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No-regret optimal control redefinition and consequences

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RÉSUMÉ.
ABSTRACT. In this paper, we redefine the notion of no-regret control introduced by J.L.Lions in [2] (original idea by Savage in statistics [4]), this new definition is based on taking spermium upon the states \( y(0,g) \) instead of taking spermium on missing data \( g \). The main interest is that this definition gives a more simple characterization of no-regret optimal control comparing to the optimality systems in [2], [5] and [6].

MOTS-CLÉS : Keywords : No-regret optimal control, low-regret optimal control, missing data problems
1. Introduction

Consider the following state equation described by

\[ Ay(v, g) = Bv + \beta g \] (1)

where \( A \in L(V, V') \) is an isomorphism, \( V \) is a Hilbert space with dual \( V' \), \( B \in L(U, V') \) is the control operator with \( U \) is a Hilbert space of controls, \( \beta \in L(G, V') \), \( G \) is also a Hilbert space of uncertainties and \( v \in U \) is the control function. Suppose that (1) is well posed in \( V \) and denote by \( y(v, g) \) its unique solution that depends on the control \( v \) and on the missing data \( g \). Associate to (1) the objective quadratic function of the form (see [3])

\[ J_0(v, g) = \|Cy(v, g) - z_d\|_H^2 + N \|v\|_U^2 \] (2)

where \( C \in L(V, H) \), \( H \) is another Hilbert space and \( z_d \) is a desired state in \( H \), \( N > 0 \). Our goal is to characterize the optimal control of (1) subject to the cost function (2) whatever the value of the uncertainty \( g \), in other words we are looking to solve

\[ \inf_{v \in U} J_0(v, g) \quad \text{for every} \quad g \in G \]

This definition doesn’t make any sense when \( G \neq \{0\} \). One thinks to look for (see [2])

\[ \inf_{v \in U} \left( \sup_{g \in G} J_0(v, g) \right) \] (3)

but we can get \( \sup_{g \in G} J_0(v, g) = +\infty \).

2. No-regret control redefinition

The last difficulty leads J.L.Lions to think about looking for controls such that

\[ J_0(v, g) \leq J_0(0, g) \quad \text{for every} \quad g \in G \]

and to define [2] :
**Definition 1** We say that $u \in U$ is a no-regret control for (1) – (2) if $u$ is a solution of

$$\inf_{v \in U} \left( \sup_{g \in G} \left( J_0(v, g) - J_0(0, g) \right) \right)$$

Here, we propose another way to define a no-regret control based on the following idea: Remark that $y(v, g) = y(v, 0) + y(0, g)$ (because of linearity in (1)) which allows us to write

$$J_0(v, g) = \|Cy(v, 0) + Cy(0, g) - zd\|^2_H + N\|v\|^2_U$$

Now, look to $J_0$ as a function of $v$ and $y(0, g)$ in other words $J_0(v, g) = J(v, y(0, g))$ where

$$J(v, y(0, g)) = \|Cy(v, 0) - zd\|^2_H + N\|v\|^2_U$$

this allows us to say that $\sup_{g \in G} J_0(v, g) = \sup_{y(0, g) \in Y} J(v, y(0, g))$ where $Y = \{y(0, g) : g \in G\} \subset V$, then solving (3) is equivalent to solve

$$\inf_{v \in U} \left( \sup_{y(0, g) \in Y} J(v, y(0, g)) \right)$$

(4)

and to redefine the no-regret control by :

**Definition 2** We say that $u \in U$ is a no-regret control for (1) – (2) if $u$ is a solution of

$$\inf_{v \in U} \left( \sup_{y(0, g) \in Y} \left( J(v, y(0, g)) - J(0, y(0, g)) \right) \right)$$

Now, we’ll try to rewrite the last quantity under inf-sup to separate the roles of $v$ and $y(0, g)$ by using the following lemma :

**Lemma 3** For every $(v, g) \in U \times G$, we have

$$J(v, y(0, g)) - J(0, y(0, g)) = J(v, 0) - J(0, 0) + 2 \langle Cy(v, 0), Cy(0, g) \rangle_H$$

(4)

$$J(v, 0) - J(0, 0) + 2 \langle Cy(v, 0), y(0, g) \rangle_Y$$

**Proof.** See [6].

The main difficulty arises in no-regret control characterization, where we do not know the structure of the set $\{v \in U : \langle Cy(v, 0), Cy(0, g) \rangle_H = 0\}$, this problem required to take another way like :
3. Low-regret control redefinition

Relax our problem by looking for controls such that
\[ J(v, y(0, g)) - J(0, y(0, g)) \leq \gamma \|y(0, g)\|_V^2 \] for every \( g \in G \), with \( \gamma > 0 \)
to get a sequence of controls \( u_\gamma \) expected to be convergent to the no-regret control \( u \).

**Definition 4** We say that \( u_\gamma \in U \) is a low-regret control for \((1) - (2)\) if \( u_\gamma \) is a solution of
\[
\inf_{v \in U} \left( \sup_{y(0,g) \in Y} J(v, y(0, g)) - J(0, y(0, g)) - \gamma \|y(0, g)\|_V^2 \right)
\]

Use (5) to get
\[
\sup_{y(0,g) \in Y} J(v, y(0, g)) - J(0, y(0, g)) - \gamma \|y(0, g)\|_V^2 \\
= J(v, 0) - J(0, 0) + \sup_{y(0,g) \in Y} \left( 2 \langle C^* Cy(v, 0) , y(0, g) \rangle_V - \gamma \|y(0, g)\|_V^2 \right) \\
\leq J(v, 0) - J(0, 0) + \sup_{y \in V} \left( 2 \langle C^* Cy(v, 0) , y \rangle_V - \gamma \|y\|_V^2 \right) \\
= J(v, 0) - J(0, 0) + \frac{1}{\gamma} \|C^* Cy(v, 0)\|_V^2
\]

Identify \( V \) and \( V' \) to obtain a new optimal control problem
\[
\inf_{v \in U} J^\gamma(v) \text{ with } J^\gamma(v) = J(v, 0) - J(0, 0) + \frac{1}{\gamma} \|C^* Cy(v, 0)\|_V^2
\]

Finally, we are inside a classical optimal control problem that depends only on the control \( v \).

4. Low-regret control and no-regret control characterization (optimality systems)

**Proposition 5** The problem \((1) - (7)\) has one solution \( u_\gamma \).
Proof. We have $J^\gamma(v) \geq -J(0,0)$ for every $v \in U$ then $d^\gamma = \inf_{v \in U} J^\gamma(v)$ exists. Let $v_n = v_n(\gamma)$ be a minimizing sequence with $J^\gamma(v_n) \to d^\gamma$ then

$$-J(0,0) \leq J(v_n,0) - J(0,0) + \frac{1}{\gamma} \|C^*Cy(v_n,0)\|_V^2 \leq d^\gamma + 1$$

from this we deduce $\|v_n\|_U \leq C_\gamma$ independent of $n$. There exists $u_\gamma \in U$ such $v_n \to u_\gamma$ in $U$. Also, $y(v_n,0) \to y(u_\gamma,0)$ by continuity w.r.t the data and from strict convexity of $J^\gamma$ we deduce that $u_\gamma$ is unique.

It stays to prove that $u_\gamma$ converges to the no-regret control $u$ when $\gamma \to 0$.

**Theorem 6** The sequence of low-regret control solution to (1) – (7) converges to the no-regret control $u$ weakly in $U$ when $\gamma \to 0$.

Proof. $u_\gamma$ is a low-regret control, then for every $v \in U$ we have

$$J(u_\gamma,0) - J(0,0) + \frac{1}{\gamma} \|C^*Cy(u_\gamma,0)\|_V^2 \leq J(v,0) - J(0,0) + \frac{1}{\gamma} \|C^*Cy(v,0)\|_V^2$$

take $v = 0$ to find

$$\|Cy(u_\gamma,g) - zd\|_H^2 + N \|u_\gamma\|_U^2 + \frac{1}{\gamma} \|C^*Cy(u_\gamma,0)\|_V^2 \leq J(0,0) = \|zd\|_H^2$$

which implies

$$\|Cy(u_\gamma,0)\|_H \leq C_\gamma \|u_\gamma\|_U \leq C_\gamma \frac{1}{\sqrt{\gamma}} \|C^*Cy(u_\gamma,0)\|_V \leq C$$

which means that the sequence $u_\gamma$ is bounded in $U$ then we can extract a subsequence still be denoted $u_\gamma$ that converges weakly to $u \in U$, it stays to prove that $u$ is a no-regret control. It’s clear that for every $v \in U$

$$J(v,g) - J(0,g) - \gamma \|y(0,g)\|_V^2 \leq J(v,g) - J(0,g)$$

for every $g \in G$

then

$$J(u_\gamma,g) - J(0,g) - \gamma \|y(0,g)\|_V^2 \leq \sup_{y(0,g) \in Y} (J(v,g) - J(0,g))$$

Make $\gamma \to 0$ to find

$$J(u,g) - J(0,g) \leq \sup_{y(0,g) \in Y} (J(v,g) - J(0,g))$$
Theorem 7 The low-regret control is characterized by

\[
\begin{align*}
A y_{\gamma} &= B u_{\gamma} \\
A^* \zeta_{\gamma} &= C^* C y_{\gamma} - z_d + \frac{1}{\gamma} (C^* C)^2 y_{\gamma} \\
B^* \zeta_{\gamma} + N u_{\gamma} &= 0 \text{ in } U
\end{align*}
\]

(9)

where \( y_{\gamma} = y(u_{\gamma}, 0) \).

Proof. A first order optimality condition gives for every \( w \in U \)

\[
(C y_{\gamma} - z_d, C y(w, 0))_H + N (u_{\gamma}, w)_U + \frac{1}{\gamma} (C^* C y_{\gamma}, C^* C y(w, 0))_V \geq 0
\]

(10)

or

\[
(C^* C y_{\gamma} - z_d + \frac{1}{\gamma} (C^* C)^2 y_{\gamma}, y(w, 0))_V + N (u_{\gamma}, w)_U \geq 0
\]

Introduce \( \zeta_{\gamma} \in V \)

\[
A^* \zeta_{\gamma} = C^* C y_{\gamma} - z_d + \frac{1}{\gamma} (C^* C)^2 y_{\gamma}
\]

then rewrite (10) as

\[
(B^* \zeta_{\gamma} + N u_{\gamma}, w)_V \geq 0
\]

But \( U \) is a vector space so \( B^* \zeta_{\gamma} + N u_{\gamma} = 0 \). \( \blacksquare \)

And now, we could give the optimality system that characterize the no-regret control \( u \).

Theorem 8 The no-regret control \( u \) solution to (1) – (7) is characterized by

\[
\begin{align*}
A y &= B u \\
A^* \zeta &= C^* C y(u, 0) - z_d + \lambda \\
B^* \zeta + N u &= 0 \text{ in } U
\end{align*}
\]

(11)

with \( \lambda \in V \).

Proof. From (8) we know that \( u_{\gamma} \to u \) in \( U \) with \( B \in L(U, V') \) we conclude that \( B u_{\gamma} \to B u \) in \( V' \), and by the optimality system (9) \( A y_{\gamma} \) is bounded in \( V' \) then weakly convergent to \( A y \) (because \( A \) is an isomorphism), pass to limit to get \( A y = B u \). By the same way \( C y_{\gamma} \) is bounded in \( H \) so \( C y(u_{\gamma}, 0) \to C y(u, 0) \) in \( H \) and \( C^* C y(u_{\gamma}, 0) \to C^* C y(u, 0) \) in \( V \) then \( C^* C y_{\gamma} + \frac{1}{\gamma} (C^* C)^2 y_{\gamma} = \left( I + \frac{1}{\gamma} C^* C \right) C^* C y_{\gamma} \) converges weakly to \( C^* C y + \lambda \)
We also know that $B^* \zeta = -Nu$, the right side converges weakly to $-Nu$, for the left side $\zeta$ is bounded in $V$ and $B^* \in L(V, U)$ (we identify $U$ to $U'$) to get $B^* \zeta \rightharpoonup B^* \zeta$.

5. Application to some optimal control problems with incomplete data

In this section, we apply the above method to a various kinds of problems (elliptic, parabolic, hyperbolic) with incomplete data, and we give an optimality system for each case.

Example 9 Here, we consider the following elliptic optimal control problem with a distributed control and a missed values on boundary

\[
\begin{align*}
-\Delta y + y &= v \quad \text{in } \Omega \\
\frac{\partial y}{\partial \nu} &= g \quad \text{on } \Gamma
\end{align*}
\]

(12)

where $\Omega$ is bounded set in $\mathbb{R}^n$ with a regular boundary $\Gamma, v \in U=L^2(\Omega), g \in G = L^2(\Gamma)$, then $y(v,g)$ is unique in $H^\frac{5}{2}(\Omega)$. With the boundary cost function

\[
J(v,g) = |y(v,g) - z_d|^2_{L^2(\Gamma)} + N \|v\|_{L^2(\Omega)}^2
\]

(13)

Note that

\[
J(v,g) - J(0,g) = J(v,0) - J(0,0) + 2 (y(v,0), y(0,0))_{L^2(\Gamma)}
\]

The low-regret control is the solution of

\[
\inf_{v \in L^2(\Omega)} J'(v) \quad \text{with } J'(v) = J(v,0) - J(0,0) + \frac{1}{\gamma} |y(v,0)|^2_{L^2(\Gamma)}
\]

A first order optimality condition gives for every $w \in L^2(\Omega)$

\[
\left( y(u_\gamma, 0) + \frac{1}{\gamma} y(u_\gamma, 0) , y(u,0) \right)_{L^2(\Gamma)} + N (u_\gamma, w)_{L^2(\Omega)} \geq 0
\]

Then, we have the following proposition.

Theorem 10 The low-regret control $u_\gamma$ is unique and characterized by

\[
\begin{align*}
-\Delta y_\gamma + y_\gamma &= u_\gamma, -\Delta \zeta_\gamma + \zeta_\gamma &= 0 \\
\frac{\partial y_\gamma}{\partial \nu} &= 0, \frac{\partial \zeta_\gamma}{\partial \nu} = y_\gamma - z_d + \frac{1}{\gamma} y_\gamma \quad \text{in } \Omega \\
\zeta_\gamma + Nu_\gamma &= 0 \quad \text{on } \Gamma \\
\zeta_\gamma + Nu_\gamma &= 0 \quad \text{in } L^2(\Omega)
\end{align*}
\]
To obtain no-regret control optimality system, adapt the proof of theorem 8 to find.

**Theorem 11** The no-regret control \( u \) is unique and characterized by
\[
\begin{cases}
-\Delta y + y = u; & -\Delta \zeta + \zeta = 0 \quad \text{in } \Omega \\
\frac{\partial y}{\partial \nu} = 0; \quad \frac{\partial \zeta}{\partial \nu} = y - z_d + \lambda & \text{on } \Gamma \\
\zeta + Nu = 0 & \text{in } L^2 (\Omega)
\end{cases}
\]
where \( \lambda \in L^2 (\Gamma) \)

**Example 12** A high order parabolic problem: it’s a fourth order equation with a distributed control and an uncertainty in the initial state, given by
\[
\begin{cases}
y + \Delta (a (x, t) \Delta y) = v & \text{in } Q \\
y = 0, \quad \frac{\partial y}{\partial \nu} = 0 & \text{on } \Sigma \\
y (0) = g & \text{in } \Omega
\end{cases}
\] (14)

Where \( \Omega \) is an open in \( \mathbb{R}^n \) with a smooth boundary \( \Gamma \), \( t \in [0; T], T > 0, Q = \Omega \times [0; T], \Sigma = \Gamma \times [0; T] \), with \( a \in L^\infty (Q) \), \( a \geq \alpha > 0 \) almost every where, \( v \in U = L^2 (Q) \), \( g \in G = L^2 (\Omega) \). The problem (14) has a unique solution in \( L^2 (0, T; H^2_0 (\Omega)) \) (see [3]). Associate to the cost function
\[
J (v, g) = \| y (v, g) - z_d \|_{L^2 (Q)}^2 + N \| v \|_{L^2 (Q)}^2
\]
We have
\[
J (v, g) - J (0, 0) = J (v, 0) - J (0, 0) + 2 \langle y (v, 0), y (0, g) \rangle_{L^2 (Q)}
\]
The low-regret control is the solution of
\[
\inf_{v \in L^2 (Q)} J^* (v) \quad \text{with } J^* (v) = J (v, 0) - J (0, 0) + \frac{1}{\gamma} \| y (v, 0) \|_{L^2 (Q)}^2
\]

**Theorem 13** The low-regret \( u_\gamma \) control is unique and characterized by
\[
\begin{cases}
y_\gamma' + \Delta (a (x, t) \Delta y_\gamma) = -\zeta_\gamma + \Delta (a (x, t) \Delta y_\gamma) = y_\gamma - z_d + \frac{1}{\gamma} y_\gamma & \text{in } Q \\
y_\gamma = 0, \quad \frac{\partial y_\gamma}{\partial \nu} = 0; \quad \zeta_\gamma = 0, \quad \frac{\partial \zeta_\gamma}{\partial \nu} = 0 & \text{on } \Sigma \\
y_\gamma (0) = 0; \quad \zeta_\gamma (T) = 0 & \text{in } \Omega
\end{cases}
\]
where \( y_\gamma = y (u_\gamma, 0) \), with
\[
\zeta_\gamma + Nu_\gamma = 0 \quad \text{in } L^2 (Q)
\]
Proof. Again, for every $w \in L^2(Q)$

$$
\left( y(u,0) - z_d + \frac{1}{\gamma} y(v,0), y(v-u,0) \right)_{L^2(Q)} + N(u,v-u)_{L^2(L^2(Q))} \geq 0
$$

Introduce

$$
\begin{cases}
-\zeta' - \Delta \zeta = y - z_d + \frac{1}{\gamma} y & \text{in } Q \\
\zeta = 0, \frac{\partial \zeta}{\partial \nu} = 0 & \text{on } \Sigma \\
\zeta(T) = 0 & \text{in } \Omega
\end{cases}
$$

to get

$$
\zeta + Nu = 0 \text{ in } L^2(Q)
$$

\[ \blacksquare \]

**Theorem 14** The no-regret $u$ control is unique and characterized by

$$
\begin{cases}
y' + \Delta(a(x,t)\Delta y) = u; -\zeta' + \Delta(a(x,t)\Delta y) = y(u,0) - z_d + \lambda & \text{in } Q \\
y = 0, \frac{\partial y}{\partial \nu} = 0; \zeta = 0, \frac{\partial \zeta}{\partial \nu} = 0 & \text{on } \Sigma \\
y(0) = 0; \zeta(T) = 0 & \text{in } \Omega
\end{cases}
$$

with

$$
\zeta + Nu = 0 \text{ in } L^2(Q)
$$

**Example 15** Let's take a hyperbolic example: a wave equation with a boundary control and a missed source

$$
\begin{cases}
y'' - \Delta y = g & \text{in } Q \\
\frac{\partial y}{\partial \nu} = v & \text{on } \Sigma \\
y(0) = 0; y'(0) = 0 & \text{in } \Omega
\end{cases}
$$

Where $\Omega$ is an open in $\mathbb{R}^n$ with a smooth boundary $\Gamma$, $t \in [0; T], T > 0, Q = \Omega \times [0; T], \Sigma = \Gamma \times [0; T], v \in U=L^2(\Sigma), g \in G = L^2(Q)$. There is a unique solution in sense of transposition $y \in L^2(Q)$ (see[3]). With the cost function

$$
J(v,g) = \|y(v,g) - z_d\|_{L^2(Q)}^2 + N\|v\|_{L^2(\Sigma)}^2
$$

Again

$$
J(v,g) - J(0,g) = J(v,0) - J(0,0) + 2(y(v,0), y(0,g))_{L^2(Q)}
$$

The low-regret control is the solution of

$$
\inf_{v \in L^2(\Sigma)} J^\gamma(v) \text{ with } J^\gamma(v) = J(v,0) - J(0,0) + \frac{1}{\gamma}\|y(v,0)\|_{L^2(Q)}^2
$$
Theorem 16  The new low-regret control \( u_\gamma \) is characterized by
\[
\begin{aligned}
\begin{cases}
\begin{aligned}
y''_\gamma - \Delta y_\gamma &= 0; \\
\zeta''_\gamma - \Delta \zeta_\gamma &= y_\gamma - z_d + \frac{1}{\gamma} y_\gamma \\
\frac{\partial y_\gamma}{\partial \nu} &= u_\gamma; \quad \frac{\partial \zeta_\gamma}{\partial \nu} = 0
\end{aligned}
\text{in } Q \\
y_\gamma (0) = 0, \ y'_\gamma (0) = 0; \ \zeta_\gamma (T) = 0, \ \zeta'_\gamma (T) = 0
\text{on } \Sigma 
\end{cases}
\end{aligned}
\]
where \( y_\gamma = y (u_\gamma, 0) \), and \( \zeta_\gamma + Nu_\gamma = 0 \text{ in } L^2 (\Sigma) \).

Proof. First order optimality condition writes for every \( v \in L^2 (\Sigma) \)
\[
\left( y_\gamma - z_d + \frac{1}{\gamma} y_\gamma, y (w, 0) \right)_{L^2 (Q)} + N (u_\gamma, w)_{L^2 (\Sigma)} \geq 0
\]
\[
\left( y (u_\gamma, 0) - z_d + \frac{1}{\gamma} y (u_\gamma, 0), y (v - u_\gamma, 0) \right)_{L^2 (\Gamma)} + N (u_\gamma, v - u_\gamma)_{L^2 (\Omega)} \geq 0
\]
Define a state \( \zeta_\gamma \) by
\[
\begin{aligned}
\begin{cases}
\zeta''_\gamma - \Delta \zeta_\gamma &= y_\gamma - z_d + \frac{1}{\gamma} y_\gamma \\
\frac{\partial \zeta_\gamma}{\partial \nu} &= 0 \\
\zeta_\gamma (T) = 0, \ \zeta'_\gamma (T) = 0
\end{cases}
\text{in } Q \text{ on } \Sigma \text{ in } \Omega
\end{aligned}
\]
to get
\[
\zeta_\gamma + Nu_\gamma = 0 \text{ in } L^2 (Q)
\]


Theorem 17  The no-regret control \( u \) is characterized by
\[
\begin{aligned}
\begin{cases}
\begin{aligned}
y'' - \Delta y &= 0; \\
\zeta'' - \Delta \zeta &= y - z_d + \lambda \\
\frac{\partial \zeta}{\partial \nu} &= \lambda w; \quad \frac{\partial \zeta}{\partial \nu} = 0
\end{aligned}
\text{in } Q \\
y (0) = 0, \ y' (0) = 0; \ \zeta (T) = 0, \ \zeta' (T) = 0
\text{on } \Sigma 
\end{cases}
\end{aligned}
\]
where \( \lambda \in L^2 (Q) \), and \( \zeta + Nu = 0 \text{ in } L^2 (\Sigma) \).

Conclusion 18  As we have seen, the redefinition of the no-regret control (an equivalent definition to the original one) and consequently the low-regret control, leads to a more simple characterization of no-regret control i.e. a more simple optimality system.
6. Bibliographie