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Lucie Baudouin * **Alexandre Seuret** * **Mohammed Safi** *

* *LAAS-CNRS, Université de Toulouse, CNRS, Toulouse, France.*

Abstract: We address the stability of a system of ordinary differential equations coupled with a transport partial differential equation, using a Lyapunov functional approach. This system can also be interpreted as a finite dimensional system subject to a state delay. Inspired from recent developments on time-delay systems, a novel method to assess stability of such a class of coupled systems is developed here. We will specifically take advantage of a polynomial approximation of the infinite dimensional state of the transport equation together with efficient integral inequalities in order to study the stability of the infinite dimensional system. The main result of this paper provides exponential stability conditions for the whole coupled system expressed in terms of linear matrix inequalities and the results are tested on academic examples.

Keywords: Transport equation, Lyapunov, integral inequalities, polynomial approximation.

1. INTRODUCTION

Distributed parameter systems represent a wide class of control systems whose state is of infinite dimension. This class of system appears in numerous applications that we will not list here. Analyzing and controlling distributed parameter systems represents an attractive area of research in applied mathematics and more recently in automatic control: see for instance Prieur (2008) Susto and Krstic (2010), Smyshlyaev et al. (2010), Smyshlyaev and Krstic (2005), Krstic et al. (2009) among many others.

We study in this document the particular situation where a finite dimensional system is coupled to a transport equation, and the main difficulty in the stability analysis we will perform is related to the infinite dimensional nature of the transport part of the whole state.

It is worth noting that the class of systems we study can be also interpreted as Time Delay Systems (TDS), which have been widely investigated in the literature (see Fridman (2014); Niculescu (2001); Gu et al. (2003)). The aim of this paper is to take advantage of some recent developments on the stability analysis of TDS in order to provide a new framework for the analysis of this system of Ordinary Differential Equations (ODEs) coupled with a transport Partial Differential Equation (PDE). The first difficulty arises from the fact that stability of TDS can be assessed using the Lyapunov-Krasovskii Theorem and analyzing the stability of our system cannot be performed using exactly the same theorem. The second difficulty lies in the infinite dimensional part of the system, which prevents from extending directly the existing methods from the finite dimension analysis. In order to provide efficient stability conditions, we will construct a Lyapunov functional by enriching the classical energy of the whole system with terms built on a polynomial approximation of the infinite dimensional state expressed using Legendre polynomials.

While polynomial approximation methods for the analysis of infinite dimension systems is not a new idea (see for instance the convex optimization and sum-of squares frameworks developed in Papachristodoulou and Peet (2006); Peet (2014) or Ahmadi et al. (2014)), the novelty of this approach relies on the use of efficient integral inequalities that are able to give a measure of the conservatism associated to the approximation. These inequalities can be interpreted as a Bessel inequality on Hilbert spaces. In previous work, e.g. Seuret and Gouaisbaut (2014, 2015), the efficiency of these inequalities for the stability analysis of TDS has been shown. Indeed, one can also read in Seuret et al. (2015) a method based on a polynomial approximation of the distributed nature of the delay, using Legendre polynomials and their properties to construct Lyapunov-Krasovskii functionals. In the present paper, where a simple transport equation replaces the delay terms (an approach also studied in e.g. Bekiaris-Liberis and Krstic (2013)), an alternative use of this new method is proposed.

In the framework of the stability analysis, the present article can be seen as a first step towards the study of more intricate PDE systems using tools inherited from TDS approaches.

Notations: Herein, $\mathbb{N}$ is the set of positive integer, $\mathbb{R}_+$ the set of non-negative reals, $\mathbb{R}^n$ the n-dimensional Euclidian space with vector norm $|\cdot|_n$ and $\mathbb{R}^{n\times m}$ the set of all $n \times m$ real matrices. If $P \in \mathbb{R}^{n\times n}$ is symmetric (i.e. $P \in \mathcal{S}_n$) and positive definite, we note either $P \succ 0$ or $P \in \mathcal{S}_n^+$ and we denote by $\lambda_{\max}(P) > 0$ its largest eigenvalue. The symmetric matrix $\left[\begin{smallmatrix} A & B \\ * & C \end{smallmatrix}\right]$ stands for $\left[\begin{smallmatrix} A & B \\ B^\top & C \end{smallmatrix}\right]$ and $diag(A, B)$ is the diagonal matrix $\left[\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right]$. Moreover, for $A \in \mathbb{R}^{n\times n}$, we

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define $\mathrm{He}(A) = A + A^\top$. The matrix I is the identity matrix and $0_{n,m}$ stands for the matrix in $\mathbb{R}^{n\times m}$ whose entries are zero. When no confusion is possible, the subscript will be omitted. For given vectors $\{u_k\}_{k=0,\dots,n}$, we denote $\mathrm{Vect}_{k=0..n} u_k$ the vector $(u_0, \dots, u_n)^\top$. Finally, $L^2(0,1;\mathbb{R}^n)$ will denote the space of square integrable functions over the interval $(0,1) \subset \mathbb{R}$ with values in $\mathbb{R}^n$.

2. FORMULATION OF THE PROBLEM

Let us consider the following coupling of a finite dimensional system in the variable X with a transport partial differential equation in the variable z, in such a way that the transport mimics a delay term in the ODE in X:

$$\begin{cases} \dot{X}(t) = AX(t) + Bz(1,t) & t > 0, \\ \partial_t z(x,t) + \rho \partial_x z(x,t) = 0, & x \in (0,1), t > 0, \\ z(0,t) = CX(t), & t > 0. \end{cases} \tag{1}$$

The pair $(X(t), z(t)) \in \mathbb{R}^n \times L^2(0,1;\mathbb{R}^m)$ is the state of the system and it satisfies compatible initial datum $(X(0), z(x,0)) = (X^0, z^0(x))$ for $x \in (0,1)$. The matrices A, B and C are constant in $\mathbb{R}^{n\times n}$, $\mathbb{R}^{n\times m}$ and $\mathbb{R}^{m\times n}$.

One should know that equation $\partial_t z + \rho \partial_x z = 0$ in (1) of unknown $z = z(x,t)$ is a simple vectorial transport PDE and if the initial data $z^0 \in L^2(0,1;\mathbb{R}^m)$ and the lateral boundary data $z(0,\cdot) = CX \in L^2(\mathbb{R}_+;\mathbb{R}^m)$ are given, it has a unique solution $z \in C(\mathbb{R}_+; L^2(0,1;\mathbb{R}^m))$ such that (see e.g. Coron (2007)), for all $t > 0$:

$$\|z(t)\|_{L^2(0,1;\mathbb{R}^m)} \le \|z^0\|_{L^2(0,1;\mathbb{R}^m)} + \|X\|_{L^2(\mathbb{R}_+;\mathbb{R}^m)}.$$

The stability of System (1) will be studied thanks to a Lyapunov functional constructed with the state (X,z) and the smart use of the projection of z over the set of polynomials of degree less than a prescribed integer $N \ge 0$. We would like to emphasize that this study aims at guaranteeing the stability of the whole system through tractable LMI tests.

2.1 Legendre Polynomials and their properties

In order to express the polynomial approximation of the infinite dimensional state z, we will work with the shifted Legendre polynomials considered over the interval $[0,1]$ and denoted $\{L_k\}_{k\in\mathbb{N}}$. The main motivation for selecting these polynomials arise from their useful properties which will be described below. Instead of giving their explicit formula, we detail here their principal constitutive properties. To begin with, the family $\{L_k\}_{k\in\mathbb{N}}$ forms an orthogonal basis of $L^2(0,1;\mathbb{R})$ since

$$\langle L_j, L_k \rangle = \int_0^1 L_j(x) L_k(x) dx = \frac{1}{2k+1} \delta_{jk},$$

where δ_{jk} denotes the Kronecker delta, equal to 1 if $j = k$ and to 0 otherwise. We denote the corresponding norm of this inner scalar product $\|L_k\| = \sqrt{\langle L_k, L_k\rangle} = 1/\sqrt{2k+1}$. The boundary values are given by:

$$L_k(0) = (-1)^k, \qquad L_k(1) = 1. \tag{2}$$

Furthermore, the following derivation formula holds:

$$L'_k(x) = \begin{cases} 0, & k = 0, \\ \sum_{j=0}^{k-1} (2j+1)(1-(-1)^{k+j}) L_j(x), & k \ge 1. \end{cases} \tag{3}$$

One can find details about Legendre polynomials in the book by Courant and Hilbert (1989).

Therefore, any element $y \in L^2(0,1;\mathbb{R})$ can be written $y(x) = \sum_{k\ge 0} \langle y, L_k\rangle L_k(x)/\|L_k\|^2$ and throughout the paper, we will denote abusively, for $z \in C(\mathbb{R}_+; L^2(0,1;\mathbb{R}^m))$,

$$z(x,t) = \sum_{k\ge 0} \langle z(t), L_k \rangle \frac{L_k(x)}{\|L_k\|^2}$$

instead of

$$z(x,t) = \begin{bmatrix} z_1(x,t) \\ \vdots \\ z_m(x,t) \end{bmatrix} = \begin{bmatrix} \sum_{k\ge0} \langle z_1(t), L_k\rangle L_k(x)/\|L_k\|^2 \\ \vdots \\ \sum_{k\ge0} \langle z_m(t), L_k\rangle L_k(x)/\|L_k\|^2 \end{bmatrix}.$$

The following property will be useful hereafter.

Property 1. Let $z \in C(\mathbb{R}_+; L^2(0,1;\mathbb{R}^m))$ satisfy the transport equation in (1). The time derivative formulas

$$\begin{aligned} \frac{d}{dt} \langle z(t), L_0 \rangle &= \rho z(0,t) - \rho z(1,t) \\ \frac{d}{dt} \langle z(t), L_k \rangle &= \rho \sum_{j=0}^{k-1} (2j+1)(1-(-1)^{k+j}) \langle z(t), L_j\rangle \\ &\quad + \rho(-1)^k z(0,t) - \rho z(1,t), \quad \forall k \in \mathbb{N}^* \end{aligned} \tag{4}$$

holds if $\partial_t z \in C(\mathbb{R}_+; L^2(0,1;\mathbb{R}^m))$.

The proof derives from the formulas (2) and (3).

2.2 Bessel-Legendre Inequality

The use of an approximation of the infinite dimensional state z (by a finite dimensional one using polynomials) will be efficient if we are able to measure the approximation error. The following lemma provides this kind of information.

Lemma 1. Let $z \in C(\mathbb{R}_+; L^2(0,1;\mathbb{R}^m))$ and $R \in \mathcal{S}_m^+$. The integral inequality

$$\int_0^1 z^\top(x,t) R z(x,t) dx \ge \sum_{k=0}^N (2k+1) \langle z(t), L_k\rangle^\top R \langle z(t), L_k \rangle \tag{5}$$

holds for all $N \in \mathbb{N}$.

Proof : It relies on the orthogonality of the Legendre polynomials and on the Bessel inequality, see e.g. Seuret and Gouaisbaut (2015). More precisely, the proof of this lemma results from the positive definiteness and the expansion of

$$\langle y_N(t), R y_N(t) \rangle = \int_0^1 y_N^\top(x,t) R y_N(x,t) dx$$

where

$$y_N(x,t) = z(x,t) - \sum_{k=0}^N \left\langle z(t), \frac{L_k}{\|L_k\|} \right\rangle \frac{L_k(x)}{\|L_k\|}$$

is the approximation error between the state z and its projection over the $N+1$ first Legendre polynomials. □

3. STABILITY ANALYSIS

3.1 Lyapunov-Krasovskii functional

Our objective is to construct a Lyapunov functional in order to narrow the proof of the stability of the complete system (1) to the resolution of a simple Linear Matrix Inequality (LMI). Since a part of the state (X, z) of the system is distributed (z being the solution of a transport equation and depending on time t and space x variables), our idea is to take advantage of an appropriate finite dimensional approximation of the state.

First of all, let us denote by E the total energy of the system:

$$E(X(t), z(t)) = |X(t)|_n^2 + \|z(t)\|^2_{L^2(0,1;\mathbb{R}^m)}.$$

In the sequel, we will use the notation $E(t) = E(X(t), z(t))$ in order to simplify the notation. Following the previous developments, N being a prescribed positive integer, we will introduce an approximate state of size $n + (N+1)m$, composed by the state of the ODE system X and the projection of the infinite dimensional state z over the set of polynomial of degree less than N. In other words, the approximate finite dimensional state vector is given by

$$\begin{bmatrix} X(t) \\ Z_N(t) \end{bmatrix} = \begin{bmatrix} X(t) \\ \mathrm{Vect}_{k=0..N} \langle z(t), L_k \rangle \end{bmatrix}.$$

Inspired by the complete Lyapunov-Krasovskii functional which is a necessary and sufficient conditions for stability for delay systems (see Gu et al. (2003)), we are looking for a candidate Lyapunov functional for system (1) of the form:

$$\begin{aligned} V_N(X(t), z(t)) &= X^\top(t) P X(t) + 2X^\top(t) \int_0^1 \mathcal{Q}(x) z(x,t) dx \\ &\quad + \int_0^1 \int_0^1 z^\top(x_1, t) \mathcal{T}(x_1, x_2) z(x_2, t) dx_1 dx_2 \\ &\quad + \int_0^1 z^\top(x,t) S z(x,t) dx + \int_0^1 \int_0^x z^\top(y,t) R z(y,t) dy dx, \end{aligned}$$

where the matrices $P \in \mathcal{S}_n^+$, $S, R \in \mathcal{S}_m^+$ and the functions $\mathcal{Q} \in L^2(0,1;\mathbb{R}^{n\times m})$ and $\mathcal{T} \in L^\infty((0,1)^2; \mathcal{S}_m)$ have to be determined. This functional is composed of four typical terms. The first quadratic term in $X(t)$ is dedicated to the state of the ODE, while the last three terms are dedicated to the state of the PDE. It is worth mentioning that the last two terms can be interpreted as the weighted energy of the transport equation and have been widely use in the literature (see for instance Coron (2007)). The term depending on the function $\mathcal{T}$ has been recently considered in the literature in Peet (2014). The term depending on $\mathcal{Q}$ is introduced in order to represent the coupling between the system of ODEs and the transport PDE.

While this class of functionals is already classical in the context of time delay systems, the interpretation of such functionals for PDEs is quite recent (see for instance the work Ahmadi et al. (2014) and Papachristodoulou and Peet (2006)). The novelty of the present paper is closely related to these works. The difference of our approach relies on the use of polynomial approximation together with efficient integral inequalities presented in Lemma 1, which is able to give a measure of the polynomial approximation.

In order to reveal the approximate state Z_N in the candidate Lyapunov functional, we select the functions Q and T as follows:

$$Q(x) = \sum_{k=0}^N Q_k L_k(x), \; T(x_1, x_2) = \sum_{i=0}^N \sum_{j=0}^N T_{ij} L_i(x_1) L_j(x_2)$$

where $\{Q_i\}_{i=0..N}$ belong to $\mathbb{R}^{n\times m}$ and $\{T_{ij} = T_{ji}^\top\}_{i,j=0..N}$ to $\mathbb{R}^{m\times m}$. Therefore we can write, with the same abuse of notation as for the energy $E(t)$,

$$\begin{aligned} V_N(t) &= \begin{bmatrix} X(t) \\ Z_N(t) \end{bmatrix} \begin{bmatrix} P & Q \\ Q^\top & T \end{bmatrix} \begin{bmatrix} X(t) \\ Z_N(t) \end{bmatrix} \\ &\quad + \int_0^1 z^\top(x,t) S z(x,t) dx + \int_0^1 \int_0^x z^\top(y,t) R z(y,t) dy dx, \end{aligned} \tag{6}$$

where

$$Q = [Q_0 \; \dots \; Q_N] \text{ in } \mathbb{R}^{n, m(N+1)}, \quad T = [T_{jk}]_{j,k=0..N} \text{ in } \mathbb{R}^{m(N+1), m(N+1)}.$$

One should notice that V_N is built from the whole state of the system (X, z) and from the approximate state vector Z_N as well. In the following subsection, conditions for exponential stability of the origin of system (1) can be obtained using the LMI framework. More particularly, we aim at proving that the functional V_N is positive definite and satisfies $\dot{V}_N(t) + 2\delta V_N(t) \le 0$ for a prescribed $\delta > 0$ and under LMIs to be determined.

3.2 Exponential stability with guaranteed decay rate

Let us set

$$\begin{aligned} S_N &= diag(S, 3S, \dots, (2N+1)S) && \text{in } \mathbb{R}^{m(N+1), m(N+1)} \\ R_N &= diag(R, 3R, \dots, (2N+1)R) && \text{in } \mathbb{R}^{m(N+1), m(N+1)} \\ \mathcal{I}_N &= diag(I_m, 3I_m, \dots, (2N+1)I_m) && \text{in } \mathbb{R}^{m(N+1), m(N+1)} \\ \mathbb{1} &= [I_m \; I_m \; \dots \; I_m]^\top && \text{in } \mathbb{R}^{m(N+1), m} \\ \mathbb{1}^* &= [I_m \; -I_m \; \dots \; (-1)^N I_m]^\top && \text{in } \mathbb{R}^{m(N+1), m} \\ A_N &= [\alpha_{jk} I_m]_{j,k=0..N} && \text{in } \mathbb{R}^{m(N+1), m(N+1)} \end{aligned}$$

with

$$\alpha_{jk} = \begin{cases} (2k+1)(1 - (-1)^{k+j}), & \text{if} \quad k \le j-1, \\ 0, & \text{if} \quad k \ge j. \end{cases}$$

We provide here a stability result for (1) based on the proposed Lyapunov functional (6) and the use of Property 1 and Lemma 1.

Theorem 1. Consider the coupled system (1) with a given transport speed $\rho > 0$. If there exist an integer $N \ge 0$, such that there exist $\delta > 0$, $P \in \mathcal{S}_n$, $Q \in \mathbb{R}^{n,(N+1)m}$ and $T \in \mathcal{S}_{(N+1)m}$ and two matrices $S, R \in \mathcal{S}_m^+$ satisfying the following LMIs

$$\Phi_N = \begin{bmatrix} P & Q \\ Q^\top & T + S_N \end{bmatrix} \succ 0, \tag{7}$$

$$\Psi_N(\rho, \delta) = \begin{bmatrix} \Psi_{11} & PB - \rho Q \mathbb{1} & \Psi_{13} \\ * & -\rho S & B^\top Q - \rho \mathbb{1}^\top T \\ * & * & \Psi_{33} \end{bmatrix} \prec 0, \tag{8}$$

$$\rho R - 2\delta(S + R) \succ 0, \tag{9}$$

where

$$\begin{aligned} \Psi_{11} &= \mathrm{He}(PA + \rho Q \mathbb{1}^* C) + \rho C^\top (S+R) C + 2\delta P \\ \Psi_{13} &= A^\top Q + \rho C^\top \mathbb{1}^{*\top} T + \rho Q A_N + 2\delta Q \\ \Psi_{33} &= \rho \mathrm{He}(T A_N) - \rho R_N + 2\delta(T + S_N + R_N), \end{aligned}$$

then the coupled system (1) is exponentially stable. Indeed, for any constant speed ρ, there exists a constant $K > 0$ such that we have a guaranteed decay rate δ for the energy:

$$E(t) \le K e^{-2\delta t} \left(|X^0|_n^2 + \|z^0\|^2_{L^2(0,1;\mathbb{R}^m)} \right), \forall t > 0. \tag{10}$$

Proof : The proof consists in showing that if the LMI conditions (7), (8) and (9) are verified for given $N \geq 0$ and $\delta > 0$, then there exist three positive scalars $\varepsilon_1, \varepsilon_2$ and ε_3 such that for all $t > 0$,

$$\varepsilon_1 E(t) \leq V_N(t) \leq \varepsilon_2 E(t), \tag{11}$$

and

$$\dot{V}_N(t) + 2\delta V_N(t) \leq -\varepsilon_3 E(t). \tag{12}$$

Indeed, afterwards, its suffices to integrate (12) in time to obtain,

$$V_N(t) \leq V_N(0)e^{-2\delta t}, \qquad \forall t \geq 0.$$

Next, with the help of (11), we can write

$$\varepsilon_1 E(t) \leq V_N(t) \leq V_N(0)e^{-2\delta t} \leq \varepsilon_2 E(0)e^{-2\delta t},$$

allowing to conclude (10).

Existence of ε_1: Since $S \succ 0$ and $\Phi_N \succ 0$, there exists a sufficiently small $\varepsilon_1 > 0$ such that

$$S \succ \varepsilon_1 I_m, \quad \Phi_N = \begin{bmatrix} P & Q \\ Q^\top & T + S_N \end{bmatrix} \succ \varepsilon_1 \begin{bmatrix} I_n & 0 \\ 0 & \mathcal{I}_N \end{bmatrix}.$$

Therefore, R being positive definite, $V_N(t)$ in (6) satisfies

$$\begin{aligned} V_N(t) \geq & \begin{bmatrix} X(t) \\ Z_N(t) \end{bmatrix}^\top \Phi_N \begin{bmatrix} X(t) \\ Z_N(t) \end{bmatrix} - Z_N^\top(t) S_N Z_N(t) \\ & + \int_0^1 z^\top(x,t) S z(x,t) dx. \end{aligned}$$

Replacing Φ_N by its lower bound depending on ε_1 and introducing ε_1 in the last integral term, we have

$$\begin{aligned} &V_N(t) \geq \varepsilon_1 |X(t)|_n^2 + \varepsilon_1 \int_0^1 z^\top(x,t) z(x,t) dx \\ &+ \int_0^1 z^\top(x,t)(S - \varepsilon_1 I_m) z(x,t) dx - Z_N^\top(t)(S_N - \varepsilon_1 \mathcal{I}_N) Z_N(t). \end{aligned}$$

Since $S - \varepsilon_1 I_m \succ 0$, Lemma 1 ensures

$$\begin{aligned} &\int_0^1 z^\top(x,t)(S - \varepsilon_1 I_m) z(x,t) dx \geq \\ &\qquad \sum_{k=0}^N (2k+1) \langle z(t), L_k \rangle^\top (S - \varepsilon_1 I_m) \langle z(t), L_k \rangle . \end{aligned}$$

This inequality can be rewritten in a more compact form using the approximate state $Z_N(t)$ and the square matrices S_N and $\mathcal{I}_m$ defined earlier as follows

$$\int_0^1 z^\top(x,t)(S - \varepsilon_1 I_m) z(x,t) dx \geq Z_N^\top(t)(S_N - \varepsilon_1 \mathcal{I}_N)_N Z_N(t).$$

We obtain a lower bound of V_N depending on the energy function $E(t)$:

$$V_N(t) \geq \varepsilon_1 |X(t)|_n^2 + \varepsilon_1 \|z(t)\|^2_{L^2(0,1;\mathbb{R}^m)} = \varepsilon_1 E(t).$$

Existence of ε_2: There exists a sufficiently large scalar $\beta > 0$ such that

$$\begin{bmatrix} P & Q \\ Q^\top & T \end{bmatrix} \preceq \beta \begin{bmatrix} I_n & 0 \\ 0 & \mathcal{I}_N \end{bmatrix},$$

yielding, under the assumptions $S \succ 0$ and $R \succ 0$, and after an integration by parts, that

$$\begin{aligned} V_N(t) \leq \beta |X(t)|_n^2 + \beta Z_N^\top(t) \mathcal{I}_N Z_N(t) & \\ + \int_0^1 z^\top(x,t)(S + R) z(x,t) dx. & \end{aligned}$$

Applying Lemma 1 to the second term of the right-hand side ensures that

$$\begin{aligned} V_N(t) &\leq \beta |X(t)|_n^2 + \int_0^1 z(t)^\top (\beta I_m + S + R) z(t) dx \\ &\leq \beta |X(t)|_n^2 + \varepsilon_2 \|z\|^2_{L^2(0,1;\mathbb{R}^m)} \leq \varepsilon_2 E(t) \end{aligned}$$

where $\varepsilon_2 = \beta + \lambda_{\max}(S) + \lambda_{\max}(R)$.

Existence of ε_3: Let us begin with a new formulation of equation (4) in Property 1, using the notations recently introduced:

$$\frac{d}{dt} Z_N(t) = \rho A_N Z_N(t) + \rho \mathbb{1}^* C X(t) - \rho \mathbb{1} z(1,t). \tag{13}$$

The proof of (12) will also partly rely on the definition of an augmented approximate vector of size $n + (N+2)m$ given by

$$\xi_N(t) = \begin{bmatrix} X(t) \\ z(1,t) \\ Z_N(t) \end{bmatrix}.$$

Let us split the computation of $\dot{V}_N$ into three terms, namely $\dot{V}_{N,1}$, $\dot{V}_{N,2}$ and $\dot{V}_{N,3}$ corresponding to each term of V_N in (6). On the one hand, using the first equation in (1) and the new formulation (13) of Property 1, we have

$$\frac{d}{dt} \begin{bmatrix} X(t) \\ Z_N(t) \end{bmatrix} = \begin{bmatrix} AX(t) + Bz(1,t) \\ \rho A_N Z_N(t) + \rho \mathbb{1}^* C X(t) - \rho \mathbb{1} z(1,t) \end{bmatrix}$$

so that we can calculate

$$\begin{aligned} &\dot{V}_{N,1}(t) \; + \; 2\delta V_{N,1}(t) \\ &\qquad = \frac{d}{dt} \left(\begin{bmatrix} X(t) \\ Z_N(t) \end{bmatrix}^\top \begin{bmatrix} P & Q \\ Q^\top & T \end{bmatrix} \begin{bmatrix} X(t) \\ Z_N(t) \end{bmatrix} \right) \\ &\qquad \quad + 2\delta \begin{bmatrix} X(t) \\ Z_N(t) \end{bmatrix}^\top \begin{bmatrix} P & Q \\ Q^\top & T \end{bmatrix} \begin{bmatrix} X(t) \\ Z_N(t) \end{bmatrix} \\ &\qquad = \xi_N^\top(t) \begin{bmatrix} \phi_1 & PB - \rho Q \mathbb{1} & \Psi_{13} \\ * & 0 & B^\top Q - \rho \mathbb{1}^\top T \\ * & * & \rho \mathrm{He}(T A_N) + 2\delta T \end{bmatrix} \xi_N(t) \end{aligned}$$

with $\phi_1 = \mathrm{He}(PA + \rho Q \mathbb{1}^* C) + 2\delta P$. On the other hand, using the transport equation of (1),

$$\begin{aligned} \dot{V}_{N,2}(t) \; &= \; \int_0^1 \partial_t \left(z^\top(x,t) S z(x,t) \right) dx \\ &= \; -\rho \int_0^1 \partial_x \left(z^\top(x,t) S z(x,t) \right) dx \\ &= \; \rho X(t)^\top C^\top S C X(t) - \rho z(1,t)^\top S z(1,t) \\ &= \; \xi_N(t)^\top \begin{bmatrix} \rho C^\top S C & 0 & 0 \\ 0 & -\rho S & 0 \\ 0 & 0 & 0 \end{bmatrix} \xi_N(t), \end{aligned}$$

and we also get

$$\begin{aligned} \dot{V}_{N,3}(t) \; &= \; \int_0^1 \int_0^x \partial_t \left(z^\top(y,t) R z(y,t) \right) dy dx \\ &= \; -\rho \int_0^1 \int_0^x \partial_y \left(z^\top(y,t) R z(y,t) \right) dy dx \\ &= \; \rho X(t)^\top C^\top R C X(t) - \rho \int_0^1 z^\top(x,t) R z(x,t) dx. \end{aligned}$$

Merging the expressions of $\dot{V}_{N,1}, \dot{V}_{N,2}$ and $\dot{V}_{N,3}$ and using the definition of the matrix $\Psi_N(\rho, \delta)$ in (8), the following expression of $\dot{V}_N + 2\delta V_N$ can be obtained (using an integration by parts for the term in R):

$$\dot{V}_N(t) + 2\delta V_N(t) = \xi_N^\top(t)\Psi_N(\rho,\delta)\xi_N(t) + Z_N^\top(t)[\rho R_N - 2\delta(S_N + R_N)]Z_N(t) - \int_0^1 z^\top(x,t)(\rho R - 2\delta(S+R))z(x,t)dx - 2\delta \int_0^1 xz^\top(x,t)Rz(x,t)dx.$$

Following the same procedure as for the existence of ε_1, the LMIs (8) and (9) ensure that there exists a sufficiently small $\varepsilon_3 > 0$ such that

$$\rho R - 2\delta(S+R) \succ \varepsilon_3 I_m, \quad \Psi_N(\rho,\delta) \prec -\varepsilon_3 \begin{bmatrix} I_n & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathcal{I}_N \end{bmatrix}.$$

Hence, injecting these inequalities into the upper bound of $\dot{V}_N(t) + 2\delta V_N(t)$ (and using also $R \succ 0$ to get rid of the last term) yields

$$\dot{V}_N(t) + 2\delta V_N(t) \le -\varepsilon_3 |X(t)|_n^2 - \varepsilon_3 \int_0^1 |z(x,t)|^2 dx + Z_N^\top(t)[\rho R_N - 2\delta(S_N + R_N) - \varepsilon_3 \mathcal{I}_N]Z_N(t) - \int_0^1 z^\top(x,t)(\rho R - 2\delta(S+R) - \varepsilon_3 I_m)z(x,t)dx.$$

Since $\rho R - 2\delta(S+R) - \varepsilon_3 I_m \in \mathcal{S}_n^+$, Lemma 1 gives:

$$-\int_0^1 z^\top(x,t)(\rho R - 2\delta(S+R) - \varepsilon_3 I_m)z(x,t)dx \le -Z_N^\top(t)[\rho R_N - 2\delta(S_N + R_N) - \varepsilon_3 \mathcal{I}_N]Z_N(t),$$

so that the Lyapunov functional V_N satisfies, for all $t > 0$

$$\dot{V}_N(t) + 2\delta V_N(t) \le -\varepsilon_3 E(t).$$

Therefore, one can conclude to the exponential stability of system (1) provided that LMIs (7)-(8)-(9) are solvable at the order N. □

3.3 Exponential stability results

The following corollary allows to guarantee the exponential stability of system (1) without considering information on the decay rate δ.

Corollary 2. Let $\rho > 0$ be a given transport speed and $N \ge 0$ an integer. If there exist matrices P, Q, R, S, T such that $P \in \mathcal{S}_n$, $Q \in \mathbb{R}^{n,(N+1)m}$, $T \in \mathcal{S}_{(N+1)m}$, S and $R \in \mathcal{S}_m^+$ satisfy $\Phi_N \succ 0$ and $\Psi_N(\rho, \delta = 0) \prec 0$, then the coupled system (1) is exponentially stable.

Proof : If the conditions $\Phi_N \succ 0$ and $\Psi(\rho, 0) \prec 0$ are verified, then equations (11) and (12) hold with $\delta = 0$. It follows that $\dot{V}_N(t) + 2\delta^* V_N(t) \le 0$ with $2\delta^* = \varepsilon_3/\varepsilon_2$ and the conclusion stems as for Theorem 1. □

4. NUMERICAL EXAMPLES

Example 1: As a first illustration of our approach, we consider system (1) with the following matrices:

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Th.1	$\delta = 0$	$\delta = 0.005$	$\delta = 0.01$	Variables
N=0	0.2284	0.2611	0.2885	16
N=1	0.1761	0.2073	0.2325	27
N=2	0.1653	0.1957	0.2202	42
N=3	0.1623	0.1925	0.2168	61
N=4	0.1620	0.1919	0.2159	84

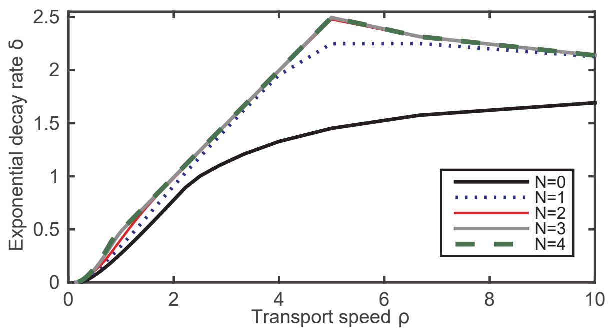
Table 1. Minimal allowable transport speed ρ_{min} in example 1.

Note first that the finite dimensional system considered alone, i.e. without the coupling with the transport PDE, is stable since the eigenvalues of $A + BC$ are -1.9 and -3. This system refers to one of the most classical time-delay example driven by the state equation

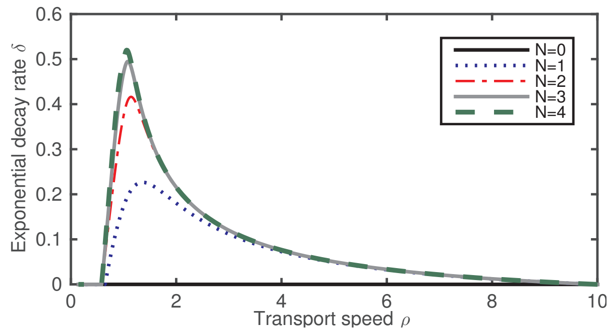
$$\dot{X}(t) = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix} X(t) + \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} X(t - 1/\rho).$$

A frequency approach can easily ensure that this system is asymptotically stable for all delay $1/\rho$ in the interval $[0,\ 6.1725]$, i.e. for all transport speed $\rho > 0.1620$.

For this example, Table 1 gathers the minimal allowable values for the transport speed ρ_{min} for several values of the required exponential decay rate δ and of the degree of the polynomial approximation N obtained solving the conditions of Theorem 1. When $\delta = 0$, one can see that Theorem 1 is able to recover an accurate (10^{-4}) estimate of the minimal transport speed, when $N = 4$. Table 1 also shows that for this example, increasing N leads to a notable reduction of the conservatism.



(a) Example 1.



(b) Example 2.

Fig. 1. Evolution of the decay rate δ with respect to ρ.

The graphs pictured in Figure 1.a show the evolution of the exponential convergence rate δ with respect to the transport speed ρ for different values of N. A notable aspect of these graphs is that Theorem 1 is able to show that the exponential decay rate δ reaches a maximum for optimal values of ρ located around $\rho = 5$. This shows that limiting the transport speed in system (1) may improve the convergence rate.

Example 2: Besides, Theorem 1 addresses also the stability of systems that may be unstable for a very high speed of transport as it is illustrated with the second example. We consider now a second set of matrices:

$$A = \begin{bmatrix} 0 & 1 \\ -2 & 0.1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = [1 \ 0] .$$

Here, one should be aware that system (1), with an arbitrarily large transport speed ρ, is not asymptotically stable since the trace of $A + BC = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0.1 \end{smallmatrix}\right]$ is strictly positive. Indeed the system has at least one eigenvalue in the right-hand-side of the complex plane.

Table 2 gathers for this example the minimal and maximal allowable transport speed ρ_{min} and ρ_{max} for several values of the required exponential decay rate δ and of the degree of the polynomial approximation N obtained solving the LMI conditions of Theorem 1. From this table, we see again that increasing the degree of approximation N delivers more accurate estimation of the minimal and maximal allowable ρ. In other words, the results obtained with a given N are included in the ones obtained with $N + 1$.

	N=0	N=1	N=2	N=3
Variables	8	12	17	23
$\delta = 0$	-	$\rho_{min} = 0.6491$	0.5840	0.5822
		$\rho_{max} = 9.9404$	9.9800	9.9800
$\delta = 0.005$	-	$\rho_{min} = 0.6569$	0.5890	0.5866
		$\rho_{max} = 9.0334$	9.0744	9.0744
$\delta = 0.01$	-	$\rho_{min} = 0.6647$	0.5939	0.5909
		$\rho_{max} = 8.2781$	8.3195	8.3195

Table 2. Minimal and maximal allowable transport speed in example 2.

The evolution of δ with respect to ρ for different values of $N = 0, 1, 2, 3, 4$ is displayed in Figure 1.b. Again, for this example, an optimal value of the transport speed is detected by Theorem 1 leading to a maximum of the convergence decay rate.

5. DISCUSSION AND CONCLUSION

In this paper we have presented a novel approach for the stability analysis of coupled ODEs - transport PDE systems issued from recent developments on time-delay systems. The method is based on the construction of a Lyapunov functional and the use of projections on Legendre polynomials. Exponential stability conditions for the complete infinite dimensional system providing an estimation of the decay rate are detailed and expressed in terms of tractable LMIs which depends explicitly on the transport velocity and on the degree N of the polynomial approximation.

This work can be interpreted as an alternative vision of the analysis provided in Seuret and Gouaisbaut (2015) and Seuret et al. (2015). In these articles, it is shown that the LMI conditions issued from this approximation method using Legendre polynomials form a hierarchy, i.e. that it is possible to prove that when increasing the degree N of approximation, the stability regions are enlarged. In other words, if there exists a solution to the LMI conditions for given ρ and δ at an order N, there also exists a solution to the same problem at the order $N + 1$.

The contribution presented in this paper constitutes a preliminary result on the stability analysis of a system including PDEs using Legendre polynomials and the Bessel-Legendre inequality. The problem treated here only covers a particular and simple example of distributed systems. Our objective is to consider a larger class of systems (coupled with ODE's or not) including, at a first stage, the heat or wave equations.

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