|^2\Big)^{2n(l+1)}\Big]}.$$

But, using the F_p^1 isometry, with $v,\ w\in\mathbb{N}^*$,

$$\begin{aligned}
\mathrm{E}\Big[\sum_{j=1}^{J_T^M}\Big(\sum_{|\alpha|\le v}|V^\alpha_{T_j-}|^2\Big)^{w}\Big] &= \mathrm{E}\Big[\int_0^t\int_{E_M\times[0,2\bar{C}]}\Big(\sum_{|\alpha|\le v}|V^\alpha_{s^-}|^2\Big)^{w}N(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u)\Big] \\
&= 2\bar{C}\mu(E_M)\,\mathrm{E}\Big[\int_0^t\Big(\sum_{|\alpha|\le v}|V^\alpha_{s^-}|^2\Big)^{w}\mathrm{d}s\Big] \\
&= 2\bar{C}\mu(E_M)\,\mathrm{E}\Big[\int_0^t\Big(\sum_{|\alpha|\le v}|\overline{V}^\alpha_{s^-}|^2\Big)^{w}\mathrm{d}s\Big] \\
&= 2\bar{C}\mu(E_M)\,\mathrm{E}\Big[\int_0^t|\bar{X}^M_{s^-}|_v^{2w}\,\mathrm{d}s\Big] \\
&= 2\bar{C}\mu(E_M)\int_0^t\mathrm{E}\big[|\bar{X}^M_s|_v^{2w}\big]\,\mathrm{d}s.
\end{aligned}$$

Now from (3.78) we have

$$\mathrm{E}\Big[\big|\bar{X}^M_t\big|_v^w\Big] \le C_{p,l,T}(1+\sqrt{\mathrm{E}\big[(J_T^M)^{vw}\big]})$$

Gathering these results, we obtain :

$$\mathrm{E}\Big[|\ln p_J|_{l+1}^{2n}\Big] \le C_{q_M,l,n,T}\Big(\mathrm{E}\big[(J_T^M)^{2n}\big]+\sqrt{\mathrm{E}\Big[\big(J_T^M\big)^{2n}\Big]}\Big(\mu(E_M)\,\mathrm{E}\Big[\big(J_T^M\big)^{4n-2}\Big]\sqrt{\mathrm{E}\big[(J_T^M)^{4n(l+1)l}\big]}\Big)^{\frac{1}{4}}\Big)$$

Since (for $t\le T$),

$$\mathrm{E}\Big[{J_t^M}^n\Big] = \underset{M\to+\infty}{O}(\lambda_M^n),$$

we have

$$\mathrm{E}\Big[|\ln p_J|_{l+1}^{2n}\Big] \le C_{q_M,l,n,T}\big(\mu(E_M)\big)^{\frac{1}{4}}\big(\mu(E_M)\big)^{2n-1+\frac{n(l+1)l}{2}} \le C_{q_M,l,n,T}\big(\mu(E_M)\big)^{n(l+2)^2}$$

•

3.6 The covariance matrix

3.6.1 Preliminaries

We consider a Poisson point measure $N(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u)$ on $\mathbb{R}^d\times\mathbb{R}$; with compensator $\mu(\mathrm{d}z)\times\mathbb{1}_{(0,\infty)}(u)\,\mathrm{d}u$ and two non-negative measurable functions $f,g:\mathbb{R}^d\to\mathbb{R}_+$. For a measurable set $B\subset\mathbb{R}^d$ we denote $B_g=\{(z,u):z\in B,u<g(z)\}\subset\mathbb{R}^d\times\mathbb{R}_+$, and we consider the process

$$N_t(\mathbb{1}_{B_g}f) \overset{\mathrm{def}}{=} \int_0^t\int_{B_g}f(z)N(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u).$$

Moreover we note $\nu_g(\mathrm{d}z)=g(z)\mu(\mathrm{d}z)$ and

$$\alpha_{g,f}(s)=\int_{\mathbb{R}^d}(1-e^{-sf(z)})\,\mathrm{d}\nu_g(\mathrm{d}z),\qquad \beta_{B,g,f}(s)=\int_{B^c}(1-e^{-sf(z)})\,\mathrm{d}\nu_g(\mathrm{d}z)$$

We have the following result.

Lemma 3.10 *Let $\phi(s)=\mathrm{E}\big[e^{-sN_t(\mathbb{1}_{B_g}f)}\big]$ the Laplace transform of the random variable $N_t(\mathbb{1}_{B_g}f)$ then we have*

$$\phi(s)=e^{-t(\alpha_{g,f}(s)-\beta_{B,g,f}(s))}.$$

Proof : From Itô's formula we have

$$\exp(N_t(\mathbb{1}_{B_g} f)) = 1 - \int_0^t \int_{\mathbb{R}^d\times\mathbb{R}_+} \exp(-s(N_{r-}(\mathbb{1}_{B_g} f)))(1-\exp(-sf(z)\mathbb{1}_{B_g}(z,u)))\,\mathrm{d}N(r,z,u)$$

and consequently

$$\mathrm{E}\left[\exp(N_t(\mathbb{1}_{B_g} f))\right] = 1 - \int_0^t \mathrm{E}\left[\exp(-s(N_{r-}(\mathbb{1}_{B_g} f)))\right]\mathrm{d}r \int_{\mathbb{R}^d\times\mathbb{R}_+} (1-\exp(-sf(z)\mathbb{1}_{B_g}(z,u)))\mu(\mathrm{d}z)\,\mathrm{d}u.$$

But

$$\begin{aligned}\int_{\mathbb{R}^d\times\mathbb{R}_+} (1-\exp(-sf(z)\mathbb{1}_{B_g}(z,u)))\mu(\mathrm{d}z)\,\mathrm{d}u &= \int_{\mathbb{R}^d\times\mathbb{R}_+} \mathbb{1}_{B_g}(z,u)(1-\exp(-sf(z)))\mu(\mathrm{d}z)\,\mathrm{d}u \\ &= \int_{\mathbb{R}^d} \mathbb{1}_B(z)(1-\exp(-sf(z))) \int_{\mathbb{R}_+} \mathbb{1}_{\{u<g(z)\}}\mu(\mathrm{d}z)\,\mathrm{d}u \\ &= \int_B (1-\exp(-sf(z)))g(z)\mu(\mathrm{d}z) = \alpha_{g,f}(s) - \beta_{B,g,f}(s),\end{aligned}$$

It follows that

$$\mathrm{E}\left[\exp(N_t(\mathbb{1}_{B_g} f))\right] = \exp(-t(\alpha_{g,f}(s) - \beta_{B,g,f}(s))).$$

□

We consider an abstract measurable space E, a measure ν on this space and a non-negative measurable function $f : E \to \mathbb{R}_+$, such that $\int f\,\mathrm{d}\nu < \infty$. For $t > 0$ and $p \geq 1$ we notice

$$\alpha_f(t) = \int_E (1-e^{-tf(a)})\,\mathrm{d}\nu(a), \qquad \text{and} \qquad I_t^p(f) = \int_0^{+\infty} s^{p-1}e^{-t\alpha_f(s)}\,\mathrm{d}s.$$

Lemma 3.11 *1. Suppose that for $p \geq 1$ and $t > 0$*

$$\liminf_{u\to\infty} \frac{1}{\ln u}\alpha_f(u) > \frac{p}{t} \tag{3.84}$$

then

$$I_t^p(f) < \infty.$$

2. A sufficient condition for (3.84) is

$$\liminf_{u\to\infty} \frac{1}{\ln u}\nu\Big(f \geq \frac{1}{u}\Big) > \frac{p}{t}. \tag{3.85}$$

In particular, if $\liminf_{u\to\infty} \frac{1}{\ln u}\nu\Big(f \geq \frac{1}{u}\Big) = \infty$, then $\forall p \geq 1$ and $\forall t > 0$,

$$I_t^p(f) < \infty.$$

We notice that if ν is finite then (3.85) cannot be satisfied.

Proof : 1) From (3.84) one can find $\varepsilon > 0$ such that as s goes to infinity $s^{p-1}e^{-t\alpha_f(s)} \leq \frac{1}{s^{1+\varepsilon}}$ and consequently $I_t^p(f) < \infty$.

2) With the notation $n(\mathrm{d}z) = \nu\circ f^{-1}(\mathrm{d}z)$, we have

$$\alpha_f(u) = \int_0^{+\infty} (1-e^{-uz})\,\mathrm{d}n(z) = \int_0^{+\infty} e^{-y} n\Big(\frac{y}{u},\infty\Big)\,\mathrm{d}y.$$

Using Fatou's lemma and (3.85), we obtain

$$\liminf_{u\to\infty} \frac{1}{\ln u}\int_0^{+\infty} e^{-y} n\Big(\frac{y}{u},\infty\Big)\,\mathrm{d}y \geq \int_0^{+\infty} e^{-y} \liminf_{u\to\infty} \frac{1}{\ln u} n\Big(\frac{y}{u},\infty\Big)\,\mathrm{d}y > \frac{p}{t}.$$

□

We consider the Poisson point measure $N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)$ on $\mathbb{R}^d \times \mathbb{R}_+$ with compensator $\mu(\mathrm{d}z) \times \mathbb{1}_{(0,\infty)}(u)\, \mathrm{d}u$. We recall that

$$N_t(\mathbb{1}_{B_g} f) \stackrel{\text{def}}{=} \int_0^t \int_{B_g} f(z) N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u),$$

for $f,\, g : \mathbb{R}^d \to \mathbb{R}_+$ and $B_g = \{(z,u) : z \in B, u < g(z)\} \subset \mathbb{R}^d \times \mathbb{R}_+$ and that (with $\nu_g(\mathrm{d}z) \stackrel{\text{def}}{=} g(z)\mu(\mathrm{d}z)$)

$$\alpha_{g,f}(s) = \int_{\mathbb{R}^d} \left(1 - e^{-sf(z)}\right) \mathrm{d}\nu_g(\mathrm{d}z), \qquad \beta_{B,g,f}(s) = \int_{B^c} \left(1 - e^{-sf(z)}\right) \mathrm{d}\nu_g(\mathrm{d}z).$$

We have the following result (with $\Gamma(p) = \int_0^{+\infty} s^{p-1} e^{-s}\, \mathrm{d}s$).

Lemma 3.12 *Let $U_t = t \int_{B^c} f(z)\, \mathrm{d}\nu_g(z)$, then, for all $p \geq 1$,*

$$\mathrm{E}\left[\frac{1}{\left(N_t(\mathbb{1}_{B_g} f) + U_t\right)^p}\right] \leq \frac{1}{\Gamma(p)} \int_0^\infty s^{p-1} \exp(-t\alpha_{g,f}(s))\, \mathrm{d}s. \tag{3.86}$$

If it is supposed that, for some $0 < \theta \leq \infty$,

$$\liminf_{a \to \infty} \frac{1}{\ln a} \nu_g\Big(f \geq \frac{1}{a}\Big) = \theta, \tag{3.87}$$

then for every $t > 0$ and $p \geq 1$ such that $\frac{p}{t} < \theta$

$$\mathrm{E}\left[\frac{1}{\left(N_t(\mathbb{1}_{B_g} f) + U_t\right)^p}\right] < \infty.$$

Proof : By a change of variables we obtain for every $\lambda > 0$,

$$\lambda^{-p} \Gamma(p) = \int_0^{+\infty} s^{p-1} e^{-\lambda s}\, \mathrm{d}s.$$

Taking the expectation in the previous equality with $\lambda = N_t(\mathbb{1}_{B_g} f) + U_t$ we obtain

$$\mathrm{E}\left[\frac{1}{\left(N_t(\mathbb{1}_{B_g} f) + U_t\right)^p}\right] = \frac{1}{\Gamma(p)} \int_0^{+\infty} s^{p-1}\, \mathrm{E}\left[\exp(-s(N_t(\mathbb{1}_{B_g} f) + U_t))\right] \mathrm{d}s.$$

Now from Lemma 3.10 we have

$$\mathrm{E}\left[\exp(-sN_t(\mathbb{1}_{B_g} f))\right] = \exp(-t(\alpha_{g,f}(s) - \beta_{B,g,f}(s))).$$

Moreover, from the definition of U_t one can easily verify that $\exp(-sU_t) \leq \exp(-t\beta_{B,g,f}(s))$ and then

$$\mathrm{E}\left[\exp(-s(N_t(\mathbb{1}_{B_g} f) + U_t))\right] \leq \exp(-t\alpha_{g,f}(s))$$

this completes the proof of (3.86). The second part of the lemma follows directly from Lemma 3.11.

□

3.6.2 The Malliavin covariance matrix

In this subsection, we prove, under some additional assumptions on p and t, that $\mathrm{E}\left[\frac{1}{|\det\, \sigma(F_M)|^p}\right]$ is bounded (uniformly on M), for the Malliavin matrix $\sigma(F_M)$ defined at the definition 1.5.

From the diffusion equation (2.44)

$$\bar{X}_t^M = x + \int_0^t \sigma(\bar{X}_s^M)\, \mathrm{d}W_s + \sum_{j=1}^{J_t^M} c_M(\bar{Z}_j, \bar{X}_{T_j-}^M) + \int_0^t g(\bar{X}_s^M)\, \mathrm{d}s,$$

let us consider the tangent flow

$$Y_t^M = \mathrm{Id} + \sum_{l=1}^m \int_0^t \nabla\sigma_l(\bar{X}_s^M) Y_s^M\, \mathrm{d}W_s^l + \sum_{j=1}^{J_t^M} \nabla_x c_M(\bar{Z}_j, \bar{X}_{T_j-}^M) Y_{T_j-}^M + \int_0^t \nabla_x g(\bar{X}_s^M) Y_s^M\, \mathrm{d}s.$$

We then define the following process (with $\nabla_x c_j = \nabla_x c_M(\bar{Z}_j, \bar{X}^M_{T_j-}))$:

$$\hat{Y}^M_t \overset{\text{def}}{=} \mathrm{Id} - \sum_{l=1}^{m} \int_0^t \hat{Y}^M_s \nabla\sigma_l(\bar{X}^M_s)\, \mathrm{d}W^l_s - \sum_{j=1}^{J^M_t} \hat{Y}^M_{T_j-} \nabla_x c_j (\mathrm{Id} + \nabla_x c_j)^{-1} + \int_0^t \hat{Y}^M_s \Big(\frac{1}{2}\sum_{l=1}^{m} \nabla\sigma_l(\bar{X}^M_s)^2 - \nabla_x g(\bar{X}^M_s)\Big)\, \mathrm{d}s.$$

Lemma 3.13 *We have, for all $t \geq 0$,*

$$Y^M_t \hat{Y}^M_t = \mathrm{Id}\,. \tag{3.88}$$

Proof : The proof is postponed in the Appendix C. •

Lemma 3.14 *Assuming hypothesis 2.1, 2.2, 2.3 we have, for $p \geq 1$, $T > t > 0$ such that $\frac{2dp}{t} < \theta$*

$$\mathrm{E}\left[\frac{1}{|\det\, \sigma(F_M)|^p}\right] \leq C_p, \tag{3.89}$$

where the constant C_p does not depend on M.

Proof :

Since

$$\begin{aligned}
\mathrm{D}_{k,r}\, \bar{X}_t = \nabla_z c_M(\bar{Z}_k, \bar{X}^M_{T_k-})\, \mathrm{D}_{k,r}\, \bar{Z}_k + \sum_{l=1}^{m} \int_0^t \nabla\sigma_l(\bar{X}^M_s)\, \mathrm{D}_{k,r}\, \bar{X}^M_s\, \mathrm{d}W^l_s \\
+ \sum_{j=k+1}^{J^M_t} \nabla_x c_M(\bar{Z}_j, \bar{X}^M_{T_j-})\, \mathrm{D}_{k,r}\, \bar{X}^M_{T_j-} + \int_0^t \nabla_x g(\bar{X}^M_s)\, \mathrm{D}_{k,r}\, \bar{X}^M_s\, \mathrm{d}s.
\end{aligned}$$

We have

$$Y^M_t = Y^M_{T_k} + \sum_{l=1}^{m} \int_{T_k}^t \nabla\sigma_l(\bar{X}^M_s) Y^M_s\, \mathrm{d}W^l_s + \sum_{j=k+1}^{J^M_t} \nabla_x c_M(\bar{Z}_j, \bar{X}^M_{T_j-}) Y^M_{T_j-} + \int_{T_k}^t \nabla_x g(\bar{X}^M_s) Y^M_s\, \mathrm{d}s$$

and, with $A_k = \nabla_z c_M(\bar{Z}_k, \bar{X}^M_{T_k-})\, \mathrm{D}_{k,r}\, \bar{Z}_k$,

$$\begin{aligned}
Y^M_t \hat{Y}^M_{T_k} A_k = \underbrace{Y^M_{T_k} \hat{Y}^M_{T_k}}_{=\mathrm{Id}} A_k + \sum_{l=1}^{m} \int_{T_k}^t \nabla\sigma_l(\bar{X}^M_s) Y^M_s \hat{Y}^M_{T_k} A_k\, \mathrm{d}W^l_s \\
+ \sum_{j=k+1}^{J^M_t} \nabla_x c_M(\bar{Z}_j, \bar{X}^M_{T_j-}) Y^M_{T_j-} \hat{Y}^M_{T_k} A_k + \int_{T_k}^t \nabla_x g(\bar{X}^M_s) Y^M_s \hat{Y}^M_{T_k} A_k\, \mathrm{d}s.
\end{aligned}$$

Then

$$\mathrm{D}_{k,r}\, \bar{X}_t = Y^M_t \hat{Y}^M_{T_k} A_k = Y^M_t \hat{Y}^M_{T_k} \nabla_z c_M(\bar{Z}_k, \bar{X}^M_{T_k-})\, \mathrm{D}_{k,r}\, \bar{Z}_k.$$

Therefore

$$\begin{aligned}
\sum_{r=1}^{d} \langle \mathrm{D}_{k,r}\, \bar{X}_t, \xi\rangle^2 &= \sum_{r=1}^{d} \langle Y^M_t \hat{Y}^M_{T_k} \nabla_z c_M(\bar{Z}_k, \bar{X}^M_{T_k-})\, \mathrm{D}_{k,r}\, \bar{Z}_k, \xi\rangle^2 \\
&= \sum_{r=1}^{d} \pi_k^2 \langle \partial_{z^r} c_M(\bar{Z}_k, \bar{X}^M_{T_k-}), (Y^M_t \hat{Y}^M_{T_k})^* \xi\rangle^2 \\
&\geq \sum_{r=1}^{d} \mathbb{1}_{B_{M-1}}(\bar{Z}_k) \langle \partial_{z^r} c(\bar{Z}_k, \bar{X}^M_{T_k-}), (Y^M_t \hat{Y}^M_{T_k})^* \xi\rangle^2
\end{aligned}$$

since $\pi_k \geq \mathbb{1}_{B_{M-1}}(\bar{Z}_k)$ and $c_M = c$ on B_{M-1} ; using Hypothesis 2.3 item 3., it follows that

$$\rho_t \geq \inf_{|\xi|=1} \sum_{r=1}^{J^M_t} \mathbb{1}_{B_{M-1}}(\bar{Z}_k) \underline{c}^2(\bar{Z}_k) |(Y^M_t \hat{Y}^M_{T_k})^* \xi|^2 \geq (\|Y^M_{T_k} \hat{Y}^M_t\|)^{-2} \sum_{r=1}^{J^M_t} \mathbb{1}_{B_{M-1}}(\bar{Z}_k) \underline{c}^2(\bar{Z}_k).$$

With $\sigma(F_M)=\sigma(\bar{X}_t^M)+U_M(t)$, we have[3]

$$\mathrm{E}\left[\left|\frac{1}{\det\sigma(F_M)}\right|^p\right]\le \mathrm{E}\left[\left|\frac{1}{\rho_t+U_M(t)}\right|^{dp}\right]\le \mathrm{E}\left[\left(\frac{1+(\|Y_{T_k}^M\hat{Y}_t^M\|)^2}{\sum_{r=1}^{J_t^M}\mathbb{1}_{B_{M-1}}(\bar{Z}_k)\underline{c}^2(\bar{Z}_k)+U_M(t)}\right)^{dp}\right]$$

Now observe that the denominator of the last fraction is equal in law to

$$\sum_{r=1}^{J_t^M}\mathbb{1}_{B_{M-1}}(Z_k)\underline{c}^2(Z_k)\mathbb{1}_{U_k<\gamma\left(Z_k,X_{T_k-}^M\right)}+U_M(t)\ge N_t(\mathbb{1}_{B_{\underline{\gamma}}^M}\underline{c}^2)+U_M(t),$$

with $B_{\underline{\gamma}}^M=\{(z,u):z\in B_{M-1},u<\underline{\gamma}(z)\}$. Assuming Hypothesis 2.3 item 3., we can apply Lemma 3.12 with $f=\underline{c}^2$ and $\mathrm{d}\nu(z)=\underline{\gamma}(z)\mu(\mathrm{d}z)$. This gives $p'\ge 1$ such that $\frac{p'}{t}<\theta$

$$\mathrm{E}\left[\left(\frac{1}{N_t(\mathbb{1}_{B_{\underline{\gamma}}^M}\underline{c}^2)+U_M(t)}\right)^{p'}\right]\le C_{p'}.$$

Finally, since the moments of $||\hat{Y}_t^M||$ are bounded uniformly on M, the result follows from Cauchy-Schwarz inequality :

$$\mathrm{E}\left[\frac{1}{|\det\,\sigma(F_M)|^p}\right]\le C_p.$$

•

3.7 Bounding the weights

Lemma 3.15 *Let $q,p\in\mathbb{N}^*$ and $T>0$. For $T>t>0$ with $\frac{4d(3q-1)}{t}<\theta$, there exists a constant $C_{p,q,T}$ such that*

$$\|H_\beta^q(F_M)\|_p\le C_{p,q,T}\lambda_M^{q^2(4q+6d+9)}. \tag{3.90}$$

Proof : Let $q\in N^*$ and $\beta=(\beta_1,\dots,\beta_q)$ a multi-index. We have to bound $H_\beta^q(F_M)\overset{\text{def}}{=}H_\beta^q(F_M,1)$; from Theorem 1.13 there exists a universal constant $C_{q,d}$ such that (recalling that $F_M\in\mathbb{R}^d$)

$$\left|H_\beta^q(F_M)\right|\le C_{q,d}\frac{1}{|\det\sigma(F_M)|^{3q-1}}\big(1+|F_M|_{q+1})^{(6d+1)q}\big)\big(1+|\mathrm{L}F_M|_{q-1}^q\big).$$

So (for $T>t>0$ such that $\frac{2d(6q-2)}{t}<\theta$; C_q will be, in the following lines, a " flying constant")

$$\begin{aligned}\mathrm{E}\left[H_\beta^q(F_M)\right]&\le C_{q,d}\sqrt{\mathrm{E}\left[\frac{1}{|\det\sigma(F_M)|^{6q-2}}\right]}\sqrt{\mathrm{E}\left[\big(1+|F_M|_{q+1})^{(6d+1)q}\big)^2\big(1+|\mathrm{L}F_M|_{q-1}^q\big)^2\right]}\\&\le C_q\Big(\mathrm{E}\left[\big(1+|F_M|_{q+1})^{(6d+1)q}\big)^4\right]\mathrm{E}\left[\big(1+|\mathrm{L}F_M|_{q-1}^q\big)^4\right]\Big)^{\frac14}\end{aligned}$$

since we know that, from Lemma 3.14, for $p\ge1$ and $T>t>0$ such that $\frac{2dp}{t}<\theta$,

$$\mathrm{E}\left[\frac{1}{|\det\,\sigma(F_M)|^p}\right]\le C_p.$$

But, from (3.76) and (3.81), there exists $C_{q,T}>0$ such that

$$\mathrm{E}\left[(|F_M|_{q+1})^{4q(6d+1)}\right]\le C_{q,T}\lambda_M^{2(q+1)q(6d+1)}\qquad\text{and}\qquad \mathrm{E}\left[|\mathrm{L}F_M|_{q-1}^{4q}\right]\le C_{q,T}\lambda_M^{4q(q+1)^2},$$

so (since $q\ge1$, $q^2\ge\frac{q(q+1)}{2}$),

$$\mathrm{E}\left[H_\beta^q(F_M)\right]\le C_{p,q,T}\lambda_M^{\frac{(q+1)q}{2}(6d+1)}\lambda_M^{q(q+1)^2}.$$

•

[3] If a, b, c and d are non-negative real numbers,

$$\frac{1}{\frac{a}{c}+d}\le\frac{1+c}{a+d}.$$

4 Joint density regularity

4.1 Introduction

We recall that we made an approximation in law F_M^x of our process X_t^x. It is clear, from its definition (*cf.* (2.45)), that the law $P_{F_M^x}$ of F_M^x possesses a smooth density : $P_{F_M^x}(\mathrm{d}y) = p_{F_M^x}(y)\,\mathrm{d}y$. Then, we have defined

$$f_M(x,y) \stackrel{\text{def}}{=} \Psi_K(x) p_{F_M^x}(y)$$

where Ψ_K is a smooth version with bounded derivatives of any order of the indicator function $\mathbb{1}_K$.

In this section we will highlight the behaviour of $f_M(x,y)$ with respect to the norm defined by (1.22), which will prove the Lemma 2.11 and, consequently, will end the proof of our main result.

4.2 Bounds for the Sobolev norms of the tangent flow and its derivatives

A simple generalisation (a little bit heavier with respect to the notations, but using the very same ideas and methods) of Proposition 3.6 gives straightforwardly the following result :

Proposition 4.1 *Let $l, q \in \mathbb{N}^*$, $p \geq 1$ and $t < T$. For every multi-index $\beta \in \{1,\dots,d\}^q$, there exists $C_{l,p,q,T} > 0$ such that*

$$\| |\partial_\beta \overline{X}_t^M|_l \|_p \leq C_{l,p,q,T} \sqrt{(\lambda_M)^l}. \tag{4.91}$$

4.3 Proof of 2.11 : an upper bound for $\|f_M\|_{2m+q,2m,p}$

We already have all the tools to prove the Proposition 2.11, which was the key for proving our main joint density result 2.1. To bound the quantity $\|f_M\|_{2m+q,2m,p}$, it is sufficient, by (1.24), to bound the quantities

$$\partial_\xi\big(f_M(x,y)\big), \qquad \xi \leq 2m+q,$$

which is the exact point of the Proposition 4.4 that we will prove now ; we will deal first with the case $\xi = 0$ (which is the point of the next proposition), to show more conveniently the method that we used.

Notation 4.2 *In all the sequel, φ_ε will represent a mollifier converging weakly, as ε tends to 0, to the Dirac distribution. We will also define*

$$\Phi_\varepsilon(x_1,\dots,x_d) \stackrel{def}{=} \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_d} \varphi_\varepsilon(t_1,\dots,t_d)\,\mathrm{d}t_1 \dots \mathrm{d}t_d. \tag{4.92}$$

Proposition 4.3 *Let $T > 0$. For every $t \in]\frac{4d(3d-1)}{\theta}, T[$*

$$f_M(x,y) \leq C_{d,T} \Psi_K(x)\Big(1 \wedge \frac{1}{(|y|-3)^{d+1}}\Big)\lambda_M^{5(d+1)^3} \tag{4.93}$$

Proof : Let us note that, formally, $p_{F_t^M}(x,y) = \mathrm{E}\left[\delta_0(F_t^M(x) - y)\right]$, where δ_0 is the Dirac distribution.

In order to work in the direction of this last representation, we will therefore consider the following approximation of f_M :

$$f_{M,\varepsilon} \stackrel{\text{def}}{=} \Psi_K(x)\,\mathrm{E}\left[\varphi_\varepsilon(F_t^M(x) - y)\Psi_2(F_t^M(x) - y)\right], \tag{4.94}$$

where φ_ε is, as we said before, a mollifier and where Ψ_2 is a smooth version (with bounded derivatives of any order) of the indicator function with respect to the ball centred at 0 with radius 2.

We will consequently look, in the first place, for an upper-bound of $f_{M,\varepsilon} = \Psi_K(x)\,\mathrm{E}\left[\varphi_\varepsilon(F_t^M(x) - y)\Psi_2(F_t^M(x) - y)\right]$; using Theorem 1.3,

$$\mathrm{E}\left[\varphi_\varepsilon(F^M - y)\Psi_2(F_t^M(x) - y)\right] = \mathrm{E}\left[\Phi_\varepsilon(F^M - y) H_M^d(F_M, \Psi_2(F_t^M(x) - y))\right]$$

It directly follows, since $\|\Phi_\varepsilon\|_\infty \leq 1$ (which is clear from the definition (4.92)) and the weight H_M does not depend on ε, the pointwise convergence of $f_{M,\varepsilon}(x,y)$ when ε tends to 0.

Using Theorem 1.13 and denoting temporarily $G_M \stackrel{\text{def}}{=} \Psi_2(F_t^M(x) - y)$,

$$\Big|H_M^d(F_M, G_M)\Big| \leq C_d |G_M|_d \frac{(1+|F_M|_{d+1})^{(6d+1)d}}{|\det\sigma(F_M)|^{3d-1}}\big(1 + |\mathrm{L}F_M|_{d-1}^d\big).$$

Following the same pattern as we did in the proof of Lemma 3.15, for $T > t > 0$ such that $\frac{2d(6q-2)}{t} < \theta$ (here $q = d$), we have

$$\begin{aligned}
\mathrm{E}\left[|H^d_M(F_M, G_M)|\right] &\leq C_d \Big(\mathrm{E}\left[\frac{1}{|\det \sigma(F_M)|^{6d-2}}\right] \mathrm{E}\left[\left(1 + |F_M|_{d+1}\right)^{4(6d+1)d}\right] \mathrm{E}\left[\left(1 + |\mathrm{L}F_M|^d_{d-1}\right)^4\right] \Big)^{\frac{1}{4}} \| |G_M|_d \|_4 \\
&\leq C'_d \Big(\mathrm{E}\left[|F_M|^{4(6d+1)d}_{d+1}\right] \mathrm{E}\left[|\mathrm{L}F_M|^{4d}_{d-1}\right] \Big)^{\frac{1}{4}} \| |G_M|_d \|_4
\end{aligned}$$

and, for M big enough (provided that $\lambda_M \to +\infty$, when $M \to +\infty$) we found that[4]

$$\mathrm{E}\left[|H^d_M(F_M, G_M)|\right] \leq C_{d,T} \lambda_M^{4(d+1)^3} \| |G_M|_d \|_4.$$

The Lemma 1.11, with $\phi(\cdot) \overset{\text{def}}{=} \Psi_2(\cdot - y)$ (and with $|\phi|_n(F) \overset{\text{def}}{=} \sup_{|\beta| \leq n} |\partial_\beta \phi(F)|$; it is clear then that $|\phi|_n(F^M_t(x)) = |\Psi_2|_n(F^M_t(x) - y))$ implies :

$$\begin{aligned}
|\phi(F^M_t)|_d &\leq C_d |\phi|_d(F^M_t)\left(1 + |F^M_t|_{1,d} + |F^M_t|^d_{1,d-1}\right) \\
&\leq C_d |\phi|_d(F^M_t)\left(1 + |\bar{X}^M_t|_d + |\bar{X}^M_t|^d_{d-1}\right).
\end{aligned}$$

So, noting that $|\Psi_2|_n(u) \leq C_n \mathbb{1}_{\{|u| \leq 3\}}$,

$$\begin{aligned}
\| |G_M|_d \|_4 &= \| |\Psi_2(F^M_t(x) - y)|_d \|_4 \\
&\leq K_d \left(\mathrm{E}\left[\mathbb{1}_{\{|F^M_t(x) - y| \leq 3\}}\right]\right)^{\frac{1}{8}} \left(\mathrm{E}\left[\left(1 + |\bar{X}^M_t|_d + |\bar{X}^M_t|^d_{d-1}\right)^8\right]\right)^{\frac{1}{8}} \\
&= C_{d,T} \left(\mathrm{P}\left[|F^M_t(x) - y| \leq 3\right]\right)^{\frac{1}{8}} (\lambda_M)^{\frac{d^2}{2}}
\end{aligned}$$

and, since

$$\begin{aligned}
\mathrm{P}\left[|F^M_t(x) - y| \leq 3)\right] &\leq \mathrm{P}\left[|F^M_t(x)| \geq |y| - 3\right] \\
&\leq \frac{\mathrm{E}\left[|F^M_t(x)|^{8(d+1)}\right]}{(|y| - 3)^{8(d+1)}} \\
&\leq \frac{K'_d}{(|y| - 3)^{8(d+1)}}
\end{aligned}$$

(we used the fact that $\mathrm{E}\left[|F^M_t(x)|^{8(d+1)}\right] = \| |F^M_t(x)|_0 \|^{8(d+1)}$ which is bounded from Proposition 3.6) so

$$\| |G_M|_d \|_4 \leq \frac{C'_{d,T} (\lambda_M)^{\frac{d^2}{2}}}{(|y| - 3)^{d+1}}.$$

So, since by definition $\|\Phi_\varepsilon\|_\infty \leq 1$,

$$\mathrm{E}\left[\varphi_\varepsilon(F^M - y)\Psi_2(F^M_t(x) - y)\right] \leq K_{d,T} \frac{\lambda_M^{4(d+1)^3} (\lambda_M)^{\frac{d^2}{2}}}{(|y| - 3)^{d+1}} \leq K_{d,T} \frac{\lambda_M^{5(d+1)^3}}{(|y| - 3)^{d+1}}.$$

And

$$f_{M,\varepsilon}(x, y) \leq C_{d,T} \Psi_K(x) \Big(1 \wedge \frac{1}{(|y| - 3)^{d+1}}\Big) \lambda_M^{5(d+1)^3}. \tag{4.95}$$

•

We are now ready to deal with the general case $|\xi| > 1$. We have to be aware that the density, conditionally on $\mathcal{G} = \sigma(T_k, k \in \mathbb{N})$, of the law of $(\overline{Z}_1, \ldots, \overline{Z}_{J^M_t})$, density given by

$$p_{J^M_t, x}(\omega, z_1, \ldots, z_{J^M_t}) = \prod_{j=1}^{J^M_t} q_M\left(z_j, \Psi_{T_j - T_{j-1}}(\overline{X}^M_{T_{j-1}})\right)$$

depends on x, which makes the differentiation more complicated.

[4] using the non-optimal inequality :

$$d((d-1)^2 + 2) + \frac{d}{2}(6d+1)(d+1) < 4(d+1)^3$$

Proposition 4.4 *Let $T > 0$, m, $q \in \mathbb{N}$. For every $t \in]\frac{4d(3q'-1)}{\theta}, T[$, with $q' = d + 2m + q$, and every multi-index ξ such that $|\xi| \leq 2m + q$,*

$$|\partial_\xi f_M(x,y)| \leq C_{d,m,q,T} \mathbb{1}_{K+1}(x)\Big(1 \wedge \frac{1}{(|y|-3)^{2m+d+1}}\Big)\lambda_M^{6(d+2m+q+1)^3}. \tag{4.96}$$

Proof :

Because of what we said concerning the density, we will separate the differentiation with respect to x and to y, and hence define two multi-indexes α and β such that

$$\partial_x^\alpha \partial_y^\beta \stackrel{\text{def}}{=} \partial_\xi.$$

Then, we will start to work on the quantity $\partial_\xi f_{M,\varepsilon}(x,y) = \partial_x^\alpha \partial_y^\beta \big(f_{M,\varepsilon}(x,y)\big)$.

It is clear that $\partial_y^\beta \big(\varphi_\varepsilon(F^M - y)\big) = (-1)^\beta \partial^\beta \varphi_\varepsilon(F^M - y)$, so

$$\begin{aligned}
\partial_x^\alpha \partial_y^\beta \big(f_{M,\varepsilon}(x,y)\big) &= \partial_x^\alpha \partial_y^\beta \Big(\Psi_K(x)\,\mathrm{E}\big[\varphi_\varepsilon(\overline{X}_t^M(x) - y)\Psi_2(\overline{X}_t^M(x) - y)\big]\Big)\\
&= \partial_x^\alpha \Big(\Psi_K(x)\,\mathrm{E}\big[\sum_{\beta'\oplus\beta''=\beta} \partial_y^{\beta'}\varphi_\varepsilon(\overline{X}_t^M(x) - y)\partial_y^{\beta''}\Psi_2(\overline{X}_t^M(x) - y)\big]\Big)\\
&= (-1)^\beta \sum_{\beta'\oplus\beta''=\beta} \partial_x^\alpha \Big(\Psi_K(x)\,\mathrm{E}\big[\partial^{\beta'}\varphi_\varepsilon(\overline{X}_t^M(x) - y)\partial^{\beta''}\Psi_2(\overline{X}_t^M(x) - y)\Big)\big]\\
&= (-1)^\beta \sum_{\beta'\oplus\beta''=\beta}\ \sum_{\alpha'\oplus\alpha''=\alpha} \partial_x^{\alpha'}\Psi_K(x)\partial_x^{\alpha''}\Big(\mathrm{E}\big[\partial^{\beta'}\varphi_\varepsilon(\overline{X}_t^M(x) - y)\partial^{\beta''}\Psi_2(\overline{X}_t^M(x) - y)\big]\Big).
\end{aligned}$$

Lemma 4.8 implies then that $\partial_x^\alpha \partial_y^\beta \big(f_{M,\varepsilon}(x,y)\big)$ converges (when ε tends to 0) and the existence of $C_{d,m,q,T} > 0$ such that

$$\partial_\xi f_{M,\varepsilon}(x,y) \leq C_{d,m,q,T} \mathbb{1}_{K+1}(x)\Big(1 \wedge \frac{1}{(|y|-3)^{2m+d+1}}\Big)\lambda_M^{6(d+2m+q+1)^3} \tag{4.97}$$

which allows us to state that $\lim_{\varepsilon\to 0} \partial_\xi f_{M,\varepsilon}(x,y) = \partial_\xi f_M(x,y)$, and letting $\varepsilon \to 0$ in (4.97), we obtain (4.96). •

In this last proof, we used the Lemma 4.8 ; to prove it, we will first need two preliminary lemmas :

Lemma 4.5 *Let $f : \mathbb{R}^d \to \mathbb{R}_+^*$ and β a multi-index, then (with the notations used in Lemma 1.6)*

$$\partial^\beta f(x) = f(x) \sum_{l=1}^{|\beta|} \sum_{M\in\mathcal{M}_{|\beta|}(l)} c_M \prod_{i=1}^{l} \partial^{M_i(\beta)} \ln f(x) \tag{4.98}$$

with $c_M \in \mathbb{N}$.

Proof : By induction on $|\beta|$.

We will just show the mechanism, which will then be rather clear for a higher range, for $|\beta| = 1,\ 2$.

For $|\beta| = 1$, let $\beta = (x_i)$; then it is clear that

$$\partial_{x_i} f(x) = f(x)\partial_{x_i} \ln f(x)$$

(so $c_M = 1$ for $M = \{\{1\}\}$).

For $|\beta| = 2$, let $\beta = (x_j, x_i)$ then

$$\begin{aligned}
\partial_{x_j}\partial_{x_i} f(x) &= \partial_{x_j}\big(f(x)\partial_{x_i}\ln f(x)\big)\\
&= \partial_{x_j} f(x)\partial_{x_i}\ln f(x) + f(x)\partial_{x_j}\partial_{x_i}\ln f(x)\\
&= \big(f(x)\partial_{x_j}\ln f(x)\big)\partial_{x_i}\ln f(x) + f(x)\partial_{x_j}\partial_{x_i}\ln f(x)\\
&= f(x)\big(\partial_{x_j}\ln f(x)\partial_{x_i}\ln f(x) + \partial_{x_j}\partial_{x_i}\ln f(x)\big)
\end{aligned}$$

(so $c_M = 1$ for $M = \{\{1,2\}\}$ or $M = \{\{1\},\{2\}\}$).

•

Lemma 4.6 *Let $l, q \in \mathbb{N}$, $p \geq 1$ and $t < T$. For every multi-index*[5] $\beta \in \{1, \ldots, d\}^q$, *there exists* $C_{l,p,q,T} > 0$ *such that*

$$\mathrm{E}\Big[\Big(\sum_{j=1}^{J_T^M} |\partial_x^\beta \bar{X}_{T_j-}^M|_v^{2w}\Big)^u\Big] \leq C_{u,v,w,|\beta|,T}(\lambda_M)^{u(vw+1)}. \tag{4.99}$$

Proof : We have

$$|\partial_x^\beta \bar{X}_{T_j-}^M|_v^2 = \sum_{|\alpha| \leq v} |\mathrm{D}_\alpha\, \partial_x^\beta \bar{X}_{T_j-}^M|^2.$$

In the very same way as we did in (3.80) (with an obvious generalisation[6] of Lemma 1.6) we can set

$$\overline{V}_t^{\alpha,\beta} \overset{\text{def}}{=} \mathrm{D}_\alpha\, \partial_x^\beta \bar{X}_t$$

with $\overline{V}_t^{\alpha,\beta}$ defined as in (3.54) ; then, there exists a process $V_t^{\alpha,\beta}$ with the same law and verifying (3.58), so we have :

$$\begin{aligned}
\mathrm{E}\Big[\Big(\sum_{j=1}^{J_T^M} |\partial_x^\beta \bar{X}_{T_j-}^M|_v^{2w}\Big)^u\Big] &= \mathrm{E}\Big[\Big(\sum_{j=1}^{J_T^M}\Big(\sum_{|\alpha|\leq v} |\mathrm{D}_\alpha\, \partial_x^\beta \bar{X}_{T_j-}^M|^2\Big)^w\Big)^u\Big] \\
&= \mathrm{E}\Big[\Big(\sum_{j=1}^{J_T^M}\Big(\sum_{|\alpha|\leq v} |\mathrm{D}_\alpha\, \partial_x^\beta \bar{X}_{T_j-}^M|^2\Big)^w\Big)^u\Big] \\
&= \mathrm{E}\Big[\Big(\sum_{j=1}^{J_T^M}\Big(\sum_{|\alpha|\leq v} |\overline{V}_{T_j-}^{\alpha,\beta}|^2\Big)^w\Big)^u\Big] \\
&= \mathrm{E}\Big[\Big(\sum_{j=1}^{J_T^M}\Big(\sum_{|\alpha|\leq v} |V_{T_j-}^{\alpha,\beta}|^2\Big)^w\Big)^u\Big] \\
&\leq \sqrt{\mathrm{E}\Big[\big(J_T^M\big)^{2u-2}\Big]}\sqrt{\mathrm{E}\Big(\Big[\sum_{j=1}^{J_T^M}\Big(\sum_{|\alpha|\leq v} |V_{T_j-}^{\alpha,\beta}|^2\Big)^{wu}\Big)^2\Big]}.
\end{aligned}$$

[5] With the convention $\{1, \ldots, d\}^0 = \{0\}$ and $\partial_x^\beta F = F$.

[6] Indeed, in this lemma, the derivatives were purely formal, therefore, we can use it directly with an operator $\mathrm{D}'_{\alpha'}$ where $\alpha' = (\alpha, \beta)$ and $\mathrm{D}'_{\alpha'} \overset{\text{def}}{=} \mathrm{D}_\alpha\, \partial_x^\beta$.

But, using the F_p^2 isometry, with $w' \in \mathbb{N}^*$, and setting $f(y) \stackrel{\text{def}}{=} \Big(\sum_{|\alpha|\le v} |V_y^{\alpha,\beta}|^2\Big)^{w'}$,

$$\begin{aligned}
&\mathrm{E}\Big[\Big(\sum_{j=1}^{J_T^M}\Big(\sum_{|\alpha|\le v}|V_{T_j-}^{\alpha,\beta}|^2\Big)^{w'}\Big)^2\Big]\\
&= \mathrm{E}\Big[\Big(\int_0^t\int_{E_M\times[0,2\bar{C}]} f(s^-)\,\mathrm{d}\tilde{N}(s,z,u) + \int_0^t\int_{E_M\times[0,2\bar{C}]} f(s^-)\,\mathrm{d}\hat{N}(s,z,u)\Big)^2\Big]\\
&\le \mathrm{E}\Big[\Big(\int_0^t\int_{E_M\times[0,2\bar{C}]} f(s^-)^2\,\mathrm{d}\tilde{N}(s,z,u)\Big] + \mathrm{E}\Big[\Big(\int_0^t\int_{E_M\times[0,2\bar{C}]} f(s^-)\,\mathrm{d}\hat{N}(s,z,u)\Big)^2\Big]\\
&\le 2\bar{C}\mu(E_M)\,\mathrm{E}\Big[\int_0^t f(s^-)^2\,\mathrm{d}s\Big] + \big(2\bar{C}\mu(E_M)\big)^2 T\,\mathrm{E}\Big[\int_0^t f(s^-)^2\,\mathrm{d}s\Big]\\
&= (2\bar{C}\mu(E_M) + \big(2\bar{C}\mu(E_M)\big)^2 T)\,\mathrm{E}\Big[\int_0^t f(s^-)^2\,\mathrm{d}s\Big]\\
&= (2\bar{C}\mu(E_M) + \big(2\bar{C}\mu(E_M)\big)^2 T)\,\mathrm{E}\Big[\int_0^t \Big(\sum_{|\alpha|\le v}|V_{s^-}^{\alpha,\beta}|^2\Big)^{2w'}\,\mathrm{d}s\Big]\\
&= (2\bar{C}\mu(E_M) + \big(2\bar{C}\mu(E_M)\big)^2 T)\,\mathrm{E}\Big[\int_0^t \Big(\sum_{|\alpha|\le v}|\overline{V}_{s^-}^{\alpha,\beta}|^2\Big)^{2w'}\,\mathrm{d}s\Big]\\
&= (2\bar{C}\mu(E_M) + \big(2\bar{C}\mu(E_M)\big)^2 T)\,\mathrm{E}\Big[\int_0^t |\partial_x^\beta \bar{X}_{s^-}^M|_v^{4w'}\,\mathrm{d}s\Big]\\
&= (2\bar{C}\mu(E_M) + \big(2\bar{C}\mu(E_M)\big)^2 T)\int_0^t \mathrm{E}\big[|\partial_x^\beta \bar{X}_s^M|_v^{4w'}\big]\,\mathrm{d}s.
\end{aligned}$$

Now from (4.1) we have (with $\lambda_M = \mu(E_M)$),

$$\mathrm{E}\Big[\big|\partial_x^\beta \bar{X}_t^M\big|_v^{4w'}\Big] \le C_{v,w',|\beta|,T}(\lambda_M)^{2vw'}$$

Gathering these results, we obtain :

$$\mathrm{E}\Big[\Big(\sum_{j=1}^{J_T^M}|\partial_x^\beta \bar{X}_{T_j-}^M|_v^{2w}\Big)^u\Big] \le C_{u,v,w,|\beta|,T}(\lambda_M)^{u-1}(\lambda_M)^{uvw+1}.$$

•

Lemma 4.7 *Let $l,q \in \mathbb{N}^*$, $p \ge 1$ and $t < T$. For every multi-index $\alpha \in \{1,\dots,d\}^q$, there exists $C_{l,p,q,T} > 0$ such that*

$$\||\partial_x^\beta \ln p_{J_t^M,x}(\overline{Z}_1,\dots,\overline{Z}_{J_t^M})|_l\|_{2p} \le C_{l,p,q,T}(\lambda_M)^{l(q+l)+1}. \tag{4.100}$$

Proof : In this proof we will denote J_t^M simply by J (and sometimes $\bar{X}_{T_j-}^M(x)$ simply by $\bar{X}_{T_j-}^M$).

Using a similar formula as (1.12) (the Malliavin derivatives were used in a formal way, so it is in particular true with the usual differential operator), we have (with $\alpha \in \{1,\dots,d\}^q$),

$$\partial_x^\alpha \phi(F) = \sum_{l=1}^n \sum_{\substack{\beta=(\beta_1,\dots,\beta_l)\\ \beta_i\in[\![1,d]\!]}} \sum_{M\in\mathcal{M}_n(l)} \partial_\beta\phi(F)\partial_x^{M_1(\alpha)}F_{\beta_1}\cdots\partial_x^{M_l(\alpha)}F_{\beta_l},$$

so, with $\alpha \stackrel{\text{def}}{=} \sum_{l=1}^n \sum_{\substack{\beta=(\beta_1,\dots,\beta_l)\\ \beta_i\in[\![1,d]\!]}} \sum_{M\in\mathcal{M}_n(l)}$,

$$\partial_x^\alpha(\ln q_M(\bar{X}_{T_j-}^M(x),\bar{Z}_j)) = \sum_{(\alpha)} \partial_\beta(\ln q_M)(\bar{X}_{T_j-}^M(x),\bar{Z}_j)\partial_x^{M_1(\alpha)}(\bar{X}_{T_j-}^M(x))_{\beta_1}\cdots\partial_x^{M_l(\alpha)}(\bar{X}_{T_j-}^M(x))_{\beta_l}.$$

By corollary 1.12, we have the existence of $C'_{q_M,l} > 0$ such that

$$|\ln q_M(F)|_l \le C_{q_M,l}\big(1 + |F|_l + |F|_{l-1}^l\big)$$

which leads to (recalling $q=|\alpha|$)

$$|\partial_x^\alpha(\ln q_M(\bar X^M_{T_j-}(x),\bar Z_j))|_l\le C_{q_M,l}\Big(1+|\bar X^M_{T_j-}|_l^2+|\bar X^M_{T_j-}|_{l-1}^{2l}+\sum_{\beta\subset\alpha}|\partial_x^\beta(\bar X^M_{T_j-}(x))|_l^{2q}\Big).$$

Then

$$\begin{aligned}|\partial_x^\alpha\ln p_J|_l&\le\sum_{j=1}^{J_T^M}|\partial_x^\alpha(\ln q_M(\bar X^M_{T_j-}(x),\bar Z_j))|_l\\&\le C_{q_M,l}\Big(\sum_{j=1}^{J_T^M}1+|\bar X^M_{T_j-}|_l^2+|\bar X^M_{T_j-}|_{l-1}^{2l}+\sum_{\beta\subset\alpha}|\partial_x^\beta(\bar X^M_{T_j-}(x))|_l^{2q}\Big)\\&\le C'_{q_M,l}\Big(J_T^M+\sum_{j=1}^{J_T^M}|\bar X^M_{T_j-}|_l^2+|\bar X^M_{T_j-}|_{l-1}^{2l}+\sum_{\beta\subset\alpha}\sum_{j=1}^{J_T^M}|\partial_x^\beta(\bar X^M_{T_j-}(x))|_l^{2q}\Big)\end{aligned}$$

it directly follows, using the Lemma 4.6,

$$\begin{aligned}\mathrm{E}\Big[|\partial_x^\alpha\ln p_J|_l^{2n}\Big]\le C_{q_M,l,n,|\alpha|}\Big(&\mathrm{E}\big[(J_T^M)^{2n}\big]+\mathrm{E}\Big[\Big(\sum_{j=1}^{J_T^M}|\bar X^M_{T_j-}|_l^2\Big)^{2n}\Big]\\&+\mathrm{E}\Big[\Big(\sum_{j=1}^{J_T^M}|\bar X^M_{T_j-}|_{l-1}^{2l}\Big)^{2n}\Big]+\sum_{\beta\subset\alpha}\mathrm{E}\Big[\Big(\sum_{j=1}^{J_T^M}|\partial_x^\beta(\bar X^M_{T_j-}(x))|_l^{2q}\Big)^{2n}\Big]\Big)\\\le C_{q_M,l,n,|\alpha|}\Big(&(\lambda_M)^{2n}+C_{n,l,T}(\lambda_M)^{2n(l+1)}+C'_{n,l,T}(\lambda_M)^{2n(l(l-1)+1)}+\sum_{\beta\subset\alpha}C_{n,l,q,|\beta|,T}(\lambda_M)^{2n(ql+1)}\Big)\\\le C_{n,l,\alpha,T}&(\lambda_M)^{2n(l(q+l)+1)}.\end{aligned}$$

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Now we can prove the wanted result :

Lemma 4.8 *Let $T>0$. Let α, β and γ multi-indexes such that $|\alpha|+|\beta|+|\gamma|\le 2m+q$. For every $t\in]\frac{4d(3q'-1)}{\theta},T[$, with $q'=d+|\beta|$, the quantity $\partial_x^\alpha\,\mathrm{E}\Big[\partial^\beta\varphi_\varepsilon(F^M-y)\partial^\gamma\Psi_2(F_t^M(x)-y)\Big]$ converges (when ε tends to 0) and*

$$\partial_x^\alpha\,\mathrm{E}\Big[\partial^\beta\varphi_\varepsilon(F^M-y)\partial^\gamma\Psi_2(F_t^M(x)-y)\Big]\le K_{d,m,q,T}\Big(1\wedge\frac{1}{(|y|-3)^{2m+d+1}}\Big)\lambda_M^{6(d+2m+q+1)^3}.\qquad(4.101)$$

Proof : (in this proof we will denote simply J_t^M by J ; let us set $\delta_M\overset{\text{def}}{=}\sqrt{U_M(t)}$ and temporarily $\Psi\overset{\text{def}}{=}\partial^\gamma\Psi_2$ and $f(u,z)\overset{\text{def}}{=}\delta_Mu+x_t(x,z_1,\dots,z_J))$

$$\begin{aligned}&\partial_x^\alpha\,\mathrm{E}\Big[\partial^\beta\varphi_\varepsilon(F^M-y)\Psi(F_t^M(x)-y)\Big]\\=&\,\mathrm{E}\Big[\partial_x^\alpha\int_{\mathbb{R}^d}\nu(\mathrm{d}u)\int_{\mathbb{R}^J}\partial^\beta\varphi_\varepsilon(\delta_Mu+x_t(x,z_1,\dots,z_J))\Psi(\delta_Mu+x_t(x,z_1,\dots,z_J))P_{J,x}(z_1,\dots,z_J)\,\mathrm{d}z_1\cdots\mathrm{d}z_J\Big]\\=&\sum_{\alpha_1\oplus\alpha_2}\mathrm{E}\Big[\int_{\mathbb{R}^d}\nu(\mathrm{d}u)\int_{\mathbb{R}^J}\partial_x^{\alpha_1}\big(\partial^\beta\varphi_\varepsilon(f(u,z))\big)\partial_x^{\alpha_2}\big(\Psi(f(u,z))\big)P_{J,x}(z_1,\dots,z_J)\,\mathrm{d}z_1\cdots\mathrm{d}z_J\Big]\\&+\sum_{\substack{\alpha_1\oplus\alpha_2\oplus\alpha_3\\\alpha_3\neq\varnothing}}\mathrm{E}\Big[\int_{\mathbb{R}^d}\nu(\mathrm{d}u)\int_{\mathbb{R}^J}\partial_x^{\alpha_1}\big(\partial^\beta\varphi_\varepsilon(f(u,z))\big)\partial_x^{\alpha_2}\big(\Psi(f(u,z))\big)\partial_x^{\alpha_3}P_{J,x}(z_1,\dots,z_J)\,\mathrm{d}z_1\cdots\mathrm{d}z_J\Big]\\=&(1)+(2).\end{aligned}$$

For the part (1), since

$$\begin{aligned}
&\mathrm{E}\Big[\int_{\mathbb{R}^d}\nu(\mathrm{d}u)\int_{\mathbb{R}^J}\partial_x^{\alpha_1}\big(\partial^\beta\varphi_\varepsilon(f(u,z))\big)\partial_x^{\alpha_2}\big(\Psi(f(u,z))\big)P_{J,x}(z_1,\dots,z_J)\,\mathrm{d}z_1\cdots\mathrm{d}z_J\Big]\\
=&\mathrm{E}\Big[\partial_x^{\alpha_1}\big(\partial^\beta\varphi_\varepsilon(F^M-y)\big)\partial_x^{\alpha_2}\big(\Psi(F^M-y)\big)\Big]\\
=&\mathrm{E}\Big[\Big(\sum_{(\alpha_1)}\partial_\beta\varphi_M(F_t^M(x)-y)\partial_{N_1(\alpha_1)}\bar{X}_{\beta_1}^{t,M}\cdots\partial_{N_l(\alpha_1)}\bar{X}_{\beta_l}^{t,M}\Big)\\
&\qquad\times\Big(\sum_{(\alpha_2)}\partial_{\beta'}\Psi(F_t^M(x)-y)\partial_{N_1'(\alpha_2)}\bar{X}_{\beta_1'}^{t,M}\cdots\partial_{N_l'(\alpha_2)}\bar{X}_{\beta_l'}^{t,M}\Big)\Big],
\end{aligned}$$

we are brought, on one hand, to prove the convergence (when ε tends to 0) and afterwards to bound the quantity

$$\mathrm{E}\Big[\partial_\beta\varphi_\varepsilon(F_t^M(x)-y)\partial_{\beta'}\partial_\gamma\Psi_2(F_t^M(x)-y)Y_{N,N'}\Big] \tag{4.102}$$

with $Y_{N,N'}\overset{\text{def}}{=}\partial_{N_1(\alpha_1)}\bar{X}_{\beta_1}^{t,M}\cdots\partial_{N_l(\alpha_1)}\bar{X}_{\beta_l}^{t,M}\partial_{N_1'(\alpha_2)}\bar{X}_{\beta_1'}^{t,M}\cdots\partial_{N_l'(\alpha_2)}\bar{X}_{\beta_l'}^{t,M}$.

On the other hand, for the part (2), Lemma 4.5 leads to

$$\partial_x^{\alpha_3}P_{J,x}=P_{J,x}\sum_{l=1}^{|\alpha_3|}\sum_{M\in\mathcal{M}_{|\alpha_3|}(l)}c_M\prod_{i=1}^{l}\partial^{M_i(\alpha_3)}\ln P_{J,x} \tag{4.103}$$

so, letting $\tilde{p}_{J,x}^{|\alpha_3|}\overset{\text{def}}{=}\sum_{l=1}^{|\alpha_3|}\sum_{M\in\mathcal{M}_{|\alpha_3|}(l)}c_M\prod_{i=1}^{l}\partial^{M_i(\alpha_3)}\ln P_{J,x}$, we have :

$$\begin{aligned}
&\mathrm{E}\Big[\int_{\mathbb{R}^d}\nu(\mathrm{d}u)\int_{\mathbb{R}^J}\partial_x^{\alpha_1}\big(\partial^\beta\varphi_\varepsilon(f(u,z))\big)\partial_x^{\alpha_2}\big(\Psi(f(u,z))\big)\partial_x^{\alpha_3}P_{J,x}(z_1,\dots,z_J)\,\mathrm{d}z_1\cdots\mathrm{d}z_J\Big]\\
=&\mathrm{E}\Big[\partial_x^{\alpha_1}\big(\partial^\beta\varphi_\varepsilon(F^M-y)\big)\partial_x^{\alpha_2}\big(\Psi(F^M-y)\tilde{p}_{J,x}^{|\alpha_3|}(\overline{Z}_1,\dots,\overline{Z}_n)\Big]\\
=&\mathrm{E}\Big[\Big(\sum_{(\alpha_1)}\partial_\beta\varphi_M(F_t^M(x)-y)\partial_{N_1(\alpha_1)}\bar{X}_{\beta_1}^{t,M}\cdots\partial_{N_l(\alpha_1)}\bar{X}_{\beta_l}^{t,M}\Big)\\
&\qquad\times\Big(\sum_{(\alpha_2)}\partial_{\beta'}\Psi(F_t^M(x)-y)\partial_{N_1'(\alpha_2)}\bar{X}_{\beta_1'}^{t,M}\cdots\partial_{N_l'(\alpha_2)}\bar{X}_{\beta_l'}^{t,M}\Big)\tilde{p}_{J,x}^{|\alpha_3|}(\overline{Z}_1,\dots,\overline{Z}_n)\Big],
\end{aligned}$$

so we are brought, again, to prove the convergence (when ε tends to 0) and afterwards to bound the quantity

$$\mathrm{E}\Big[\partial_\beta\varphi_\varepsilon(F_t^M(x)-y)\partial_{\beta'}\partial_\gamma\Psi_2(F_t^M(x)-y)\tilde{Y}_{N,N'}\Big] \tag{4.104}$$

with $\tilde{Y}_{N,N'}\overset{\text{def}}{=}\partial_{N_1(\alpha_1)}\bar{X}_{\beta_1}^{t,M}\cdots\partial_{N_l(\alpha_1)}\bar{X}_{\beta_l}^{t,M}\partial_{N_1'(\alpha_2)}\bar{X}_{\beta_1'}^{t,M}\cdots\partial_{N_l'(\alpha_2)}\bar{X}_{\beta_l'}^{t,M}\tilde{p}_{J,x}^{|\alpha_3|}(\overline{Z}_1,\dots,\overline{Z}_n)$.

We can see a similar structure between (4.102) and (4.104) : for the moment we will treat them at the same time ; we will temporarily denote by Y either $Y_{N,N'}$ or $\tilde{Y}_{N,N'}$, and keep working on

$$\mathrm{E}\Big[\partial_\beta\varphi_\varepsilon(F_t^M(x)-y)\partial_{\beta'}\partial_\gamma\Psi_2(F_t^M(x)-y)Y\Big]. \tag{4.105}$$

Letting $G_M\overset{\text{def}}{=}\partial_{\beta'}\partial_\gamma\Psi_2(F_t^M(x)-y)Y$, using Theorem 1.3, we have

$$\mathrm{E}\big[\partial_\beta\varphi_\varepsilon(F^M-y)\partial_{\beta'}\partial_\gamma\Psi_2(F_t^M(x)-y)Y\big]=\mathrm{E}\big[\Phi_\varepsilon(F^M-y)H_M^{d+|\beta|}(F_M,G_M)\big]. \tag{4.106}$$

It directly follows, since $\|\Phi_\varepsilon\|_\infty\leq 1$ and the weight H_M does not depend on ε, the pointwise convergence of $\partial_\alpha\big(f_{M,\varepsilon}(x,y)\big)$ when ε tends to 0.

Now, with Theorem 1.13, and following the same pattern as we did in the proof of Lemma 3.15, for $T>t>0$ such that $\frac{4d(3q'-1)}{t}<\theta$ (recalling $q'=d+|\beta|$),

$$\begin{aligned}
&\mathrm{E}\Big[H_M^{d+|\beta|}(F_M,G_M)\Big]\\
&\qquad\leq C_d\Big(\mathrm{E}\Big[\big(1+|\mathrm{L}F_M|_{q'-1}^{q'}\big)^4\Big]\Big)^{\frac14}\Big(\mathrm{E}\Big[\big(1+|F_M|_{q'+1})^{(6d+1)q'}\big)^4\Big]\Big)^{\frac14}\||G_M|_{q'}\|_4.
\end{aligned}$$

So, for M big enough (provided that $\lambda_M \to +\infty$, when $M \to +\infty$) we find that

$$\begin{aligned}\mathrm{E}\left[H_M^{d+|\beta|}(F_M, G_M)\right] &\leq C_{d,q',T}\lambda_M^{((q'-1)^2+2)q'+\frac{d}{2}(6d+1)(d+1)}\| |G_M|_{q'}\|_4 \\ &\leq C_{d,q',T}\lambda_M^{(q'+1)^3+3(d+1)^3}\| |G_M|_{q'}\|_4\end{aligned}$$

Moreover (with Cauchy-Swartz and 1.10)

$$\begin{aligned}\| |G_M|_{q'}\|_8 &\leq C_{q'}\| |\partial_{\beta'}\partial_\gamma \Psi_2(F_t^M(x)-y)|_{q'}|Y|_{q'}\|_4 \\ &\leq C_{q'}\| |\partial_{\beta'}\partial_\gamma \Psi_2(F_t^M(x)-y)|_{q'}\|_8 \| |Y|_{q'}\|_8.\end{aligned}$$

With the fact that, for all multi-index τ

$$\partial_\tau \Psi_2(u) \leq C\mathbb{1}_{|u|\leq 3} \tag{4.107}$$

We can show that, as we already did in the proof of Proposition 4.3, for all β, β', γ of length less than $2m+q$,

$$\| |\partial_{\beta'}\partial_\gamma \Psi_2(F_t^M(x)-y)|_{q'}\|_8 \leq \frac{K'_{d,m,q}(\lambda_M)^{\frac{q'^2}{2}}}{(|y|-3)^{2m+d+1}}. \tag{4.108}$$

using 4.7, 4.1 (and also 1.10), it appears that

$$\| |Y_{N,N'}|_{q'}\|_p \leq C(\lambda_M)^{\frac{q'}{2}|\alpha|} \leq C(\lambda_M)^{\frac{q'}{2}(2m+q)}$$

and

$$\| |\tilde{Y}_{N,N'}|_{q'}\|_p \leq C_{l,p,q,T}(\lambda_M)^{\frac{q'}{2}|\alpha|}(\lambda_M)^{|\alpha|(q'(q'+|\alpha|)+1)} \leq C_{l,p,q,T}(\lambda_M)^{(2m+q)(q'^2+(2m+q+1)q'+1)}$$

Since $\lambda_M \to +\infty$, we have finally

$$\partial_x^\alpha \mathrm{E}\left[\partial^\beta \varphi_\varepsilon(F^M-y)\partial^\gamma \Psi_2(F_t^M(x)-y)\right] \leq K_{d,m,q,T}\Big(1 \wedge \frac{1}{(|y|-3)^{2m+d+1}}\Big)\lambda_M^{A(d,m,q,q')},$$

with

$$A(d,m,q,q') \stackrel{\text{def}}{=} (q'+1)^3 + 3(d+1)^3 + \frac{q'^2}{2} + (2m+q)(q'^2+(2m+q+1)q'+1).$$

Now, setting $\overline{q} \stackrel{\text{def}}{=} d+2m+q \geq q'$,

$$A(d,m,q,q') \leq 4(\overline{q}+1)^3 + 2\overline{q}^3 + \frac{3}{2}\overline{q}^2 + \overline{q} \leq 6(\overline{q}+1)^3.$$

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5 Regenerative scheme and Harris-recurrence

We assume that the process X_t, solution of (**??**), admits a transition density for any time $t > 0$, denoted $p_t(x, y)$, which is strictly positive and continuous in x and y ; a criteria for such a situation is given by Theorem 2.1.

As a consequence, for any $t > 0$ and any compact set C, there exists a probability measure ν and a constant $\alpha > 0$ such that the local Doeblin condition is verified :

$$P_t(x, \mathrm{d}y) \geq \alpha \mathbb{1}_C(x)\nu(\mathrm{d}y). \tag{5.109}$$

In order to obtain some ergodic result over the stochastic process X_t, the main heuristic idea is to approximate, when $t \to +\infty$, the quantity $\frac{1}{t}\int_0^t f(X_s)\,\mathrm{d}s$ by $\frac{1}{n}\sum_{i=1}^n \int_{R_i}^{R_{i+1}} f(X_s)\,\mathrm{d}s$, where the r.v. R_i are to be defined and where the r.v. $\int_{R_i}^{R_{i+1}} f(X_s)\,\mathrm{d}s$ would be i.i.d which will allow to conclude by applying the strong law of large numbers.

To do so in a rigorous way, we will follow the path developed by Eva Löcherbach in [9], *Ergodicity and speed of convergence to equilibrium for diffusion processes*, 2013 (*cf.* also Ikeda, Nagasawa and Watanabe (1966) [8]). First, given a càdlàg Markov process Y_t, we will define the notion of regeneration times :

Definition 5.1 *A sequence $(R_n)_{n\geq 1}$ is called generalized sequence of* ***regeneration times****, if*

1. *$R_n \uparrow \infty$ as $n \to +\infty$.*
2. *$R_{n+1} = R_n + R_1 \circ \vartheta_{R_n}$ (ϑ is the shift operator defined by $\vartheta_t f = f(t + .)$, where $f : \mathbb{R}_+ \to E$ is càdlàg).*
3. *$Y_{R_n+.}$ is independent of $\mathcal{F}^Y_{S_r-}$.*
4. *At regeneration times, the process starts afresh from $Y_{R_n} \sim \nu(\mathrm{d}y)$.*
5. *The trajectories $(Y_{R_n+s}, 0 \leq s \leq R_{n+1} - R_n)_n$ are 2-independent , i.e. $(Y_{R_n+s}, 0 \leq s \leq R_{n+1} - R_n)$ and $(Y_{R_m+s}, 0 \leq s \leq R_{m+1} - R_m)$ are independent if and only if $|m - n| \geq 2$.*

These regeneration times do not exist for the original solution X_t, but they exist for a version of the process on an extended probability space, rich enough to support the driving Brownian motion, the Poisson measure and an i.i.d sequence of uniform random variable $(U_n)_{n\geq 1}$.

We will construct, then, a stochastic process $(Y_t)_{t\geq 0}$ on this richer probability space, equal in law to $(X_t)_{t\geq 0}$.

First we will fix a compact C and a time parameter $t_* > 0$ such that (5.109) is true :

$$P_{t_*}(x, \mathrm{d}y) \geq \alpha \mathbb{1}_C(x)\nu(\mathrm{d}y).$$

Then we set $Y_t = X_t$ for all $0 \leq t \leq \tilde{S}_1$, where

$$\tilde{S}_1 \overset{\text{def}}{=} \inf\{t \geq t_* : X_t \in C\} \qquad \text{and} \qquad \tilde{R}_1 \overset{\text{def}}{=} \tilde{S}_1 + t_*.$$

At time $\tilde{S}_1$, we choose U_1, the first of the uniform random variables. If $U_1 \leq \alpha$, we choose

$$Y_{\tilde{R}_1} \sim \nu(\mathrm{d}y). \tag{5.110}$$

Else, if $U_1 > \alpha$, given $Y_{\tilde{S}_1} = x$, we choose

$$Y_{\tilde{R}_1} \sim \frac{P_{t_*}(x, \mathrm{d}y) - \alpha\nu(\mathrm{d}y)}{1 - \alpha}. \tag{5.111}$$

Finally, given $Y_{\tilde{R}_1} = y$, we fill in the missing trajectory $(Y_t)_{t\in]\tilde{S}_1,\tilde{R}_1[}$ between time $\tilde{S}_1$ and time $\tilde{R}_1$ according to the diffusion bridge law

$$\frac{p_{t-\tilde{S}_1}(x, z)p_{\tilde{R}_1-t}(z, y)}{p_{t_*}(x, y)}\,\mathrm{d}z. \tag{5.112}$$

Notice that by construction, if we do not care about the exact choice of the auxiliary random variable U_1, then we have that $(Y_t)_{t\leq \tilde{R}_1} \overset{\mathcal{L}}{=} (X_t)_{t\leq \tilde{R}_1}$.

We continue this construction after time $\tilde{R}_1$: choose Y_t equal to X_t for all $t \in]\tilde{R}_1, \tilde{S}_2]$ where

$$\tilde{S}_2 \stackrel{\text{def}}{=} \inf\{t > \tilde{R}_1 : X_t \in C\} \qquad \text{and} \qquad \tilde{R}_2 \stackrel{\text{def}}{=} \tilde{S}_2 + t_*.$$

At time $\tilde{S}_2$, we choose U_2 in order to realize the choice of $Y_{\tilde{R}_2}$ according to the splitting of the transition kernel P_{t_*}, as in (5.110) and (5.111). More generally, the construction is therefore achieved along the sequence of stopping times

$$\tilde{S}_{n+1} \stackrel{\text{def}}{=} \inf\{t > \tilde{R}_n : X_t \in C\} \qquad \text{and} \qquad \tilde{R}_{n+1} \stackrel{\text{def}}{=} \tilde{S}_{n+1} + t_*, \qquad n \geq 1,$$

where during each $]\tilde{R}_n, \tilde{S}_{n+1}]$, Y follows the original solution of the SDE, whereas the intervals $[\tilde{S}_{n+1}, \tilde{R}_{n+1}]$ are used to construct the splitting. In particular, every time that we may choose a transition according to (5.110), we introduce a regeneration event for the process Y, and therefore the following two sequences of generalized stopping times will play a role. Firstly,

$$S_1 = \inf\{\tilde{S}_n : U_n \leq \alpha\}, \quad \ldots, \quad S_n = \inf\{\tilde{S}_m > S_{n-1} : U_m \leq \alpha\}, \qquad n \geq 2,$$

and secondly,

$$R_n \stackrel{\text{def}}{=} S_n + t_*, \qquad n \geq 1.$$

The above construction of the process X, since at each time $\tilde{S}_n$, a projection into the future is made.

Let $\tilde{N}_t \stackrel{\text{def}}{=} \sup\{n : U_n \leq t\}$ and

$$\mathcal{F}_t^Y = \sigma\{Y_s, s \leq t, U_n, Y_{\tilde{R}_n}, n \leq \tilde{N}_t\}, \qquad t \geq 0,$$

be the canonical filtration of the process Y. The sequence of $(\mathcal{F}_t^Y)_{t\geq 0}$-stopping times $(R_n)_{n\geq 1}$ is a generalized sequence of regeneration times as it was defined in Definition 5.1.

Remark 5.1 *The trajectories of Y are not the same as those of the original solution X of the SDE. However, by definition, the Harris-recurrence is only a property in law. As a consequence, if, for a given set A, we succeed to show that almost surely, Y visits it infinitely often, the same is automatically true for X as well.*

We can now state the theorem we were looking for (the demonstration of it is directly taken from Eva Löcherbach's lecture, *Ergodicity and speed of convergence to equilibrium for diffusion processes*, and we give it here only for the convenience of the reader) :

Theorem 5.2 *If for all $x \in \mathbb{R}^d$ we have $\mathrm{P}_x[R_1 < \infty] = 1$, then the process X is recurrent in the sense of Harris.*

Proof :

Define a measure π on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ by

$$\pi(A) \stackrel{\text{def}}{=} \mathrm{E}\Big[\int_{R_1}^{R_2} \mathbb{1}(Y_s)\,\mathrm{d}s\Big], \qquad A \in \mathcal{B}(\mathbb{R}^d).$$

For any $n \geq 2$, put $\xi_n \stackrel{\text{def}}{=} \int_{R_{n-1}}^{R_n} \mathbb{1}(Y_s)\,\mathrm{d}s$. By construction, the random variables ξ_{2n}, $n \geq 1$, are i.i.d. and so are, on the other hand, as well, the random variables ξ_{2n+1}. Put

$$N_t = \sup\{n : R_n \leq t\}$$

and observe that $N_t \to \infty$ as $t \to \infty$. Hence, applying the strong law of large numbers separately to the sequence $(\xi_{2n})_{n\geq 1}$ and the sequence $(\xi_{2n+1})_{n\geq 1}$, we have that

$$\lim_{t\to\infty} \frac{\int_0^t \mathbb{1}(Y_s)\,\mathrm{d}s}{N_t} = \pi(A)$$

P_x-almost surely, for any $x \in \mathbb{R}^d$. This implies that any set A such that $\pi(A) > 0$ is visited infinitely often by the process Y almost surely. Thus, we have the recurrence property also for the process X, for any set A such that $\pi(A) > 0$. Then, by a deep theorem of Azéma, Duflo and Revuz (1969) [1], see also Theorem 1.2. of Höpfner and Löcherbach (2003) [7], the process is indeed Harris.

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A Miscellaneous

A.1 Regularisation

Let us define on $\mathbb{R}$ the following function:

$$\varphi(x) \stackrel{\text{def}}{=} \begin{cases} \alpha \exp(-\frac{1}{1-x^2}) & \text{si } |x| < 1 \\ 0 & \text{si } |x| \geq 1 \end{cases}$$

where α is chosen in order to have $\int_{\mathbb{R}} \varphi(x)\,\mathrm{d}x = 1$. Numerically, $\alpha \approx (0,44399)^{-1}$.

Proposition A.1 *$\varphi : \mathbb{R} \to \mathbb{R}$ is $\mathcal{C}^\infty$ compact support function.*

We then define $\varphi_\varepsilon : \mathbb{R} \to \mathbb{R}$ by:

$$\varphi_\varepsilon(x) \stackrel{\text{def}}{=} \frac{1}{\varepsilon}\varphi\left(\frac{x}{\varepsilon}\right), \qquad \forall x \in \mathbb{R}.$$

Thus defined, function φ_ε converges weakly, as ε tends to 0, to the Dirac distribution ; it is a mollifier: for every continuous function f,

1. $\lim_{\varepsilon \to 0} f * \varphi_\varepsilon(x) = f(x)$
2. $f * \varphi_\varepsilon$ is $\mathcal{C}^\infty$

(with $f * \varphi_\varepsilon(x) \stackrel{\text{def}}{=} \int_{\mathbb{R}} f(y)\varphi_\varepsilon(x-y)\,\mathrm{d}y)$).

Proposition A.2 *$\varphi_\varepsilon : \mathbb{R} \to \mathbb{R}$ remains a $\mathcal{C}^\infty$ compact support function and there exists $M > 0$ such that:*

$$\forall x \in \mathbb{R}, \qquad |\varphi'_\varepsilon(x)| \leq \frac{M}{\varepsilon^2} \quad \textit{and} \quad |\varphi''_\varepsilon(x)| \leq \frac{M}{\varepsilon^3}.$$

Proof : If $|x| \geq \varepsilon$,

$$\varphi'_\varepsilon(x) = \varphi''_\varepsilon(x) = 0.$$

Elsewhere, $\varphi_\varepsilon(x) = \frac{\alpha}{\varepsilon}\exp\left(-\frac{1}{1-\left(\frac{x}{\varepsilon}\right)^2}\right)$ and

$$\begin{aligned} \varphi'_\varepsilon(x) &= \frac{\alpha}{\varepsilon}\exp\left(-\frac{1}{1-\left(\frac{x}{\varepsilon}\right)^2}\right) \times \frac{-2x}{\varepsilon^2} \times \frac{1}{\left(1-\left(\frac{x}{\varepsilon}\right)^2\right)^2} \\ &= \frac{-2\alpha}{\varepsilon^2}\exp\left(-\frac{1}{1-\left(\frac{x}{\varepsilon}\right)^2}\right)\frac{\frac{x}{\varepsilon}}{\left(1-\left(\frac{x}{\varepsilon}\right)^2\right)^2}. \end{aligned}$$

The function $y \mapsto \exp\left(-\frac{1}{1-y^2}\right)\frac{y}{(1-y^2)^2}$ is defined and continuous (considering a continuous extension for $y = \pm 1$) on $\mathbb{R}$ and bounded, as we can easily prove by considering its limits when y tends respectively to $\pm\infty$, -1 and 1.
Denoting now by M_1 an upperbound and letting $C_1 = 2\alpha M_1$, we obtain the first assumption.

Now,

$$\begin{aligned} \varphi''_\varepsilon(x) &= \frac{\alpha}{\varepsilon^3}\exp\left(-\frac{1}{1-\left(\frac{x}{\varepsilon}\right)^2}\right)\left(\frac{-2x}{\varepsilon^2\left(1-\left(\frac{x}{\varepsilon}\right)^2\right)^2} \times \frac{-2x}{\left(1-\left(\frac{x}{\varepsilon}\right)^2\right)^2} + \frac{\partial}{\partial x}\left(\frac{-2x}{\left(1-\left(\frac{x}{\varepsilon}\right)^2\right)^2}\right)\right) \\ &= \frac{\alpha}{\varepsilon^3}\exp\left(-\frac{1}{1-\left(\frac{x}{\varepsilon}\right)^2}\right)\left(\frac{4\left(\frac{x}{\varepsilon}\right)^2}{\left(1-\left(\frac{x}{\varepsilon}\right)^2\right)^4} - \frac{2}{\left(1-\left(\frac{x}{\varepsilon}\right)^2\right)^2} + \frac{4x}{\varepsilon^2} \times \frac{-2x}{\left(1-\left(\frac{x}{\varepsilon}\right)^2\right)^3}\right). \end{aligned}$$

Similarly, the function $y \mapsto \exp\left(-\frac{1}{1-y^2}\right)\left(\frac{4y^2}{(1-y^2)^4} - \frac{2}{(1-y^2)^2} - \frac{8y^2}{(1-y^2)^3}\right)$ is *bounded* on $\mathbb{R}$ by a constant M_2. With $C_2 = \alpha M_2$ and $M = \max(C_1, C_2)$ we then obtain the last property. •

We then define $h_\varepsilon : \mathbb{R} \to \mathbb{R}$ by:

$$h_\varepsilon(x) \stackrel{\text{def}}{=} |x| \vee 2\varepsilon$$

and $\phi_\varepsilon : \mathbb{R} \to \mathbb{R}$ by:

$$\phi_\varepsilon(x) = h_\varepsilon * \varphi_\varepsilon(x).$$

Proposition A.3 *1. ϕ_ε converges pointwise to the absolute value function $x \mapsto |x|$ and*

$$\phi_\varepsilon(x) = \begin{cases} 2\varepsilon & \textit{if} \quad |x| \le \varepsilon \\ |x| & \textit{if} \quad |x| > 3\varepsilon \end{cases}$$

and

$$0 \le \phi_\varepsilon(x) \le 4\varepsilon \quad \textit{if} \quad |x| \in]\varepsilon, 3\varepsilon].$$

2. There exists $C > 0$ such that

$$\forall x \in \mathbb{R}, \qquad |\phi_\varepsilon'(x)| \le C \qquad \textit{and} \qquad |\phi_\varepsilon''(x)| \le \frac{C}{\varepsilon}\mathbb{1}_{|x|\le 3\varepsilon}.$$

Proof : First, let us remark that, since ϕ_ε is defined by a convolution, it is a smooth function and, for all $n \in \mathbb{N}^*$, $\phi_\varepsilon^{(n)}(x) = h_\varepsilon * \varphi_\varepsilon^{(n)}(x)$.

We will now prove the two items by dividing the problem into the following three cases.

- **Case $|x| \le \varepsilon$:**

  We have

$$\begin{aligned} \phi_\varepsilon(x) = h_\varepsilon * \varphi_\varepsilon(x) &= \int_{-2\varepsilon}^{2\varepsilon} \varphi_\varepsilon(x-y) h_\varepsilon(y)\,\mathrm{d}y \\ &= 2\varepsilon \int_{-2\varepsilon}^{2\varepsilon} \varphi_\varepsilon(x-y)\,\mathrm{d}y \\ &= 2\varepsilon \underbrace{\int_{\mathbb{R}} \varphi_\varepsilon(z)\,\mathrm{d}z}_{=1} = 2\varepsilon. \end{aligned}$$

  Hence, for all $|x| < \varepsilon$,

$$\phi_\varepsilon'(x) = \phi_\varepsilon''(x) = 0,$$

  (which remains true when $|x| = \varepsilon$, since ϕ_ε is a smooth function).

- **Case $|x| > 3\varepsilon$:**

  noticing that $z \mapsto z\varphi_\varepsilon(z)$ is an odd function (with compact support),

$$\begin{aligned} \phi_\varepsilon(x) = \int_{\mathbb{R}} h_\varepsilon(x-z)\varphi_\varepsilon(z)\,\mathrm{d}z &= \int_{|z|\le\varepsilon} h_\varepsilon(x-z)\varphi_\varepsilon(z)\,\mathrm{d}z \\ &= \int_{|z|\le\varepsilon} |x-z|\varphi_\varepsilon(z)\,\mathrm{d}z \\ &= \int_{|z|\le\varepsilon} \mathrm{sign}(x)(x-z)\varphi_\varepsilon(z)\,\mathrm{d}z \\ &= \mathrm{sign}(x) \times \Big(x - \underbrace{\int_{\mathbb{R}} \varphi_\varepsilon(z) z\,\mathrm{d}z}_{=0}\Big) = |x|. \end{aligned}$$

  Hence, for all $|x| > 3\varepsilon$,

$$\phi_\varepsilon'(x) = \pm 1 \qquad \text{and} \qquad \phi_\varepsilon''(x) = 0,$$

  (and obviously again, it remains true when $|x| = 3\varepsilon$).

- **Case $|x| \in]\varepsilon, 3\varepsilon]$:**

  Let g be a continuous function (with compact support), then

$$\begin{aligned} h_\varepsilon * g(x) = \int_{\mathbb{R}} g(x-y) h_\varepsilon(y)\,\mathrm{d}y &= \int_{-4\varepsilon}^{4\varepsilon} g(x-y) h_\varepsilon(y)\,\mathrm{d}y \\ &\le 4\varepsilon \int_{\mathbb{R}} |g(z)|\,\mathrm{d}z. \end{aligned}$$

With $g \overset{\text{def}}{=} \varphi_\varepsilon$, $\int_{\mathbb{R}} |g(z)|\,\mathrm{d}z = 1$ and it follows that

$$\phi_\varepsilon(x) \leq 4\varepsilon \ ;$$

with respectively $g \overset{\text{def}}{=} \varphi'_\varepsilon$ and $g \overset{\text{def}}{=} \varphi''_\varepsilon$, and using the Proposition A.2, it follows that

$$\phi'_\varepsilon(x) = h_\varepsilon * \varphi'_\varepsilon \leq 4\varepsilon \int_{-\varepsilon}^{\varepsilon} \frac{M}{\varepsilon^2}\,\mathrm{d}y = 8M \ ;$$

and

$$\phi''_\varepsilon(x) = h_\varepsilon * \varphi''_\varepsilon \leq 4\varepsilon \int_{-\varepsilon}^{\varepsilon} \frac{M}{\varepsilon^3}\,\mathrm{d}y = \frac{8M}{\varepsilon}.$$

Gathering the three cases, we may conclude. •

A.2 Gronwall's lemma

In all this work we used the following version of the Gronwall's lemma:

Proposition A.4 *If a measurable function $g : [0,T] \to \mathbb{R}^+$ is such that*

1. $G = \sup_{t\in[0,T]} g(t) < +\infty$ *;*
2. *for all $t \in [0,T]$,*

$$g(t) \leq A + B\int_0^t g(s)\,\mathrm{d}s$$

then, for all $t \in [0,T]$,

$$g(t) \leq A\exp(Bt).$$

Proof : It is easy to obtain by induction that, for every $n \in \mathbb{N}^*$,

$$g(t) \leq A\Big(1 + \sum_{k=1}^{n-1} \frac{(Bt)^k}{k!} + B^n \int_0^t \int_0^{t_1} \dots \int_0^{t_{n-1}} g(t_n)\,\mathrm{d}t_n \cdots \mathrm{d}t_1\,\mathrm{d}t\Big),$$

which implies

$$g(t) \leq A\Big(1 + \sum_{k=1}^{n-1} \frac{(Bt)^k}{k!} + G\frac{(Bt)^n}{n!}\Big).$$

Since $\lim_{n\to+\infty} G\frac{(Bt)^n}{n!} = 0$, the assertion follows. •

A.3 Moment inequalities

In this subsection of the appendix we prove some moment inequalities which we used in Subsection **??**, so the notations introduced in that subsection prevail.

We assume in this subsection that $\mu(E) < \infty$. This is just to simplify the notations — in concrete applications we will replace μ by $\mathbb{1}_G\mu$. Then we consider an index set Λ and we denote by α the elements of Λ. Moreover we consider a family of processes $V_t^\alpha \in \mathbb{R}^d, \alpha \in \Lambda$ which verify the following equation

$$\begin{aligned} V_t^\alpha \quad = \quad & V_0^\alpha + \sum_{l=1}^m \int_0^t (H_l^\alpha(s) + \langle \nabla\sigma_l(X_s), V_s^\alpha\rangle)\,\mathrm{d}W_s^l \\ & + \int_0^t (h^\alpha(s) + \langle \nabla b(X_s), V_s^\alpha\rangle)\,\mathrm{d}s \\ & + \int_0^t \int_{E\times(0,1)} (Q^\alpha(s-,z) + \big\langle \nabla_x c(X_{s-},z), V_{s-}^\alpha\big\rangle)\mathbb{1}_{\{u\leq\gamma(X_{s-},z)\}}\,\mathrm{d}N(s,u,z). \end{aligned} \tag{A.113}$$

Here $H_l^\alpha, h_l^\alpha$and Q^α are adapted càdlàg processes which verify

$$\int_0^T (|H_l^\alpha(s)|^2 + |h^\alpha(s)| + \int_E |Q^\alpha(s,z)|\,\overline{\gamma}(z)d\mu(z))\,\mathrm{d}s < \infty.$$

(Where the functions σ, b, c and γ are the ones introduced in Subsection **??**.) So the corresponding stochastic integrals in (A.113) make sense.

Proposition A.5 *We suppose that*

$$|Q^{\alpha}(s,z)| \leq \overline{q}(z)\,|R^{\alpha}_s| \tag{A.114}$$

for some adapted càdlàg process R^{α} and some measurable function $\overline{q} : E \to \mathbb{R}_+$ and we denote

$$\begin{aligned} \widehat{c}_1(p) &= \int_E (\overline{q}(z) + \overline{c}_{(1)}(z))(1+\overline{q}(z))^{2p}\overline{\gamma}(z)d\mu(z), \\ \widehat{c}_2(p) &= \int_E (\overline{q}(z) + \overline{c}_{(1)}(z))(1+\overline{c}_{(1)}(z))^{2p}\overline{\gamma}(z)d\mu(z). \end{aligned} \tag{A.115}$$

For every $p \in \mathbb{N}$ there exists a universal constant C_p such that $0 \leq t \leq T$

$$\begin{aligned} \mathrm{E}\left[|V^{\alpha}_t|^{2p}\right] \leq & \exp\left(C_p t(1 + \|\nabla\sigma\|^{2p}_{\infty} + \|\nabla b\|^{2p}_{\infty} + \widehat{c}_2(p))\right) \\ & \times \left(|V^{\alpha}_0|^{2p} + C_p \int_0^t \mathrm{E}\left[\sum_{l=1}^m |H^{\alpha}_l(s)|^{2p} + |h^{\alpha}(s)|^{2p} + \widehat{c}_1(p)\left|R^{\alpha}_{s-}\right|^{2p}\right]\mathrm{d}s\right) \end{aligned} \tag{A.116}$$

Proof : Using Itô's formula for $f(x) = x^{2p}$, we obtain[7]

$$(V^{\alpha}_t)^{2p} = (V^{\alpha}_0)^{2p} + M^{\alpha}_t + I^{\alpha}_t + J^{\alpha}_t$$

with

$$M^{\alpha}_t = \sum_{l=1}^m \int_0^t 2p(V^{\alpha}_s)^{2p-1}(H^{\alpha}_l(s) + \langle\nabla\sigma_l(X_s), V^{\alpha}_s\rangle)\,\mathrm{d}W^l_s,$$

$$\begin{aligned} I^{\alpha}_t =& \sum_{l=1}^m \int_0^t p(2p-1)(V^{\alpha}_s)^{2p-2}\sum_{l=1}^m (H^{\alpha}_l(s) + \langle\nabla\sigma_l(X_s), V^{\alpha}_s\rangle)^2\,\mathrm{d}s \\ & + 2p\int_0^t (V^{\alpha}_s)^{2p-1}(h^{\alpha}(s) + \langle\nabla b(X_s), V^{\alpha}_s\rangle)\,\mathrm{d}s \end{aligned}$$

and

$$J^{\alpha}_t = \int_0^t \int_{E\times(0,1)} \left(\left(V^{\alpha}_{s-} + Q^{\alpha}(s-,z) + \left\langle\nabla_x c(X_{s-},z), V^{\alpha}_{s-}\right\rangle\right)^{2p} - \left(V^{\alpha}_{s-}\right)^{2p}\right)\mathbb{1}_{\{u\leq\gamma(X_{s-},z)\}}\,\mathrm{d}N(s,u,z).$$

Using the trivial inequality $a^u b^v \leq a^{u+v} + b^{u+v}$ we obtain

$$\begin{aligned} \mathrm{E}\left[|I^{\alpha}_t|\right] \leq & C_p\int_0^t \mathrm{E}\left[\sum_{l=1}^m |H^{\alpha}_l(s)|^{2p} + |h^{\alpha}(s)|^{2p}\right]\mathrm{d}s \\ & + C_p(1 + \|\nabla\sigma\|^{2p}_{\infty} + \|\nabla b\|^{2p}_{\infty})\int_0^t \mathrm{E}\left[\left|V^{\alpha}_s\right|^{2p}\right]\mathrm{d}s. \end{aligned}$$

We estimate now J^{α}_t. Using the elementary inequality

$$(a+b)^{2p} - a^{2p} \leq C_p\,|b|\,(|a|^{2p-1} + |b|^{2p-1})$$

we obtain

$$\begin{aligned} & \left|V^{\alpha}_{s-} + Q^{\alpha}(s-,z) + \left\langle\nabla_x c(X_{s-},z), V^{\alpha}_{s-}\right\rangle\right|^{2p} - \left|V^{\alpha}_{s-}\right|^{2p} \\ \leq \quad & C_p(|Q^{\alpha}(s-,z)| + \overline{c}_{(1)}(z)\left|V^{\alpha}_{s-}\right|)(\left|V^{\alpha}_{s-}\right|^{2p-1}(1 + \overline{c}^{2p-1}_{(1)}(z)) + |Q^{\alpha}(s-,z)|^{2p-1}). \end{aligned}$$

Recall that $|Q^{\alpha}(s-,z)| \leq \overline{q}(z)\left|R^{\alpha}_{s-}\right|$ so the above term is upper bounded by

$$\begin{aligned} & C_p(\overline{q}(z)\left|R^{\alpha}_{s-}\right| + \overline{c}_{(1)}(z)\left|V^{\alpha}_{s-}\right|)(\left|V^{\alpha}_{s-}\right|^{2p-1}(1 + \overline{c}_{(1)}(z))^{2p-1} + \left|\overline{q}(z)R^{\alpha}_{s-}\right|^{2p-1}) \\ \leq \quad & C_p(\overline{q}(z) + \overline{c}_{(1)}(z))(\left|R^{\alpha}_{s-}\right| + \left|V^{\alpha}_{s-}\right|)(((1 + \overline{c}_{(1)}(z))^{2p-1}\left|V^{\alpha}_{s-}\right|^{2p-1} + \left|\overline{q}(z)R^{\alpha}_{s-}\right|^{2p-1}) \end{aligned}$$

We use once again the inequality $a^u b^v \leq a^{u+v} + b^{u+v}$ and we upper bound the above term by

$$C_p(\overline{q}(z) + \overline{c}_{(1)}(z))(\left|R^{\alpha}_{s-}\right|^{2p}(1 + \overline{q}(z))^{2p} + (1 + \overline{c}_{(1)}(z))^{2p}\left|V^{\alpha}_{s-}\right|^{2p}).$$

[7] For $x = (x_i)_{1\leq i\leq d} \in \mathbb{R}^d$ we simply denote $(x^{2p}_i)_{1\leq i\leq d}$ by x^{2p}.

It follows that

$$\begin{aligned}\mathrm{E}\left[\,|J_t^\alpha|\,\right] \leq & C_p \int_E (\overline{q}(z)+\overline{c}_{(1)}(z))(1+\overline{q}(z))^{2p}\overline{\gamma}(z)d\mu(z)\int_0^t \mathrm{E}\left[\left|R_{s-}^\alpha\right|^{2p}\right]\mathrm{d}s \\ & + C_p \int_E (\overline{q}(z)+\overline{c}_{(1)}(z))(1+\overline{c}_{(1)}(z))^{2p}\overline{\gamma}(z)d\mu(z)\int_0^t \mathrm{E}\left[\left|V_{s-}^\alpha\right|^{2p}\right]\mathrm{d}s.\end{aligned}$$

Since M_t^α is a martingale we obtain

$$\mathrm{E}\left[\,(V_t^\alpha)^{2p}\,\right] = \mathrm{E}\left[\,(V_0^\alpha)^{2p}\,\right] + \mathrm{E}\left[I_t^\alpha\right] + \mathrm{E}\left[J_t^\alpha\right]$$

and (we recall the notation in (A.115))

$$\begin{aligned}\mathrm{E}\left[\,|V_t^\alpha|^{2p}\,\right] \leq & |V_0^\alpha|^{2p} + \mathrm{E}\left[\,|I_t^\alpha|\,\right] + \mathrm{E}\left[\,|J_t^\alpha|\,\right] \\ \leq & |V_0^\alpha|^{2p} + C_p\int_0^t \mathrm{E}\left[\sum_{l=1}^m |H_l^\alpha(s)|^{2p} + |h^\alpha(s)|^{2p} + \widehat{c}_1(p)\left|R_{s-}^\alpha\right|^{2p}\right]\mathrm{d}s \\ & + C_p(1+\|\nabla\sigma\|_\infty^{2p} + \|\nabla b\|_\infty^{2p} + \widehat{c}_2(p))\int_0^t \mathrm{E}\left[\left|V_s^\alpha\right|^{2p}\right]\mathrm{d}s.\end{aligned}$$

The Gronwall's lemma then gives

$$\begin{aligned}\mathrm{E}\left[\,|V_t^\alpha|^{2p}\,\right] \leq & \exp\left(C_p t(1+\|\nabla\sigma\|_\infty^{2p} + \|\nabla b\|_\infty^{2p} + \widehat{c}_2(p))\right) \\ & \times\left(|V_0^\alpha|^{2p} + C_p\int_0^t \mathrm{E}\left[\sum_{l=1}^m |H_l^\alpha(s)|^{2p} + |h^\alpha(s)|^{2p} + \widehat{c}_1(p)\left|R_{s-}^\alpha\right|^{2p}\right]\mathrm{d}s\right).\end{aligned}$$

•

A.4 Proof of (2.40)

$$\begin{aligned}I &= \mathrm{E}\left[f(X_{T_k}^M)\mathbb{1}_{\{U_k\geq\gamma(Z_k,X_{T_k-}^M)\}}|X_{T_k-}^M = x\right] \\ &= \int_{\langle X_{T_k-}^M = x\rangle} f(X_{T_k}^M)\mathbb{1}_{\{U_k\geq\gamma(Z_k,X_{T_k-}^M)\}}\frac{\mathrm{dP}}{\mathrm{P}(X_{T_k-}^M = x)} \\ &= \int_{\langle X_{T_k-}^M = x\rangle\cap\langle U_k\geq\gamma(Z_k,X_{T_k-}^M)\rangle} f(X_{T_k}^M)\frac{\mathrm{dP}}{\mathrm{P}(X_{T_k-}^M = x)}\end{aligned}$$

On the event $\langle X_{T_k-}^M = x\rangle \cap \langle U_k \geq \gamma(Z_k, X_{T_k-}^M)\rangle$ we have $X_{T_k}^M = x$, so I becomes

$$\begin{aligned}I &= f(x)\frac{\mathrm{P}(\langle X_{T_k-}^M = x\rangle\cap\langle U_k\geq\gamma(Z_k,X_{T_k-}^M)\rangle)}{\mathrm{P}(X_{T_k-}^M = x)} \\ &= f(x)\frac{\mathrm{P}(\langle X_{T_k-}^M = x\rangle\cap\langle U_k\geq\gamma(Z_k,x)\rangle)}{\mathrm{P}(X_{T_k-}^M = x)} \\ &= f(x)\,\mathrm{P}(U_k\geq\gamma(Z_k,x)) && \text{(since } U_k \text{ and } X_{T_k-}^M \text{ are independent)} \\ &= f(x)\frac{1}{\mu(B_{M+1})2\overline{c}}\int_{B_{M+1}}\int_0^{2\overline{c}}\mathbb{1}_{\{u\geq\gamma(z,x)\}}\,\mathrm{d}\mu(z)\,\mathrm{d}u \\ &= f(x)\frac{1}{\mu(B_{M+1})}\int_{B_{M+1}}\left(1-\frac{\gamma(z,x)}{2\overline{c}}\right)\mathrm{d}\mu(z) \\ &= f(x)\theta_{M,\gamma}(x) = f(x)\theta_{M,\gamma}(x)\int_{\mathbb{R}^d}\phi(z-z_M^*)\,\mathrm{d}z && \left(\textstyle\int\phi = 1 \text{ and } \phi(B(0,1)^c)=0\right) \\ &= \int_{|z|>M+1} f(x+c_M(z,x))\phi(z-z_M^*)\theta_{M,\gamma}(x)\,\mathrm{d}z && \text{(since } c_M(z,x)=0 \text{ if } |z|>M+1) \\ &= \int_{|z|>M+1} f(x+c_M(z,x))q_M(z,x)\,\mathrm{d}z.\end{aligned}$$

On the other hand:

$$
\begin{aligned}
J &= \mathrm{E}\left[f(X^M_{T_k})\mathbb{1}_{\{U_k<\gamma(Z_k,X^M_{T_k-})\}}|X^M_{T_k-}=x\right] \\
&= \int_{\langle X^M_{T_k-}=x\rangle} f(X^M_{T_k})\mathbb{1}_{\{U_k<\gamma(Z_k,X^M_{T_k-})\}}\frac{\mathrm{dP}}{\mathrm{P}(X^M_{T_k-}=x)} \\
&= \int_{\langle X^M_{T_k-}=x\rangle\cap\langle U_k\geq\gamma(Z_k,X^M_{T_k-})\rangle} f(x+c_M(Z_k,x))\frac{\mathrm{dP}}{\mathrm{P}(X^M_{T_k-}=x)} \\
&= \int_\Omega f(x+c_M(Z_k,x))\mathbb{1}_{\{U_k<\gamma(Z_k,x)\}}\,\mathrm{dP} \qquad \text{(since (U_k,Z_k) and $X^M_{T_k-}$ are independent)} \\
&= \frac{1}{\mu(B_{M+1})2\overline{c}}\int_{B_{M+1}}\int_0^{2\overline{c}} f(x+c_M(z,x))\mathbb{1}_{\{u<\gamma(z,x)\}}\,\mathrm{d}\mu(z)\,\mathrm{d}u \\
&= \frac{1}{\mu(B_{M+1})}\int_{B_{M+1}} f(x+c_M(z,x))\frac{\gamma(z,x)}{2\overline{c}}\,\mathrm{d}\mu(z) \qquad \text{(with $\mathrm{d}\mu(z)=h(z)\,\mathrm{d}z$)}
\end{aligned}
$$

We finally have

$$
\mathrm{E}\left[f(X^M_{T_k})|X^M_{T_k-}=x\right] = \int_{\mathbb{R}^d} f(x+c_M(z,x))q_M(z,x)\,\mathrm{d}z.
$$

Now $\sigma(\overline{X}_{T_{k+1}-})\subset\mathcal{G}_k$, so

$$
\begin{aligned}
&\mathrm{E}\left[f(\overline{X}^M_{T_{k+1}})|\overline{X}^M_{T_{k+1}-}=x\right] \\
&\quad= \mathrm{E}\left[f(\overline{X}^M_{T_{k+1}-}+c_M(\overline{Z}_{k+1},\overline{X}^M_{T_{k+1}-}))|\overline{X}^M_{T_{k+1}-}=x\right] \\
&\quad= \mathrm{E}\left[\,\mathrm{E}\left[f(\overline{X}^M_{T_{k+1}-}+c_M(\overline{Z}_{k+1},\overline{X}^M_{T_{k+1}-}))|\mathcal{G}_k\right]|\overline{X}^M_{T_{k+1}-}=x\right] \\
&\quad= \mathrm{E}\left[\int_{\mathbb{R}^d} f(\overline{X}^M_{T_{k+1}-}+c_M(z,\overline{X}^M_{T_{k+1}-}))q_M(\overline{X}^M_{T_{k+1}-},z)\,\mathrm{d}z|\overline{X}^M_{T_{k+1}-}=x\right] \\
&\quad= \int_{\mathbb{R}^d}\int_{\langle\overline{X}^M_{T_{k+1}-}=x\rangle} f(\overline{X}^M_{T_{k+1}-}+c_M(z,\overline{X}^M_{T_{k+1}-}))q_M(\overline{X}^M_{T_{k+1}-},z)\frac{\mathrm{dP}}{\mathrm{P}(\overline{X}^M_{T_{k+1}-}=x)}\,\mathrm{d}z \\
&\quad= \int_{\mathbb{R}^d} f(x+c_M(z,x))q_M(z,x)\,\mathrm{d}z
\end{aligned}
$$

So we have:

$$
\mathrm{E}\left[f(X^M_{T_k})|X^M_{T_k-}=x\right] = \mathrm{E}\left[f(\overline{X}^M_{T_k})|\overline{X}^M_{T_k-}=x\right] \tag{A.117}
$$

We can prove now that the processes X^M_t and $\overline{X}^M_t$ are sharing the same law.

A.5 X^M_t and $\overline{X}^M_t$ share the same law

- First, if $0\leq t<T_1$, $x_t=\Psi_t(x)$ and then $\overline{X}^M_t=\Psi_t(x)\sim X^M_t$.
- Moreover, if $\overline{X}^M_{T_k-}\sim X^M_{T_k-}$ we have $\overline{X}^M_{T_k}\sim X^M_{T_k}$: recalling that, if $\phi(x):=\mathrm{E}\left[Y|X=x\right]$, we have $\int_A\phi(x)\,\mathrm{dP}_X(x)=\int_{\langle X\in A\rangle}Y\,\mathrm{d}P$, since $\mathrm{P}_{\overline{X}^M_{T_k-}}=\mathrm{P}_{X^M_{T_k-}}$ the relation (A.117) leads to

$$
\mathrm{E}\left[f(X^M_{T_k})\right] = \mathrm{E}\left[f(\overline{X}^M_{T_k})\right]
$$

- Finally, if $T_k\leq t<T_{k+1}$, $\overline{X}^M_t=\Psi_{t-T_k}(\overline{X}^M_{T_k})\sim\Psi_{t-T_k}(X^M_{T_k})=X^M_t$.

B Sobolev norms of X^M_t and its derivatives.

We have used, within the proof of the Lemma 3.1 that X^M_t has moments of any order. The proof of that result follows the same pattern of this same lemma, although a bit simpler. The following result is the equation (3.60) in the special case $k=0$.

Proposition B.1 *Let $M \in \mathbb{N}^*$. For all $T > 0$ and $p \geq 1$, there exists a constant $C_{T,p} > 0$ (which does not depend on M) such that*

$$\mathrm{E}\left[|X_t^M|^{2p}\right] \leq C_{T,p}. \tag{B.118}$$

Proof :

We localize our problem by using the sequence $(\tau_K^M)_{K\in\mathbb{N}^*}$ of stopping times defined by

$$\tau_K^M \stackrel{\text{def}}{=} \inf\{t > 0 \ : \ |X_t^M| \geq K\}. \tag{B.119}$$

We can prove that a.s. $\lim_{K\to\infty} \tau_K^M = \infty$:
From the hypothesis made on the coefficients of X_t^M, it is clear that, for all $t \geq 0$,

$$\mathrm{E}\big[\sup_{s\leq t} |X_s^M|\big] < \infty. \tag{B.120}$$

We have, for $t \geq 0$

$$\begin{aligned}\lim_{K\to\infty} \mathrm{P}(\tau_K^M < t) &= \lim_{K\to\infty} \mathrm{P}(\sup_{s\leq t} |X_s^M| > K) \\ &\leq \lim_{K\to\infty} \frac{1}{K}\, \mathrm{E}\big[\sup_{s\leq t} |X_s^M|\big] = 0.\end{aligned}$$

$(\tau_K^M)_{K\in\mathbb{N}^*}$ tends to ∞ in probability and so, there exists a subsequence (that we will continue to denote by $(\tau_K^M)_{K\in\mathbb{N}^*}$) which tends to ∞ a.s.

If we admit for the moment the Lemma B.2, we know that there exists a constant $C_{p,T}$ which does not depend on K and M and such that, for all $0 \leq t \leq T$,

$$\mathrm{E}\left[|X^M_{t\wedge\tau_K^M}|^{2p}\right] \leq C_{T,p}.$$

The monotone convergence theorem implies then

$$\mathrm{E}\left[|X_t^M|^{2p}\right] = \sup_K \mathrm{E}\left[|X_t^M|^{2p} \mathbb{1}_{\tau_K^M > t}\right]$$

and

$$\sup_K \mathrm{E}\left[|X_t^M|^{2p} \mathbb{1}_{\tau_K^M > t}\right] = \sup_K \mathrm{E}\left[|X^M_{t\wedge\tau_K^M}|^{2p}\right] \leq C_{T,p}.$$

•

Lemma B.2 *Let $M \in \mathbb{N}^*$ and a sequence $(\tau_K^M)_{K\in\mathbb{N}^*}$ of stopping times defined by* (B.119)*. There exists a constant $C_{p,T}$, which does not depend on K and M, and such that, for all $0 \leq t \leq T$,*

$$\mathrm{E}\left[|X^M_{t\wedge\tau_K^M}|^{2p}\right] \leq C_{T,p}.$$

Proof :

Recalling the definition (2.35) of X_t^M:

$$X_t^M = x + \int_0^t \sigma(X_s^M)\,\mathrm{d}W_s + \int_0^t \int_E c_M(z, X_{s-}^M) \mathbb{1}_{\{u\leq\gamma(z,X_{s-}^M)\}} N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) + \int_0^t g(X_s^M)\,\mathrm{d}s,$$

we have, for a single component (omitting for a moment the parameter M in order to simplify the notations), applying Itô's formula with $f(x) = x^{2p}$ with respect to every component of the process $X^M_{t\wedge\tau_K^M}$,

$$\begin{aligned}(X^i_{t\wedge\tau_K^M})^{2p} = (X_0^i)^{2p} &+ \sum_{l=1}^m \int_0^{t\wedge\tau_K^M} 2p(X_s^i)^{2p-1}\sigma_{il}(X_s)\,\mathrm{d}W_s^l \\ &+ 2p\int_0^{t\wedge\tau_K^M} (X_s^i)^{2p-1} g_i(X_s)\,\mathrm{d}s \\ &+ p(2p-1)\sum_{l=1}^m \int_0^{t\wedge\tau_K^M} (X_s^i)^{2p-2}\big(\sigma_{il}(X_s)\big)^2\,\mathrm{d}s \\ &+ \int_0^{t\wedge\tau_K^M} \int_E \Big(X^i_{s^-} + c_M(z, X^M_{s-})\mathbb{1}_{\{u\leq\gamma(z,X^M_{s-})\}}\Big)^{2p} - (X^i_{s^-})^{2p} N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)\end{aligned}$$

We now take the expectation with respect to the Brownian motion (*i.e.* conditionally with respect to all the other random quantities):

$$\begin{aligned}
\mathrm{E}_W\left[(X^i_{t\wedge\tau^M_K})^{2p}\right] =& \mathrm{E}_W\left[(X^i_0)^{2p}\right] + 2p\int_0^{t\wedge\tau^M_K}\mathrm{E}_W\left[(X^i_s)^{2p-1}g_i(X_s)\right]\mathrm{d}s\\
&+ p(2p-1)\sum_{l=1}^m\int_0^{t\wedge\tau^M_K}\mathrm{E}_W\left[(X^i_s)^{2p-2}\big(\sigma_{il}(X_s)\big)^2\right]\mathrm{d}s\\
&+\int_0^{t\wedge\tau^M_K}\int_E\mathrm{E}_W\left[\Big(X^i_{s^-}+c_M(z,X^M_{s-})\mathbb{1}_{\{u\le\gamma(z,X^M_{s-})\}}\Big)^{2p}-(X^i_{s^-})^{2p}\right]N(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u)
\end{aligned}$$

Since $s\le t\wedge\tau^M_K$, we have $X_s = X_{s^M}$, and obviously $t\ge t\wedge\tau^M_K$, so we have

$$\begin{aligned}
\mathrm{E}_W\left[|X^i_{t\wedge\tau^M_K}|^{2p}\right] \le& \mathrm{E}_W\left[|X^i_0|^{2p}\right] + 2p\int_0^{t}\mathrm{E}_W\left[|X^i_{s\wedge\tau^M_K}|^{2p-1}|g_i(X_s)|\right]\mathrm{d}s\\
&+ p(2p-1)\sum_{l=1}^m\int_0^{t}\mathrm{E}_W\left[|X^i_{s\wedge\tau^M_K}|^{2p-2}\big|\sigma_{il}(X_s)\big|^2\right]\mathrm{d}s\\
&+\int_0^{t\wedge\tau^M_K}\int_E\mathrm{E}_W\left[\left|\Big(X^i_{s^-}+c_M(z,X^M_{s-})\mathbb{1}_{\{u\le\gamma(z,X^M_{s-})\}}\Big)^{2p}-(X^i_{s^-})^{2p}\right|\right]N(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u)
\end{aligned}$$

It follows (since $(a+b)^{2n}-a^{2n}\le(|a|+|b|)^{2n}-a^{2n}$, for all $a,b\in\mathbb{R}$ and using the inequality $x^uy^v\le x^{u+v}+y^{u+v}$)

$$\begin{aligned}
\mathrm{E}_W\left[|X_{t\wedge\tau^M_K}|^{2p}\right] \le& \mathrm{E}_W\left[|X_0|^{2p}\right] + 2p\int_0^{t}\mathrm{E}_W\left[|X_{s\wedge\tau^M_K}|^{2p}\right]\mathrm{d}s + 2pT\|g^{2p}\|_\infty\\
&+ p(2p-1)m\int_0^{t}\mathrm{E}_W\left[|X_{s\wedge\tau^M_K}|^{2p}\right]\mathrm{d}s + p(2p-1)mT\|\sigma^{2p}\|_\infty\\
&+\int_0^{t\wedge\tau^M_K}\int_E\mathrm{E}_W\left[\Big(|X_{s^-}|+\overline{c}(z)\mathbb{1}_{\{u\le\overline{\gamma}\}}\Big)^{2p}-|X_{s^-}|^{2p}\right]N(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u);
\end{aligned}$$

using now $|a^{2p}-b^{2p}|\le|a-b|(a+b)^{2p-1}$, we have $\Big(|X_{s^-}|+\overline{c}(z)\mathbb{1}_{\{u\le\overline{\gamma}\}}\Big)^{2p}-|X_{s^-}|^{2p}\le 2^{2p}\overline{c}(z)\mathbb{1}_{\{u\le\overline{\gamma}\}}|X_{s^-}|^{2p}$, so,

$$\begin{aligned}
\mathrm{E}_W\left[|X_{t\wedge\tau^M_K}|^{2p}\right] \le& C_T + A_p\int_0^{t}\mathrm{E}_W\left[|X_{s\wedge\tau^M_K}|^{2p}\right]\mathrm{d}s + B_p\int_0^{t\wedge\tau^M_K}\int_E\overline{c}(z)\mathbb{1}_{\{u\le\overline{\gamma}\}}\,\mathrm{E}_W\left[|X_{s^-}|^{2p}\right]N(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u)\\
\le& C_T + A_p\int_0^{t}\mathrm{E}_W\left[|X_{s\wedge\tau^M_K}|^{2p}\right]\mathrm{d}s + B_p\int_0^{t}\int_E\overline{c}(z)\mathbb{1}_{\{u\le\overline{\gamma}\}}\,\mathrm{E}_W\left[|X_{(s\wedge\tau^M_K)^-}|^{2p}\right]N(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u).
\end{aligned}$$

With

$$\theta^K_t \stackrel{\text{def}}{=} \sup_{0\le u\le t}\mathrm{E}_W\left[|X^M_{u\wedge\tau^M_K}|^{2p}\right],$$

we then have

$$\theta^K_t\le C_T + A_p\int_0^t\theta^K_s\,\mathrm{d}s + B_p\int_0^t\int_E\overline{c}(z)\mathbb{1}_{\{u\le\overline{\gamma}\}}\theta^K_{s^-}N(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u).$$

With

$$R_1 \stackrel{\text{def}}{=} \int_0^t\int_E\overline{c}(z)\mathbb{1}_{\{u\le\overline{\gamma}\}}\,\mathrm{d}\mu(z)\,\mathrm{d}u$$

we have

$$\begin{aligned}
\mathrm{E}\left[\theta^K_t\right] \le& C_T + A_p\,\mathrm{E}\left[\int_0^t\theta^K_s\,\mathrm{d}s\right] + B_p\,\mathrm{E}\left[\int_0^t\int_E\overline{c}(z)\mathbb{1}_{\{u\le\overline{\gamma}\}}\theta^K_{s^-}\,\mathrm{d}N(s,z,u)\right]\\
\le& C_T + A_p\int_0^t\mathrm{E}\left[\theta^K_s\right]\mathrm{d}s + B_p\int_0^t\int_E\overline{c}(z)\mathbb{1}_{\{u\le\overline{\gamma}\}}\,\mathrm{E}\left[\theta^K_{s^-}\right]\mathrm{d}s\mu(\mathrm{d}z)\,\mathrm{d}u\\
\le& C_T + (A_p+R_1)\int_0^t\mathrm{E}\left[\theta^K_s\right]\mathrm{d}s.
\end{aligned}$$

The Gronwall's lemma ends then the proof.

•

C Tangent flow

Let us recall the expression of the tangent flow Y_t^M:

$$Y_t^M = \mathrm{Id} + \sum_{l=1}^{m} \int_0^t \nabla\sigma_l(\bar{X}_s^M) Y_s^M \,\mathrm{d}W_s^l + \sum_{j=1}^{J_t^M} \nabla_x c_M(\bar{Z}_j, \bar{X}_{T_j-}^M) Y_{T_j-}^M + \int_0^t \nabla_x g(\bar{X}_s^M) Y_s^M \,\mathrm{d}s.$$

We have then defined the following process (with $\nabla_x c_j = \nabla_x c_M(\bar{Z}_j, \bar{X}_{T_j-}^M)$):

$$\hat{Y}_t^M \stackrel{\mathrm{def}}{=} \mathrm{Id} - \sum_{l=1}^{m} \int_0^t \hat{Y}_s^M \nabla\sigma_l(\bar{X}_s^M) \,\mathrm{d}W_s^l - \sum_{j=1}^{J_t^M} \hat{Y}_{T_j-}^M \nabla_x c_j (\mathrm{Id} + \nabla_x c_j)^{-1} + \int_0^t \hat{Y}_s^M \Big(\frac{1}{2} \sum_{l=1}^{m} \nabla\sigma_l(\bar{X}_s^M)^2 - \nabla_x g(\bar{X}_s^M) \Big) \,\mathrm{d}s,$$

and stated that:

Lemma C.1 *For all $t \geq 0$,*

$$Y_t^M \hat{Y}_t^M = \mathrm{Id}\,. \tag{C.121}$$

Proof :

Step 1

Let us consider (for the moment $m = 1$) the stochastic process sharing the same law as Y_t (that we will continue to denote Y_t, in order to simplify the notations) defined by

$$Y_t = \mathrm{Id} + \int_0^t \Sigma Y_s \,\mathrm{d}W_s + \sum_{j=1}^{J_t^M} C_j Y_{T_j-}^M + \int_0^t G Y_s \,\mathrm{d}s$$

with

$$C_j = \nabla_x c_M(Z_j, X_{T_j-}^M) \mathbb{1}_{\{U_j \leq \gamma(Z_j, X_{T_j-}^M)\}}$$

and let us set

$$\hat{Y}_t = \mathrm{Id} + \int_0^t \hat{Y}_s A \,\mathrm{d}W_s + \sum_{j=1}^{J_t^M} \hat{Y}_{T_j-}^M H_j + \int_0^t \hat{Y}_s B \,\mathrm{d}s$$

that is

$$Y_t^{i,j} = \delta_{i,j} + \sum_{h=1}^{d} \int_0^t \Sigma_{i,h} Y_s^{h,j} \,\mathrm{d}W_s + \sum_{k'=1}^{J_t^M} \sum_{h=1}^{d} C_{k'}^{i,h} Y_{T_{k'}-}^{h,j} + \sum_{h=1}^{d} \int_0^t G_{i,h} Y_s^{h,j} \,\mathrm{d}s$$

and

$$\hat{Y}_t^{i,j} = \delta_{i,j} + \sum_{h=1}^{d} \int_0^t \hat{Y}_s^{i,h} A_{h,j} \,\mathrm{d}W_s + \sum_{k'=1}^{J_t^M} \sum_{h=1}^{d} \hat{Y}_{T_{k'}-}^{i,h} H_{k'}^{h,j} + \sum_{h=1}^{d} \int_0^t \hat{Y}_s^{i,h} B_{h,j} \,\mathrm{d}s$$

We have $(\hat{Y}_t Y_t - \mathrm{Id})_{p,q} = \sum_{n=1}^{d} \hat{Y}_t^{p,n} Y_t^{n,q} - \delta_{p,q}$. Using Itô's formula it follows:

$$\begin{aligned}
&\hat{Y}_t^{p,n} Y_t^{n,q} - \delta_{p,q,n} \\
&\quad = \int_0^t Y_s^{n,q} \sum_{h=1}^{d} \hat{Y}_s^{p,h} A_{h,n} \,\mathrm{d}W_s + \int_0^t \hat{Y}_s^{p,n} \sum_{h=1}^{d} \Sigma_{n,h} Y_s^{h,q} \,\mathrm{d}W_s \\
&\quad + \frac{1}{2} \int_0^t \left(\sum_{g=1}^{d} \hat{Y}_s^{p,g} A_{g,n} \right) \left(\sum_{h=1}^{d} \Sigma_{n,h} Y_s^{h,q} \right) \\
&\quad + \int_0^t Y_s^{n,q} \sum_{h=1}^{d} \hat{Y}_s^{p,h} B_{h,n} \,\mathrm{d}s + \int_0^t \hat{Y}_s^{p,n} \sum_{h=1}^{d} G_{n,h} Y_s^{h,q} \,\mathrm{d}s \\
&\quad + \int_0^t \int_E \Big(Y_{s-}^{n,q} + \sum_{h=1}^{d} C_{s-}^{n,h}(z,u) Y_{s-}^{h,q} \Big) \Big(\hat{Y}_{s-}^{p,n} \sum_{h=1}^{d} \hat{Y}_{s-}^{p,h} H_{s-}^{h,n}(z,u) \Big) - Y_{s-}^{n,q} \hat{Y}_{s-}^{p,n} N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)
\end{aligned}$$

The integrated term with respect to $\mathrm{d}W_s$ of $(\hat{Y}_t Y_t - \mathrm{Id})_{p,q}$ has the following form:

$$\begin{aligned}\sum_{n=1}^{d}\sum_{h=1}^{d} Y_s^{n,q}\hat{Y}_s^{p,h}A_{h,n} + \sum_{n=1}^{d}\sum_{h=1}^{d}\hat{Y}_s^{p,n}Y_s^{h,q}\Sigma_{n,h} &= \sum_{n=1}^{d}\sum_{h=1}^{d} Y_s^{h,q}\hat{Y}_s^{p,n}A_{n,h} + \sum_{n=1}^{d}\sum_{h=1}^{d}\hat{Y}_s^{p,n}Y_s^{h,q}\Sigma_{n,h} \\ &= \sum_{n=1}^{d}\sum_{h=1}^{d} Y_s^{h,q}\hat{Y}_s^{p,n}(A_{n,h}+\Sigma_{n,h}).\end{aligned}$$

This term is null for

$$A = -\Sigma$$

which we will suppose in the following of this proof.

For the third term

$$\begin{aligned}\sum_{n=1}^{d}\left(\sum_{g=1}^{d}\hat{Y}_s^{p,g}A_{g,n}\right)\left(\sum_{h=1}^{d}\Sigma_{n,h}Y_s^{h,q}\right) &= \sum_{n=1}^{d}\sum_{g=1}^{d}\left(\sum_{h=1}^{d}\hat{Y}_s^{p,g}A_{g,n}\Sigma_{n,h}Y_s^{h,q}\right) \\ &= \sum_{g=1}^{d}\sum_{h=1}^{d}\left(\sum_{n=1}^{d}A_{g,n}\Sigma_{n,h}\right)\hat{Y}_s^{p,g}Y_s^{h,q} \\ &= \sum_{n=1}^{d}\sum_{h=1}^{d}\left(\sum_{g=1}^{d}A_{n,g}\Sigma_{g,h}\right)\hat{Y}_s^{p,n}Y_s^{h,q} \\ &= \sum_{n=1}^{d}\sum_{h=1}^{d}(A\Sigma)_{n,h}\,\hat{Y}_s^{p,n}Y_s^{h,q}\end{aligned}$$

The integrated term with respect to $\mathrm{d}s$ of $(\hat{Y}_t Y_t - \mathrm{Id})_{p,q}$ has the following form:

$$\begin{aligned}&\frac{1}{2}\sum_{n=1}^{d}\sum_{h=1}^{d}(A\Sigma)_{n,h}\,\hat{Y}_s^{p,n}Y_s^{h,q} + \sum_{n=1}^{d}\sum_{h=1}^{d}Y_s^{n,q}\hat{Y}_s^{p,h}B_{h,n} + \sum_{n=1}^{d}\sum_{h=1}^{d}\hat{Y}_s^{p,n}G_{n,h}Y_s^{h,q} \\ &\quad = \frac{1}{2}\sum_{n=1}^{d}\sum_{h=1}^{d}(A\Sigma)_{n,h}\,\hat{Y}_s^{p,n}Y_s^{h,q} + \sum_{n=1}^{d}\sum_{h=1}^{d}Y_s^{h,q}\hat{Y}_s^{p,n}B_{n,h} + \sum_{n=1}^{d}\sum_{h=1}^{d}\hat{Y}_s^{p,n}G_{n,h}Y_s^{h,q} \\ &\quad = \sum_{n=1}^{d}\sum_{h=1}^{d}\left(\frac{1}{2}(A\Sigma)_{n,h} + B_{n,h} + G_{n,h}\right)\hat{Y}_s^{p,n}Y_s^{h,q} \\ &\quad = \sum_{n=1}^{d}\sum_{h=1}^{d}\left(\frac{1}{2}A\Sigma + B + G\right)_{n,h}\hat{Y}_s^{p,n}Y_s^{h,q}.\end{aligned}$$

This term is null if (with $A = -\Sigma$)

$$B = \frac{1}{2}\Sigma^2 - G.$$

Step 2

Multidimensional Brownian case:

$$Y_t = \mathrm{Id} + \int_0^t \Sigma_1 Y_s\,\mathrm{d}W_s^1 + \cdots + \int_0^t \Sigma_m Y_s\,\mathrm{d}W_s^n + \sum_{j=1}^{J_t^M} C_j Y_{T_j-}^M + \int_0^t GY_s\,\mathrm{d}s$$

and let us set

$$\hat{Y}_t = \mathrm{Id} + \int_0^t \hat{Y}_s A_1\,\mathrm{d}W_s^1 + \cdots + \int_0^t \hat{Y}_s A_m\,\mathrm{d}W_s^n + \sum_{j=1}^{J_t^M}\hat{Y}_{T_j-}^M H_j + \int_0^t \hat{Y}_s B\,\mathrm{d}s$$

that is

$$Y_t^{i,j} = \delta_{i,j} + \sum_{l=1}^{m}\sum_{h=1}^{d}\int_0^t \Sigma_{i,h}^l Y_s^{h,j}\,\mathrm{d}W_s^l + \sum_{k'=1}^{J_t^M}\sum_{h=1}^{d} C_{k'}^{i,h}Y_{T_{k'}-}^{h,j} + \sum_{h=1}^{d}\int_0^t G_{i,h}Y_s^{h,j}\,\mathrm{d}s$$

and

$$\hat{Y}_t^{i,j} = \delta_{i,j} + \sum_{l=1}^{m}\sum_{h=1}^{d}\int_0^t \hat{Y}_s^{i,h} A_{h,j}^l \,\mathrm{d}W_s^l + \sum_{k'=1}^{J_t^M}\sum_{h=1}^{d} \hat{Y}_{T_{k'}-}^{i,h} H_{k'}^{h,j} + \sum_{h=1}^{d}\int_0^t \hat{Y}_s^{i,h} B_{h,j}\,\mathrm{d}s$$

and

$$\begin{aligned}
\hat{Y}_t^{p,n} Y_t^{n,q} - \delta_{p,q,n} = &\int_0^t Y_s^{n,q} \sum_{l=1}^{m}\sum_{h=1}^{d} \hat{Y}_s^{p,h} A_{h,n}^l \,\mathrm{d}W_s^l + \int_0^t \hat{Y}_s^{p,n} \sum_{l=1}^{m}\sum_{h=1}^{d} \Sigma_{n,h}^l Y_s^{h,q}\,\mathrm{d}W_s^l \\
&+ \sum_{l=1}^{m} \frac{1}{2}\int_0^t \left(\sum_{g=1}^{d} \hat{Y}_s^{p,g} A_{g,n}^l\right)\left(\sum_{h=1}^{d} \Sigma_{n,h}^l Y_s^{h,q}\right) \\
&+ \int_0^t Y_s^{n,q} \sum_{h=1}^{d} \hat{Y}_s^{p,h} B_{h,n}\,\mathrm{d}s + \int_0^t \hat{Y}_s^{p,n} \sum_{h=1}^{d} G_{n,h} Y_s^{h,q}\,\mathrm{d}s \\
&+ \int_0^t \int_E \left(Y_{s-}^{n,q} + \sum_{h=1}^{d} C_{s-}^{n,h}(z,u) Y_{s-}^{h,q}\right)\left(\hat{Y}_{s-}^{p,n} \sum_{h=1}^{d} \hat{Y}_{s-}^{p,h} H_{s-}^{h,n}(z,u)\right) - Y_{s-}^{n,q}\hat{Y}_{s-}^{p,n} N(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u)
\end{aligned}$$

The same computations, by a straight superposition, give us then, for all $l \in [\![0,m]\!]$,

$$A_i = -\Sigma_i$$

and

$$B = \frac{1}{2}\left(\sum_{i=1} \Sigma_i^2\right) - G.$$

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