Non-linear normal modes and invariant manifolds

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1. INTRODUCTION

Small-amplitude motions of dynamic systems (structural, fluid, control, etc.) about an equilibrium state are modeled by linear differential equations which have constant coefficients. These are typically obtained by a Taylor series expansion of the forces about the equilibrium point. Under quite general circumstances these equations admit a set of special solutions, called normal mode motions, in which each system component moves with the same frequency and with a fixed ratio amongst the displacements of the components (for a conservative system; for a non-conservative system all displacements and velocities are linearly related to a single displacement/velocity pair). The relative amplitudes of the motion of the components during such an oscillation are given by the eigenvectors (or the eigenfunctions for continuous systems) of an operator which arise in the differential equations of motion. The corresponding eigenvalues contain information regarding the frequency of the motion and how quickly it decays due to dissipation. A general motion of the system can be written as a linear combination of "eigenmotions," i.e., the normal modes, with the coefficients in the combination being exponential functions of the eigenvalues multiplying time and with the displacements being given by the corresponding eigenvectors. (This is simply the principle of superposition.) An important feature of these eigenmotions is that if the motion is started such that only a subset of them is active, for example only one, then the remaining eigenmotions remain quiescent for all time. This is the *invariance* property of the normal modes.

In many cases the linearized model of the system breaks down, and one must account for non-linear effects to capture the system's dynamics properly. For example, this happens in systems with dry friction, clearances and, more commonly, if motion amplitudes become sufficiently large. In such cases one must keep higher order terms in the series expansion of the forces, leading to *non-linear* differential equations of motion which contain polynomial terms (or piecewise-linear terms for systems with friction or clearances) which arise from non-linear effects of the forces. In general, one has no hope of attacking the non-linear system in the same way as the linearized system, although much work has been done for special cases, such as conservative systems, especially those with special symmetries. The review paper of Rosenberg [1] and the recent thesis of Vakakis [2] contain an almost complete account of the history of the subject. In this note we describe a fundamentally new way to handle these problems [3].

2. THE MAIN IDEA

The method described herein is geometric in nature and utilizes the theory of invariant manifolds for dynamical systems. It contains all previous methods as special cases.

Furthermore, it is a constructive technique so that one can actually use it to *solve* non-linear problems.

The method was suggested by the center manifold technique [4], which is used in bifurcation analysis. At critical parameter values center manifold theory provides a series approximation of an invariant manifold (essentially a hypersurface) on which a bifurcation occurs, and gives the differential equations on that manifold which describe the dynamics near the bifurcation point. In the present context the linear normal modes, which for oscillatory systems are represented by hyperplanes in the phase space, are extended into their non-linear counterparts by using similar ideas. These *non-linear invariant manifolds* are tangent to the linear modal hyperplanes at the operating point.

Our formulation is in terms of a system of $2N$ first order ordinary differential equations (ODEs) which arise from a set of N second order, oscillator-type ODEs. We are in the process of generalizing the method to handle systems of first order equations, but this is not given here. We assume that the equations of motion are of the form ($i = 1, 2, \ldots, N$):

$$\dot{x}_i = y_i, \qquad \dot{y}_i = f_i(x_1, x_2, \ldots, x_N; y_1, y_2, \ldots, y_N), \tag{1}$$

where (x_i, y_i) represent displacement and velocity co-ordinates as measured about some equilibrium condition, and the functions f_i, which are the forces and moments acting on the system normalized by the respective inertias, are to be expanded in a Taylor series about the equilibrium point $(x_i, y_i) = (0, 0)$. We now assume that there exists (at least) one motion for which all co-ordinates behave in a similar manner; that is, they oscillate with the same frequency and/or decay (or grow if the equilibrium is unstable) at the same rate. In the general non-linear case, the frequency will be amplitude dependent and the decay rate is not exponential. Such a motion can be expressed as a functional dependence in which all displacements and velocities are related to a single pair of displacement and velocity, which we choose arbitrarily here as the first one. In order to implement this, we write $u = x_1$ and $v = y_1$ and express the other x_is and y_is in terms of u and v:

$$x_i = X_i(u, v), \qquad y_i = Y_i(u, v) \tag{2}$$

This equation is that of a constraint surface of dimension 2 (or, equivalently, of co-dimension $2N-2$) in the $2N$-dimensional phase space. Any motion which satisfies the equations of motion, the above-mentioned tangency condition and this constraint is defined to be a *non-linear normal mode.* In other words, a motion in a normal mode takes place on the *non-linear invariant manifold* defined by equation (2).

A set of equations which can be solved for this constraint surface, that is for the X_is and Y_is, can be obtained by requiring that solutions satisfy both the equations of motion and the constraint conditions. Taking the time derivative of the constraint equations and using the chain rule for differentiation yields:

$$\dot{x}_i = \frac{\partial X_i}{\partial u}\dot{u} + \frac{\partial X_i}{\partial v}\dot{v}, \qquad \dot{y}_i = \frac{\partial Y_i}{\partial u}\dot{u} + \frac{\partial Y_i}{\partial v}\dot{v}. \tag{3}$$

Next, we substitute the equations of motion for $\dot{x}_i$ and $\dot{y}_i$, and replace x_i and y_i everywhere by X_i and Y_i to obtain the $2N-2$ equations, which can be solved for X_i and Y_i ($i = 1, 2, \ldots, N$):

$$Y_i(u, v) = \frac{\partial X_i(u, v)}{\partial u} v + \frac{\partial X_i(u, v)}{\partial v} f_1(u, X_2(u, v), \ldots, X_N(u, v); v, Y_2(u, v), \ldots, Y_N(u, v))$$

$$f_i(u, X_2(u, v), \ldots, X_N(u, v); v, Y_2(u, v), \ldots, Y_N(u, v))$$
$$= \frac{\partial Y_i(u, v)}{\partial u} v + \frac{\partial Y_i(u, v)}{\partial v} f_1(u, X_2(u, v), \ldots, X_N(u, v); v, Y_2(u, v), \ldots, Y_N(u, v))). \tag{4}$$

In general, these functional equations are at least as difficult to solve as the original differential equations, but they do allow for an approximate solution in the form of a series expansion. This provides exactly the information needed for dealing with first order non-linear effects, and is perfectly consistent with the expansion of the non-linear forces about the operating point. In order to carry this out we expand the X_is and Y_is as follows:

$$\begin{aligned} X_i(u, v) &= a_{0i} + a_{1i}u + a_{2i}v + a_{3i}u^2 + a_{4i}uv + a_{5i}v^2 \\ &\quad + a_{6i}u^3 + a_{7i}u^2v + a_{8i}uv^2 + a_{9i}v^3 + \cdots, \\ Y_i(u, v) &= b_{0i} + b_{1i}u + b_{2i}v + b_{3i}u^2 + b_{4i}uv + b_{5i}v^2 \\ &\quad + b_{6i}u^3 + b_{7i}u^2v + b_{8i}uv^2 + b_{9i}v^3 + \cdots, \end{aligned} \tag{5}$$

where terms up to order three are sufficient to obtain first order non-linear effects in nearly all cases. These expansions are substituted into the equations for the X_is and Y_is, equations (4), which are then expanded in powers of u and v. Like powers of these variables are then gathered together and are matched. These matching conditions provide equations from which the coefficients a_{ji} and b_{ji} can be solved.

Once the coefficients have been solved for, the dynamics on an invariant manifold, i.e., the normal mode's dynamics, can be generated by simply substituting the X_is and Y_is for x_i and y_i in the first pair of equations of motion; that is, the ones for x_1 and y_1. This results in the following modal dynamic equation:

$$\dot{u} = v, \qquad \dot{v} = f_1(u, X_2(u, v), \cdots, X_N(u, v); v, Y_2(u, v), \cdots, Y_N(u, v)). \tag{6}$$

where (u, v) represent the variables on the invariant manifold and correspond to projections of the modal dynamics on to (x_1, y_1). There will be N such equations at each equilibrium point, one for each mode, and they contain the effects of non-linearities up to the order taken in the f_is and in the X_is and the Y_is.

This procedure provides the geometric structure of the non-linear normal modes near the equilibrium point. It results in *decoupled, single-degree-of-freedom non-linear oscillator* equations, each of which represents the dynamics of the system on an invariant, two-dimensional subspace which is tangent to the linear normal mode eigenspace at the equilibrium point.

3. DISCUSSION

If the non-linearities have shifted the equilibrium position, or generated an additional one, this will be captured by a non-zero solution, or multiple solutions for the a_{0i}s, respectively (note that the b_{0i}s will generally be zero for this class of problems unless there is a rigid-body mode of motion). The coefficients a_{1i}, a_{2i}, b_{1i} and b_{2i} represent the linear modal amplitude ratios, and are solved for from a set of coupled algebraic (in fact, quadratic) equations which are generated in a manner which is completely different from that of the traditional approach. (Note that for an undamped system $a_{2i} = b_{1i} = 0$ and $a_{1i} = b_{2i}$.) There will be N real solutions of these equations, one for each mode, at each equilibrium point. We are quite close to completing a proof that the linear modes obtained from these equations are strictly equivalent to those obtained by solving the standard eigenvalue problem. The coefficients of the non-linear terms represent the bending of the invariant manifold which arises from the non-linearities in the forces f_i. In general, there should be a unique real solution of the equations for these coefficients for each mode. We have shown that these equations are *linear* in the unknowns, which makes it very inexpensive (much more so than solving an eigenvalue problem for the linear modes!) to obtain the non-linear terms in the series approximations for the non-linear modes.

Singularities occur in the coefficients of the non-linear terms if the system has an internal resonance. In the case of two resonant modes the invariant manifold associated with those modes will be four-dimensional, and on it the dynamics will be governed by two oscillators which cannot be uncoupled. Our procedure can be modified in a straightforward way to handle such cases.

Another case in which the series expansion (5) will fail to yield the non-linear normal modes is when the phenomenon of *mode localization* occurs [5]. For example, we expect localized oscillations to take place in a weakly coupled system, the symmetry of which is broken by non-linear effects. When localization occurs some coefficients in equation (5) will become large, indicating the failure of the expansion procedure and also the high sensitivity of the system to symmetry-breaking non-linear perturbations. To obtain approximations of the *localized normal modes* we can use the same methodology but we will need to expand X_i and Y_i in terms of another parameter, *e.g.*, the weak coupling.

An important feature of our method is that, by using the first order formulation, systems with damping can be handled without difficulty. *This aspect is entirely new*; all previous methods for non-linear normal modes have dealt with conservative systems, or systems in which the dissipative forces have very special symmetries. The methodology described in this note will allow for transient dynamic analysis to be carried out, since the non-linear mode with the slowest decay rate will dominate the dynamics of a structure as its vibrations decay to the equilibrium.

We have derived the general equations for the coefficients in the expansions of the non-linear modes in terms of the coefficients in the expansions of the forces, for a general N-degree-of-freedom system. We have also implemented this technique on several two-degree-of-freedom non-linear systems. These results are currently being written into a full-length paper [6].

ACKNOWLEDGMENT

This work is supported in part by grants from the National Science Foundation (MSS-8915453 for S.W.S. and MSS-8913196 for C.P.).

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