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UNITARY STABLE RANKS AND NORM-ONE RANKS

RAYMOND MORTINI

ABSTRACT. In the context of commutative C^* -algebras we solve a problem related to a question of M. Rieffel by showing that the all-units rank and the norm-one rank coincide with the topological stable rank. We also introduce the notion of unitary M -stable rank for an arbitrary commutative unital ring and compare it with the Bass stable rank. In case of uniform algebras, a sufficient condition for norm-one reducibility is given.

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INTRODUCTION

Let C be a C^* algebra with identity. Given a pair (a, b) of elements in C for which $aC + bC = C$, one can conclude from the work of Robertson [9] that there exist two units u and v in C^{-1} with $ua + vb = \mathbf{1}$ if and only if A has dense invertible group. In that case there even exists a unitary element $u \in C$ (that is an element satisfying $uu^* = u^*u = \mathbf{1}$) such that $a + ub \in C^{-1}$. In his groundbreaking paper [8, p. 307], Mark Rieffel posed the problem whether there is an analogue for C^* -algebras C with $\text{tsr } C = n$. This question was re-asked in [1]. We shall give a positive answer to weaker versions of this question in context of the algebra $C(X, \mathbb{K})$ of $\mathbb{K}$ -valued continuous functions on a compact Hausdorff space X , where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. To this end we give several possible ways of extending the definition of the unit-1-stable rank (see [2]) from pairs (a, b) to $(n+1)$ -tuples. Some of them were briefly mentioned in [1]. Generally speaking, we replace “unitary” elements in C (which correspond to unimodular functions in $C(X, \mathbb{K})$) either by invertible elements (called units) or by norm-one elements. The original question by Rieffel remains unanswered, though.

Let R be a commutative unital ring. Then

$$U_n(R) = \{\mathbf{f} = (f_1, \dots, f_n) \in R^n : \sum_{j=1}^n Rf_j = R\}$$

is the set of invertible n -tuples. If R carries a topology, then the *topological stable rank*, $\text{tsr } R$, of R is the smallest integer n for which $U_n(R)$ is dense in R^n (or infinity if $U_n(R)$ is never dense). This concept was introduced

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by Rieffel [8]. It is well known that within the realm of commutative unital Banach algebras A one has $\text{bsr } A \leq \text{tsr } A$, where $\text{bsr } A$ is the Bass stable rank of A . Recall that this item is defined to be the smallest integer n for which any $(\mathbf{f}, g) \in U_{n+1}(R)$ is reducible in the sense that there exists $\mathbf{x} \in R^n$ such that $\mathbf{f} + \mathbf{x}g \in U_n(R)$.

Let us recall the following easy fact, which was one of the motivations for dubbing these items “stable ranks” (they satisfy certain stabilizing properties):

Proposition 0.1. *Let A be a commutative unital algebra. Suppose that $\text{bsr } A = n$, $n < \infty$, and let $m \geq n$. Then every invertible $(m+1)$ -tuple $(\mathbf{f}, g) \in A^{m+1}$ is reducible.*

As usual, a Q -algebra is a commutative unital topological algebra over $\mathbb{K}$ for which the set A^{-1} of units is open. If, additionally, inversion $x \rightarrow x^{-1}$ is a continuous operation on A^{-1} , then we call A a cQ -algebra. The following interesting characterization of the topological stable rank (see [1, p. 52]) is the key to our results.

Theorem 0.2. *Let $A = (A, |\cdot|)$ be a normed cQ -algebra. For $\mathbf{a} = (a_1, \dots, a_n) \in A^n$, let $\|\mathbf{a}\| = \sum_{j=1}^n |a_j|$ be a fixed norm on the product space. Then the following assertions are equivalent:*

- (1) $\text{tsr } A \leq n$;
- (2) For every $(\mathbf{a}, g) \in U_{n+1}(A)$ there is $\mathbf{v} \in U_n(A)$ and $\mathbf{y} \in A^n$ such that
 - i) $\|\mathbf{v} - \mathbf{a}\| < \varepsilon$,
 - ii) $\mathbf{v} = \mathbf{a} + \mathbf{y}g$.

1. THE UNITARY STABLE RANKS

We begin with two possible extensions of the definition of the unit-1-stable rank. Recall that a commutative unital ring has the *unit-1-stable rank* if for every invertible pair $(a, b) \in U_2(R)$ there exist $u, v \in R^{-1}$ such that $au + bv = \mathbf{1}$. In that case one says that (a, b) is *totally reducible*.

Definition 1.1. Let R be a commutative unital ring.

- (1) The *unitary M -stable rank*¹, $\text{usr } R$, of R is the smallest integer n such that for every $(\mathbf{a}, b) \in U_{n+1}(R)$ there is $\mathbf{u} \in U_n(R)$ such that $\mathbf{a} + \mathbf{u}b \in U_n(R)$. If there exists no such n , then we put $\text{usr } R = \infty$.
- (2) The *all-units rank*, $\text{aur } R$, of R is the smallest integer n such that for every $(\mathbf{a}, b) \in U_{n+1}(R)$ there are $u_j \in R^{-1}$ such that $\mathbf{a} + \mathbf{u}b \in U_n(R)$, where $\mathbf{u} = (u_1, \dots, u_n)$. If there exists no such n , then we put $\text{aur } R = \infty$.

Note that $\text{bsr } R \leq \text{usr } R \leq \text{aur } R$ is a trivial estimate. Thus, if $\text{aur } R = 1$, then $\text{bsr } R = \text{usr } R = \text{aur } R = 1$, and this holds if and only if R has the unit-1-stable rank.

¹in order to distinguish our stable rank here from the one given in [6], I added my initial M here

Theorem 1.2. *Let R be a commutative unital ring. Then*

- (1) *The unitary M -stable rank has the stabilizing property; that is if $\text{usr } R = n < \infty$, and if $m \geq n$ then, for any $(\mathbf{f}, g) \in U_{m+1}(R)$ there is $\mathbf{u} \in U_m(R)$ such that $\mathbf{f} + \mathbf{u}g \in U_m(R)$.*
- (2) $\text{bsr } R \leq \text{usr } R \leq \text{bsr } R + 1$.

Both cases in (2) can occur.

- I don't know whether the all-units rank has the stabilizing property.

Proof. (1) We may assume that $m \geq n + 1$. Let $(f_1, \dots, f_m, g) \in U_{m+1}(R)$. Then $(f_1, \dots, f_n, f_{n+1} + g, \dots, f_m + g, g) \in U_{m+1}(A)$, too. Hence, there is $(a_1, \dots, a_{m+1}) \in R^{m+1}$ such that

$$(1.1) \quad \sum_{j=1}^n a_j f_j + \left(\sum_{j=n+1}^m a_j (f_j + g) + a_{m+1} g \right) = \mathbf{1}.$$

Put $h := \sum_{j=n+1}^m a_j (f_j + g) + a_{m+1} g$. Then

$$(f_1, \dots, f_n, h) \in U_{n+1}(R).$$

Since $\text{bsr } R \leq \text{usr } R = n$, there exists ² $(x_1, \dots, x_n) \in A^n$ such that

$$(f_1 + x_1 h, \dots, f_n + x_n h) \in U_n(A);$$

that is

$$\sum_{j=1}^n y_j (f_j + x_j h) = \mathbf{1} \text{ for some } (y_1, \dots, y_n) \in A^n.$$

We claim that

$$(f_1 + x_1 a_{m+1} g, \dots, f_n + x_n a_{m+1} g, f_{n+1} + g, \dots, f_m + g) \in U_m(A).$$

To show this, note that h has the form $h = r + a_{m+1} g$, where $r \in I_A(f_{n+1} + g, \dots, f_m + g)$. Hence

$$\begin{aligned} \mathbf{1} &= \sum_{j=1}^n y_j (f_j + x_j a_{m+1} g) + \sum_{j=1}^n y_j x_j r \\ &\in I_A(f_1 + x_1 a_{m+1} g, \dots, f_n + x_n a_{m+1} g, f_{n+1} + g, \dots, f_m + g). \end{aligned}$$

If we put $u_j = x_j a_{m+1}$ for $j = 1, \dots, n$ and $u_j = \mathbf{1}$ for $j = n+1, \dots, m$, then we see that $\mathbf{f} + \mathbf{c}g \in U_m(A)$, where $\mathbf{c} = (c_1, \dots, c_m)$. Moreover, $\mathbf{c} \in U_m(R)$, since at least one coordinate is $\mathbf{1}$.

(2) Since the first inequality $\text{bsr } R \leq \text{usr } R$ is obvious, it remains to show that $\text{usr } R \leq \text{bsr } R + 1$. But this follows from the proof of part (1) by putting $m = n + 1$, where $n = \text{bsr } R$.

Since $\text{tsr } C([0, 1], \mathbb{C}) = 1$, we may approximate the solution (x, y) to $xa + yb = 1$ by an invertible pair (u, v) . Hence $ua + vb$ is invertible again. So $\text{usr } C([0, 1], \mathbb{C}) = 1 = \text{bsr } C([0, 1], \mathbb{C})$. By [7], (z, f) is not totally reducible for every $f \in A(\mathbb{D})$ with $f(0) \neq 0$. Hence $\text{usr } A(\mathbb{D}) \geq 2$. But $\text{bsr } A(\mathbb{D}) = 1$, ([5]). Hence $\text{usr } A(\mathbb{D}) = 2$. $\square$

² Here we may use Proposition 0.1 or directly the assumption $\text{usr } R = n$.

Here is a first relation of the unitary M -stable rank to the topological stable rank.

Proposition 1.3. *Let A be a Q -algebra. Then $\text{bsr } A \leq \text{usr } A \leq \text{tsr } A$.*

• I don't know whether $\text{usr } A \leq \text{aur } A \leq \text{tsr } A$ or $\text{usr } A \leq \text{tsr } A \leq \text{aur } A$ always holds for normed Q -algebras.

Proof. The first inequality, $\text{bsr } A \leq \text{usr } A$ is trivial. Now suppose that $n := \text{tsr } A < \infty$. Let $(a, b) \in U_{n+1}(A)$. Then there is $x \in A^n$ and $y \in A$ such that $x \cdot a + yb = 1$. Since $\text{tsr } A \leq n$, there is a net $(u_\lambda) \in U_n(A)$ converging to x . Since A is a topological algebra, $v_\lambda := u_\lambda \cdot a + yb$ tends to 1 . The openness of the set of units of A now implies that $v_\lambda \in A^{-1}$ whenever λ is large. We fix some of these λ . If $u_\lambda = (u_1, \dots, u_n)$, then the ideal $I_A(u_1, \dots, u_n)$ coincides with A . Hence there is $y_\lambda \in A^n$ such that $y = u_\lambda \cdot y_\lambda$. Thus

$$v_\lambda = u_\lambda \cdot (a + y_\lambda b) \in A^{-1}.$$

Since $\text{tsr } A = n$, we may approximate y_λ by $w_\lambda \in U_n(A)$. Hence $u_\lambda \cdot (a + w_\lambda b) \in A^{-1}$ whenever w_λ is sufficiently close to y_λ . We conclude that $a + w_\lambda b \in U_n(A)$ and so $\text{usr } A \leq n$. $\square$

The preceding result shows that in case of a Q -algebra A , $\text{tsr } A = 1$ is a sufficient condition for $\text{usr } A = 1$.

2. THE SMALL-NORM AND THE NORM-ONE RANKS

The following two concepts are briefly mentioned in [1].

Definition 2.1. Let $A = (A, \|\cdot\|)$ be a normed algebra.

- (1) A is said to have the *norm-one rank* n (denoted by $\text{nor } A$) if n is the smallest integer (or infinity) such that for every $(f, g) \in U_{n+1}(A)$ there is $c = (c_1, \dots, c_n) \in A^n$ such that $\|c_j\| = 1$ and

$$f + c g \in U_n(A).$$

- (2) A is said to have the *small-norm rank* n (denoted by $\text{snr } A$) if n is the smallest integer (or infinity) such that for every $\varepsilon > 0$ and every $(f, g) \in U_{n+1}(A)$ there is $a = (a_1, \dots, a_n) \in A^n$ such that $\|a_j\| < \varepsilon$ and

$$f + a g \in U_n(A).$$

- I don't know whether these ranks have the stabilizing property.

Let $S_A = \{a \in A : \|a\| = 1\}$ be the unit sphere in A . The following relations now hold between the different ranks. The striking point is that the norm-one rank is bigger than the topological stable rank. This result is due to Badea [1]. We re-present here for the reader's convenience the simple proof.

Proposition 2.2 (Badea). *Let $A = (A, \|\cdot\|)$ be normed cQ -algebra and $\|a\| := \sum_{j=1}^n \|a_j\|$, $a \in A^n$. Then*

$$\text{bsr } A \leq \text{usr } A \leq \text{tsr } A \leq \text{snr } A \leq \text{nor } A.$$

Proof. The first two inequalities are dealt with in Proposition 1.3. To show $\text{tsr } A \leq \text{snr } A \leq \text{nor } A$, we will use Theorem 0.2. So suppose that $n := \text{nor } A < \infty$. Let $(\mathbf{a}, a_{n+1}) \in U_{n+1}(A)$. Then, for every $k \in \mathbb{N}^*$, $(\mathbf{a}, (1/k)a_{n+1}) \in U_{n+1}(A)$. By hypothesis, there is $\mathbf{d}_k \in A^n \cap (S_A)^n$ (depending on k), such that

$$\mathbf{a} + \mathbf{d}_k \frac{a_{n+1}}{k} \in U_n(A).$$

Now given $\varepsilon > 0$, choose $k = k(\varepsilon)$ so big that

$$\max \left\{ \frac{1}{\varepsilon}, \frac{\|\mathbf{a}_{n+1}\|}{\varepsilon} n \right\} < k(\varepsilon).$$

Let $\mathbf{x} := \mathbf{d}_{k(\varepsilon)}/k(\varepsilon)$. Then $\|x_j\| \leq \varepsilon$ for $j = 1, \dots, n$ and

$$\mathbf{v} := \mathbf{a} + \mathbf{x} a_{n+1} \in U_n(A).$$

Thus $\text{snr } A \leq n$. Moreover, since $\|\mathbf{v} - \mathbf{a}\| < \varepsilon$, we conclude from Theorem 0.2, that $\text{tsr } A \leq \text{snr } A$. $\square$

Proposition 2.3. *Let A be normed cQ -algebra. Then*

$$\text{bsr } A \leq \text{usr } A \leq \text{aur } A \leq \text{snr } A \leq \text{nor } A.$$

Proof. In view of Theorem 2.2 it only remains to show that $\text{aur } A \leq \text{snr } A$. Since A^{-1} is open, we may choose $\delta > 0$ so that for all $a \in A$, $\|a - \mathbf{1}\| < \delta$ implies $a \in A^{-1}$. Suppose now that $n := \text{snr } A < \infty$. Let $(\mathbf{f}, g) \in U_{n+1}(A)$ and put $\mathbf{e} := (\mathbf{1}, \dots, \mathbf{1})$. Then $(\mathbf{f} - \mathbf{e}g, g) \in U_{n+1}(A)$. Given $0 < \varepsilon < \delta$, there is, by assumption, $\mathbf{x} = (x_1, \dots, x_n) \in A^n$ with $\|x_j\| \leq \varepsilon$, such that

$$(\mathbf{f} - \mathbf{e}g) + \mathbf{x}g \in U_n(A).$$

Hence $\mathbf{f} + (\mathbf{x} - \mathbf{e})g \in U_n(A)$. But $a_j := \mathbf{1} - x_j \in A^{-1}$, because $\|a_j - \mathbf{1}\| = \|x_j\| < \varepsilon < \delta$. Hence $\text{aur } A \leq n$. $\square$

Our main goal in this subsection is to determine the norm-one rank of $C(X, \mathbb{K})$. To this end, we need a refinement of Theorem 0.2 (in case of the algebra $A = C(X, \mathbb{K})$). This refinement will say that in the equation $\mathbf{f} + \mathbf{y}g \in U_n(C(X, \mathbb{K}))$, $n = \text{tsr } C(X, \mathbb{K})$, we can actually choose $\mathbf{y} = (y_1, \dots, y_n)$ in such a way that all its components y_j have norm as small as we wish (in Badea's result we had $\|y_j g\|_\infty < \varepsilon$).

Proposition 2.4. *Let X be a compact Hausdorff space. Then*

$$\text{snr } C(X, \mathbb{K}) = \text{tsr } C(X, \mathbb{K}).$$

Proof. In view of Proposition 2.2, it remains to show that $\text{snr } C(X, \mathbb{K}) \leq \text{tsr } C(X, \mathbb{K})$. So let $n := \text{tsr } C(X, \mathbb{K}) < \infty$ and fix $(\mathbf{f}, g) \in U_{n+1}(C(X, \mathbb{K}))$.

Case 1 $Z(g) = \emptyset$. Then g is invertible and $(g^{-1}\mathbf{f}, 1) \in U_{n+1}(C(X, \mathbb{K}))$. By Theorem 0.2, for every $\varepsilon > 0$, there is $\mathbf{y} = (y_1, \dots, y_n) \in C(X, \mathbb{K})^n$, $\|y_j \cdot 1\|_\infty \leq \varepsilon$, such that

$$g^{-1}\mathbf{f} + \mathbf{y} \cdot 1 \in U_n(C(X, \mathbb{K})).$$

Hence $\mathbf{f} + \mathbf{y}g \in U_n(C(X, \mathbb{K}))$.

Case 2 $Z(g) \neq \emptyset$. Choose an open neighborhood U of $Z(g)$ such that $\mathbf{f} \neq \mathbf{0}$ on U . Let V, W be two open sets satisfying $Z(g) \subseteq W \subseteq \overline{W} \subseteq V \subseteq \overline{V} \subseteq U$. Since X is normal, there is $\phi \in C(X, [0, 1])$ which

$$\phi \equiv 0 \text{ on } \overline{V} \text{ and } \phi = 1 \text{ on } X \setminus U.$$

Then $\overline{V} \subseteq Z(\phi) \subseteq U$. We deduce that $(\mathbf{f}, \phi) \in U_{n+1}(C(X, \mathbb{K}))$. Let $\varepsilon > 0$ and

$$\delta := \min\{|g(x)| : x \in X \setminus W\}.$$

Note that $\delta > 0$. Since, by assumption, $\text{tsr } C(X, \mathbb{K}) = n$, we may use Theorem 0.2 to get a function $\mathbf{h} = (h_1, \dots, h_n) \in C(X, \mathbb{K}^n)$ with

$$\mathbf{u} := \mathbf{f} + \mathbf{h}\phi \in U_n(C(X, \mathbb{K})) \text{ and } \|h_j\phi\|_\infty \leq \varepsilon\delta.$$

Now we define a function $\mathbf{a} = (a_1, \dots, a_n)$ by

$$a_j = \begin{cases} \frac{1}{g}(u_j - f_j) & \text{on } X \setminus W \\ 0 & \text{on } V \end{cases}$$

Since $\mathbf{u} = \mathbf{f}$ on $V \supseteq \overline{W}$, we conclude that $\mathbf{a}$ is well-defined and hence continuous. Moreover,

$$|a_j| \leq \begin{cases} \frac{1}{\delta} \varepsilon\delta = \varepsilon & \text{on } X \setminus W \\ 0 & \text{on } V. \end{cases}$$

Thus $\|a_j\|_\infty \leq \varepsilon$. Finally

$$\mathbf{f} + \mathbf{a}g = \begin{cases} \mathbf{f} + (\mathbf{u} - \mathbf{f}) = \mathbf{u} & \text{on } X \setminus W \\ \mathbf{f} + \mathbf{0} = \mathbf{u} & \text{on } V. \end{cases}$$

In other words, $\mathbf{f} + \mathbf{a}g = \mathbf{u} \in U_n(C(X, \mathbb{K}))$. $\square$

Theorem 2.5. *Let X be a compact Hausdorff space. Then*

$$\text{bsr } C(X, \mathbb{K}) = \text{tsr } C(X, \mathbb{K}) = \text{nor } C(X, \mathbb{K}).$$

Proof. By Vasershtein's result [10], we already have $\text{bsr } C(X, \mathbb{K}) = \text{tsr } C(X, \mathbb{K})$.

In view of Proposition 2.2, it suffices to show that $\text{nor } C(X, \mathbb{K}) \leq \text{tsr } C(X, \mathbb{K})$.

Let $A = C(X, \mathbb{K})$ and $n := \text{tsr } A$.

Case 1 $n = 1$. Let $(f, g) \in U_2(A)$. First suppose that $Z(g) = \emptyset$. Since $\text{tsr } A = 1$, there is $u \in A^{-1}$ such that $\|g^{-1}f - u\|_\infty \leq 1/2$. Now

$$g^{-1}f + \frac{u}{|u|} \neq 0 \text{ on } X,$$

because

$$g^{-1}f + \frac{u}{|u|} = (g^{-1}f - u) + u(1 + \frac{1}{|u|}) = (g^{-1}f - u) + \frac{u}{|u|} (1 + |u|),$$

and the second summand has modulus strictly bigger than 1. Hence

$$f + \frac{u}{|u|}g \in U_1(A).$$

If $Z(g) \neq \emptyset$, we use Proposition 2.4 to conclude that there is $a \in A$ with $u := f + ag \in U_1(A)$ and $\|a\|_\infty < 1/2$. Approximating a by an invertible function we may assume that a already is invertible. Since $f \neq 0$ on $Z(g)$, say $|f| > \delta > 0$ on $Z(g)$, we may choose two open sets U and V such that

$$Z(g) \subseteq U \subseteq \overline{U} \subseteq V \subseteq \overline{V} \subseteq \{x \in X : |g| < \delta/2\} \cap \{x \in X : |f(x)| > \delta\}.$$

Let $x_0 \in U$. We will construct a function $\phi \in A$ such that

$$|(a\phi)(x_0)| = 1 \text{ and } \|a\phi\|_\infty \leq 1$$

and $f + (a\phi)g \neq 0$ on X . To this end, let $\psi \in C(X, [0, 1])$ satisfy

$$\psi \equiv 0 \text{ on } X \setminus V \text{ and } \psi = 1 \text{ on } U$$

and let ϕ be defined by

$$\phi = \frac{1}{a}\psi + (1 - \psi).$$

Then ϕ does the job. In fact,

- $(a\phi)(x_0) = \psi(x_0) + a(x_0) \cdot 0 = 1$;
- $|a\phi| \leq \psi + |a|(1 - \psi) \leq \psi + (1 - \psi) = 1$;
- $|f + (a\phi)g| = |f + ag| = |u| > 0$ on $X \setminus V$ and
- $|f + (a\phi)g| \geq |f| - |a\phi||g| \geq \delta - 1 \cdot |g| \geq \delta/2 > 0$ on V .

We conclude that $\text{nor } A = 1$. So the case $n = 1$ is settled completely.

Case 2 $\text{tsr } A = n < \infty$.

For $\mathbf{f} = (f_1, \dots, f_n) \in A^n$, set $\|\mathbf{f}\| = \sqrt{\sum_{j=1}^n \|f_j\|_\infty^2}$ and $|\mathbf{f}| := \sqrt{\sum_{j=1}^n |f_j|^2}$.

Note that $|\mathbf{f}| \leq \|\mathbf{f}\|$.

Let $(\mathbf{f}, g) \in U_{n+1}(A)$. We first assume that $Z(g) \neq \emptyset$. By Theorem 2.4, there is $\mathbf{y} = (y_1, \dots, y_n) \in A^n$ with $\mathbf{u} := \mathbf{f} + \mathbf{y}g \in U_n(A)$ and $|y_j| \leq 1/2$. Since $\mathbf{f} \neq \mathbf{0}$ on $Z(g)$, say $|\mathbf{f}| > \delta > 0$ on $Z(g)$, we may choose two open sets U and V such that

$$Z(g) \subseteq U \subseteq \overline{U} \subseteq V \subseteq \overline{V} \subseteq \{x \in X : |g| < \delta/(2\sqrt{n})\} \cap \{x \in X : |\mathbf{f}(x)| > \delta\}.$$

Fix $x_0 \in Z(g)$. As above, let $\psi \in C(X, [0, 1])$ satisfy

$$\psi \equiv 0 \text{ on } X \setminus V \text{ and } \psi = 1 \text{ on } U.$$

For $j = 1, \dots, n$, let v_j be defined by

$$v_j = \psi + y_j(1 - \psi),$$

and put $\mathbf{v} = (v_1, \dots, v_n)$. We claim that

$$\mathbf{f} + \mathbf{v}g \in U_n(A) \text{ and } \|v_j\|_\infty = 1.$$

In fact,

- $|v_j| \leq \psi + (1/2)(1 - \psi) \leq 1$;
- $|v_j(x_0)| = \psi(x_0) = 1$; hence $\|v_j\|_\infty = 1$;
- $|\mathbf{f} + \mathbf{v}g| = |\mathbf{f} + \mathbf{y}g| = |\mathbf{u}| > 0$ on $X \setminus V$;
- $|\mathbf{f} + \mathbf{v}g| \geq |\mathbf{f}| - |g||\mathbf{v}| \geq \delta - \sqrt{n}\delta/(2\sqrt{n}) = \delta/2$ on V .

Suppose now that $Z(g) = \emptyset$ and let $(\mathbf{f}, g) \in U_{n+1}(A)$, $n \geq 2$ (the case $n = 1$ was done in the preceding paragraph). Then $(g^{-1}\mathbf{f}, 1) \in U_{n+1}(A)$ and it suffices to prove the existence of $\mathbf{v} = (v_1, \dots, v_n) \in A^n$ such that $\|v_j\|_\infty = 1$ and

$$g^{-1}\mathbf{f} + \mathbf{v} \in U_n(A).$$

Let $\mathbf{F} := g^{-1}\mathbf{f}$ and denote the coordinates of $\mathbf{F}$ by F_j . Since $\text{tsr } A = n$, there is $\mathbf{u} = (u_1, \dots, u_n) \in U_n(A)$ such that

$$\|\mathbf{F} - \mathbf{u}\| < 1/2.$$

We shall proceed inductively, with respect to the length of invertible sub-tuples of $\mathbf{u}$, and will frequently use the following type of estimates. Let $\tilde{\mathbf{u}} := (u_1, \dots, u_m) \in U_m(A)$ and

$$\mathbf{v} = (v_1, \dots, v_m) := \left(\frac{u_1}{|\tilde{\mathbf{u}}|}, \dots, \frac{u_m}{|\tilde{\mathbf{u}}|} \right).$$

The hypothesis $\tilde{\mathbf{u}} \in U_m(A)$ (or equivalently $|\tilde{\mathbf{u}}| \geq \delta > 0$ on X) implies that $\mathbf{v} \in A^m$ and each coordinate of $\mathbf{v}$ has norm less than 1 (may be strict). Moreover, if $\tilde{\mathbf{F}} = (F_1, \dots, F_m)$, then

$$(2.1) \quad \tilde{\mathbf{F}} + \mathbf{v} \in U_m(A),$$

because

$$\begin{aligned} |\tilde{\mathbf{F}} + \mathbf{v}| &= |(\tilde{\mathbf{F}} - \tilde{\mathbf{u}}) + (\tilde{\mathbf{u}} + \mathbf{v})| \\ &\geq |\tilde{\mathbf{u}} + \mathbf{v}| - |\tilde{\mathbf{F}} - \tilde{\mathbf{u}}| \\ &= (1 + |\tilde{\mathbf{u}}|) - |\tilde{\mathbf{F}} - \tilde{\mathbf{u}}| \\ &\geq 1 - \|\mathbf{F} - \mathbf{u}\| \geq 1/2. \end{aligned}$$

• If $u_1 \in A^{-1} = U_1(A)$ then, by the paragraph above for $m = 1$, we see that $F_1 + u_1/|u_1| \in A^{-1}$. (Note that $|F_1 - u_1| < 1/2$). Hence

$$\left(F_1 + \frac{u_1}{|u_1|}, F_2 + 1, \dots, F_n + 1 \right) \in U_n(A).$$

• If $\mathbf{u}_{1,2} := (u_1, u_2) \in U_2(A)$, but neither u_1 nor u_2 is in $U_1(A)$, then there are $x_j \in X$ such that $u_j(x_j) = 0$, ($j = 1, 2$). Hence, the coordinates of

$$\mathbf{v}_{1,2} := \left(\frac{u_1}{\sqrt{|u_1|^2 + |u_2|^2}}, \frac{u_2}{\sqrt{|u_1|^2 + |u_2|^2}} \right),$$

have norm 1. Moreover, by (2.1).

$$\mathbf{H}_{1,2} := \mathbf{F}_{1,2} + \mathbf{v}_{1,2} := (F_1 + v_1, F_2 + v_2) \in U_2(A),$$

and so

$$(F_1 + v_1, F_2 + v_2, F_3 + 1, \dots, F_n + 1) \in U_n(A).$$

• If $\mathbf{u}_{1,2,3} := (u_1, u_2, u_3) \in U_3(A)$, but neither (u_1, u_2) , (u_1, u_3) nor (u_2, u_3) in $U_2(A)$, then there are $x_{1,2}, x_{1,3}, x_{2,3} \in X$ such that $u_i(x_{i,j}) = 0$, ($i = 1, 2$),

$u_i(x_{1,3}) = 0$, ($i = 1, 3$), and $u_i(x_{2,3}) = 0$, ($i = 2, 3$). Hence, the coordinates of

$$\mathbf{v}_{1,2,3} := \left(\frac{u_1}{\sqrt{|u_1|^2 + |u_2|^2 + |u_3|^2}}, \frac{u_2}{\sqrt{|u_1|^2 + |u_2|^2 + |u_3|^2}}, \frac{u_3}{\sqrt{|u_1|^2 + |u_2|^2 + |u_3|^2}} \right),$$

have norm 1. Moreover, by (2.1).

$$\mathbf{H}_{1,2,3} := \mathbf{F}_{1,2,3} + \mathbf{v}_{1,2,3} := (F_1 + v_1, F_2 + v_2, F_3 + v_3) \in U_3(A),$$

and so

$$(F_1 + v_1, \dots, F_3 + v_3, F_4 + 1, \dots, F_n + 1) \in U_n(A).$$

Now we proceed inductively up to the n -th step. Since $\mathbf{u} \in U_n(A)$, we may assume (by the induction hypothesis), that no subtuple of order $n - 1$ is invertible. Then we may choose $x_j \in \bigcap_{k \neq j} Z(u_k) \neq \emptyset$, $j = 1, \dots, n$. Consequently, the coordinates of

$$\mathbf{v} := \frac{\mathbf{u}}{|\mathbf{u}|} = \left(\frac{u_1}{|\mathbf{u}|}, \dots, \frac{u_n}{|\mathbf{u}|} \right)$$

each have norm one. Since $\mathbf{F} + \mathbf{v} \in U_n(A)$ (by (2.1)), we are done.

Case 3 $\text{tsr } A = \infty$. By Theorem 2.2, nor A cannot be finite in that case. Hence we deduce from all the three cases above that $\text{nor } A \leq \text{tsr } A \leq \text{nor } A$, and so we have equality of all the three stable ranks for $C(X, \mathbb{K})$. $\square$

A combination of the previous results now yields:

Corollary 2.6. *Let X be a compact Hausdorff space and $A = C(X, \mathbb{K})$. Then*

$$\text{bsr } A = \text{usr } A = \text{aur } A = \text{tsr } A = \text{snr } A = \text{nor } A.$$

Recall that in the context of the algebras $C(X, \mathbb{K})$, the original question by Rieffel reads as follows:

- Given $(\mathbf{f}, g) \in U_{n+1}(C(X, \mathbb{K}))$, when does there exist $\mathbf{u} = (u_1, \dots, u_n) \in C(X, \mathbb{K}^n)$ with $\mathbf{f} + \mathbf{u}g \in U_n(C(X, \mathbb{K}))$ such that all the components u_j of $\mathbf{u}$ have modulus one? It remains unanswered.

3. GENERAL UNIFORM ALGEBRAS

Given a commutative unital normed algebra A , let us call an $(n+1)$ -tuple $(\mathbf{f}, g) \in U_{n+1}(A)$ *norm-one reducible*, if there exists $\mathbf{c} = (c_1, \dots, c_n) \in A^n$ such that $\|c_j\| = 1$ and $\mathbf{f} + \mathbf{c}g \in U_n(A)$. In the previous section we have shown that in $C(X, \mathbb{K})$ every invertible $(n+1)$ -tuple is norm-one reducible, provided $\text{tsr } C(X, \mathbb{K}) = n$. Using those ideas, we give a sufficient condition on tuples to be norm-one reducible in an arbitrary uniform algebra. The proof is based on the theory of (weak) peak-points and the following function theoretic Lemma from [4, p. 491]. Recall that a point $x \in X$ is a *weak peak point* for a uniformly closed subalgebra A of $C(X, \mathbb{C})$ if $\{x\}$ is an intersection of peak-sets (these are closed subsets E of X for which there exists $f \in A$ such that $f(\xi) = 1$ if $\xi \in E$ and $|f(\xi)| < 1$ if $\xi \in X \setminus E$).

Lemma 3.1. *Let $0 < \eta < 1$ and $0 < \varepsilon < 1$. Then there exists an automorphism L of the unit disk with fixed points -1 and 1 , and a positive zero a such that the image of $\{z \in \overline{\mathbb{D}} : |z - 1| > \eta\}$ under L is contained in $\{w \in \overline{\mathbb{D}} : |w + 1| < \varepsilon\}$.*

Proposition 3.2. *Let A be a uniform algebra. We view A as a uniformly closed subalgebra of $C(X, \mathbb{C})$, where $X = M(A)$. Suppose that $n := \text{snr } A < \infty$ and let $(\mathbf{f}, g) \in U_{n+1}(A)$. Then $(\mathbf{f}, g)$ is norm-one reducible if $Z(g)$ meets the Shilov boundary.*

Proof. Recall that by Proposition 2.3 that $\text{snr } A \leq \text{nor } A$. If $f_j \equiv 0$ on X for every j , then $(0 + 1 \cdot g, \dots, 0 + 1 \cdot g) \in U_n(A)$ is a solution to our norm-controlled reducibility. So we may assume that not all the f_j are the zero functions. If $g \equiv 0$, then $\mathbf{f} \in U_n(A)$ and we take $\mathbf{f} + \mathbf{e} \cdot g$ as a solution, where $\mathbf{e} = (\mathbf{1}, \dots, \mathbf{1})$.

Let $E = \partial A$ be the Shilov boundary of A . By our assumption, $Z(g) \cap E \neq \emptyset$. Since $\text{snr } A = n$, there is $\mathbf{y} = (y_1, \dots, y_n) \in A^n$ with $\mathbf{u} := \mathbf{f} + \mathbf{y}g \in U_n(A)$ and $\|y_j\| < 1/2$. Let $x_0 \in Z(g) \cap E$. Since $\mathbf{f} \neq \mathbf{0}$ on $Z(g)$, say $|\mathbf{f}| > \delta > 0$ on $Z(g)$, we may choose two open sets U and V such that

$$x_0 \subseteq U \subseteq \overline{U} \subseteq V \subseteq \overline{V} \subseteq \{x \in X : |g| < \delta/(2\sqrt{n})\} \cap \{x \in X : |\mathbf{f}(x)| > \delta\}.$$

Because E is the closure of the set of weak-peak points [3], $U \cap E$ contains such a point x_1 . Hence, there is a peak-set S such that $x_1 \in S \subseteq U$. Choose a peak function $q \in A$ associated with S . Let $m \in \mathbb{N}$, $m \geq 2$, be so big that on $X \setminus V$ the function $\Phi := [(1 + q)/2]^m$ satisfies

$$|\Phi| \leq 1/2.$$

Let $\eta > 0$ be such that

$$\{z \in \mathbb{D} : |z| \leq 1/2\} \subseteq \{z \in \overline{\mathbb{D}} : |z - 1| > \eta\},$$

and put

$$\varepsilon := \frac{\delta'}{4\sqrt{n}\|g\|_\infty},$$

where $\delta' := \min_X |\mathbf{u}|$. Consider the Möbius transform of Lemma 3.1 with $L(1) = 1$, $L(-1) = -1$,

$$L(\{z \in \overline{\mathbb{D}} : |z - 1| > \eta\}) \subseteq \{w \in \overline{\mathbb{D}} : |w + 1| < \varepsilon\}.$$

Then $\psi := (1 + L \circ \Phi)/2$ again is a peak function in A associated with S (note that the membership in A is given by the functional calculus: $\sigma(\Phi) \subseteq \overline{\mathbb{D}}$ and L holomorphic in a neighborhood of $\overline{\mathbb{D}}$). Due to the choice of our parameters, $\psi \sim 0$ on $X \setminus V$; more precisely,

$$|\psi| \leq \frac{\delta'}{8\sqrt{n}\|g\|_\infty}.$$

For $j = 1, \dots, n$, let v_j be defined by

$$v_j = \psi^2 + y_j(1 - \psi)^2 = \left(\frac{1 + L \circ \Phi}{2}\right)^2 + y_j \left(\frac{1 - L \circ \Phi}{2}\right)^2,$$

and put $\mathbf{v} = (v_1, \dots, v_n)$. Then $\mathbf{v} \in A^n$. We claim that

$$\mathbf{f} + \mathbf{v}g \in U_n(A) \text{ and } \|v_j\|_\infty = 1.$$

In fact, since $x_1 \in S \cap E$, $|v_j(x_1)| = 1$. Moreover if $p := L \circ \Phi$,

$$\begin{aligned} |v_j| &\leq \left| \frac{1+p}{2} \right|^2 + |y_j| \left| \frac{1-p}{2} \right|^2 \\ &\leq \left| \frac{1+p}{2} \right|^2 + 1 \cdot \left| \frac{1-p}{2} \right|^2 \\ &\leq \frac{1}{4} \left((1+|p|^2 + 2\operatorname{Re} p) + (1+|p|^2 - 2\operatorname{Re} p) \right) \\ &\leq \frac{1}{4} \cdot 4 = 1 \end{aligned}$$

Moreover

- $|\mathbf{f} + \mathbf{v}g| \geq |\mathbf{f}| - |\mathbf{v}| |g| \geq \delta - \sqrt{n} \delta / (2\sqrt{n}) = \delta/2$ on V and
- $|\mathbf{f} + \mathbf{v}g| \geq |\mathbf{f} + \mathbf{y}g| - |\mathbf{v} - \mathbf{y}| |g| = |\mathbf{u}| - |\mathbf{v} - \mathbf{y}| |g|$ on $X \setminus V$.

But $v_j - y_j = \psi^2 + y_j(1 + \psi^2 - 2\psi) - y_j = \psi^2 + y_j\psi^2 - 2\psi y_j = \psi(\psi + y_j\psi - 2y_j)$. Hence, on $X \setminus V$,

$$|v_j - y_j| \leq 4|\psi| \leq 4 \frac{\delta'}{8\|g\|_\infty \sqrt{n}}.$$

Consequently, on $X \setminus V$,

$$|\mathbf{f} + \mathbf{v}g| \geq \delta' - \|g\|_\infty \frac{\delta'}{2\|g\|_\infty \sqrt{n}} \sqrt{n} = \delta'/2 > 0.$$

□

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