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SHIFTED CONVOLUTION OF CUSP-FORMS WITH θ -SERIES

GUANGSHI LÜ, JIE WU & WENGUANG ZHAI

ABSTRACT. In this paper we apply simple approaches to improve a recent result due to Luo, concerning a shifted convolution sum involving the Fourier coefficients of cusp forms with those of theta series. In order to explore what should be the best possible bound, a mean square result of this shifted convolution sum is also established.

1. INTRODUCTION

Suppose $\lambda_1(n)$ and $\lambda_2(n)$ are two arithmetic functions, $b \geq 0$ is an integer. It is a classical and important problem in analytic number theory to study the shifted convolution sum

$$\sum_{n \leq x} \lambda_1(n) \lambda_2(n+b).$$

There are a large number of papers in this direction. For example, when $\lambda_1(n) = \lambda_2(n) = \tau(n)$, the Dirichlet divisor function, this problem is the so-called additive divisor problem [1, 6, 8, 9, 12, 18, 19]. For other interesting cases, see [14, 15, 7].

Recently Luo [16] investigated a mixed shifted convolution sum

$$(1.1) \quad \sum_{n \geq 1} \lambda_f(n+b) r_\ell(n) \phi(n),$$

where $\lambda_f(n)$ is the n th normalized Hecke eigenvalue of a holomorphic cusp form of weight k and level N , $b \geq 0$ is a fixed integer,

$$r_\ell(n) := |\{(n_1, \dots, n_\ell) \in \mathbb{Z}^\ell : n_1^2 + \dots + n_\ell^2 = n\}|$$

and $\phi(t)$ is a smooth function with support in $[x/2, 5x/2]$, satisfying $\phi^{(j)}(t) \ll_j (x/J)^{-j}$ for all real numbers $t \in \mathbb{R}$, all integers $j \geq 0$ and $1 \leq J \leq x^\beta$ with $\beta < 1$. In order to estimate the sum (1.1), he first established a Voronoi formula for $r_\ell(n)$ and then combined this formula with the upper bound for the Salié sum to derive

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the following result [16, Theorem] : For any $\varepsilon > 0$, the inequality

$$(1.2) \quad \sum_{n \geq 1} \lambda_f(n+b)r_\ell(n)\phi(n) \ll \begin{cases} x^{\ell/2-(\ell-1)/4+\varepsilon} J^{g_\ell} & \text{if } 4 \nmid \ell \text{ or } b \neq 0 \\ x^{\ell/2-(\ell-1)/4+\varepsilon} (x^{1/4+\varepsilon} + J^{g_\ell}) & \text{if } 4 \mid \ell \text{ and } b = 0 \end{cases}$$

holds for all $x \geq 2$, where g_ℓ is the smallest integer such that $g_\ell \geq (\ell+1)/2$ [†] and the implied constant depends on f , b , ℓ and ε . With a suitable choice of $\phi(t)$ and J , the inequality (1.2) implies immediately the following result [16, Corollary] : For $\ell \geq 2$, $k \geq \ell/2 + 3$ and $\varepsilon > 0$, the inequality

$$(1.3) \quad \mathcal{S}_{f,b,\ell}(x) := \sum_{n \leq x} \lambda_f(n+b)r_\ell(n) \ll_{f,b,\ell,\varepsilon} x^{\ell/2-\vartheta_\ell+\varepsilon}$$

holds for all $x \geq 2$, where $\vartheta_\ell := (\ell-1)/(4g_\ell+4)$. In particular

$$\vartheta_2 = \frac{1}{12}, \quad \vartheta_3 = \frac{1}{6}, \quad \vartheta_4 = \frac{3}{16}, \quad \vartheta_5 = \frac{1}{4}, \quad \vartheta_6 = \frac{1}{4}, \quad \vartheta_\ell \leq \frac{\ell-1}{2\ell+6} < \frac{1}{2} \quad (\ell \geq 7).$$

The first aim of this paper is to propose a better bound for $\mathcal{S}_{f,b,\ell}(x)$.

Theorem 1. *Let f be a cusp form of weight k and level N and let $\ell \geq 2$ be an integer. For any $\varepsilon > 0$, we have*

$$(1.4) \quad \mathcal{S}_{f,b,\ell}(x) := \sum_{n \leq x} \lambda_f(n+b)r_\ell(n) \ll_{f,\ell,\varepsilon} x^{\ell/2-\vartheta_\ell+\varepsilon}$$

uniformly for $x \geq 2$ and $0 \leq b \leq x$, where

$$\vartheta_3 = \frac{1}{4}, \quad \vartheta_4 = \frac{1}{2}, \quad \vartheta_5 = \frac{1}{2}, \quad \vartheta_\ell = \frac{2}{3} \quad (\ell \geq 6).$$

In addition, if we assume $N = 1$, then (1.4) holds for $\ell = 2$ with $\vartheta_2 = \frac{1}{6}$.

Remark 1. Theorem 1 improves Luo's (1.3) in three directions; it enlarges the exponent ϑ_ℓ , relaxes the restricted condition $k \geq \ell/2 + 3$ and removes the dependence of b . Our method is completely different from that of [16]. Our new idea is to explore the regularity of $r_\ell(n)$ by the circle method in analytic number theory and Siegel's mass formula. Thanking to these classic tools of analytic number theory, we can show that the influence of $r_\ell(n)$ to the bound [22, (1.10) and Theorem 2]

$$(1.5) \quad \sum_{n \leq x} \lambda_f(n) \ll_f x^{1/3} (\log x)^{2/(\sqrt{\pi}\Gamma(5/2))-1}$$

is rather little. More precisely by using the circle method we shall give a very simple proof of $\vartheta_\ell = \frac{1}{2}$ for $\ell \geq 3$. Further when $\ell \geq 6$, with the help of Siegel's mass formula we shall prove a better exponent $\theta_\ell = \frac{2}{3}$, which means that our bound for $\mathcal{S}_{f,b,\ell}(x)$ is of the same quality as (1.5). Finally the case of $\ell = 2$ requires a more delicate consideration (see the end of Section 3).

[†]It seems that Luo's proof only could show that g_ℓ is the smallest integer such that $g_\ell > (\ell+1)/2$. This means that $\vartheta_3 = \frac{1}{8}$ and $\vartheta_5 = \frac{1}{5}$ instead of $\vartheta_3 = \frac{1}{6}$ and $\vartheta_5 = \frac{1}{4}$.

Remark 2. In fact our method allows us to consider a more general case. Let $\ell \geq 2$, $\mathbf{y} := (y_1, \dots, y_\ell) \in \mathbb{Z}^\ell$ and $\mathbf{A} = (a_{ij})$ be an integral matrix such that $a_{ii} \equiv 0 \pmod{2}$ for $1 \leq i \leq \ell$. The positive definite quadratic form $Q(\mathbf{y})$ is defined by $Q(\mathbf{y}) = \frac{1}{2} \mathbf{y}^t \mathbf{A} \mathbf{y}$. For each $n \geq 1$, define

$$r(n, Q) := |\{\mathbf{y} \in \mathbb{Z}^\ell : Q(\mathbf{y}) = n\}|.$$

Thus $r_\ell(n) = r(n, Q_0)$ with $Q_0 = y_1^2 + \dots + y_\ell^2$. Similar to $\mathcal{S}_{f,b,\ell}(x)$, we define

$$(1.6) \quad \mathcal{S}_{f,b,Q}(x) := \sum_{n \leq x} \lambda_f(n+b) r(n, Q).$$

Define $\mathcal{F}$ to be a class of cusp forms, which consists of holomorphic cusp forms with respect to any finite volume discrete subgroup (such that ∞ is a singular cusp of width 1), any positive real weight and any multiplier systems, as well as Maass cusp forms of any weight and any level [11, 2, 4]. Since the estimate (2.1) below holds for all $f \in \mathcal{F}$ ([10, Theorem 5.3] and [11, Theorem 8.1]), our proof of Theorem 1 implies that

$$\mathcal{S}_{f,b,Q}(x) \ll_{f,Q,\varepsilon} x^{\ell/2-1/2+\varepsilon}$$

holds uniformly for $1 \leq b \leq x$, provided $\ell \geq 5$.

A natural question is what should be the best bound for $\mathcal{S}_{f,b,\ell}(x)$. It is well-known that [3, Theorem 2])

$$\sum_{n \leq x} \lambda_f(n) = \Omega_{\pm} \left(x^{1/4} \exp \left\{ \frac{D(\log \log x)^{1/4}}{(\log \log \log x)^{3/4}} \right\} \right)$$

where $D > 0$ a constant. A slight weaker Ω -result

$$\sum_{n \leq x} \lambda_f(n) = \Omega(x^{1/4})$$

is supported by the asymptotic formula (see [21])

$$\int_1^X \left| \sum_{n \leq x} \lambda_f(n) \right|^2 dx = C_f X^{3/2} + O(X(\log X)^2),$$

for $X \geq 2$, where C_f is a positive constant. On the other hand, we have $r_\ell(n) \asymp n^{\ell/2-1}$ ($\ell \geq 3$). From the above facts it seems reasonable to propose the following conjecture.

Conjecture. *Let f be a cusp form of weight k and level N . Let $b \geq 0$ and $\ell \geq 3$ be two integers. For any $\varepsilon > 0$, we have*

$$(1.7) \quad \mathcal{S}_{f,b,\ell}(x) \ll_{f,b,\ell,\varepsilon} x^{\ell/2-3/4+\varepsilon}$$

for $x \rightarrow \infty$.

It seems rather difficult to establish (1.7). However we can prove that the bound (1.7) is true on average for $\ell \geq 5$ (see (1.8) and (1.9) below). For this, we shall study the square mean of $\mathcal{S}_{f,b,\ell}(x)$. At this time, our method also allows us to deal with the more general case $\mathcal{S}_{f,b,Q}(x)$.

Our second aim is to prove the following asymptotic formula.

Theorem 2. *Let f be a cusp form of weight k and level N . Let $Q(\mathbf{y})$ be a positive definite quadratic form defined as above. If $\ell \geq 6$, then we have*

$$(1.8) \quad \begin{aligned} \int_1^X |\mathcal{S}_{f,b,Q}(x)|^2 dx &= C_{f,b,Q} \int_1^X x^{\ell-2} (x+b)^{1/2} dx + O_{f,Q}(X^{\ell-7/12} (\log X)^{1/2}) \\ &= \frac{C_{f,b,Q}}{\ell-1/2} X^{\ell-1/2} + O_{f,Q}(bX^{\ell-3/2} + X^{\ell-7/12} (\log X)^{1/2}) \end{aligned}$$

uniformly for $X \geq 2$ and $0 \leq b \leq X$, where the constant $C_{f,b,Q}$ is defined by (4.12) below, and the implied constants depend on f and Q .

Remark 3. In Section 3, we shall prove also that the inequality

$$(1.9) \quad \int_1^X |\mathcal{S}_{f,b,5}(x)|^2 dx \ll_{f,\varepsilon} X^{\ell-1/2+\varepsilon}$$

holds uniformly for $X \geq 1$ and $0 \leq b \leq X$.

2. PRELIMINARY LEMMAS

In order to prove our theorems, we need the following lemmas.

Lemma 2.1. *Let f be a cusp form of weight k and level N .*

(a) *The estimate*

$$(2.1) \quad \sum_{n \leq x} \lambda_f(n) e(\alpha n) \ll_f x^{1/2} \log x$$

holds uniformly for $\alpha \in \mathbb{R}$, where $e(u) := e^{2\pi i u}$.

(b) *For any $\varepsilon > 0$, we have*

$$(2.2) \quad \sum_{\substack{n \leq x \\ n \equiv a \pmod{q}}} \lambda_f(n) \ll_{f,\varepsilon} x^{1/2-1/6+\varepsilon}$$

uniformly for $(q, aN) = 1$ and $q \leq x^{2/3}$.

Proof. See for example, [10, Theorem 5.3] and [20]. □

Lemma 2.2. *Let f be a cusp form of weight k and level N , $(h, q) = 1$ and*

$$(2.3) \quad A(x, h/q) := \sum'_{n \leq x} \lambda_f(n) e_q(hn),$$

where $e_q(m) := e(m/q)$ and $\sum'_{n \leq x}$ means that if x is an integer, the term $n = x$ should be halved. Then for any $\varepsilon > 0$ we have

$$A(x, h/q) = \frac{q^{1/2} x^{1/4}}{\sqrt{2}\pi} \sum_{n \leq M} \frac{\lambda_f(n)}{n^{3/4}} e_q(-\bar{h}n) \cos\left(\frac{4\pi\sqrt{nx}}{q} - \frac{\pi}{4}\right) + O_{f,\varepsilon}\left(\frac{qx^{1/2+\varepsilon}}{M^{1/2}}\right)$$

uniformly for $1 \leq q \leq x$ and $1 \leq M \ll x$, where $\bar{h}$ satisfies $h\bar{h} \equiv 1 \pmod{q}$. In particular for any $\varepsilon > 0$ we have

$$(2.4) \quad A(x, h/q) \ll_{f, \varepsilon} q^{2/3} x^{1/3+\varepsilon}$$

uniformly for $x \geq 1$ and $q \geq 1$. Here the implied constants depend on f and ε only.

Proof. The first assertion is [13, Theorem 1.1]. The bound (2.4) is trivial if $q \geq x$. When $1 \leq q \leq x$, it follows from the first assertion with $M = (q^2 x)^{1/3}$. See also, [17, (2.7)]. $\square$

Lemma 2.3. *Let $A(x, h/q)$ be defined as in (2.3).*

(i) *We have*

$$(2.5) \quad A_1(x, h/q) := \int_1^x A(u, h/q) du \ll_f q^{3/2} x^{3/4}$$

uniformly for $x \geq 1$ and $1 \leq q \leq x^{1/2}$.

(ii) *For any $\varepsilon > 0$ we have*

$$\int_1^X |A(x, h/q)|^2 dx = \frac{1}{(4k+2)\pi^2} \sum_{n=1}^{\infty} \frac{|\lambda_f(n)|^2}{n^{3/2}} q X^{3/2} + O_{f, \varepsilon}(q^{3/2} X^{5/4+\varepsilon} + q^2 X^{1+\varepsilon})$$

uniformly for $X \geq 1$ and $q \geq 1$.

Proof. They are (1.6.12) and (1.5.23) of [13], respectively. $\square$

Lemma 2.4. *Let $\ell \geq 2$, $\mathbf{y} := (y_1, \dots, y_\ell) \in \mathbb{Z}^\ell$ and $\mathbf{A} = (a_{ij})$ be an integral matrix such that $a_{ii} \equiv 0 \pmod{2}$ for $1 \leq i \leq \ell$. The positive definite quadratic form $Q(\mathbf{y})$ is defined by $Q(\mathbf{y}) = \frac{1}{2} \mathbf{y}^t \mathbf{A} \mathbf{y}$. For each $n \geq 1$, define*

$$r(n, Q) := |\{\mathbf{y} \in \mathbb{Z}^\ell : Q(\mathbf{y}) = n\}|.$$

Then for $\ell \geq 4$ we have

$$(2.6) \quad r(n, Q) = \sigma_Q n^{\ell/2-1} \sum_{q=1}^{\infty} \sum_{h=1}^q{}^* S\left(\frac{hQ}{q}\right) \frac{e_q(-hn)}{q^\ell} + O(n^{\ell/4-\delta_\ell+\varepsilon}),$$

where

$$S(Q) := \sum_{0 \leq y_1, \dots, y_\ell \leq q-1} e(Q(\mathbf{y})), \quad \sigma_Q := \frac{(2\pi)^{\ell/2}}{\Gamma(\ell/2) \sqrt{|\mathbf{A}|}}, \quad \delta_\ell := \begin{cases} \frac{1}{4} & \text{if } \ell \text{ is odd,} \\ \frac{1}{2} & \text{if } \ell \text{ is even,} \end{cases}$$

and $\sum^*$ means the sum is over $1 \leq h \leq q$ with $(h, q) = 1$. Furthermore we have

$$(2.7) \quad S(hQ/q) \ll q^{\ell/2} \quad ((h, q) = 1),$$

Proof. See for example, Theorem 11.2 of [10]. $\square$

3. PROOF OF THEOREM 1

3.1. Case of $\ell \geq 3$.

Let

$$F(\alpha) := \sum_{n \leq x} \lambda_f(n+b) e(-\alpha n) \quad \text{and} \quad S(\alpha) := \sum_{|m| \leq x^{1/2}} e(\alpha m^2).$$

Then

$$\begin{aligned} \int_0^1 F(\alpha) S(\alpha)^\ell d\alpha &= \sum_{n \leq x} \lambda_f(n+b) \sum_{|m_1| \leq x^{1/2}} \cdots \sum_{|m_\ell| \leq x^{1/2}} \int_0^1 e((m_1^2 + \cdots + m_\ell^2 - n)\alpha) d\alpha \\ &= \sum_{n \leq x} \lambda_f(n+b) \sum_{|m_1| \leq x^{1/2}} \cdots \sum_{\substack{|m_\ell| \leq x^{1/2} \\ m_1^2 + \cdots + m_\ell^2 = n}} 1 \\ &= \mathcal{S}_{f,b,\ell}(x). \end{aligned}$$

Firstly Lemma 2.1 allows us to deduce

$$F(\alpha) = e(\alpha b) \sum_{b < n \leq x+b} \lambda_f(n) e(-\alpha n) \ll x^{1/2} \log x$$

uniformly for $x \geq 2$ and $0 \leq b \leq x$. On the other hand, it is easy to see that

$$\begin{aligned} \int_0^1 |S(\alpha)|^2 d\alpha &= \sum_{\substack{|m| \leq x^{1/2} \\ m^2 = n^2}} \sum_{|n| \leq x^{1/2}} 1 \ll x^{1/2}, \\ \int_0^1 |S(\alpha)|^{2d} d\alpha &= \sum_{\substack{|m_1| \leq x^{1/2}, \dots, |m_d| \leq x^{1/2} \\ m_1^2 + \cdots + m_d^2 = n_1^2 + \cdots + n_d^2}} \sum_{\substack{|n_1| \leq x^{1/2}, \dots, |n_d| \leq x^{1/2}}} 1 \leq \sum_{n \leq dx} r_d(n)^2 \ll x^{d-1} \end{aligned}$$

for $d \geq 2$. From the above estimates and the Cauchy inequality we get

$$\mathcal{S}_{f,b,3}(x) \ll x^{1/2} (\log x) \left(\int_0^1 |S(\alpha)|^2 d\alpha \int_0^1 |S(\alpha)|^4 d\alpha \right)^{1/2} \ll x^{3/2-1/4} \log x$$

and for $\ell \geq 4$

$$\begin{aligned} \mathcal{S}_{f,b,\ell}(x) &\ll x^{1/2} (\log x) \left(\int_0^1 |S(\alpha)|^4 d\alpha \int_0^1 |S(\alpha)|^{2(\ell-2)} d\alpha \right)^{1/2} \\ &\ll x^{\ell/2-1/2} \log x, \end{aligned}$$

namely we can take $\vartheta_3 = \frac{1}{4}$ and $\vartheta_\ell = \frac{1}{2}$ for $\ell \geq 4$.

3.2. Case of $\ell \geq 6$.

When $\ell \geq 6$, we can apply Lemmas 2.2 and 2.4 to improve the exponent $\vartheta_\ell = \frac{1}{2}$. From Lemma 2.4 with $Q_0(\mathbf{y}) = y_1^2 + \cdots + y_\ell^2$ we get

$$(3.1) \quad r_\ell(n) = \sigma_{Q_0} n^{\ell/2-1} \sum_{q=1}^{\infty} \sum_{h=1}^q S\left(\frac{hQ_0}{q}\right) \frac{e_q(-hn)}{q^\ell} + O_\varepsilon(n^{\ell/4-\delta_\ell+\varepsilon}).$$

In view of (2.4) in Lemma 2.2 and the bound (2.7) we easily deduce, via a simple integration by parts, that

$$\begin{aligned} \mathcal{S}_{f,b,\ell}(x) &\ll \sum_{q \geq 1} \frac{1}{q^{\ell/2}} \sum_{\substack{1 \leq h \leq q \\ (h,q)=1}} \left| \sum_{1+b \leq n \leq x+b} n^{\ell/2-1} \lambda_f(n) e_q(-hn) \right| + x^{\ell/4-\delta_\ell+1+\varepsilon} \\ &\ll_{f,\varepsilon} x^{\ell/2-2/3+\varepsilon} \sum_{q \geq 1} \frac{1}{q^{\ell/2-5/3}} + x^{\ell/4-\delta_\ell+1+\varepsilon} \\ &\ll_{f,\varepsilon} x^{\ell/2-2/3+\varepsilon} \quad (\text{recall } \ell \geq 6). \end{aligned}$$

3.3. Case of $\ell = 2$.

We need the following Lemmas.

Lemma 3.1. *Let $m \geq 2$ be a positive integer. There is an arithmetic function $h_m(n)$ such that*

$$(3.2) \quad h_m(n) = 0 \quad \text{if } \exists p \text{ such that } p \mid n \text{ and } p \nmid m,$$

$$(3.3) \quad |h_m(n)| \leq \tau(m)\tau_4(n) \quad \text{if } n \mid m^\infty,$$

$$(3.4) \quad \lambda_f(mn) = \sum_{d \mid n} h_m(d) \lambda_f(n/d),$$

where $\tau_k(n)$ denotes the number of solutions of $n = n_1 \cdots n_k$ with positive numbers $n_1, \dots, n_k$, and $\tau(n) := \tau_2(n)$.

Proof. As usual, we denote by $v_p(n)$ the p -adic valuation of n . By using the formula

$$\lambda_f(mn) = \prod_{p \mid d} \lambda_f(p^{v_p(n)+v_p(m)}) \prod_{p \nmid d} \lambda_f(p^{v_p(n)}),$$

we can write, for $\Re s > 1$,

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\lambda_f(mn)}{n^s} &= \prod_{p \mid m} \sum_{\nu=0}^{\infty} \frac{\lambda_f(p^{\nu+v_p(m)})}{p^{\nu s}} \prod_{p \nmid m} \sum_{\nu=0}^{\infty} \frac{\lambda_f(p^\nu)}{p^{\nu s}} \\ &= L(s, f) \prod_{p \mid m} \left(\sum_{\nu=0}^{\infty} \frac{\lambda_f(p^{\nu+v_p(m)})}{p^{\nu s}} \right) \left(\sum_{\nu=0}^{\infty} \frac{\lambda_f(p^\nu)}{p^{\nu s}} \right)^{-1}. \end{aligned}$$

This implies that there is an arithmetic function $h_m(n)$ such that (3.2) and (3.4) hold.

Next we prove (3.3). If $p \mid m$, it is easy to see that

$$h_m(p^\nu) = \begin{cases} \lambda_f(p^{v_p(m)}) & \text{if } \nu = 0, \\ \lambda_f(p^{v_p(m)+1}) - \lambda_f(p^{v_p(m)}) \lambda_f(p) & \text{if } \nu = 1, \\ \lambda_f(p^{v_p(m)+\nu}) - \lambda_f(p^{v_p(m)+\nu-1}) \lambda_f(p) + \lambda_f(p^{v_p(m)+\nu-2}) & \text{if } \nu \geq 2. \end{cases}$$

In view of Degline's bound $|\lambda_f(n)| \leq \tau(n)$, for any $n \mid m^\infty$ we have

$$\begin{aligned} |h_m(n)| &= \prod_{p^\nu \parallel n} |h_m(p^\nu)| \leq \prod_{p^\nu \parallel n} 4(v_p(m) + \nu) \\ &\leq \prod_{p^\nu \parallel n} (v_p(m) + 1) \binom{\nu + 3}{3} \leq \tau(m) \tau_4(n). \end{aligned}$$

This completes the proof. $\square$

Lemma 3.2. *Let f be a cusp form of weight k and level 1. Then the estimate*

$$(3.5) \quad S_f(x; a, q) := \sum_{\substack{n \leq x \\ n \equiv a \pmod{q}}} \lambda_f(n) \ll_f x^{1/3+\varepsilon}$$

holds uniformly for $x \geq 1$ and $q \geq a \geq 1$.

Proof. Note that (2.2) and (3.5) is trivial for $q > x^{2/3}$. Now suppose $q \leq x^{2/3}$. Writting $m := (a, q)$, $a = ma_1$ and $q = mq_1$ such that $(a_1, q_1) = 1$ and using (3.4) of Lemma 3.1, we have

$$\begin{aligned} S_f(x; a, q) &= \sum_{\substack{n \leq x/m \\ n \equiv a_1 \pmod{q_1}}} \lambda_f(mn) \\ &= \sum_{\substack{n_1 n_2 \leq x/m \\ n_1 n_2 \equiv a_1 \pmod{q_1}}} h_m(n_1) \lambda_f(n_2) \\ &= \sum_{n_1 \leq x/m} h_m(n_1) \sum_{\substack{n_2 \leq x/mn_1 \\ n_2 \equiv a_1 \bar{n}_1 \pmod{q_1}}} \lambda_f(n_2), \end{aligned}$$

where $n_1 \bar{n}_1 \equiv 1 \pmod{q_1}$. Noting that $(a_1 \bar{n}_1, q_1) = 1$, we can apply (2.2) with $N = 1$ and (3.2)-(3.3) of Lemma 3.1 to write

$$\begin{aligned} (3.6) \quad S_f(x; a, q) &\ll \tau(m) \left(\frac{x}{m}\right)^{1/3+\varepsilon} \sum_{n_1 \mid m^\infty} \frac{\tau_4(n_1)}{n_1^{1/3+\varepsilon}} \\ &\ll \tau(m) \left(\frac{x}{m}\right)^{1/3+\varepsilon} \prod_{p \mid m} \left(1 - \frac{1}{p^{1/3}}\right)^{-4} \\ &\ll x^{1/3+\varepsilon}. \end{aligned}$$

This completes the proof. $\square$

Now we prove Theorem 1 for $\ell = 2$. By the classical expression

$$r_2(n) = 4 \sum_{d \mid n} \chi(d)$$

($\chi(n)$ is the non trivial Dirichlet character modulo 4), we can write

$$(3.7) \quad \mathcal{S}_{f,b,2}(x) = 4S_1 + 4S_2 - 4S_3,$$

where

$$\begin{aligned} S_1 &:= \sum_{d \leq \sqrt{x}} \sum_{dm \leq x} \chi(d) \lambda_f(dm + b), \\ S_2 &:= \sum_{m \leq \sqrt{x}} \sum_{dm \leq x} \chi(d) \lambda_f(dm + b), \\ S_3 &:= \sum_{d \leq \sqrt{x}} \sum_{m \leq \sqrt{x}} \chi(d) \lambda_f(dm + b). \end{aligned}$$

By Lemma 3.2, we have

$$\begin{aligned} (3.8) \quad S_1 &= \sum_{d \leq \sqrt{x}} \chi(d) \sum_{\substack{n \leq x+b \\ n \equiv b \pmod{d}}} \lambda_f(n) \\ &\ll \sum_{d \leq \sqrt{x}} (x+b)^{1/3+\varepsilon} \ll x^{5/6+\varepsilon}. \end{aligned}$$

Note that $\chi(d) = 1$ if $d \equiv 1 \pmod{4}$, $\chi(d) = -1$ if $d \equiv 3 \pmod{4}$ and $\chi(d) = 0$ if $2 \mid d$, we have by Lemma 3.2 that

$$\begin{aligned} (3.9) \quad S_2 &= \sum_{m \leq \sqrt{x}} \left(\sum_{(4d+1)m \leq x} \lambda_f((4d+1)m + b) - \sum_{(4d+3)m \leq x} \lambda_f((4d+3)m + b) \right) \\ &= \sum_{m \leq \sqrt{x}} \left(\sum_{\substack{n \leq x+b \\ n \equiv m+b \pmod{4m}}} \lambda_f(n) - \sum_{\substack{n \leq x+b \\ n \equiv 3m+b \pmod{4m}}} \lambda_f(n) \right) \\ &\ll \sum_{m \leq \sqrt{x}} (x+b)^{1/3+\varepsilon} \ll x^{5/6+\varepsilon}. \end{aligned}$$

For S_3 we have by Lemma 3.2 again that

$$\begin{aligned} (3.10) \quad S_3 &= \sum_{d \leq \sqrt{x}} \chi(d) \sum_{\substack{n \leq d\sqrt{x}+b \\ n \equiv b \pmod{d}}} \lambda_f(n) \\ &\ll \sum_{d \leq \sqrt{x}} (d\sqrt{x} + b)^{1/3+\varepsilon} \ll x^{5/6+\varepsilon}. \end{aligned}$$

Now Theorem 1 for the case $\ell = 2$ follows from (3.7)-(3.10).

4. PROOF OF THEOREM 2 AND (1.9)

Suppose $Q(y)$ is the quadratic form defined in Lemma 2.4 for $\ell \geq 6$, and $Q(\mathbf{y}) = y_1^2 + \cdots + y_5^2$ for $\ell = 5$. By Deligne's inequality $|\lambda_f(n)| \leq \tau(n)$ and the bound $r(n, Q) \ll n^{\ell/2-1}$, we have, with the notation $\mathcal{L} := \log(2X)$,

$$(4.1) \quad \int_1^{\sqrt{X}} |\mathcal{S}_{f,b,Q}(x)|^2 dx \ll \int_1^{\sqrt{X}} x^\ell (\log x)^2 dx \ll X^{(\ell+1)/2} \mathcal{L}^2 \ll X^{\ell-2}.$$

It suffices for us to evaluate $\int_{\sqrt{X}}^X |\mathcal{S}_{f,b,Q}(x)|^2 dx$. For any $x \in [X^{1/2}, X] \setminus \mathbb{Z}$, Lemma 2.4 and [5, p. 256, (2.25), p. 270] allow us to write

$$(4.2) \quad \mathcal{S}_{f,b,Q}(x) = \sigma_Q \sum_{q=1}^{\infty} \sum_{h=1}^q {}^* S\left(\frac{hQ}{q}\right) \frac{e_q(bh)}{q^\ell} A_{\ell,b}(x, -h/q) + O(\delta_Q x^{\ell/4 - \delta_\ell + 1 + \varepsilon}),$$

where

$$A_{\ell,b}(x, -h/q) := \sum_{1+b \leq n \leq x+b} (n-b)^{\ell/2-1} \lambda_f(n) e_q(-hn),$$

$$\delta_Q := \begin{cases} 1 & \text{if } \ell \geq 6, \\ 0 & \text{if } \ell = 5 \text{ and } Q(\mathbf{y}) = y_1^2 + \cdots + y_5^2. \end{cases}$$

By partial summation, we can deduce, with the notation (2.3),

$$(4.3) \quad A_{\ell,b}(x, -h/q) = x^{\ell/2-1} A(x+b, -h/q) - (\ell/2-1)I$$

with

$$(4.4) \quad I := \int_b^{x+b} (u-b)^{\ell/2-2} A(u, -h/q) du.$$

It is easy to see that Lemma 2.1 implies that

$$A_{\ell,b}(x, -h/q) \ll X^{\ell/2-1/2} \mathcal{L}$$

uniformly for $1 \leq x \leq X$, $0 \leq b \leq X$ and $(h, n) = 1$. By this and (2.7), we have

$$\sum_{q > X^{1/2}} \frac{1}{q^\ell} \sum_{h=1}^q {}^* \left| S\left(\frac{hQ}{q}\right) A_{\ell,b}(x, -h/q) \right| \ll X^{\ell/4+1/2} \mathcal{L}.$$

Thus we can write (4.2) as, for $x \in [X^{1/2}, X] \setminus \mathbb{Z}$,

$$(4.5) \quad \begin{aligned} \mathcal{S}_{f,b,Q}(x) &= \sigma_Q \sum_{q \leq X^{1/2}} \sum_{h=1}^q {}^* S\left(\frac{hQ}{q}\right) \frac{e_q(bh)}{q^\ell} A_{\ell,b}(x, -h/q) \\ &\quad + O(X^{\ell/4+1/2} \mathcal{L} + \delta_Q X^{\ell/4 - \delta_\ell + 1 + \varepsilon}). \end{aligned}$$

Suppose now $1 \leq q \leq X^{1/2}$ and we estimate the integral I defined as in (4.4). If $q^2 \leq b$, by partial integration and Lemma 2.3 we have, with the notation (2.5),

$$(4.6) \quad \begin{aligned} I &= x^{\ell/2-2} A_1(x+b, -h/q) - \frac{\ell-4}{2} \int_b^{x+b} (u-b)^{\ell/2-3} A_1(u, -h/q) du \\ &\ll q^{3/2} X^{\ell/2-5/4}. \end{aligned}$$

If $q^2 > b$, we can write

$$(4.7) \quad \begin{aligned} I &= \int_b^{q^2} A(u, -h/q) (u-b)^{\ell/2-2} du + \int_{q^2}^{x+b} A(u, -h/q) (u-b)^{\ell/2-2} du \\ &\ll q^{\ell-1} \log(2q) + q^{3/2} X^{\ell/2-5/4} \ll q^{3/2} X^{\ell/2-5/4} \mathcal{L}, \end{aligned}$$

where we estimated the first integral by Lemma 2.1 and the second as in (4.6). In view of (4.6), (4.7) and (2.7), the contribution of I to the last sum in (4.5) is

$$\begin{aligned} &\ll X^{\ell/2-5/4} \mathcal{L} \sum_{q \leq X^{1/2}} q^{-(\ell-5)/2} \\ &\ll R_\ell(X) := \begin{cases} X^{\ell/2-1+\varepsilon} & \text{if } \ell \geq 6, \\ X^{\ell/2-3/4+\varepsilon} & \text{if } \ell = 5. \end{cases} \end{aligned}$$

Since $\ell/4 - \delta_\ell + 1 \leq \ell/2 - 1$ ($\ell \geq 6$), thus we can write (4.5) as, for $x \in [X^{1/2}, X] \setminus \mathbb{Z}$,

$$(4.8) \quad \mathcal{S}_{f,b,Q}(x) = \sigma_Q x^{\ell/2-1} \sum_{q \leq X^{1/2}} \sum_{h=1}^q S\left(\frac{hQ}{q}\right) \frac{e_q(bh)}{q^\ell} A(x+b, -h/q) + O(R_\ell(X)).$$

Now we split the sum over $q \leq X^{1/2}$ into two parts according to $q \leq X^{1/6}$ or $X^{1/6} < q \leq X^{1/2}$. In view of $X^{1/2} \leq x$, we can apply Lemma 2.2 to the sum $\sum_{q \leq X^{1/6}}$ by taking $M = x+b$. It is easy to see that the contribution of the error term to $\mathcal{S}_{f,b,Q}(x)$ is $O(X^{\ell/2-1+\varepsilon})$ if $\ell \geq 6$ and $O(X^{19/12})$ if $\ell = 5$. Thus the formula (4.8) becomes

$$(4.9) \quad \mathcal{S}_{f,b,Q}(x) = S_1(x) + S_2(x) + O(R_\ell(X)),$$

where

$$\begin{aligned} S_1(x) &:= \frac{\sigma_Q}{\sqrt{2\pi}} x^{\ell/2-1} (x+b)^{1/4} \sum_{q \leq X^{1/6}} \sum_{h=1}^q \sum_{n \leq x+b} \psi(h, n, q) \cos(g_{n,q}(x)), \\ S_2(x) &:= \sigma_Q x^{\ell/2-1} \sum_{X^{1/6} < q \leq X^{1/2}} \sum_{h=1}^q S\left(\frac{hQ}{q}\right) \frac{e_q(bh)}{q^\ell} A(x+b, -h/q), \end{aligned}$$

and

$$\psi(h, n, q) := \frac{\lambda_f(n) e_q(bh - \bar{h}n)}{n^{3/4} q^{\ell-1/2}} S\left(\frac{hQ}{q}\right), \quad g_{n,q}(x) := \frac{4\pi \sqrt{n(x+b)}}{q} - \frac{\pi}{4}.$$

We first evaluate the integral $\int_{\sqrt{X}}^X |S_1(x)|^2 dx$. By the elementary formula

$$\cos u \cos v = \frac{\cos(u-v) + \cos(u+v)}{2},$$

we can write

$$(4.10) \quad |S_1(x)|^2 = S_{11}(x) + S_{12}(x) + S_{13}(x),$$

where $h_j \overline{h_j} \equiv 1 \pmod{q_j}$ for $j = 1, 2$, and

$$\begin{aligned}
S_{11}(x) &:= \left(\frac{\sigma_Q}{2\pi}\right)^2 x^{\ell-2} (x+b)^{1/2} \sum_{q_1 \leq X^{1/6}} \sum_{h_1=1}^{q_1*} \sum_{n_1 \leq x+b} \sum_{q_2 \leq X^{1/6}} \sum_{h_2=1}^{q_2*} \sum_{n_2 \leq x+b} \\
&\quad \times \psi(h_1, n_1, q_1) \overline{\psi(h_2, n_2, q_2)}, \\
S_{12}(x) &:= \left(\frac{\sigma_Q}{2\pi}\right)^2 x^{\ell-2} (x+b)^{1/2} \sum_{q_1 \leq X^{1/6}} \sum_{h_1=1}^{q_1*} \sum_{n_1 \leq x+b} \sum_{q_2 \leq X^{1/6}} \sum_{h_2=1}^{q_2*} \sum_{n_2 \leq x+b} \\
&\quad \times \psi(h_1, n_1, q_1) \overline{\psi(h_2, n_2, q_2)} \cos(g_{n_1, q_1}(x) - g_{n_2, q_2}(x)), \\
S_{13}(x) &:= \left(\frac{\sigma_Q}{2\pi}\right)^2 x^{\ell-2} (x+b)^{1/2} \sum_{q_1 \leq X^{1/6}} \sum_{h_1=1}^{q_1*} \sum_{n_1 \leq x+b} \sum_{q_2 \leq X^{1/6}} \sum_{h_2=1}^{q_2*} \sum_{n_2 \leq x+b} \\
&\quad \times \psi(h_1, n_1, q_1) \overline{\psi(h_2, n_2, q_2)} \cos(g_{n_1, q_1}(x) + g_{n_2, q_2}(x)).
\end{aligned}$$

The relation $\sqrt{n_1}/q_1 = \sqrt{n_2}/q_2$ implies that n_1 and n_2 have the same square-free factor. We write $n_j = d_j^2 m$ with $\mu(m) \neq 0$ for $j = 1, 2$. Recalling (2.7), Deligne's well-known bound $|\lambda_f(n)| \leq \tau(n)$ and the trivial inequality $\tau(uv) \leq \tau(u)\tau(v)$, we have

$$\begin{aligned}
(4.11) \quad \psi(h_1, n_1, q_1) \overline{\psi(h_2, n_2, q_2)} &\ll \frac{\tau(d_1^2 m) \tau(d_2^2 m)}{(d_1 d_2 m)^{3/2} (q_1 q_2)^{\ell/2-1/2}} \\
&\ll \frac{\tau(d_1)^2 \tau(d_2)^2 \tau(m)^2}{(d_1 d_2 m)^{3/2} (q_1 q_2)^{\ell/2-1/2}}.
\end{aligned}$$

The above estimate (4.11) implies that the infinite series

$$(4.12) \quad C_{f,b,Q} := \left(\frac{\sigma_Q}{2\pi}\right)^2 \sum_{q_1 \geq 1} \sum_{1 \leq h_1 \leq q_1}^* \sum_{n_1 \geq 1} \sum_{q_2 \geq 1} \sum_{1 \leq h_2 \leq q_2}^* \sum_{n_2 \geq 1} \psi(h_1, n_1, q_1) \overline{\psi(h_2, n_2, q_2)}$$

$\sqrt{n_1}/q_1 = \sqrt{n_2}/q_2$

converges absolutely provided $\ell \geq 4$ and

$$(4.13) \quad C_{f,b,Q} \ll_{f,Q} 1 \quad (\forall b).$$

If a quadruple (q_1, q_2, n_1, n_2) satisfies $\sqrt{n_1}/q_1 = \sqrt{n_2}/q_2$ but is not included in $S_{11}(x)$, then we must have that $q_1 > X^{1/6}$, or $q_2 > X^{1/6}$, or $n_1 > x+b \geq X^{1/2}$, or $n_2 > x+b \geq X^{1/2}$. Thus $n_1 q_2^2 = n_2 q_1^2 > X^{1/3}$. In view of the following elementary estimates

$$\begin{aligned}
(4.14) \quad t_\ell(m) &:= \sum_{m=nq^2} \tau(n) q^{-\ell/2+3} \leq \tilde{t}_\ell(m) := \begin{cases} \tau_3(m) & \text{if } \ell \geq 6, \\ m^{1/4} \tau_3(m) & \text{if } \ell = 5, \end{cases} \\
&\sum_{n \leq u} \tau_3(n)^2 \ll u(\log u)^8 \quad (u \geq 2),
\end{aligned}$$

we can derive, for all $x \in [X^{1/2}, X] \setminus \mathbb{Z}$,

$$\begin{aligned}
 (4.15) \quad \left| \frac{S_{11}(x)}{x^{\ell-2}(x+b)^{1/2}} - C_{f,b,Q} \right| &\ll \sum_{\substack{n_1, n_2, q_1, q_2 \\ n_2 q_1^2 = n_1 q_2^2 > X^{1/3}}} \frac{\tau(n_1)\tau(n_2)}{(n_1 q_2^2 n_2 q_1^2)^{3/4} (q_1 q_2)^{\ell/2-3}} \\
 &\ll \sum_{m > X^{1/3}} \frac{t_\ell(m)^2}{m^{3/2}} \leq \sum_{m > X^{1/3}} \frac{\tilde{t}_\ell(m)^2}{m^{3/2}} \\
 &\ll \begin{cases} X^{-1/6} \mathcal{L}^8 & \text{if } \ell \geq 6, \\ X^{-1/12} \mathcal{L}^8 & \text{if } \ell = 5. \end{cases}
 \end{aligned}$$

On the other hand, we have, in view of (4.13),

$$C_{f,b,Q} \int_{\sqrt{X}}^X x^{\ell-2}(x+b)^{1/2} dx = C_{f,b,Q} \int_1^X x^{\ell-2}(x+b)^{1/2} dx + O(X^{\ell/2})$$

uniformly for $X \geq 1$ and $0 \leq b \leq X$. And hence we obtain

$$(4.16) \quad \int_{\sqrt{X}}^X S_{11}(x) dx = C_{f,b,Q} \int_1^X x^{\ell-2}(x+b)^{1/2} dx + \begin{cases} O(X^{\ell-2/3} \mathcal{L}^8) & \text{if } \ell \geq 6, \\ O(X^{\ell-7/12} \mathcal{L}^8) & \text{if } \ell = 5. \end{cases}$$

By the first derivative test, (4.11) and (2.7), we have (note $\ell \geq 5$)

$$\begin{aligned}
 (4.17) \quad \int_{X^{1/2}}^X |S_{13}(x)| dx &\ll X^{\ell-1} \sum_{\substack{n_1, n_2 \leq x+b \\ q_1, q_2 \leq X^{1/6}}} \frac{\tau(n_1)\tau(n_2)}{(n_1 n_2)^{3/4} (q_1 q_2)^{\ell/2-3/2}} \frac{1}{\sqrt{n_1}/q_1 + \sqrt{n_2}/q_2} \\
 &\ll X^{\ell-1} \sum_{\substack{n_1, n_2 \leq x+b \\ q_1, q_2 \leq X^{1/6}}} \frac{\tau(n_1)\tau(n_2)}{n_1 n_2 (q_1 q_2)^{\ell/2-2}} \\
 &\ll X^{\ell-5/6} \mathcal{L}^6,
 \end{aligned}$$

where we used the inequality $u + v \geq 2\sqrt{uv}$ for $u, v > 0$.

Similarly by the first derivative test again

$$(4.18) \quad \int_{X^{1/2}}^X |S_{12}(x)| dx \ll X^{\ell-1} (\Sigma_{11} + \Sigma_{12}),$$

where

$$\begin{aligned}
 \Sigma_{11} &:= \sum_{\substack{n_1, n_2 \leq x+b, q_1, q_2 \leq X^{1/6} \\ |\sqrt{n_1}/q_1 - \sqrt{n_2}/q_2| \geq \frac{1}{10}(\sqrt{n_1 n_2}/q_1 q_2)^{1/2}}} \frac{\tau(n_1)\tau(n_2)}{(n_1 n_2)^{3/4} (q_1 q_2)^{\ell/2-3/2}} \frac{1}{|\sqrt{n_1}/q_1 - \sqrt{n_2}/q_2|}, \\
 \Sigma_{12} &:= \sum_{\substack{n_1, n_2 \leq x+b, q_1, q_2 \leq X^{1/6} \\ 0 < |\sqrt{n_1}/q_1 - \sqrt{n_2}/q_2| < \frac{1}{10}(\sqrt{n_1 n_2}/q_1 q_2)^{1/2}}} \frac{\tau(n_1)\tau(n_2)}{(n_1 n_2)^{3/4} (q_1 q_2)^{\ell/2-3/2}} \frac{1}{|\sqrt{n_1}/q_1 - \sqrt{n_2}/q_2|}.
 \end{aligned}$$

As in the argument of (4.17) we get easily that (recall $\ell \geq 5$)

$$(4.19) \quad \Sigma_{11} \ll \sum_{n_1, n_2 \leq x+b, q_1, q_2 \leq X^{1/6}} \frac{\tau(n_1)\tau(n_2)}{n_1 n_2 (q_1 q_2)^{\ell/2-2}} \ll X^{1/6} \mathcal{L}^6.$$

If $|\sqrt{n_1}/q_1 - \sqrt{n_2}/q_2| < \frac{1}{10}(\sqrt{n_1 n_2}/q_1 q_2)^{1/2}$ holds, then it follows that

$$\sqrt{n_1} q_2 \asymp \sqrt{n_2} q_1 \quad \text{and} \quad \sqrt{n_1} q_2 + \sqrt{n_2} q_1 \asymp (n_1 n_2)^{1/4} (q_1 q_2)^{1/2}.$$

So in view of these facts, by a similar argument as in (4.15) we have, for $\ell \geq 6$,

$$\begin{aligned} \Sigma_{12} &\ll \sum_{\substack{q_1, q_2 \leq X^{1/6}; n_1, n_2 \leq x+b \\ n_1 q_2^2 - n_2 q_1^2 \neq 0}} \frac{d(n_1) d(n_2)}{(n_1 n_2 q_1^2 q_2^2)^{1/2} (q_1 q_2)^{\ell/2-4} |n_1 q_2^2 - n_2 q_1^2|} \\ &\ll X^{1/3} \sum_{\substack{m_1, m_2 \leq X^{1/6}(X+b) \\ m_1 \neq m_2}} \frac{d_3(m_1) d_3(m_2)}{(m_1 m_2)^{1/2} |m_1 - m_2|} \\ &\ll X^{1/3} \sum_{1 \leq h \leq X^{1/6}(X+b)} \frac{1}{h} \sum_{m \leq X^{1/6}(X+b)} \frac{d_3(m) d_3(m+h)}{(m(m+h))^{1/2}}. \end{aligned}$$

By applying the Cauchy inequality to the sum over m and (4.14), we can deduce

$$(4.20) \quad \Sigma_{12} \ll X^{1/3} \mathcal{L}^9 \quad (\ell \geq 6).$$

When $\ell = 5$, the similar estimate with $X^{1/2}$ in place of $X^{1/3}$ holds. Inserting (4.19) and (4.20) into (4.18), we find that

$$(4.21) \quad \int_{\sqrt{X}}^X |S_{12}(x)| dx \ll R_\ell^*(X) := \begin{cases} X^{\ell-2/3} \mathcal{L}^9 & \text{if } \ell \geq 6, \\ X^{\ell-1/2} \mathcal{L}^9 & \text{if } \ell = 5. \end{cases}$$

Combining (4.16), (4.21) and (4.17) with (4.10) yields

$$(4.22) \quad \int_{\sqrt{X}}^X |S_1(x)|^2 dx = C_{f,b,Q} \int_1^X x^{\ell-2} (x+b)^{1/2} dx + O(R_\ell^*(X)).$$

Finally we estimate $\int_{\sqrt{X}}^X |S_2(x)|^2 dx$. By (2.7) and Cauchy's inequality, it follows

$$\begin{aligned} |S_2(x)|^2 &\ll x^{\ell-2} \left(\sum_{X^{1/6} < q \leq X^{1/2}} q^{-\ell/2+1} |A(x+b, -h/q)| \right)^2 \\ &\ll X^{\ell-2} \mathcal{L} \sum_{X^{1/6} < q \leq X^{1/2}} q^{-\ell+3} |A(x+b, -h/q)|^2. \end{aligned}$$

From this and Lemma 2.3(ii), we deduce (recall $\ell \geq 5$ and $0 \leq b \leq X$)

$$(4.23) \quad \begin{aligned} \int_{\sqrt{X}}^X |S_2(x)|^2 dx &\ll X^{\ell-2} \mathcal{L} \sum_{X^{1/6} < q \leq X^{1/2}} q^{-\ell+3} (qX^{3/2} + q^{3/2} X^{5/4+\varepsilon} + q^2 X^{1+\varepsilon}) \\ &\ll \begin{cases} X^{\ell-2/3} \mathcal{L} & \text{if } \ell \geq 6, \\ X^{\ell-1/2+\varepsilon} & \text{if } \ell = 5. \end{cases} \end{aligned}$$

On the other hand, (4.22) implies immediately

$$\int_{\sqrt{X}}^X |S_1(x)|^2 dx \ll X^{\ell-1/2}.$$

In view of this and (4.23), a simple application of Cauchy's inequality yields

$$(4.24) \quad \int_{\sqrt{X}}^X |S_1(x)S_2(x)| dx \ll \begin{cases} X^{\ell-7/12} \mathcal{L}^{1/2} & \text{if } \ell \geq 6, \\ X^{\ell-1/2+\varepsilon} & \text{if } \ell = 5. \end{cases}$$

Now the first assertion of Theorem 2 and (1.9) follow from (4.1), (4.9), (4.22), (4.23) and (4.24). The second is an immediate consequence by noticing that

$$(x+b)^{1/2} = x^{1/2} + O(bx^{-1/2})$$

valid uniformly for $x \geq 1$ and $b \geq 1$.

$$\begin{aligned} \sum_{p_1+p_2+p_3=N} \tau(p_1) &= \sum_{p_1+p_2+p_3=N} \tau(N-p_2-p_3) \\ &\ll N^{15/2} \exp \{ -C\sqrt{\log N} \} \end{aligned}$$

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