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An estimator for the tail index of an integrated conditional Pareto-Weibull-type model

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Abstract. We introduce a nonparametric regression estimator for a tail heaviness parameter in an integrated conditional Pareto-Weibull-type model. The estimator is based on local log excesses over a high random threshold. Asymptotic properties are derived under proper regularity conditions.

Key words and phrases: Extremes, local estimation, regression, tail index.

1 Introduction

In the recent years, a lot of attention in extreme value theory has been devoted to situations where the variable of interest $Y$ is observed together with a random covariate $X$. Goegebeur et al. (2014) introduced an estimator for the conditional extreme value index $\gamma(x)$ when $\gamma(x) > 0$, while de Wet et al. (2015) introduced an estimator for the conditional Weibull-tail coefficient. In both of these cases, a weighted average of the log-excesses over a threshold is used, where the threshold is considered to be non-random. The aim of the present paper is to construct an estimator that can be used for both conditional Weibull-tail distributions and Pareto-type

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distributions. To this end, we use a two parameter family of distributions, which contain both the Pareto-type distributions and the Weibull-tail distributions. The estimator is based on a random threshold, as was also done in Stupfler (2013), who introduced an estimator for the conditional extreme value index $\gamma(x)$ with $\gamma(x) \in \mathbb{R}$.

Let $F(y; x) := \mathbb{P}(Y \leq y|X = x)$, the conditional response distribution function, and $F(y; x) := 1 - F(y; x)$. Assume

\[ F(y; x) = \exp \left( -D^{\tau(x)}(\ln H(y; x)) \right), \]

where

- $y > y^*(x)$ with $y^*(x) > 0$,
- $D^{\tau(x)}(y) = \int_1^y u^{\tau(x)-1}du$, with $\tau(x) \in [0, 1]$,
- $H$ is an increasing function that satisfies $H^+(t; x) := \inf\{y : H(y; x) \geq t\} = \ell^\theta(x)\ell(t; x)$, where $\theta(x) > 0$, and $\ell$ is a slowly varying function at infinity, i.e. $\ell(\lambda y; x) \ell(y; x) \to 1$ as $y \to \infty$ for all $\lambda > 0$.

As noted in Gardes et al. (2011), this model includes Weibull-tail distributions with Weibull-tail coefficient $\theta(x)$ if $\tau(x) = 0$, and Pareto-type tails with extreme value index $\theta(x)$ if $\tau(x) = 1$, while $\tau(x) \in (0, 1)$ is an intermediate class of distributions. In the following, we let $(X_i, Y_i)$, $i = 1, \ldots, n$, be independent copies of the random vector $(X, Y) \in \mathbb{R}^q \times \mathbb{R}_+$ with $q \geq 1$, where the conditional distribution of $Y$ given $X = x$ satisfies (1). Furthermore, let $x \in \mathbb{R}^q$ be arbitrary and denote by $B(x, h)$, the ball with center $x$ and radius $h$, i.e. $B(x, h) := \{z \in \mathbb{R}^q : d(x, z) \leq h\}$, with $d(x, z)$ being the distance between $x$ and $z$. The number of observations in the ball is given by $N_{n, x, h} := \sum_{i=1}^n \mathbb{I}_{\{X_i \in B(x, h)\}}$, where $\mathbb{I}_\{}$ is the indicator function, and denote by $n_x$ the expected number of observations in $B(x, h)$, i.e. $n_x := n\mathbb{P}(X \in B(x, h))$.

Conditional on $N_{n, x, h} = p$, $p \geq 1$, we introduce $Z_j$, $j = 1, \ldots, p$, as the response variables for which the covariate $X_j$ is in the ball $B(x, h)$, and denote by $Z_{1, p} \leq \ldots \leq Z_{p, p}$ the associated
order statistics. In this setting we define our estimator of $\theta(x)$ as
\[
\hat{\theta}(k_x; x) := \frac{1}{\mu_{\tau(x)}} \frac{1}{k_x} \sum_{i=1}^{k_x} \left[ \ln Z_{p-i+1} - \ln Z_{p-k_x} \right]
\]
with
\[
\mu_{\tau(x)}(t) := \int_{0}^{\infty} (D_{\tau(x)}(u + t) - D_{\tau(x)}(t)) \exp(-u) du,
\]
and assuming that $k_x \in \{1, \ldots, p - 1\}$. This estimator is an adaptation of the estimator proposed by Gardes et al. (2011) to the regression context. It consists mainly in averaging the log-spacings between the upper order statistics of the response variables for which the covariates are in the ball centered at $x$.

In the following, we will let $U_h(t; x)$ and $U(t; x)$ be the tail quantile functions corresponding to the conditional distribution function $F_h(y; x) := P(Y \leq y | X \in B(x, h))$ and $F(y; x)$, respectively, i.e. $U_h(\cdot; x) := (1/F_h(\cdot; x))$ and $U(\cdot; x) := (1/F(\cdot; x))$, where the superscript $\leftarrow$ denotes the generalised inverse as introduced above. In order to control the difference between $U_h(t; x)$ and $U(t; x)$, we define $\omega(u, v, x, h) := \sup_{z \in [u, v]} |\log U_h(z; x) - \log U(z; x)|$, with $u \leq v$.

The asymptotic properties of $\hat{\theta}(k_x; x)$ will be examined under the following second order condition.

**Assumption A(\rho(x))** There exist $\rho(x) < 0$ and $b(y; x) \to 0$ for $y \to \infty$ such that
\[
\ln \frac{\ell(\lambda y; x)}{\ell(y; x)} = b(y; x)D_{\rho(x)}(\lambda)(1 + o(1)),
\]
where $o(1)$ is uniform on $\lambda \in [1, \infty)$.

Note that this assumption immediately implies that the function $|b(y; x)|$ is regularly varying with index $\rho(x)$.

## 2 Asymptotic properties

In this section we examine the asymptotic properties of our estimator. We start by establishing the consistency of $\hat{\theta}(k_x; x)$. 

Theorem 1 Assume that $\mathcal{F}(:, x)$ satisfies (1) and that $A(\rho(x))$ holds. If $n_x \to \infty$, $k_x \to \infty$ and $\frac{k_x}{n_x} \to 0$ in such a way that for some $\delta > 0$,

$$
\frac{1}{\mu_{\tau}(x)} \left( \ln \frac{n_x}{k_x} \right) \omega \left( \frac{n_x}{(1+\delta)k_x}, \frac{1+\delta}{x}, h \right) \to 0,
$$

then

$$
\hat{\theta}(k_x; x) \xrightarrow{p} \theta(x).
$$

Proof: Let $I_x := \mathbb{N} \cap [(1 - n_x^{-1/4})n_x, (1 + n_x^{-1/4})n_x]$. According to Lemma 1 in Stupfler (2013), one has that $\mathbb{P}(N_{n,x,h} \in I_x) \to 1$ as $n_x \to \infty$. For any $t > 0$, define the event

$$
S(t; x) := \left\{ |\hat{\theta}(k_x; x) - \theta(x)| > t \right\}.
$$

Note that after applying the law of total probability one obtains the inequality

$$
\mathbb{P}(S(t; x)) \leq \sup_{p \in I_x} \mathbb{P}(S(t; x)|N_{n,x,h} = p) + \mathbb{P}(N_{n,x,h} \notin I_x).
$$

We have thus to show that $\sup_{p \in I_x} \mathbb{P}(S(t; x)|N_{n,x,h} = p) \to 0$.

To this aim, let $T_i, i = 1, \ldots, p$, be unit Pareto random variables, with $T_{1,p} \leq \ldots \leq T_{p,p}$ the associated order statistics. Given $N_{n,x,h} = p \geq 1$, the distribution of the random vector $(Z_1, \ldots, Z_p)$, is the same as that of the random vector $(U_h(T_1; x), \ldots, U_h(T_p; x))$; see Lemma 2 in Stupfler (2013). Thus, denoting

$$
\hat{\theta}(k_x; x) := \frac{1}{\mu_{\tau}(x)} \left( \ln \frac{n_x}{k_x} \right) \frac{1}{k_x} \sum_{i=1}^{k_x} \left[ \ln U_h(T_{p-i+1,p}; x) - \ln U_h(T_{p-k_x,p}; x) \right],
$$

$$
\tilde{\theta}(k_x; x) := \frac{1}{\mu_{\tau}(x)} \left( \ln \frac{n_x}{k_x} \right) \frac{1}{k_x} \sum_{i=1}^{k_x} \left[ \ln U(T_{p-i+1,p}; x) - \ln U(T_{p-k_x,p}; x) \right],
$$

and

$$
R_p(x) := \frac{1}{\mu_{\tau}(x)} \left( \ln \frac{n_x}{k_x} \right) \frac{1}{k_x} \sum_{i=1}^{k_x} \left[ \ln U_h(T_{p-i+1,p}; x) - \ln U_h(T_{p-k_x,p}; x) - \ln U(T_{p-i+1,p}; x) - \ln U(T_{p-k_x,p}; x) \right],
$$

we have

$$
\mathbb{P}(S(t; x)|N_{n,x,h} = p) = \mathbb{P}\left( |\hat{\theta}(k_x; x) - \theta(x)| > t \right) \leq \mathbb{P}\left( |\tilde{\theta}(k_x; x) - \theta(x)| > \frac{t}{2} \right) + \mathbb{P}\left( |R_p(x)| > \frac{t}{2} \right).
$$

(2)
The two probabilities on the right-hand side of (2) are now studied separately. Concerning the first one, note that, with $T_i^* (p) := \frac{T_{p-k_x,p} - 1}{T_{p-k_x,p}}$, $i = 1, \ldots, k_x$,

\[
\tilde{\theta}(k_x; x) = \theta(x) \frac{1}{\mu_{\tau(x)}} \left( \ln \frac{p}{k_x} \right) \sum_{i=1}^{k_x} \left[ D_{\tau(x)} \left( \ln T_{p-k_x,p} + \ln T_i^* (p) \right) - D_{\tau(x)} \left( \ln T_{p-k_x,p} \right) \right] \\
+ \frac{1}{\mu_{\tau(x)}} \left( \ln \frac{p}{k_x} \right) \sum_{i=1}^{k_x} \ln \left( \exp \left( D_{\tau(x)} \left( \ln T_{p-k_x,p} + \ln T_i^* (p) \right) ; x \right) \right) \\
=: \tilde{\theta}_1(k_x; x) + \tilde{\theta}_2(k_x; x).
\]

For the sequel, it is important to keep in mind that $(T_{k_x-i+1}^* (p), i = 1, \ldots, k_x) \overset{D}{=} (T_{1, k_x}, \ldots, T_{k_x, k_x})$, independently of $T_{p-k_x,p}$. Application of a Taylor series expansion to $\tilde{\theta}_1(k_x; x)$ gives

\[
\tilde{\theta}_1(k_x; x) = \theta(x) \frac{1}{\ln \frac{p}{k_x}} \sum_{i=1}^{k_x} \ln \left( \frac{p}{k_x} \right) \sum_{i=1}^{k_x} \ln \left( \frac{p}{k_x} \right) \\
+ \frac{\theta(x) - 1}{2} \left( \ln T_{p-k_x,p} + \ln \tilde{T}_i(p) \right)^{\tau(x)-2} \left( \ln T_i^* (p) \right)^2 \\
=: \tilde{\theta}_{11}(k_x; x) + \tilde{\theta}_{12}(k_x; x)
\]

where $\ln \tilde{T}_i(p)$ is a random value between 0 and $\ln T_i^* (p)$. The cases $\tau(x) = 1$ and $\tau(x) \neq 1$ can now be studied separately. If $\tau(x) = 1$, we have that $\tilde{\theta}_{11}(k_x; x) = \theta(x) \frac{1}{k_x} \sum_{i=1}^{k_x} \ln T_i^* (p)$ and $\tilde{\theta}_{12}(k_x; x) = 0$, and thus for any $t > 0$

\[
\sup_{p \in I_x} \mathbb{P} \left( \left| \tilde{\theta}_1(k_x; x) - \theta(x) \right| > t \right) = \sup_{p \in I_x} \mathbb{P} \left( \left| \theta(x) \frac{1}{k_x} \sum_{i=1}^{k_x} \ln T_i^* (p) - \theta(x) \right| > t \right) \\
= \sup_{p \in I_x} \mathbb{P} \left( \left| \theta(x) \frac{1}{k_x} \sum_{i=1}^{k_x} \ln T_{k_x-i+1,k_x} - \theta(x) \right| > t \right) \\
= \mathbb{P} \left( \left| \theta(x) \frac{1}{k_x} \sum_{i=1}^{k_x} \ln T_i - \theta(x) \right| > t \right) \\
\rightarrow 0,
\]

by the law of large numbers. Otherwise, if $\tau(x) < 1$, by combining Lemma 6 in Stupfler (2013) with our Lemmas 1 and 3, we deduce that

\[
\sup_{p \in I_x} \mathbb{P} \left( \left| \tilde{\theta}_{11}(k_x; x) - \theta(x) \right| > t \right) \rightarrow 0,
\]
while concerning $\tilde{\theta}_{12}(k_x; x)$,
\[
\left| \tilde{\theta}_{12}(k_x; x) \right| \leq \frac{\theta(x)}{2} \left( \ln T_{p-k_x,p} \right)^{-1} \left( \ln T_{p-k_x,p} \right)^{\tau(x)-1} \left( \ln \frac{p}{k_x} \right)^{\tau(x)-1} \mu_{\tau(x)} \left( \ln \frac{p}{k_x} \right) \sum_{i=1}^{k_x} \left( \ln T_{i}^*(p) \right)^2.
\]

Using again the law of large numbers combining with the convergence $\sup_{p \in I_{x}} \mathbb{P} \left( (\ln T_{p-k_x,p})^{-1} > t \right) \rightarrow 0$ and our Lemma 3, we deduce that
\[
\sup_{p \in I_{x}} \mathbb{P} \left( \left| \tilde{\theta}_{12}(k_x; x) \right| > t \right) \rightarrow 0.
\]

This leads also for $\tau(x) < 1$ to
\[
\sup_{p \in I_{x}} \mathbb{P} \left( \left| \tilde{\theta}_{1}(k_x; x) - \theta(x) \right| > t \right) \rightarrow 0. \tag{3}
\]

Concerning now $\tilde{\theta}_{2}(k_x; x)$, we have to use assumption $A(\rho(x))$ which ensures that
\[
\tilde{\theta}_{2}(k_x; x) = \frac{1}{\mu_{\tau(x)} \left( \ln \frac{p}{k_x} \right)} \left( \frac{1}{k_x} \sum_{i=1}^{k_x} \ln \ell \left( \exp \left( D_{\tau(x)} \left( \ln \left( T_{p-k_x,p} \right) + \ln T_{i}^*(p) \right) \right) - D_{\tau(x)} \left( \ln \left( T_{p-k_x,p} \right) \right) \right) \ell \left( \exp \left( D_{\tau(x)} \left( \ln \left( T_{p-k_x,p} \right) \right) \right) \right) \right) \frac{b(\exp\left(D_{\tau(x)}\left(\ln\left(T_{p-k_x,p}\right)\right)\right);x)}{\mu_{\tau(x)} \left( \ln \frac{p}{k_x} \right)}
\]
\[
= \frac{1}{k_x} \sum_{i=1}^{k_x} D_{\rho(x)} \left( \exp \left( D_{\tau(x)} \left( \ln \left( T_{p-k_x,p} \right) + \ln T_{i}^*(p) \right) \right) - D_{\tau(x)} \left( \ln \left( T_{p-k_x,p} \right) \right) \right) \left( 1 + \delta_{n} \right)
\]
where $\delta_{n} \xrightarrow{p} 0$ uniformly in $i$ and $p$. An application of the mean value theorem, shows that
\[
D_{\rho(x)} \left( \exp \left( D_{\tau(x)} \left( \ln \left( T_{p-k_x,p} \right) + \ln T_{i}^*(p) \right) \right) - D_{\tau(x)} \left( \ln \left( T_{p-k_x,p} \right) \right) \right) = \left[ \exp \left( D_{\tau(x)} \left( \ln \tilde{T}_{i}(p) + \ln T_{p-k_x,p} \right) - D_{\tau(x)} \left( \ln T_{p-k_x,p} \right) \right) \right]^{\rho(x)} \left( \ln \tilde{T}_{i}(p) + \ln T_{p-k_x,p} \right)^{\tau(x)-1} \ln T_{i}^*(p),
\]
where $\ln \tilde{T}_{i}(p)$ is a random value between 0 and $\ln T_{i}^*(p)$. Since
\[
\left[ \exp \left( D_{\tau(x)} \left( \ln \tilde{T}_{i}(p) + \ln T_{p-k_x,p} \right) - D_{\tau(x)} \left( \ln T_{p-k_x,p} \right) \right) \right]^{\rho(x)} \leq 1,
\]
it follows that
\[
\left| \tilde{\theta}_{2}(k_x; x) \right| \leq \left| 1 + \delta_{n} \right| \frac{\left( \ln T_{p-k_x,p} \right)^{\tau(x)-1} \left( \ln \frac{p}{k_x} \right)^{\tau(x)-1} b \left( \exp \left( D_{\tau(x)} \left( \ln T_{p-k_x,p} \right) \right) ; x \right) \frac{1}{k_x} \sum_{i=1}^{k_x} \ln T_{i}^*(p) \right|.
\]
Clearly,
\[
\sup_{p \in I_p} \mathbb{P} \left( |(1 + \delta) - 1| > t \right) \to 0
\]
and
\[
\sup_{p \in I_p} \mathbb{P} \left( |b \left( \exp \left( \frac{D_r(x)}{\ln T_{p-k_x,p}} \right) \right) > t \right) \to 0,
\]
(observe that \(b(\exp(D_r(x))\ln y); x)\) is regularly varying at infinity, and apply Lemma 6 of Stupfler, 2013), from which we deduce that
\[
\sup_{p \in I_p} \mathbb{P} \left( \left| \hat{\theta}_2(k_x; x) \right| > t \right) \to 0
\]
according to our Lemma 3. Finally, coming back to \(R_p(x)\), we have
\[
|R_p(x)| \leq \frac{2\omega(T_{p-k_x,p}, T_{p,p}, x, h)}{\omega(x)} \frac{\mu_r(x)}{\mu_r(x)}, \quad (4)
\]
Since \(\omega(u,v,x,h)\) is a decreasing function in \(u\) and an increasing function in \(v\), it is clear that for all \(t > 0\),
\[
\left\{ \frac{2\omega \left( \frac{n_x}{(1+\delta)k_x}, \frac{n_x}{(1+\delta)k_x}, x, h \right)}{\mu_r(x)} \leq t \right\} \cap \left\{ T_{p-k_x,p} \geq \frac{n_x}{(1+\delta)k_x} \right\} \cap \left\{ T_{p,p} \leq \frac{n_x}{1+\delta} \right\} \leq \left\{ \frac{2\omega(T_{p-k_x,p}, T_{p,p}, x, h)}{\mu_r(x)} \leq t \right\}.
\]
By considering the complementary event, we have
\[
\left\{ \frac{2\omega \left( \frac{n_x}{(1+\delta)k_x}, \frac{n_x}{(1+\delta)k_x}, x, h \right)}{\mu_r(x)} > t \right\} \subseteq \left\{ \frac{2\omega \left( \frac{n_x}{(1+\delta)k_x}, \frac{n_x}{(1+\delta)k_x}, x, h \right)}{\mu_r(x)} > t \right\} \cup \left\{ T_{p-k_x,p} < \frac{n_x}{(1+\delta)k_x} \right\} \cup \left\{ T_{p,p} > \frac{n_x}{1+\delta} \right\}.
\]
Taking \(n_x\) sufficiently large, under the assumption of Theorem 1, we have
\[
\sup_{p \in I_p} \mathbb{P} \left( \left| \frac{2\omega(T_{p-k_x,p}, T_{p,p}, x, h)}{\mu_r(x)} \right| > t \right) \leq \sup_{p \in I_p} \mathbb{P} \left( T_{p-k_x,p} < \frac{n_x}{(1+\delta)k_x} \right) + \sup_{p \in I_p} \mathbb{P} \left( T_{p,p} > \frac{n_x}{1+\delta} \right)
\]
\[\to 0,
\]
by Lemma 6 in Stupfler (2013) and using the properties of the largest order statistic \(T_{p,p}\). This ensures then under our Lemma 2 that
\[
\sup_{p \in I_p} \mathbb{P} \left( \left| R_p(x) \right| > t \right) \to 0.
\]
Combining the above results, Theorem 1 follows.

Now we establish the asymptotic normality of \(\hat{\theta}(k_x; x)\), when properly normalised.
Theorem 2 Assume that $F(.; x)$ satisfies (1) and that $A(\rho(x))$ holds. If $n_x \to \infty$, $k_x \to \infty$ and $k_x/n_x \to 0$ in such a way that for some $\delta > 0$,

$$
\frac{\sqrt{k_x}}{\mu_\tau(x)} \left( \ln \frac{n_x}{k_x} \right) \omega \left( n_x, \frac{n_x^{1+\delta}}{x, h} \right) \to 0,
$$

and if additionally

$$
\sqrt{k_x} b \left( \exp \left( D_\tau(x) \left( \ln \frac{n_x}{k_x} \right) \right) \right) \to \lambda \in \mathbb{R}
$$

and for $\tau(x) < 1$

$$
\frac{\sqrt{k_x}}{\ln \frac{n_x}{k_x}} \to 0
$$

then

$$
\sqrt{k_x} \left( \tilde{\theta}(k_x; x) - \theta(x) \right) \xrightarrow{D} \mathcal{N} \left( \frac{\lambda}{1 - \rho(x)} \mathbb{I}_{\{\tau(x) = 1\}} + \lambda \mathbb{I}_{\{\tau(x) < 1\}}, \theta^2(x) \right).
$$

Proof: Given $N_{n,x,h} = p \geq 1$, the distribution of $\sqrt{k_x} (\tilde{\theta}(k_x; x) - \theta(x))$ is the same as that of $\sqrt{k_x} (\tilde{\theta}(k_x; x) - \theta(x))$. Thus according to Lemma 5 in Stupfler (2013), it is sufficient to prove that the latter has the same distribution as a triangular array of the form

$$
D_n + \varphi_np
$$

where $D_n \xrightarrow{D} \mathcal{N} \left( \frac{\lambda}{1 - \rho(x)} \mathbb{I}_{\{\tau(x) = 1\}} + \lambda \mathbb{I}_{\{\tau(x) < 1\}}, \theta^2(x) \right)$ and $\sup_{p \in I_x} \mathbb{P}(\varphi_np > t) \to 0$ for all $t > 0$, as $n_x \to \infty$. We can use the same decomposition of $\tilde{\theta}(k_x; x)$ as in the proof of Theorem 1, that is in terms of $\tilde{\theta}_{11}(k_x; x)$, $\tilde{\theta}_{12}(k_x; x)$, $\tilde{\theta}_2(k_x; x)$ and $R_p(x)$. Expanding further on the term $\tilde{\theta}_{11}(k_x; x)$ gives

$$
\tilde{\theta}_{11}(k_x; x) = \tilde{\theta}(x) \frac{1}{k_x} \sum_{i=1}^{k_x} \ln T_i + \theta(x) \left[ \frac{\ln T_p k_x}{\ln k_x} \frac{\tau(x)-1}{\tau(x)-1} \left( \frac{\ln p}{k_x} \right)^{\tau(x)-1} \mu_\tau(x) \left( \ln \frac{p}{k_x} \right) \right] \frac{1}{k_x} \sum_{i=1}^{k_x} \ln T_i
$$

$$
= \tilde{\theta}_{111}(k_x; x) + \tilde{\theta}_{112}(k_x; x).
$$

The first term $\tilde{\theta}_{111}(k_x; x)$ can be dealt with directly with the central limit theorem

$$
\sqrt{k_x} \left( \tilde{\theta}_{111}(k_x; x) - \theta(x) \right) \xrightarrow{D} \mathcal{N} \left( 0, \theta^2(x) \right).
$$
Note that $\tilde{\theta}_{112}(k_x; x) = 0$ if $\tau(x) = 1$, so we only need to consider the case $\tau(x) < 1$. For $\tilde{\theta}_{112}(k_x; x)$, we have thus to show that for all $t > 0$

$$
\sup_{p \in I_x} \mathbb{P} \left( \sqrt{k_x} \left| \left( \frac{\ln T_p - k_x}{\ln p/k_x} \right)^{\tau(x) - 1} - 1 \right| > t \right) \to 0.
$$

From the mean value theorem we get

$$
\sup_{p \in I_x} \mathbb{P} \left( \sqrt{k_x} \left| \left( \frac{\ln T_p - k_x}{\ln p/k_x} \right)^{\tau(x) - 1} - 1 \right| > t \right)
\leq \sup_{p \in I_x} \mathbb{P} \left( \left( 1 - \frac{\ln(k_x p T_p - k_x p)}{\ln(p/k_x)} \right)^{\tau(x) - 2} \frac{\sqrt{k_x}}{\ln(1 - n_x^{-1/4}) n_x/k_x} \left| \ln \left( \frac{k_x p T_p - k_x p}{p} \right) \right| > t \right).
$$

Taylor’s theorem gives

$$
\sup_{p \in I_x} \mathbb{P} \left( \left| \sum_{p \in I_x} \left( \left( 1 - \frac{\ln(k_x p T_p - k_x p)}{\ln(p/k_x)} \right)^{\tau(x) - 2} \frac{\sqrt{k_x}}{\ln(1 - n_x^{-1/4}) n_x/k_x} \left| \ln \left( \frac{k_x p T_p - k_x p}{p} \right) \right| > t \right)\right)
\leq \sup_{p \in I_x} \mathbb{P} \left( \left| \sum_{p \in I_x} \left( \left( 1 - \frac{\ln(k_x p T_p - k_x p)}{\ln(p/k_x)} \right)^{\tau(x) - 2} \frac{\sqrt{k_x}}{\ln(1 - n_x^{-1/4}) n_x/k_x} \left| \ln \left( \frac{k_x p T_p - k_x p}{p} \right) \right| > t \right)\right).
$$

Concerning now the term $\tilde{\theta}_{12}(k_x; x)$ (which only needs to be considered in case $\tau(x) < 1$), remark that

$$
\left| \sqrt{k_x} \tilde{\theta}_{12}(k_x; x) \right| \leq \left| \frac{\theta(x) \sqrt{k_x}}{2} \ln \frac{\ln T_p - k_x}{\ln p/k_x} \ln \frac{\ln T_p - k_x}{\ln p/k_x} \left( \frac{\ln T_p - k_x}{\ln p/k_x} \right)^{\tau(x) - 1} \left( \frac{\ln p}{\ln T_p - k_x} \right)^{\tau(x) - 1} \frac{1}{\mu_{\tau(x)}(\ln p/k_x)} \right| \left( \ln T_p^{*}(p) \right)^{2}.
$$

Combining again Lemma 6 in Stupfler (2013) with our Lemmas 1 and 3 together with our assumptions, we infer that

$$
\sup_{p \in I_x} \mathbb{P} \left( \left| \sqrt{k_x} \tilde{\theta}_{12}(k_x; x) \right| > t \right) \to 0.
$$
For $\tilde{\theta}_2(k_x; x)$, we need also to distinguish between the two cases $\tau(x) = 1$ and $\tau(x) < 1$. We first consider the case $\tau(x) = 1$, where we use the fact that $b(.; x)$ is regularly varying at infinity combining with Lemma 6 in Stupfler (2013) and the law of large numbers according to which

$$\limsup_{p \in I_x} \mathbb{P} \left( \left| \frac{1}{k_x} \sum_{i=1}^{k_x} \left( T^*_i(p)^{\rho(x)} - 1 \right) \right| > \lambda \right) = \mathbb{P} \left( \left| \frac{1}{k_x} \sum_{i=1}^{k_x} \frac{T_{i}^{\rho(x)} - 1}{\rho(x)} \right| > \lambda \right) \rightarrow 0.$$ 

The convergence

$$\limsup_{p \in I_x} \mathbb{P} \left( \sqrt{k_x} \tilde{\theta}_2(k_x; x) - \frac{\lambda}{1 - \rho(x)} \right) \rightarrow 0$$

then follows from our assumptions and our Lemma 3. In the case where $\tau(x) < 1$, using the same arguments as in the proof of Theorem 1, we have the following decomposition

$$\tilde{\theta}_2(k_x; x) =: \tilde{\theta}_{21}(k_x; x) + \tilde{\theta}_{22}(k_x; x) + \tilde{\theta}_{23}(k_x; x),$$

where

$$\tilde{\theta}_{21}(k_x; x) := (1 + \delta_n) b \left( \exp \left( D_{\tau(x)} \left( \ln T_{p-k_x,p} \right) \right) \right) \cdot \frac{\left( \ln T_{p-k_x,p} \right)^{\tau(x) - 1}}{\mu_{\tau(x)}} \cdot \frac{1}{k_x} \sum_{i=1}^{k_x} \ln T^*_i(p)$$

$$\tilde{\theta}_{22}(k_x; x) := (1 + \delta_n) b \left( \exp \left( D_{\tau(x)} \left( \ln T_{p-k_x,p} \right) \right) \right) \cdot \frac{1}{k_x} \sum_{i=1}^{k_x} \ln T^*_i(p)$$

and

$$\tilde{\theta}_{23}(k_x; x) := (1 + \delta_n) b \left( \exp \left( D_{\tau(x)} \left( \ln T_{p-k_x,p} \right) \right) \right) \cdot \frac{\left( \ln T_{p-k_x,p} \right)^{\tau(x) - 1}}{\mu_{\tau(x)}} \cdot \frac{1}{k_x} \sum_{i=1}^{k_x} \ln T^*_i(p) \left\{ e^{\rho(x)} \left[ D_{\tau(x)} \left( \ln \tilde{T}_i(p) + \ln T_{p-k_x,p} \right) \right] - 1 \right\}.$$
For what concerns the remainder term $R_p(x)$, using the same arguments as in the proof of Theorem 1, we get for all $t > 0$, that

$$
\left\{ \left\| \sqrt{k_x} \frac{2\omega(T_{p-k_x,p}, T_{p,p}, x, h)}{\mu_\tau(x) \left( \ln \frac{n_x}{k_x} \right)} \right\| > t \right\} \subseteq \left\{ \left\| \sqrt{k_x} \frac{2\omega \left( \frac{n_x}{(1+\delta)k_x}, n_x^{1+\delta}, x, h \right)}{\mu_\tau(x) \left( \ln \frac{n_x}{k_x} \right)} \right\| > t \right\} \cup \left\{ T_{p-k_x,p} < \frac{n_x}{(1+\delta)k_x} \right\}
$$

Taking now $n_x$ sufficiently large, this implies by assumption that

$$
\sup_{p \in I_x} \mathbb{P} \left( \left\| \sqrt{k_x} R_p(x) \right\| > t \right) \leq \sup_{p \in I_x} \mathbb{P} \left( T_{p-k_x,p} < \frac{n_x}{(1+\delta)k_x} \right) + \sup_{p \in I_x} \mathbb{P} \left( T_{p,p} > n_x^{1+\delta} \right) \rightarrow 0.
$$

This convergence combined with (4) and Lemma 2 ensures that

$$
\sup_{p \in I_x} \mathbb{P} \left( \left\| \sqrt{k_x} R_p(x) \right\| > t \right) \rightarrow 0.
$$

Combining all these convergences yield our Theorem 2.

**Appendix**

In this section we introduce some lemmas which are useful for establishing the main results.

**Lemma 1** Assume that $n_x \rightarrow \infty$, $k_x \rightarrow \infty$ such that $\frac{k_x}{n_x} \rightarrow 0$. If $\tau(x) < 1$, then there exist a constant $C > 0$, such that

$$
\sup_{p \in I_x} \left| \left( \ln \frac{p}{k_x} \right)^{\tau(x)-1} \mu_\tau(x) \left( \ln \frac{p}{k_x} \right) - 1 \right| \leq C \left( \ln \frac{n_x}{k_x} \right)^{-1}.
$$

**Proof:** First note that we have $\mu_\tau(x)(y) = y^{\tau(x)-1} + \tilde{R}(y)$, with

$$
\tilde{R}(y) := \frac{\tau(x) - 1}{2} y^{\tau(x)-2} \int_0^\infty (1 + \xi)^{\tau(x)-2} u^2 e^{-u} du,
$$

where $\xi$ is a value between 0 and $\frac{\tau(x)}{2}$. Hence $|\tilde{R}(y)| \leq y^{\tau(x)-2}$. Consequently

$$
\left| \left( \ln \frac{p}{k_x} \right)^{\tau(x)-1} \mu_\tau(x) \left( \ln \frac{p}{k_x} \right) - 1 \right| = \left| \frac{\tilde{R} \left( \ln \frac{p}{k_x} \right)}{\left( \ln \frac{p}{k_x} \right)^{\tau(x)-1} + \tilde{R} \left( \ln \frac{p}{k_x} \right)} \right| \leq \left( \ln \frac{p}{k_x} \right)^{-1} \left( 1 + O \left( \left( \ln \frac{p}{k_x} \right)^{-1} \right) \right)^{-1}.
$$
Since
\[
\sup_{p \in I_x} \left( \ln \frac{p}{k_x} \right)^{-1} \leq \left( \ln \frac{n_x \left( 1 - n_x^{-\frac{1}{2}} \right)}{k_x} \right)^{-1},
\]
the result easily follows.

\textbf{Lemma 2} Assume that \( n_x \to \infty, k_x \to \infty \) such that \( \frac{k_x}{n_x} \to 0 \). Then
\[
\frac{\mu_{\tau(x)} \left( \ln \frac{p}{k_x} \right)}{\mu_{\tau(x)} \left( \ln \frac{n_x}{k_x} \right)} \to 1
\]
uniformly in \( p \in I_x \).

\textbf{Proof}: We start by rewriting the term \( \frac{\mu_{\tau(x)} \left( \ln \frac{p}{k_x} \right)}{\mu_{\tau(x)} \left( \ln \frac{n_x}{k_x} \right)} - 1 \) as
\[
\frac{\mu_{\tau(x)} \left( \ln \frac{p}{k_x} \right)}{\mu_{\tau(x)} \left( \ln \frac{n_x}{k_x} \right)} - 1 = \left( \frac{\mu_{\tau(x)} \left( \ln \frac{p}{k_x} \right)}{\mu_{\tau(x)} \left( \ln \frac{n_x}{k_x} \right)} - 1 \right) \left( \frac{\ln \frac{p}{k_x}}{\mu_{\tau(x)} \left( \ln \frac{n_x}{k_x} \right)} - 1 \right).
\]

According to Lemma 2 in Gardes et al. (2011), \( \mu_{\tau(x)} \left( \ln \frac{n_x}{k_x} \right) \sim \left( \ln \frac{n_x}{k_x} \right)^{\tau(x)-1} \). Thus, using a Taylor series expansion combining with the fact that uniformly in \( p \in I_x \), \( \ln \frac{p}{n_x} \to 0 \), we have
\[
\left( \frac{\ln \frac{p}{k_x}}{\mu_{\tau(x)} \left( \ln \frac{n_x}{k_x} \right)} - 1 \right) \sim \left( \ln \left( 1 + \frac{p}{n_x} \right) \right)^{\tau(x)-1} - 1 \to 0 \quad (5)
\]
uniformly in \( p \in I_x \). Moreover, from the proof of Lemma 1, we know that
\[
\left| \frac{\mu_{\tau(x)} \left( \ln \frac{p}{k_x} \right)}{\mu_{\tau(x)} \left( \ln \frac{n_x}{k_x} \right)} - 1 \right| = \frac{\hat{R} \left( \ln \frac{p}{k_x} \right)}{\left( \ln \frac{p}{k_x} \right)^{\tau(x)-1}} \leq \left( \ln \frac{p}{k_x} \right)^{-1} \to 0 \quad (6)
\]
uniformly in \( p \in I_x \). Combining (5) and (6), our Lemma 2 follows.

\textbf{Lemma 3} Assume that \( I_n \) is some index set, and, for \( p \in I_n \) let \( (X_n(p))_n \) and \( (Y_n(p))_n \) be sequences of random variables. If for all \( \varepsilon > 0 \) and some \( x, y \in \mathbb{R} \),
\[
\sup_{p \in I_n} \mathbb{P}(\|X_n(p) - x\| > \varepsilon) \to 0
\]
and

\[
\sup_{p \in I_n} \mathbb{P}(\{|Y_n(p) - y| > \varepsilon\}) \to 0
\]

as \( n \to \infty \), then

\[
\sup_{p \in I_n} \mathbb{P}(\{|X_n(p)Y_n(p) - xy| > \varepsilon\}) \to 0
\]

as \( n \to \infty \).

**Proof:** Note that for all \( p \in I_n \),

\[
\{|X_n(p)Y_n(p) - xy| > \varepsilon\} \subseteq \{|(X_n(p) - x)| > 1\} \cup \left\{|(Y_n(p) - y)| > \frac{\varepsilon}{3}\right\} \\
\quad \cup \left\{|y(X_n(p) - x)| > \frac{\varepsilon}{3}\right\} \cup \left\{|x(Y_n(p) - y)| > \frac{\varepsilon}{3}\right\} .
\]

Lemma 3 then follows using the subadditivity property of a probability measure.

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