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The $R_2$ measure for totally positive algebraic integers

V. Flammang
UMR CNRS 7502, IECL, Université de Lorraine, site de Metz, Département de Mathématiques, UFR MIM, Ile du Saulcy, CS 50128, 57045 METZ cedex 01, FRANCE
E-mail address : valerie.flammang@univ-lorraine.fr

Abstract
Let $\alpha$ be a totally positive algebraic integer of degree $d$, i.e., whose all conjugates $\alpha_1 = \alpha, \ldots, \alpha_d$ are positive real numbers. We study the set $R_2$ of the quantities $\left(\prod_{i=1}^{d}(1 + \alpha_i^2)^{1/2}\right)^{1/d}$. We first show that $\sqrt{2}$ is the smallest point of $R_2$. Then, we prove that there exists a number $l$ such that $R_2$ is dense in $(l, \infty)$. Finally, using the method of auxiliary functions, we find the six smallest points of $R_2$ in $(\sqrt{2}, l)$. The polynomials involved in the auxiliary function are found by our recursive algorithm.

1 Introduction
Let $P(x) = a_0 x^d + \cdots + a_d = a_0(x - \alpha_1)\cdots(x - \alpha_d)$, $a_0 \neq 0$, $P \neq x$, be a polynomial with complex coefficients. M. Langevin [La] defined three families of measures of polynomials which are, for $p > 0$:

$$M_p(P) = \left( \int_0^1 |P(e^{2\pi it})|^p dt \right)^{1/p},$$

$$L_p(P) = \left( \sum_{i=1}^{d} |a_i|^p \right)^{1/p},$$

$$R_p(P) = |a_0| \prod_{i=1}^{d} (1 + |\alpha_i|^p)^{1/p}.$$
Note that \( \lim_{p \to 0} M(P) = \exp \left( \int_0^1 \log |P(e^{2\pi i t})| dt \right) \) is the well known Mahler measure of \( P \) and \( L_1(P) \) is the well known length of \( P \).

In this paper, we are interested in the \( R_2 \) measure of \( P \) which is \( R_2(P) = |a_0| \prod_{i=1}^d (1 + |a_i|^2)^{1/2} \). If \( \alpha \) is an algebraic integer, the \( R_2 \) measure of \( \alpha \) is the \( R_2 \) measure of its minimal polynomial. The absolute \( R_2 \) measure of \( \alpha \) is the quantity \( r_2(\alpha) = R_2(\alpha)^{1/\deg(\alpha)} \).

From a well known theorem of Kronecker [Kr], it is easy to prove that, if \( \alpha \) is an algebraic integer, \( r_2(\alpha) = \sqrt{2} \) if and only if \( \alpha \) is a root of unity.

Now, we suppose that \( \alpha \) is a totally positive algebraic integer (all its conjugates are positive real numbers). We have

**Theorem 1.** If \( \alpha \) is a nonzero totally positive algebraic integer then \( r_2(\alpha) \geq \sqrt{2} \). The equality holds if and only if \( \alpha = 1 \).

The result comes immediately from the following inequality due to K. Mahler:

\[
\left( \prod_{i=1}^d (u_i + v_i) \right)^{1/d} \geq \left( \prod_{i=1}^d u_i \right)^{1/d} + \left( \prod_{i=1}^d v_i \right)^{1/d} \text{ for } u_i, v_i > 0.
\]

In order to study the structure of the set \( R_2 \) of the quantities \( r_2(\alpha) \), we show the following

**Theorem 2.** \( R_2 \) is dense in \((l, \infty)\) where \( l = \lim_{n \to \infty} r_2(\beta_n^2) \).

The \( \beta_n^2 \) were defined by C.J. Smyth [Sm1] as follows:

\[
\begin{align*}
\beta_0^2 &= 1 \\
\beta_n^2 &= \beta_{n+1}^2 + \beta_{n+1}^{-2} - 2
\end{align*}
\]

\( \beta_n^2 \) is a totally positive algebraic integer of degree \( 2^n \).

Towards determining the structure of \( R_2 \) in the gap \((\sqrt{2}, l)\), we prove the following

**Theorem 3.** If \( \alpha \) is a totally positive algebraic integer whose minimal polynomial is different from \( x - 1, x^2 - 3x + 1, x^4 - 7x^3 + 13x^2 - 7x + 1, x^8 - 15x^7 + 83x^6 - 220x^5 + 303x^4 - 220x^3 + 83x^2 - 15x + 1, x^6 - 11x^5 + 41x^4 - 63x^3 + 41x^2 - 11x + 1 \) and \( x^8 - 15x^7 + 84x^6 - 225x^5 + 311x^4 - 225x^3 + 84x^2 - 15x + 1 \), then we have:
Corollary 4. The six smallest points of \( R_2 \) in \((\sqrt{2}, l)\) are:

1. \( r_2(x-1) = r_2(\beta_0^2) \),
2. \( r_2(x^2 - 3x + 1) = r_2(\beta_1^2) \),
3. \( r_2(x^4 - 7x^3 + 13x^2 - 7x + 1) = r_2(\beta_2^2) \),
4. \( r_2(x^8 - 15x^7 + 83x^6 - 220x^5 + 303x^4 - 220x^3 + 83x^2 - 15x + 1 ) = r_2(\beta_3^2) \),
5. \( r_2(x^6 - 11x^5 + 41x^4 - 63x^3 + 41x^2 - 11x + 1) \),
6. \( r_2(x^8 - 15x^7 + 84x^6 - 225x^5 + 311x^4 - 225x^3 + 84x^2 - 15x + 1) \).

We conjecture that the next point has minimal polynomial \( x^{14} - 27x^{13} + 308x^{12} - 1963x^{11} + 7790x^{10} - 20307x^9 + 35763x^8 - 43131x^7 + 35763x^6 - 20307x^5 + 7790x^4 - 1963x^3 + 308x^2 - 27x + 1 \) and \( R_2 \) measure 1.8698925.

Section 2 deals with the denseness of the set \( R_2 \). In Section 3, we describe the method of explicit auxiliary functions. We link these functions with the integer transfinite diameter. Then, we explain the recursive algorithm which enables us to obtain the constant of Theorem 2. All the computations were done on a MacBookPro with the languages Pascal and Pari.

2 Denseness of the set \( R_2 \)

2.1 Study of the sequence \((r_2(\beta_n^2))_{n \geq 0}\)

We first prove the following

Lemma 5.

\[
r_2(\beta_n^2) = \left(2 \prod_{i=1}^{n-1} (1 + \lambda_i)^{1/2}\right)^{1/2} \quad \text{where} \quad \lambda_0 = \frac{1}{2} \quad \text{and} \quad \lambda_{i+1} = \frac{\lambda_i}{(1 + \lambda_i)^2} \quad \text{for} \quad i \geq 0.
\]

Proof

For \( n \geq 0 \), we put \( \gamma_n = \beta_n^2 \) so, \( \gamma_n = \gamma_{n+1} + \gamma_{n+1}^{-1} - 2 \) and \( \gamma_{n+1}^2 + \gamma_{n+1}^{-2} = \gamma_n^2 + 4\gamma_n + 2 \).

Therefore, we can write: \( R_2(\beta_n^2) = R_2(\gamma_n) = \prod_{i=1}^{2^n}(1 + \gamma_{n,i}^2)^{1/2} \) where,

for \( 1 \leq i \leq 2^n \), \( \gamma_{n,i} \) denote the conjugates of \( \gamma_n \).

Then, we have:

\[
R_2(\beta_n^2) = \prod_{i=1}^{2^n-1} \left((1 + \gamma_{n,i}^2)(1 + \gamma_{n,i}^{-2})\right)^{1/2} = \prod_{i=1}^{2^n-1} \left(2 + \gamma_{n,i}^2 + \gamma_{n,i}^{-2}\right)^{1/2} = \prod_{i=1}^{2^n-1} (2 + \gamma_{n,i}^2 + \gamma_{n,i}^{-2})^{1/2}.
\]
\[
\gamma_{n-1,i}^2 + 4\gamma_{n-1,i} + 2)^{1/2} = \prod_{i=1}^{2^{n-1}} (\gamma_{n-1,i} + 2).
\]

Finally, we have \( R_2(\beta_n^2) = 2^{2^{n-1}} \prod_{i=1}^{2^{n-1}} (1 + 1/2\gamma_{n-1,i}) \).

Then the result comes immediately from the more general following lemma that we proved in [F]:

**Lemma 6.** The notations are the same as previously. Then we have
\[
\Gamma_{n,0} = \left( \prod_{i=0}^n (1 + \lambda_i)^{1/2} \right)^{2^n} \text{ where } \lambda_0 = \frac{1}{2} \text{ and } \lambda_{i+1} = \frac{\lambda_i}{(1 + \lambda_i)^2} \text{ for } i \geq 0.
\]

The lemma shows that the sequence \( (r_2(\beta_n^2))_{n \geq 0} \) is increasing. Furthermore, as \( \log(1 + x) \leq x \) for all \( x \geq 0 \), we have \( \log r_2(\beta_n^2) \leq \frac{1}{2} + \sum_{i=0}^{n-1} \frac{\lambda_i}{2^i} \). The series \( \sum_{i=0}^{n-1} \frac{\lambda_i}{2^i} \) is convergent because \( 0 \leq \lambda_i \leq 1 \) for \( i \geq 0 \).

Thus, the sequence \( (r_2(\beta_n^2))_{n \geq 0} \) is also convergent and its limit is \( l = 1.874348 \ldots \).

Note that \( l \) gives an upper bound for the first accumulation point of \( R_2 \).

## 2.2 Proof of Theorem 2

The proof and notations follow those of C.J. Smyth in [Sm1]. For a given function \( g : [0, \infty) \rightarrow \mathbb{R} \), put \( M(g) \) the set of all means
\[
M_g(\alpha) = \frac{1}{d} \sum_{i=1}^d g(|\alpha_i|)
\]
for \( \alpha \) a totally real algebraic integer, i.e., all its conjugates \( \alpha_1 = \alpha, \ldots, \alpha_d \) are real numbers. When the limits exist, put \( a(g) = \lim_{n \rightarrow \infty} M_g(\beta_n) \) and \( c(g) = \lim_{n \rightarrow \infty} M_g(2\cos(2\pi/n)) \).

Here a convenient choice for \( g \) is \( g : x \rightarrow \frac{1}{2} \log(1 + x^4) \) because then \( M_g(\alpha) = \log r_2(\alpha^2) \).

The proof consists in two parts.

### 2.2.1 First step of the proof

C.J. Smyth [Sm1] proved the following

**Theorem 7.** Let \( g : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) be a monotonic increasing function, zero on \([0, 1] \) such that
\[
\lim_{x \rightarrow \infty} g(x + 1)/g(x) = 1
\]
and the values of \(\log_2 g(2k + 1) \mod 1\) (k=0,1,2,...) are everywhere dense in (0,1).

Then the limit \(a(g)\) exists and \(\mathcal{M}(g)\) is dense in \((a(g), \infty)\).

We replace the function \(g\) by the function \(g^*\) which satisfies the hypothesis of Theorem 7:
\[
g^*(x) = \begin{cases} 
  g(x) + g(1/x) & \text{if } x > 1 \\
  0 & \text{if } 0 \leq x \leq 1 
\end{cases}
\]

As \(\beta_{n,i}^{-1}\) or \(-\beta_{n,i}^{-1}\) is a conjugate of \(\beta_{n,i}\), we have:
\[
\mathcal{M}_g(\beta_n) = \frac{1}{2^n} \sum_{i=1}^{2^n} g(\beta_{n,i}) = \frac{1}{2^n} \sum_{i=1}^{2^n-1} (g(\beta_{n,i}) + g(\beta_{n,i}^{-1})) = M_{g^*}(\beta_n).
\]

Thus, the existence of \(a(g^*)\) implies those of \(a(g)\) and \(a(g^*) = a(g)\).

It is easy to see that \(g^*\) satisfies the first hypothesis of Theorem 7. So, it is sufficient to study the denseness of the set \(\mathcal{F} = \{\log_2 g(2k + 1) \mod 1, k \in \mathbb{N}\}\).

Let \(t \in [0, 1]\) and \(\epsilon > 0\). Does there exist \(f \in \mathcal{F}\) such that \(|f - t| < \epsilon\)? We search for \(n\) and \(k\) satisfying:
\[
| \log_2 g^*(2k + 1) - t - n | < \epsilon
\]
i.e.,
\[
| \log_2 g^*(2k + 1) - t' - n \log 2 | < \epsilon'
\]

The uniform continuity of the function \(\log\) on \([1, \infty)\) gives:
\[
\forall \epsilon' > 0, \exists \eta(\epsilon') \text{ such that } \forall x, y > 0, |x - y| < \eta(\epsilon') \Rightarrow |\log x - \log y| < \epsilon'.
\]

We choose \(n\) such that \(2^{-n} < \eta(\epsilon')\) and \(k\) such that \(|(2k+1) - (g^*)^{-1}(2^n e')| \leq 1\). As \((g^*)'\) is bounded by 1, the mean value Theorem for \(g^*\) on \((1, \infty)\) gives:
\[
|g^*(2k + 1) - 2^n e'| \leq 1,
\]
i.e.,
\[
|2^{-n} g^*(2k + 1) - e'| \leq 2^{-n} < \eta(\epsilon')
\]
and the inequality (2.1) follows immediately. Thus, we have proved that \(\mathcal{M}(g)\) is dense in \((a(g^*), \infty) = (a(g), \infty)\).
2.2.2 Second step of the proof

C.J. Smyth [Sm1] established the following

**Theorem 8.** Let \( g : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) be a function such that \( \lim_{x \to \infty} g(x) = \infty \) and which satisfies a Lipschitz condition

\[
|g(x) - g(y)| < B(\lambda)|x - y|
\]

for \( x, y \in [0, \lambda] \), for each \( \lambda > 0 \). Then \( \mathcal{M}(g) \) is dense on \( (c(g), \infty) \), where

\[
c(g) = \frac{2}{\pi} \int_{0}^{\pi/2} g(2 \cos \theta) d\theta.
\]

It is easy to see that, for our function \( g \), the Lipschitz condition is satisfied for \( B(\lambda) = 4\lambda^3 \).

2.2.3 Conclusion

We have shown that \( \mathcal{M}(g) \) is dense on \( (\min(a(g), c(g)), \infty) \) which means that \( \mathcal{R}_2 \) is dense on \( (l, \infty) \), where \( l = \lim_{n \to \infty} r_2(\beta_n^2) = 1.874348 \ldots \).

2.3 Proof of Theorem 3

2.4 The explicit auxiliary function

The auxiliary function involved in Theorem 3 is of the following type:

\[
(2.2) \quad \text{for } x > 0, \quad f(x) = \frac{1}{2} \log(1 + x^2) - c_0 \log x - \sum_{1 \leq j \leq J} c_j \log |Q_j(x)|
\]

where the \( c_j \) are positive real numbers and the polynomials \( Q_j \) are non zero polynomials in \( \mathbb{Z}[x] \).

Let \( \alpha \) be a totally positive algebraic integer with conjugates \( \alpha_1 = \alpha, \ldots, \alpha_d \) and minimal polynomial \( P \). Then we have

\[
\sum_{i=1}^{d} f(\alpha_i) \geq md
\]

where \( m \) denotes the minimum of the function \( f \), i.e.,

\[
\log \mathcal{R}_2(\alpha) \geq md + \sum_{1 \leq j \leq J} c_j \log |\prod_{i=1}^{d} Q_j(\alpha_i)|.
\]
We assume that \( P \) does not divide any \( Q_j \), then \( \prod_{i=1}^{d} Q_j(\alpha_i) \) is a nonzero integer because it is the resultant of \( P \) and \( Q_j \).

Therefore, if \( \alpha \) is not a root of \( Q_j \), we have

\[
 r_2(\alpha) \geq e^m.
\]

It is possible to reduce the domain of study of the function \( f \). If we consider the function \( g(x) = 1/2[f(x) + f(1/x)] \), we get a minimum greater or equal to those given by \( f \). But \( g \) is invariant under the application \( x \rightarrow 1/x \) so it is sufficient to study \( g \) on \((0, 1)\). Thus, without loss of generality, we can limit our study to auxiliary functions invariant under this transformation. This implies that we can take for \( Q_j \) to be reciprocal polynomials, i.e., \( Q_j(x) = x^{\deg Q_j} Q_j(1/x) \). The condition \( f(x) = f(1/x) \) gives

\[
 2c_0 + \sum_{1 \leq j \leq J} c_j \deg Q_j = 1.
\]

We denote \( \deg Q_j = 2d_j \) for \( 1 \leq j \leq J \).

On \((0,1)\), the auxiliary function \( f \) can be written

\[
 f(x) = \frac{1}{2} \log x + \frac{1}{2} \log(x+1/x) - c_0 \log x - \sum_{1 \leq j \leq J} c_j \log \left| \frac{Q_j(x)}{x^{d_j}} \right| - \sum_{1 \leq j \leq J} c_j \log x^{d_j} \geq m.
\]

Thus, if we put \( y = x + 1/x - 2 \), \( f(x) \) becomes

\[
 \text{for } y > 0, \quad g(y) = \frac{1}{2} \log(y + 2) - \sum_{1 \leq j \leq J} c_j \log |U_j(y)| \geq m
\]

where \( \deg(U_j) = d_j \).

The main problem is to find a good list of polynomials \( Q_j \) which gives a value of \( m \) as large as possible. Thus, we link the auxiliary function with the integer transfinite diameter in order to find the polynomials with our recursive algorithm.

### 2.5 Auxiliary functions and integer transfinite diameter

In this section, we shall need the following definition:

Let \( K \) be a compact subset of \( \mathbb{C} \).
If $\varphi$ is a positive function defined on $K$, the $\varphi$-integer transfinite diameter of $K$ is defined as

$$t_{Z,\varphi}(K) = \lim_{n \to \infty} \inf_{P \in \mathbb{Z}[Y]} \inf_{y \in K} \sup_{n \geq 1} \left( |P(y)|^{\frac{1}{n \varphi(y)}} \right).$$

This weighted version of the integer transfinite diameter was introduced by F. Amoroso [A] and is an important tool in the study of rational approximations of logarithms of rational numbers.

Inside the auxiliary function (2.2), we replace the numbers $c_j$ by rational numbers. Then we can write:

$$\text{(2.3) for } y > 0, \quad f(y) = \frac{1}{2} \log(y + 2) - \frac{t}{r} \log |Q(y)| \geq m$$

where $Q \in \mathbb{Z}[Y]$ is of degree $r$ and $t$ is a positive real number. We want to get a function whose minimum $m$ is as large as possible. Thus we search a polynomial $Q \in \mathbb{Z}[Y]$ such that

$$\sup_{y > 0} |Q(y)|^{t/r}(y + 2)^{-1/2} \leq e^{-m}.$$ 

If we suppose that $t$ is fixed, it is clear that we need an effective upper bound for the quantity

$$t_{Z,\varphi}((0, \infty)) = \lim_{r \to +\infty} \inf_{P \in \mathbb{Z}[Y]} \inf_{y > 0} \sup_{r \geq 1} \left( |P(y)|^{\frac{1}{r \varphi(y)}} \right)$$

where we use the weight $\varphi(y) = (y + 2)^{-1/2}$.

Even if we replace the compact subset $K$ by the infinite interval $(0, \infty)$, the weight $\varphi$ ensures that the quantity $t_{Z,\varphi}((0, \infty))$ is finite.

### 2.6 Construction of the auxiliary function

The improvement compared with Wu’s algorithm is that our polynomials are obtained by induction. Suppose that we have $Q_1, Q_2, ..., Q_J$. Then we use the semi-infinite linear programming (introduced in number theory by C. J. Smyth [Sm2]) to optimize $f$ for this set of polynomials (i.e., to get the greatest possible $m$). We obtain the numbers $c_1, c_2, ..., c_J$ and $f$ in the form (2.3) with $t = \sum_{j=1}^{J} c_j \deg(Q_j)$. 
For several value of $k$, we seek a polynomial $R(y) = \sum_{i=0}^{k} a_i y^i \in \mathbb{Z}[y]$ such that

$$
\sup_{y>0} |Q(y)R(y)|^{1/2} (y + 2)^{-1/2} \leq e^{-m},
$$
i.e., such that

$$
\sup_{y>0} |Q(y)R(y)|(y + 2)^{-(r+k)/2t}
$$
is as small as possible.

We apply LLL algorithm to the linear forms in $a_0, \ldots, a_k$

$$Q(y_i)R(y_i)(y_i + 2)^{-(r+k)/2t}$$

where $y_i$ are control points uniformly distributed in the interval $[0,70]$, including the points where $f$ has its least local minima. We get a polynomial $R$ whose factors $R_j$ are good candidates to enlarge the set of polynomials $(Q_1, Q_2, \ldots, Q_J)$. We only keep the polynomials $R_j$ which have a nonzero coefficient $c_j$ in the new optimized auxiliary function $f$. After optimization, some previous polynomials $Q_j$ may have a zero coefficient and are removed.

In order to get the constant of Theorem 3, we take $k$ from 4 to 15 successively.

The polynomials $Q_j$ of degree $d_j$ and the coefficients $c_j$ involved in the auxiliary function of Theorem 3 are listed in the Table 1 below. Only polynomials numbered 1, 2, 4, 6, 9 and 13 from the list have $r_2$ measure less than the constant in the theorem.
<table>
<thead>
<tr>
<th>$j$</th>
<th>$c_j$</th>
<th>$d_j$</th>
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References


