OPTIMAL TIME-DECAY ESTIMATES FOR THE COMPRESSIBLE NAVIER-STOKES EQUATIONS IN THE CRITICAL L p FRAMEWORK

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Abstract. The global existence issue for the isentropic compressible Navier-Stokes equations in the critical regularity framework has been addressed in [7] more than fifteen years ago. However, whether (optimal) time-decay rates could be shown in general critical spaces and any dimension $d \geq 2$ has remained an open question. Here we give a positive answer to that issue not only in the L^2 critical framework of [7] but also in the more general L^p critical framework of [3, 6, 14]. More precisely, we show that under a mild additional decay assumption that is satisfied if the low frequencies of the initial data are in e.g. $L^{p/2}(\mathbb{R}^d)$, the L^p norm (the slightly stronger $\dot{B}^0_{p,1}$ norm in fact) of the critical global solutions decays like $t^{-d(\frac{1}{p}-\frac{1}{4})}$ for $t \to +\infty$, exactly as firstly observed by A. Matsumura and T. Nishida in [23] in the case $p=2$ and $d=3$, for solutions with high Sobolev regularity.

Our method relies on refined time weighted inequalities in the Fourier space, and is likely to be effective for other hyperbolic/parabolic systems that are encountered in fluid mechanics or mathematical physics.

1. Introduction

In Eulerian coordinates, the motion of a general barotropic compressible fluid in the whole space $\mathbb{R}^d$ is governed by the following Navier-Stokes system:

$$\begin{cases} \partial_t \varrho + \mathrm{div}\,(\varrho u) = 0, \\ \partial_t(\varrho u) + \mathrm{div}\,(\varrho u \otimes u) - \mathrm{div}\big(2\mu D(u) + \lambda\,\mathrm{div}\,u\,\mathrm{Id}\big) + \nabla \Pi = 0. \end{cases} \tag{1.1}$$

Here $u = u(t,x) \in \mathbb{R}^d$ (with $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$) stands for the velocity field and $\varrho = \varrho(t,x) \in \mathbb{R}_+$ is the density. The barotropic assumption means that $\Pi \triangleq P(\varrho)$ for some given function P (that will be taken suitably smooth in all that follows). The notation $D(u) \triangleq \frac{1}{2}(D_x u + {}^T D_x u)$ stands for the *deformation tensor*, and div is the divergence operator with respect to the space variable. The density-dependent functions λ and μ (the *bulk* and *shear viscosities*) are supposed to be smooth enough and to satisfy

$$\mu > 0 \quad \text{and} \quad \nu \triangleq \lambda + 2\mu > 0. \tag{1.2}$$

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System (1.1) is supplemented with initial data

$$(\varrho, u)|_{t=0} = (\varrho_0, u_0), \tag{1.3}$$

and we focus on solutions going to some constant state $(\varrho_\infty, 0)$ with $\varrho_\infty > 0$, at infinity.

As in many works dedicated to nonlinear evolutionary PDEs, *scaling invariance* will play a fundamental role in our paper. The reason why is that whenever such an invariance exists, suitable critical quantities (that is, having the same scaling invariance as the system under consideration) control the possible finite time blow-up, and the global existence of strong solutions. In our situation, we observe that (1.1) is invariant by the transformation

$$\varrho(t,x) \rightsquigarrow \varrho(\ell^2 t, \ell x), \quad u(t,x) \rightsquigarrow \ell u(\ell^2 t, \ell x), \qquad \ell > 0, \tag{1.4}$$

up to a change of the pressure term Π into $\ell^2\Pi$. Therefore we expect critical norms or spaces for investigating (1.1), to have the scaling invariance (1.4).

As observed by the first author in [7], one may solve (1.1) in critical homogeneous Besov spaces of type $\dot B^s_{2,1}$ (see Def. 2.1). In that context, in accordance with (1.4), with the scaling properties of Besov spaces (see (A.1) below) and with the conditions at ∞, criticality means that $a_0 \triangleq \varrho_0 - \varrho_\infty$ and u_0 have to be taken in $\dot B^{d/2}_{2,1}$ and $\dot B^{d/2-1}_{2,1}$, respectively. Besides smallness however, in order to achieve *global existence*, an additional condition has to be prescribed on the low frequencies of the density. This comes from the fact that the scaling invariance in (1.4) modifies the (low order) pressure term. Schematically, in the low-frequency regime, the first order terms of (1.1) predominate and hyperbolic energy methods are thus expected to be appropriate. In particular it is suitable to work at the same level of regularity for $a \triangleq \varrho - \varrho_\infty$ and u, that is $\dot B^{\frac d2-1}_{2,1}$ (the influence of the viscous term $\mathcal{A}u$ is decisive, though, as it supplies parabolic decay estimates for both a and u in low frequencies). To handle the high-frequency part of the solution, the main difficulty comes from the convection term in the density equation, as it may cause a loss of one derivative. This is overcome in [7] by performing an energy method on the mixed type system (1.1) after spectral localization.

The result of [7] has been extended to Besov spaces that are not related to L^2. The original proof of [3] and [6] relies on a paralinearized version of System (1.1) combined with a Lagrangian change of variables after spectral localization. In a recent paper [14], B. Haspot achieved essentially the same result by means of a more elementary approach which is based on the use of Hoff's viscous effective flux [15] (see also [11] for global results in more general spaces, in the density dependent viscosity coefficients case). This eventually leads to the following statement[1]:

Theorem 1.1. *Let $d \geq 2$ and p satisfying*

$$2 \leq p \leq \min(4, 2d/(d-2)) \quad \textit{and, additionally,} \quad p \neq 4 \quad \textit{if} \quad d = 2. \tag{1.5}$$

Assume that $P'(\varrho_\infty) > 0$ and that (1.2) is fulfilled. There exists a constant $c = c(p,d,\lambda,\mu,P,\varrho_\infty)$ such that if $a_0 \triangleq \varrho_0 - \varrho_\infty$ is in $\dot B^{\frac dp}_{p,1}$, if u_0 is in $\dot B^{\frac dp-1}_{p,1}$ and if in

[1]Throughout the paper z^ℓ and z^h designate the low and high frequency parts of any tempered distribution z, that is $\mathcal{F}(z^\ell) \triangleq \psi\,\mathcal{F}z$ and $z^h \triangleq z - z^\ell$ where ψ is a suitable smooth compactly supported function, equal to 1 in a neighborhood of 0.

addition $(a_0^\ell, u_0^\ell) \in \dot{B}_{2,1}^{\frac{d}{2}-1}$ *with*

$$\mathcal{X}_{p,0} \triangleq \|(a_0,u_0)\|^\ell_{\dot{B}_{2,1}^{\frac{d}{2}-1}} + \|a_0\|^h_{\dot{B}_{p,1}^{\frac{d}{p}}} + \|u_0\|^h_{\dot{B}_{p,1}^{\frac{d}{p}-1}} \leq c \tag{1.6}$$

then (1.1) *has a unique global-in-time solution* (ϱ, u) *with* $\varrho = \varrho_\infty + a$ *and* (a,u) *in the space* X_p *defined by*[2]*:*

$$(a,u)^\ell \in \widetilde{\mathcal{C}}_b(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}-1}) \cap L^1(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}+1}), \quad a^h \in \widetilde{\mathcal{C}}_b(\mathbb{R}_+; \dot{B}_{p,1}^{\frac{d}{p}}) \cap L^1(\mathbb{R}_+; \dot{B}_{p,1}^{\frac{d}{p}}),$$
$$u^h \in \widetilde{\mathcal{C}}_b(\mathbb{R}_+; \dot{B}_{p,1}^{\frac{d}{p}-1}) \cap L^1(\mathbb{R}_+; \dot{B}_{p,1}^{\frac{d}{p}+1}).$$

Furthermore, we have for some constant $C = C(p,d,\lambda,\mu,P,\varrho_\infty)$,

$$\mathcal{X}_p \leq C\mathcal{X}_{p,0}, \tag{1.7}$$

with

$$\mathcal{X}_p \triangleq \|(a,u)\|^\ell_{\widetilde{L}^\infty(\dot{B}_{2,1}^{\frac{d}{2}-1})} + \|(a,u)\|^\ell_{L^1(\dot{B}_{2,1}^{\frac{d}{2}+1})} + \|a\|^h_{\widetilde{L}^\infty(\dot{B}_{p,1}^{\frac{d}{p}}) \cap L^1(\dot{B}_{p,1}^{\frac{d}{p}})} + \|u\|^h_{\widetilde{L}^\infty(\dot{B}_{p,1}^{\frac{d}{p}-1}) \cap L^1(\dot{B}_{p,1}^{\frac{d}{p}+1})}. \tag{1.8}$$

One may wonder how the global strong solutions constructed above look like for large time. Although providing an accurate long-time asymptotic description is still out of reach, a number of results concerning the time decay rates of smooth global solutions – sometimes referred to as $L^q - L^r$ decay estimates – are available. In this direction, the first achievement is due to Matsumura and Nishida [23, 24] in the 80ies. There, in the 3D case, the authors proved the global existence of classical solutions to (1.1) supplemented with data (ϱ_0, u_0) which are small perturbations in $L^1(\mathbb{R}^3) \times H^3(\mathbb{R}^3)$ of $(\varrho_\infty, 0)$, and established the following fundamental decay estimate:

$$\|(\varrho - \varrho_\infty, u)(t)\|_{L^2} \leq C\langle t\rangle^{-\frac{3}{4}} \quad \text{with} \quad \langle t\rangle \triangleq \sqrt{1+t^2}. \tag{1.9}$$

The decay rate in (1.9) (which is the same as for the heat equation with data in $L^1(\mathbb{R}^3)$) turns out to be the optimal one for the linearized system (1.1) about $(\varrho_\infty, 0)$. For that reason, it is often referred to as the optimal time-decay rate.

Shortly after Matsumura and Nishida, still for data with high Sobolev regularity, Ponce obtained in [27] the following optimal L^p decay rates for (1.1):

$$\|\nabla^k(\varrho - \varrho_\infty, u)(t)\|_{L^p} \leq C\langle t\rangle^{-\frac{d}{2}(1-\frac{1}{p})-\frac{k}{2}}, \quad 2 \leq p \leq \infty, \quad 0 \leq k \leq 2, \quad d = 2,3. \tag{1.10}$$

Similar results have been established in some situations where the fluid domain is not $\mathbb{R}^d$: the half-space or exterior domain cases have been investigated by Kagei and Kobayashi in [18, 19], Kobayashi in [20], and Shibata and Kobayashi in [21].

To find out what kind of asymptotic behavior is likely to be true for the global strong solutions of the compressible Navier-Stokes equations constructed above, it is natural to first investigate the decay properties of the linearized system (1.1) about $(\varrho_\infty, 0)$. As observed by different authors, this is strongly connected to the information given by wave propagation. In that respect, one may mention the work by Zeng in [31] dedicated

[2] The subspace $\widetilde{\mathcal{C}}_b(\mathbb{R}_+; \dot{B}^s_{q,1})$ of $\mathcal{C}_b(\mathbb{R}_+; \dot{B}^s_{q,1})$ is defined in (A.10), and the norms $\|\cdot\|_{\widetilde{L}^\infty(\dot{B}^s_{p,1})}$ are introduced just below Definition 2.1.

the one-dimensional case, and the detailed analysis of the Green function for the multi-dimensional case carried out by Hoff and Zumbrun in [16, 17], that leads to L^p decay rates towards diffusion waves that are the same as in (1.10). In [22], Liu and Wang exhibited pointwise estimates of diffusion waves with the optimal time-decay rate in odd dimension (as having Huygens's principle plays an important role therein). Let us finally mention the recent work by Guo and Wang in [12] that uses homogeneous Sobolev norms of *negative order* and allows to get optimal rates without resorting to time-decay properties of the linear system.

2. Main results

Let us emphasize that all the aforementioned works concern solutions with *high Sobolev regularity.* The optimal time-decay estimates issue for (1.1) in the critical regularity framework has been addressed only very recently, by Okita in [26]. There, thanks to a smart modification of the method of [7], Inequality (1.10) with $k = 0$ is proved in the L^2 critical framework in dimension $d \geq 3$ provided the data are additionally in some superspace of L^1. In the survey paper [10], the first author proposed another description of the time decay which allows to handle dimension $d \geq 2$ in the L^2 critical framework.

Our aim here is to develop the method of [10] so as to establish optimal decay results *in the general L^p critical framework of Theorem 1.1 and in any dimension $d \geq 2$.* As a by-product, we shall actually obtain a very accurate description of the decay rates, not only for Lebesgue spaces, but also for a full family of Besov norms with negative or positive regularity indices.

Before writing out the main statement of our paper, we need to introduce some notation and definition. To start with, we need a *Littlewood-Paley decomposition.* To this end, we fix some smooth radial non increasing function χ supported in $B(0, \frac{4}{3})$ and with value 1 on $B(0, \frac{3}{4})$, then set $\varphi(\xi) = \chi(\xi/2) - \chi(\xi)$ so that

$$\sum_{k\in\mathbb{Z}} \varphi(2^{-k}\cdot) = 1 \quad \text{in} \quad \mathbb{R}^d \setminus \{0\} \quad \text{and} \quad \operatorname{Supp}\varphi \subset \big\{\xi \in \mathbb{R}^d : 3/4 \leq |\xi| \leq 8/3\big\}\cdot$$

The homogeneous dyadic blocks $\dot{\Delta}_k$ are defined by

$$\dot{\Delta}_k u \triangleq \varphi(2^{-k}D)u = \mathcal{F}^{-1}(\varphi(2^{-k}\cdot)\mathcal{F}u) = 2^{kd}h(2^k\cdot) \star u \quad \text{with} \quad h \triangleq \mathcal{F}^{-1}\varphi.$$

The Littlewood-Paley decomposition of a general tempered distribution f reads

$$f = \sum_{k\in\mathbb{Z}} \dot{\Delta}_k f. \tag{2.11}$$

As it holds only modulo polynomials, it is convenient to consider only tempered distributions f such that

$$\lim_{k\to-\infty} \|\dot{S}_k f\|_{L^\infty} = 0, \tag{2.12}$$

where $\dot{S}_k f$ stands for the low frequency cut-off defined by $\dot{S}_k f \triangleq \chi(2^{-k}D)f$. Indeed, for those distributions, (2.11) holds true in $\mathcal{S}'(\mathbb{R}^d)$.

Let us now turn to the definition of the main functional spaces and norms that will come into play in our paper.

Definition 2.1. *For $s\in\mathbb{R}$ and $1\leq p,r\leq\infty$, the homogeneous Besov space $\dot B^s_{p,r}$ is the set of tempered distributions f satisfying* (2.12) *and*

$$\|f\|_{\dot B^s_{p,r}}\triangleq\Big\|\Big(2^{ks}\|\dot\Delta_k f\|_{L^p}\Big)\Big\|_{\ell^r(\mathbb{Z})}<\infty.$$

In the case where f depends also on the time variable, we shall often consider the subspace $\widetilde L^\infty_T(\dot B^s_{p,1})$ of those functions of $L^\infty(0,T;\dot B^s_{p,1})$ such that

$$\|f\|_{\widetilde L^\infty_T(\dot B^s_{p,1})}\triangleq\sum_{k\in\mathbb{Z}}2^{ks}\sup_{t\in[0,T]}\|\dot\Delta_k f(t,\cdot)\|_{L^p}<\infty.$$

Restricting the above norms to the low or high frequencies parts of distributions will be fundamental in our approach. To this end, we fix some suitable integer k_0 (the value of which will follow from the proof of the main theorem) and put[3]

$$\|f\|^\ell_{\dot B^s_{p,1}}\triangleq\sum_{k\leq k_0}2^{ks}\|\dot\Delta_k f\|_{L^p}\quad\text{and}\quad\|f\|^h_{\dot B^s_{p,1}}\triangleq\sum_{k\geq k_0-1}2^{ks}\|\dot\Delta_k f\|_{L^p},$$
$$\|f\|^\ell_{\widetilde L^\infty_T(\dot B^s_{p,1})}\triangleq\sum_{k\leq k_0}2^{ks}\|\dot\Delta_k f\|_{L^\infty_T(L^p)}\quad\text{and}\quad\|f\|^h_{\widetilde L^\infty_T(\dot B^s_{p,1})}\triangleq\sum_{k\geq k_0-1}2^{ks}\|\dot\Delta_k f\|_{L^\infty_T(L^p)},$$

where, for any Banach space X, we denote by $L^\infty_T(X)\triangleq L^\infty([0,T];X)$ the set of essentially bounded mesurable functions from $[0,T]$ to X.

Finally, we agree that throughout the paper C stands for a positive harmless "constant", the meaning of which is clear from the context. Similarly, $f\lesssim g$ means that $f\leq Cg$ and $f\approx g$ means that $f\lesssim g$ and $g\lesssim f$. It will be also understood that $\|(f,g)\|_X\triangleq\|f\|_X+\|g\|_X$ for all $f,g\in X$.

To simplify the presentation, it is wise to perform a suitable rescaling so as to reduce our study to the case where, at infinity, the density ϱ_∞, the sound speed $c_\infty\triangleq\sqrt{P'(\varrho_\infty)}$ and the total viscosity $\nu_\infty\triangleq\lambda_\infty+2\mu_\infty$ (with $\lambda_\infty\triangleq\lambda(\varrho_\infty)$ and $\mu_\infty\triangleq\mu(\varrho_\infty)$) are equal to 1. This may be done by making the change of unknowns:

$$\widetilde a(t,x)\triangleq\frac{\varrho}{\varrho_\infty}\bigg(\frac{\nu_\infty}{\varrho_\infty c_\infty^2}\,t,\frac{\nu_\infty}{\varrho_\infty c_\infty}\,x\bigg)-1\quad\text{and}\quad\widetilde u(t,x)\triangleq\frac{u}{c_\infty}\bigg(\frac{\nu_\infty}{\varrho_\infty c_\infty^2}\,t,\frac{\nu_\infty}{\varrho_\infty c_\infty}\,x\bigg)\cdot\tag{2.13}$$

Assuming that[4] $\varrho_\infty=1$, $P'(\varrho_\infty)=1$ and $\nu_\infty=1$, our main result is the following one.

Theorem 2.1. *Let $d\geq 2$ and p satisfying Condition* (1.5). *Let (ϱ_0,u_0) fulfill the assumptions of Theorem 1.1, and denote by (ϱ,u) the corresponding global solution of System* (1.1). *There exists a positive constant $c=c(p,d,\lambda,\mu,P)$ such that if in addition*

$$\mathcal{D}_{p,0}\triangleq\|(a_0,u_0)\|^\ell_{\dot B^{-s_0}_{2,\infty}}\leq c\quad\textit{with}\quad s_0\triangleq d\bigg(\frac{2}{p}-\frac12\bigg),\tag{2.14}$$

then we have for all $t\geq 0$,

$$\mathcal{D}_p(t)\lesssim\big(\mathcal{D}_{p,0}+\|(\nabla a_0,u_0)\|^h_{\dot B^{\frac dp-1}_{p,1}}\big),\tag{2.15}$$

[3]Note that for technical reasons, we need a small overlap between low and high frequencies.

[4]For the statement of decay estimates in the general case, the reader is referred to Section 4.

where the norm $\mathcal{D}_p(t)$ is defined by

$$\mathcal{D}_p(t) \triangleq \sup_{s\in(-s_0,2]} \|\langle\tau\rangle^{\frac{s_0+s}{2}}(a,u)\|^{\ell}_{L^\infty_t(\dot B^s_{2,1})} + \|\langle\tau\rangle^{\alpha}(\nabla a,u)\|^h_{\widetilde L^\infty_t(\dot B^{\frac dp-1}_{p,1})} + \|\tau\nabla u\|^h_{\widetilde L^\infty_t(\dot B^{\frac dp}_{p,1})} \tag{2.16}$$

and $\alpha \triangleq \frac{s_0}{2}+\min(2,\frac d4+\frac12-\varepsilon)$ *for some arbitrarily small* $\varepsilon>0$.

Some comments are in order.

(1) There is some freedom in the choice of s_0. In the standard case $p=2$ and for regular solutions, it is usual to assume that the data are in L^1 which, by critical embedding corresponds to $s_0=-d/2$. This value of s_0 is relevant in other contexts like the Boltzmann equation (see the work by Sohinger and Strain [28]), or hyperbolic systems with dissipation (see the paper by the second author and Kawashima [29]).

The reason why the space L^1 is natural when working in L^2-type framework is just because products of two terms in L^2, are in L^1. In our L^p framework, the similar heuristics would bring us to replace L^1 by $L^{p/2}$. Choosing s_0 as above corresponds exactly to the critical embedding $L^{p/2}\hookrightarrow \dot B^{-s_0}_{2,\infty}$.

(2) The decay rate for the low frequencies of the solution (first term of $\mathcal{D}_p$) is optimal inasmuch as it corresponds to the one of the linearized system (1.1) about $(\varrho_\infty,0)$ for general data in $\dot B^{-s_0}_{2,\infty}$. The last term of $\mathcal{D}_p$ is consistent with the critical functional framework given by the bulk regularity of the velocity. Finally, the (maximal) value of α in the second term of $\mathcal{D}_p$ may be guessed from the fact that in order to close the estimates, we need $\|u^\ell\cdot\nabla u^h\|_{\dot B^{\frac dp-1}_{p,1}}$ to decay like $\tau^{-\alpha}$, while the decay of ∇u^h in $\dot B^{\frac dp}_{p,1}$ is only τ^{-1}. Then applying the following product law in Besov spaces:

$$\|\tau^\alpha u^\ell\cdot\nabla u^h\|_{\widetilde L^\infty_t(\dot B^{\frac dp-1}_{p,1})}\lesssim \|\tau^{\alpha-1}u^\ell\|_{\widetilde L^\infty_t(\dot B^{\frac d2-1}_{2,1})}\|\tau\nabla u\|_{\widetilde L^\infty_t(\dot B^{\frac dp}_{p,1})},$$

and using the low frequency decay rate for u gives us the constraint $\alpha-1<\frac12(s_0+\frac d2-1)$ (at least if $\frac d2-1\leq 2$).

(3) If replacing (2.14) by the stronger assumption $\|(a_0,u_0)\|^\ell_{\dot B^{-s_0}_{2,1}}\leq c$, then one can take $\varepsilon=0$ and change the first term of $\mathcal{D}_p(t)$ for the slightly stronger norm $\sup_{s\in(-s_0,2]}\|\langle\tau\rangle^{\frac{s+s_0}{2}}(a,u)\|^\ell_{\widetilde L^\infty_t(\dot B^s_{2,1})}$.

(4) In physical dimensions $d=2,3$, Condition (1.5) allows us to consider the case $p>d$, so that the regularity exponent $d/p-1$ for the velocity becomes negative. Our result thus applies to *large* highly oscillating initial velocities (see [3, 6] for more explanation).

(5) Our functional $\mathcal{D}_p$ has been worked out to encode enough decay information to handle all the nonlinear terms. Having a more accurate description than in [26] and, in particular, exhibiting gain of regularity and decay altogether (last term of $\mathcal{D}_p$) is the key to getting optimal decay estimates in dimension $d=2$ and for $p>2$. Let us also emphasize that one may deduce L^q-L^r decay estimates in the spirit of (1.10) from the expression of $\mathcal{D}_p$ (see Corollaries 4.1 and 4.2 below).

(6) Last but not least, our approach is very robust : suitable modifications of the definition of $\mathcal{D}_p$ should allow to prove optimal decay estimates in critical spaces for other hyperbolic-parabolic systems arising in fluid mechanics models.

We end this section with an overview of our strategy. The starting point is to rewrite System (1.1) as the linearized compressible Navier-Stokes equations about $(1,0)$, looking at the nonlinearities as source terms. More concretely, we consider

$$\left\{\begin{array}{l}\partial_t a + \operatorname{div} u = f,\\ \partial_t u - \mathcal{A}u + \nabla a = g,\end{array}\right. \tag{2.17}$$

with $f \triangleq -\operatorname{div}(au)$, $\mathcal{A} \triangleq \mu_\infty\Delta + (\lambda_\infty+\mu_\infty)\nabla\operatorname{div}$ such that $\mu_\infty > 0$ and $\lambda_\infty + 2\mu_\infty = 1$,

$$g \triangleq -u\cdot\nabla u - I(a)\mathcal{A}u - k(a)\nabla a + \frac{1}{1+a}\operatorname{div}\big(2\widetilde{\mu}(a)D(u) + \widetilde{\lambda}(a)\operatorname{div} u\,\mathrm{Id}\big),$$

where[5]

$$I(a) \triangleq \frac{a}{1+a},\quad k(a) \triangleq \frac{P'(1+a)}{1+a} - 1,\quad \widetilde{\mu}(a) \triangleq \mu(1+a) - \mu(1) \ \text{ and } \ \widetilde{\lambda}(a) \triangleq \lambda(1+a) - \lambda(1).$$

In the case of high Sobolev regularity, the basic method to prove (1.9) is to take advantage of the corresponding $L^1 - L^2$ estimates for the semi-group generated by the left-hand side of (2.17), treating the terms f and g by means of Duhamel formula, and accepting loss of derivatives as the case may be. In the critical regularity framework however, one cannot afford any loss of regularity for the high frequency part of the solution (and some terms like $u\cdot\nabla a$ induce a loss of one derivative as one cannot expect any smoothing for a, solution of a transport equation). As regards the well-posedness issue in the L^2 critical framework (that is Theorem 1.1 with $p=2$), that difficulty has been overcome in [7] thanks to an appropriate energy method after spectral localization of the mixed hyperbolic-parabolic system (2.17) *including the convection terms*. As pointed out by Okita in [26], if assuming in addition that the initial data are in L^1 (or rather in the larger Besov space $\dot B^0_{1,\infty}$) then the same arguments lead to optimal time-decay estimates in the L^2 critical framework if $d \geq 3$.

To prove Theorem 2.1 in its full generality, one has to proceed differently: on one hand, using Okita's time decay functional does not allow to cover the two-dimensional case, and on the other hand it is not adapted to the general L^p setting. To get round the difficulty arising from the regularity-loss for the high-frequency part of the solution in the L^p critical framework, we shall follow the approach that has been used recently by Haspot [14] to prove Theorem 1.1. It is based on the observation that, at leading order, both the divergence-free part of u and the so-called *effective velocity* w (which is another name for the *viscous effective flux* of D. Hoff in [15]) fulfill some constant coefficient heat equation, while a satisfies a damped transport equation. Now, to cover all dimensions $d \geq 2$ and values of p satisfying (1.5), we also need to include an additional decay information for the high frequencies of the velocity in the definition of the decay functional $\mathcal{D}_p$ (last term of (2.16)).

Another difficulty if $p > 2$, compared to the L^2 case, is that one cannot expect interaction between high frequencies to provide any L^1 information on the low frequencies.

[5] In our analysis, the exact value of functions k, $\widetilde{\lambda}$, $\widetilde{\mu}$ and even I will not matter : we only need those functions to be smooth enough and to vanish at 0.

Indeed, let us look at the term $u\cdot\nabla a$ as an example. Decomposing a and u into their low and high frequency parts, we get

$$(u\cdot\nabla a)^\ell=(u^\ell\cdot\nabla a^\ell)^\ell+(u^h\cdot\nabla a^\ell)^\ell+(u^\ell\cdot\nabla a^h)^\ell+(u^h\cdot\nabla a^h)^\ell. \tag{2.18}$$

As pointed out in Theorem 1.1, the high-frequency part (a^h,u^h) lies in an L^p-type space. Hence one cannot bound the last term of (2.18) in a functional space with integrability index below $p/2$. A similar difficulty arises for the second and third terms which, at most, are in L^r with $1/r=1/p+1/2$. The strategy will thus be to bound the low frequencies of $u\cdot\nabla a$ in $L^{p/2}$, and to resort to $L^{p/2}-L^2$ type decay estimates instead of L^1-L^2 estimates. This heuristics turns out to work if $p\leq d$. Unfortunately, if $p>d$ (a case that may occur in physical dimension $d=2,3$), the low frequency part of some nonlinear terms need not be in $L^{p/2}$. The remedy is to perform estimates in the negative Besov space $\dot B^{-d(\frac2p-\frac12)}_{2,\infty}$, which corresponds to the following critical embedding:

$$L^{\frac p2}(\mathbb{R}^d)\hookrightarrow\dot B^0_{\frac p2,\infty}(\mathbb{R}^d)\hookrightarrow\dot B^{-d(\frac2p-\frac12)}_{2,\infty}(\mathbb{R}^d),$$

and matches the low frequency assumption (2.14). This requires our using some non so-classical product estimates in Besov spaces where the low frequency cut-off is crucial, see Proposition A.2.

The rest of the paper unfolds as follows. Section 3 is devoted to the proof of Theorem 2.1. In Section 4, we point out some consequences of our main theorem : optimal decay estimates in Lebesgue spaces in the spirit of (1.9), and explicit dependency with respect to the Mach and Reynolds number of (1.1) (recall that, so far, we set all parameters of the system to 1 for simplicity). Some material concerning Besov spaces, paradifferential calculus, product and commutator estimates is recalled in Appendix.

3. The proof of time-decay estimates

We here prove Theorem 2.1, taking for granted the global existence result of Theorem 1.1. We proceed in three steps, according to the three terms of the time-weighted functional $\mathcal{D}_p$ defined in (2.16).

Step 1 combines the low frequency decay properties of the semi-group defined by the left-hand side of (2.17), and Duhamel principle to handle the nonlinear terms. In that step, having Condition (2.14) is fundamental, as it rules the decay rate of the low-frequency part of $\mathcal{D}_p$ (and thus indirectly of high frequency terms, owing to nonlinear interaction). The proof for $p>d$ turns out to be more involved as for $p\leq d$ because the space $\dot B^{\frac dp-1}_{p,1}$ is no longer (locally) embedded in L^p so that one cannot resort directly to the obvious product law $L^p\times L^p\to L^{\frac p2}$ to treat nonlinearities.

In the second step, in order to exhibit the decay of the high frequencies part of the solution, we introduce (after B. Haspot in [13, 14]) the *effective velocity* $w\triangleq\nabla(-\Delta)^{-1}(a-\operatorname{div}u)$. This is motivated by the observation that if (2.17) is written in terms of a, w and of the divergence free part $\mathcal{P}u$ of u, then, up to low order terms, a satisfies a *damped* transport equation, and both w and $\mathcal{P}u$ satisfy a heat equation. Applying a suitable energy method after spectral localization enables us to avoid the loss of one derivative coming from the convection terms, and to take advantage of the nice decay properties provided by the heat and damped transport equations.

In the last step, we establish gain of regularity and decay altogether for the high frequencies of the velocity. That step strongly relies on the maximal regularity estimates for the Lamé semi-group which are the same as that for heat semi-group (see the remark that follows Prop. A.6), and is fundamental to get our main result in any dimension $d \geq 2$ and for all indices p satisfying (1.5).

In what follows, we shall use repeatedly that for $0 < \sigma_1 \leq \sigma_2$ with $\sigma_2 > 1$, we have

$$\int_0^t \langle t-\tau\rangle^{-\sigma_1}\langle\tau\rangle^{-\sigma_2}\,d\tau \lesssim \langle t\rangle^{-\sigma_1}. \tag{3.1}$$

Step 1: Bounds for the low frequencies. Let $(E(t))_{t\geq 0}$ be the semi-group associated to the left-hand side of (2.17). We get after spectral localization[6] that for all $k \in \mathbb{Z}$,

$$\begin{pmatrix} a_k(t) \\ u_k(t) \end{pmatrix} = E(t)\begin{pmatrix} a_{0,k} \\ u_{0,k} \end{pmatrix} + \int_0^t E(t-\tau)\begin{pmatrix} f_k(\tau) \\ g_k(\tau) \end{pmatrix} d\tau \tag{3.2}$$

where f and g have been defined in (2.17).

From an explicit computation of the action of $E(t)$ in Fourier variables (see e.g. [3]), we know that for any $k_0 \in \mathbb{Z}$, there exist two positive constants c_0 and C depending only on k_0 and such that

$$|\mathcal{F}(E(t)U)(\xi)| \leq Ce^{-c_0 t|\xi|^2}|\mathcal{F}U(\xi)| \quad \text{for all} \quad |\xi| \leq 2^{k_0}. \tag{3.3}$$

Therefore, using Parseval equality and the definition of $\dot\Delta_k$, we get for all $k \leq k_0$,

$$\|E(t)\dot\Delta_k U\|_{L^2} \leq Ce^{-\frac{c_0}{4}2^{2k}t}\|\dot\Delta_k U\|_{L^2}. \tag{3.4}$$

Hence, multiplying by $t^{\frac{s_0+s}{2}}2^{ks}$ and summing up on $k \leq k_0$,

$$t^{\frac{s_0+s}{2}}\sum_{k\leq k_0} 2^{ks}\|E(t)\dot\Delta_k U\|_{L^2} \lesssim \|U\|^{\ell}_{\dot B^{-s_0}_{2,\infty}}\sum_{k\leq k_0}(\sqrt{t}\,2^k)^{s_0+s}\,e^{-\frac{c_0}{4}(\sqrt{t}\,2^k)^2}. \tag{3.5}$$

Due to the following fact: for any $\sigma > 0$ there exists a constant C_σ so that

$$\sup_{t\geq 0}\sum_{k\in\mathbb{Z}} t^{\frac{\sigma}{2}}2^{k\sigma}e^{-\frac{c_0}{4}2^{2k}t} \leq C_\sigma, \tag{3.6}$$

we get from (3.5) that for $s + s_0 > 0$,

$$\sup_{t\geq 0}\, t^{\frac{s_0+s}{2}}\|E(t)U\|^{\ell}_{\dot B^{s}_{2,1}} \lesssim \|U\|^{\ell}_{\dot B^{-s_0}_{2,\infty}}. \tag{3.7}$$

In addition, it is clear that for $s + s_0 > 0$,

$$\|E(t)U\|^{\ell}_{\dot B^{s}_{2,1}} \lesssim \|U\|^{\ell}_{\dot B^{-s_0}_{2,\infty}}\Big(\sum_{k\leq k_0} 2^{k(s_0+s)}\Big) \lesssim \|U\|^{\ell}_{\dot B^{-s_0}_{2,\infty}}. \tag{3.8}$$

Therefore, setting $\langle t\rangle \triangleq \sqrt{1+t^2}$, we arrive at

$$\sup_{t\geq 0}\langle t\rangle^{\frac{s_0+s}{2}}\|E(t)U\|^{\ell}_{\dot B^{s}_{2,1}} \lesssim \|U\|^{\ell}_{\dot B^{-s_0}_{2,\infty}}, \tag{3.9}$$

and thus, taking advantage of Duhamel's formula,

$$\Big\|\int_0^t E(t-\tau)(f(\tau),g(\tau))\,d\tau\Big\|^{\ell}_{\dot B^{s}_{2,1}} \lesssim \int_0^t \langle t-\tau\rangle^{-\frac{s_0+s}{2}}\|(f(\tau),g(\tau))\|^{\ell}_{\dot B^{-s_0}_{2,\infty}}\,d\tau. \tag{3.10}$$

[6]Throughout, we set $z_k \triangleq \dot\Delta_k z$ for any tempered distribution z and $k \in \mathbb{Z}$.

We claim that if p fulfills (1.5), then we have for all $t\geq 0$,

$$\int_0^t \langle t-\tau\rangle^{-\frac{s_0+s}{2}}\|(f,g)(\tau)\|^{\ell}_{\dot B^{-s_0}_{2,\infty}}\,d\tau\lesssim \langle t\rangle^{-\frac{s_0+s}{2}}\big(\mathcal{D}_p^2(t)+\mathcal{X}_p^2(t)\big), \tag{3.11}$$

where $\mathcal{X}_p$ and $\mathcal{D}_p$ have been defined in (1.8) and (2.16), respectively.

Proving our claim requires different arguments depending on whether $p\leq d$ or $p>d$. Let us start with the easier case $2\leq p\leq d$. Then, owing to the embedding $L^{p/2}\hookrightarrow \dot B^{-s_0}_{2,\infty}$, it suffices to establish that

$$\int_0^t \langle t-\tau\rangle^{-\frac{s_0+s}{2}}\|(f,g)(\tau)\|_{L^{p/2}}\,d\tau\lesssim \langle t\rangle^{-\frac{s_0+s}{2}}\big(\mathcal{D}_p^2(t)+\mathcal{X}_p^2(t)\big). \tag{3.12}$$

Now, to bound the term with f, we use the decomposition

$$f=u\cdot\nabla a+a\,\mathrm{div}\,u^\ell+a\,\mathrm{div}\,u^h,\quad \text{with}\ \ u^\ell\triangleq\sum_{k<k_0}\dot\Delta_k u\ \ \text{and}\ \ u^h\triangleq u-u^\ell. \tag{3.13}$$

It follows from Hölder inequality that

$$\begin{aligned}\int_0^t \langle t-\tau\rangle^{-\frac{s_0+s}{2}}\|(u\cdot\nabla a)(\tau)\|_{L^{p/2}}\,d\tau\\ \leq\Big(\sup_{0\leq\tau\leq t}\langle\tau\rangle^{\frac{s_0}{2}-\frac{d}{2p}+\frac d4}\|u(\tau)\|_{L^p}\Big)\Big(\sup_{0\leq\tau\leq t}\langle\tau\rangle^{\frac{s_0}{2}-\frac{d}{2p}+\frac d4+\frac12}\|\nabla a(\tau)\|_{L^p}\Big)\\ \times\int_0^t\langle t-\tau\rangle^{-\frac{s_0+s}{2}}\langle\tau\rangle^{-s_0+\frac dp-\frac{d+1}{2}}\,d\tau.\end{aligned} \tag{3.14}$$

By Minkowski's inequality, we have

$$\|u\|_{L^p}\leq\|u^\ell\|_{L^p}+\|u^h\|_{L^p}, \tag{3.15}$$

and embedding (see the Appendix) together with the definition of u^ℓ and u^h imply that

$$\|u^\ell\|_{L^p}\lesssim\|u\|^{\ell}_{\dot B^{\frac d2-\frac dp}_{2,1}}\quad\text{and}\quad \|u^h\|_{L^p}\lesssim\|u\|^{h}_{\dot B^{\frac dp-1}_{p,1}}\quad\text{if}\ \ 2\leq p\leq d. \tag{3.16}$$

Combining (3.15)–(3.16) and using the definition of α and of $\mathcal{D}_p(t)$ thus yields

$$\sup_{0\leq\tau\leq t}\langle\tau\rangle^{\frac{s_0}{2}-\frac{d}{2p}+\frac d4}\|u(\tau)\|_{L^p}\lesssim\mathcal{D}_p(t). \tag{3.17}$$

Similarly, we can get

$$\sup_{0\leq\tau\leq t}\langle\tau\rangle^{\frac{s_0}{2}-\frac{d}{2p}+\frac d4}\|a(\tau)\|_{L^p}+\sup_{0\leq\tau\leq t}\langle\tau\rangle^{\frac{s_0}{2}-\frac{d}{2p}+\frac d4+\frac12}\|\nabla a(\tau)\|_{L^p}\lesssim\mathcal{D}_p(t). \tag{3.18}$$

Because $2\leq p\leq d$ and $s_0=\frac{2d}{p}-\frac d2$, we arrive for all $-s_0<s\leq 2$ at

$$\frac{s_0}{2}+\frac s2\leq\frac dp-\frac d4+1\leq s_0-\frac dp+\frac{d+1}{2}=\frac dp+\frac12\cdot \tag{3.19}$$

Since $d/p+1/2>1$, we get from (3.1),

$$\int_0^t\langle t-\tau\rangle^{-\frac{s_0+s}{2}}\langle\tau\rangle^{-s_0+\frac dp-\frac{d+1}{2}}\,d\tau\lesssim\langle t\rangle^{-\frac{s_0+s}{2}}. \tag{3.20}$$

Hence, it follows from (3.14), (3.17), (3.18) that

$$\int_0^t\langle t-\tau\rangle^{-\frac{s_0+s}{2}}\|(u\cdot\nabla a)(\tau)\|_{L^{p/2}}\,d\tau\lesssim\langle t\rangle^{-\frac{s_0+s}{2}}\mathcal{D}_p^2(t). \tag{3.21}$$

Bounding the term with $a\,\mathrm{div}\,u^\ell$ is similar: we get

$$\int_0^t \langle t-\tau\rangle^{-\frac{s_0+s}{2}}\|(a\,\mathrm{div}\,u^\ell)(\tau)\|_{L^{p/2}}\,d\tau \lesssim \langle t\rangle^{-\frac{s_0+s}{2}}\mathcal{D}_p^2(t). \tag{3.22}$$

Regarding the term with $a\,\mathrm{div}\,u^h$, we use that if $t\geq 2$,

$$\int_0^t \langle t-\tau\rangle^{-\frac{s_0+s}{2}}\|(a\,\mathrm{div}\,u^h)(\tau)\|_{L^{p/2}}\,d\tau$$
$$\leq \int_0^t \langle t-\tau\rangle^{-\frac{s_0+s}{2}}\|a(\tau)\|_{L^p}\|\mathrm{div}\,u^h(\tau)\|_{L^p}\,d\tau = \Big(\int_0^1+\int_1^t\Big)(\cdots)d\tau \triangleq I_1+I_2.$$

Remembering the definitions of $\mathcal{X}_p(t)$ and $\mathcal{D}_p(t)$, we can obtain

$$\begin{aligned} I_1 &\lesssim \langle t\rangle^{-\frac{s_0+s}{2}}\Big(\sup_{0\leq\tau\leq 1}\|a(\tau)\|_{L^p}\Big)\int_0^1\|\mathrm{div}\,u^h(\tau)\|_{L^p}\,d\tau \\ &\lesssim \langle t\rangle^{-\frac{s_0+s}{2}}\mathcal{D}_p(1)\mathcal{X}_p(1) \end{aligned} \tag{3.23}$$

and, using the fact that $\langle\tau\rangle\approx\tau$ when $\tau\geq 1$,

$$\begin{aligned} I_2 &= \int_1^t \langle t-\tau\rangle^{-\frac{s_0+s}{2}}\langle\tau\rangle^{-1-\frac{s_0}{2}}\big(\langle\tau\rangle^{\frac{s_0}{2}}\|a(\tau)\|_{L^p}\big)\big(\tau\|\mathrm{div}\,u^h(\tau)\|_{L^p}\big)\,d\tau \\ &\leq \sup_{1\leq\tau\leq t}\Big(\langle\tau\rangle^{\frac{s_0}{2}}\|a(\tau)\|_{L^p}\Big)\sup_{1\leq\tau\leq t}\Big(\tau\|\mathrm{div}\,u^h(\tau)\|_{L^p}\Big)\int_1^t\langle t-\tau\rangle^{-\frac{s_0+s}{2}}\langle\tau\rangle^{-1-\frac{s_0}{2}}\,d\tau \\ &\lesssim \langle t\rangle^{-\frac{s_0+s}{2}}\mathcal{D}_p^2(t). \end{aligned}$$

Therefore, for $t\geq 2$, we arrive at

$$\int_0^t \langle t-\tau\rangle^{-\frac{s_0+s}{2}}\|(a\,\mathrm{div}\,u^h)(\tau)\|_{L^{p/2}}\,d\tau \lesssim \langle t\rangle^{-\frac{s_0+s}{2}}\Big(\mathcal{D}_p(t)\mathcal{X}_p(t)+\mathcal{D}_p^2(t)\Big)\cdot \tag{3.24}$$

The case $t\leq 2$ is obvious as $\langle t\rangle\approx 1$ and $\langle t-\tau\rangle\approx 1$ for $0\leq\tau\leq t\leq 2$, and

$$\int_0^t \|a\,\mathrm{div}\,u^h\|_{L^{p/2}}\,d\tau \leq \|a\|_{L_t^\infty(L^p)}\|\mathrm{div}\,u^h\|_{L_t^1(L^p)} \lesssim \mathcal{D}_p(t)\mathcal{X}_p(t). \tag{3.25}$$

Next, in order to bound the term of (3.12) corresponding to g, we use the decomposition $g=g^1+g^2+g^3+g^4$ with $g^1\triangleq -u\cdot\nabla u$, $g^2\triangleq -k(a)\nabla a$,

$$\begin{aligned} g^3 &\triangleq 2\,\frac{\widetilde{\mu}(a)}{1+a}\,\mathrm{div}\,D(u)+\frac{\widetilde{\lambda}(a)}{1+a}\,\nabla\mathrm{div}\,u - I(a)\mathcal{A}u \\ \text{and}\qquad g^4 &\triangleq 2\,\frac{\widetilde{\mu}'(a)}{1+a}D(u)\cdot\nabla a+\frac{\widetilde{\lambda}'(a)}{1+a}\,\mathrm{div}\,u\,\nabla a. \end{aligned}$$

The terms with g^1 and g^2 may be handled as f above: $k(a)\nabla a$ and $u\cdot\nabla u^\ell$ may be treated as $u\cdot\nabla a$, and $u\cdot\nabla u^h$, as $a\,\mathrm{div}\,u^h$. To handle the viscous term g^3, we see that it suffices to bound

$$K(a)\nabla^2 u^\ell \quad\text{and}\quad K(a)\nabla^2 u^h, \tag{3.26}$$

where K stands for some smooth function vanishing at 0.

To bound $K(a)\nabla^2 u^\ell$, we write that

$$\int_0^t \langle t-\tau\rangle^{-\frac{s_0+s}{2}}\|K(a)\nabla^2 u^\ell\|_{L^{p/2}}\,d\tau \lesssim \int_0^t \langle t-\tau\rangle^{-\frac{s_0+s}{2}}\langle\tau\rangle^{-1-s_0}\,d\tau$$
$$\times\Big(\sup_{\tau\in[0,t]}\langle\tau\rangle^{\frac{s_0}{2}}\|a(\tau)\|_{L^p}\Big)\Big(\sup_{\tau\in[0,t]}\langle\tau\rangle^{\frac{s_0}{2}+1}\|\nabla^2 u^\ell(\tau)\|_{L^p}\Big)$$
$$\lesssim \langle t\rangle^{-\frac{s_0+s}{2}}\mathcal{D}_p^2(t), \tag{3.27}$$

where we used (3.18), the fact that $\frac{d}{4}-\frac{d}{2p}\geq 0$, and

$$\|\nabla^2 u^\ell(\tau)\|_{L^p}\lesssim \|u\|^\ell_{\dot B^{\frac d2-\frac dp+2}_{2,1}}\lesssim \|u\|^\ell_{\dot B^2_{2,1}}.$$

To bound the term involving $K(a)\nabla^2 u^h$, we consider the cases $t\geq 2$ and $t\leq 2$ separately. If $t\geq 2$ then we write

$$\int_0^t \langle t-\tau\rangle^{-\frac{s_0+s}{2}}\|K(a)\nabla^2 u^h\|_{L^{p/2}}\,d\tau \lesssim \int_0^t \langle t-\tau\rangle^{-\frac{s_0+s}{2}}\|a(\tau)\|_{L^p}\|\nabla^2 u^h(\tau)\|_{L^p}\,d\tau$$
$$=\Big(\int_0^1+\int_1^t\Big)(\cdots)d\tau \triangleq J_1+J_2.$$

Now, because $d/p-1\geq 0$, we have by embedding,

$$J_1\lesssim \langle t\rangle^{-\frac{s_0+s}{2}}\Big(\sup_{0\leq\tau\leq 1}\|a(\tau)\|_{L^p}\Big)\int_0^1\|\nabla^2 u^h\|_{L^p}\,d\tau\lesssim \langle t\rangle^{-\frac{s_0+s}{2}}\mathcal{D}_p(1)\mathcal{X}_p(1)$$

and

$$J_2\lesssim \sup_{0\leq\tau\leq t}\Big(\langle\tau\rangle^{\frac{s_0}{2}}\|a(\tau)\|_{L^p}\Big)\sup_{0\leq\tau\leq t}\Big(\tau\|\nabla^2 u^h(\tau)\|_{L^p}\Big)\int_1^t\langle t-\tau\rangle^{-\frac{s_0+s}{2}}\langle\tau\rangle^{-1-\frac{s_0}{2}}\,d\tau$$
$$\lesssim \langle t\rangle^{-\frac{s_0+s}{2}}\mathcal{D}_p^2(t).$$

Therefore, we end up with

$$\int_0^t \langle t-\tau\rangle^{-\frac{s_0+s}{2}}\|K(a)\nabla^2 u^h\|_{L^{p/2}}\,d\tau\lesssim \langle t\rangle^{-\frac{s_0+s}{2}}\Big(\mathcal{D}_p(t)\mathcal{X}_p(t)+\mathcal{D}_p^2(t)\Big)\quad\text{for all }\ t\geq 2.$$

In the case $t\leq 2$, we have $\langle t\rangle\approx 1$ and $\langle t-\tau\rangle\approx 1$ for $0\leq\tau\leq t\leq 2$, and

$$\int_0^t \langle t-\tau\rangle^{-\frac{s_0+s}{2}}\|K(a)\nabla^2 u^h\|_{L^{p/2}}\,d\tau\lesssim\int_0^t\|a\|_{L^p}\|\nabla^2 u^h\|_{L^p}\,d\tau\lesssim \mathcal{D}_p(t)\mathcal{X}_p(t). \tag{3.28}$$

To bound g^4, it suffices to consider $\nabla F(a)\otimes\nabla u$ where F is some smooth function vanishing at 0. Once again, it is convenient to split that term into

$$\nabla F(a)\otimes\nabla u=\nabla F(a)\otimes\nabla u^\ell+\nabla F(a)\otimes\nabla u^h.$$

For bounding the first term, one may proceed as for proving (3.27) (using (3.18)):

$$\int_0^t \langle t-\tau\rangle^{-\frac{s_0+s}{2}}\|\nabla F(a)\otimes\nabla u^\ell\|_{L^{p/2}}\,d\tau \lesssim \int_0^t\langle t-\tau\rangle^{-\frac{s_0+s}{2}}\langle\tau\rangle^{-1-s_0}\,d\tau$$
$$\times\Big(\sup_{\tau\in[0,t]}\langle\tau\rangle^{\frac{s_0}{2}+\frac12}\|\nabla a(\tau)\|_{L^p}\Big)\Big(\sup_{\tau\in[0,t]}\langle\tau\rangle^{s_0+\frac12}\|\nabla u^\ell(\tau)\|_{L^p}\Big)$$
$$\lesssim \langle t\rangle^{-\frac{s_0+s}{2}}\mathcal{D}_p^2(t).$$

To handle the term $\nabla F(a)\otimes\nabla u^h$, we consider the cases $t\geq 2$ and $t\leq 2$ separately. If $t\geq 2$ then we write

$$\begin{aligned}\int_0^t \langle t-\tau\rangle^{-\frac{s_0+s}{2}}\|\nabla F(a)\otimes\nabla u^h\|_{L^{p/2}}\,d\tau &\lesssim \int_0^t \langle t-\tau\rangle^{-\frac{s_0+s}{2}}\|\nabla a(\tau)\|_{L^p}\|\nabla u^h(\tau)\|_{L^p}\,d\tau\\ &=\Big(\int_0^1+\int_1^t\Big)(\cdots)d\tau \triangleq K_1+K_2.\end{aligned}$$

It is clear that

$$K_1\lesssim \langle t\rangle^{-\frac{s_0+s}{2}}\Big(\sup_{0\leq\tau\leq 1}\|\nabla a(\tau)\|_{L^p}\Big)\int_0^1\|\nabla u^h\|_{L^p}\,d\tau\lesssim \langle t\rangle^{-\frac{s_0+s}{2}}\mathcal{D}_p(1)\mathcal{X}_p(1)$$

and that

$$K_2\lesssim \sup_{1\leq\tau\leq t}\Big(\langle\tau\rangle^{\frac{s_0}{2}+\frac12}\|\nabla a(\tau)\|_{L^p}\Big)\sup_{1\leq\tau\leq t}\Big(\tau^{\frac12}\|\nabla u^h(\tau)\|_{L^p}\Big)\int_1^t\langle t-\tau\rangle^{-\frac{s_0+s}{2}}\langle\tau\rangle^{-1-\frac{s_0}{2}}\,d\tau.$$

Note that for $\tau\geq 1$, we have

$$\tau^{\frac12}\|\nabla u^h(\tau)\|_{L^p}\lesssim \tau\|\nabla u(\tau)\|^h_{\dot B^{\frac dp}_{p,1}}.$$

Hence one can conclude thanks to (3.18) that

$$K_2\lesssim \langle t\rangle^{-\frac{s_0+s}{2}}\mathcal{D}_p^2(t)\quad\text{for all}\ \ t\geq 2.$$

The (easy) case $t\leq 2$ is left to the reader, which completes the proof of (3.12), and thus of (3.11), for $p\leq d$.

Let us now prove (3.11) in the case $p>d$ (that can occur only if $d=2,3$). The idea is to replace the Hölder inequality $\|FG\|_{L^{p/2}}\leq\|F\|_{L^p}\|G\|_{L^p}$ with the following two inequalities:

$$\|FG\|_{\dot B^{-s_0}_{2,\infty}}\lesssim\|F\|_{\dot B^{1-\frac dp}_{p,1}}\|G\|_{\dot B^{\frac d2-1}_{2,1}},\tag{3.29}$$

$$\|FG\|^\ell_{\dot B^{-\frac dp}_{2,\infty}}\lesssim\|F\|_{\dot B^{\frac dp-1}_{p,1}}\|G\|_{\dot B^{1-\frac dp}_{2,1}},\tag{3.30}$$

which stem from Proposition A.1 (second item) and Besov embedding.

Using Inequality (3.29) turns out to be appropriate for handling the terms:

$$u\cdot\nabla a^\ell,\quad a\mathrm{div}\,u^\ell,\quad u\cdot\nabla u^\ell,\quad k(a)\nabla a^\ell\ \text{ and }\quad K(a)\nabla^2u^\ell,$$

while (3.30) will be used for $\nabla(F(a))\otimes\nabla u^\ell$.

We claim that

$$\|(a,u)(\tau)\|_{\dot B^{1-\frac dp}_{p,1}}\lesssim\langle\tau\rangle^{-\frac12}\mathcal{D}_p(\tau).\tag{3.31}$$

and that

$$\|a(\tau)\|_{\dot B^{\frac dp}_{p,1}}\lesssim\langle\tau\rangle^{-\frac dp}\mathcal{D}_p(\tau).\tag{3.32}$$

Indeed, we have by embedding and definition of s_0,

$$\|(a,u)^\ell(\tau)\|_{\dot B^{1-\frac dp}_{p,1}}\lesssim\|(a,u)^\ell(\tau)\|_{\dot B^{1-s_0}_{2,1}}\lesssim\langle\tau\rangle^{-\frac12}\mathcal{D}_p(\tau),\tag{3.33}$$

and, by interpolation, because $p > d$,

$$\|u^h\|_{\dot B^{1-\frac dp}_{p,1}} \lesssim \|u^h\|^{\frac dp}_{\dot B^{\frac dp-1}_{p,1}} \|u^h\|^{1-\frac dp}_{\dot B^{\frac dp+1}_{p,1}}. \tag{3.34}$$

Hence, using the definition of the second and third terms of $\mathcal{D}_p$,

$$\|u^h(\tau)\|_{\dot B^{1-\frac dp}_{p,1}} \lesssim \langle\tau\rangle^{-(1+\frac dp(\alpha-1))}\mathcal{D}_p(\tau). \tag{3.35}$$

As obviously $\alpha \geq 1$, putting (3.33) and (3.35) together yields (3.31) for u. Finally, because $p \leq 2d$,

$$\|a^h(\tau)\|_{\dot B^{1-\frac dp}_{p,1}} \lesssim \|a^h(\tau)\|_{\dot B^{\frac dp}_{p,1}} \lesssim \langle\tau\rangle^{-\alpha}\mathcal{D}_p(\tau),$$

and Inequality (3.31) is thus also fulfilled by a^h.

For proving (3.32), we notice that by embedding and because $\frac d2 \leq 2$ in the case we are interested in,

$$\|a^\ell(\tau)\|_{\dot B^{\frac dp}_{p,1}} \lesssim \|a^\ell(\tau)\|_{\dot B^{\frac d2}_{2,1}} \lesssim \langle\tau\rangle^{-\frac dp}\mathcal{D}_p(\tau),$$

and, because $\alpha \geq 1 \geq \frac dp$,

$$\|a^h(\tau)\|_{\dot B^{\frac dp}_{p,1}} \lesssim \langle\tau\rangle^{-\frac dp}\mathcal{D}_p(\tau).$$

Now, thanks to (3.29) and (3.31), one can thus write

$$\begin{aligned}\int_0^t \langle t-\tau\rangle^{-\frac{s_0+s}{2}} \|(u\cdot\nabla a^\ell)(\tau)\|^\ell_{\dot B^{-s_0}_{2,\infty}}\,d\tau &\lesssim \int_0^t \langle t-\tau\rangle^{-\frac{s_0+s}{2}} \|u\|_{\dot B^{1-\frac dp}_{p,1}} \|\nabla a^\ell\|_{\dot B^{\frac d2-1}_{2,1}}\,d\tau\\ &\lesssim \mathcal{D}_p^2(t)\int_0^t \langle t-\tau\rangle^{-\frac{s_0+s}{2}}\langle\tau\rangle^{-(\frac dp+\frac12)}\,d\tau.\end{aligned}$$

For all $-s_0 < s \leq 2$, we have

$$0 < \frac{s_0+s}{2} \leq \frac dp - \frac d4 + 1 \leq \frac dp + \frac12\cdot \tag{3.36}$$

As $d/p + 1/2 > 1$ (because $p < 2d$), Inequality (3.1) thus implies that

$$\int_0^t \langle t-\tau\rangle^{-\frac{s_0+s}{2}} \|(u\cdot\nabla a^\ell)(\tau)\|^\ell_{\dot B^{-s_0}_{2,\infty}}\,d\tau \lesssim \langle t\rangle^{-\frac{s_0+s}{2}}\mathcal{D}_p^2(t). \tag{3.37}$$

The terms $a\,\mathrm{div}\, u^\ell$, $u\cdot\nabla u^\ell$ and $k(a)\nabla a^\ell$ are similar. Regarding the term $K(a)\nabla^2u^\ell$, we just have to write that, thanks to (3.29) and Bernstein inequality,

$$\begin{aligned}&\int_0^t \langle t-\tau\rangle^{-\frac{s_0+s}{2}} \|K(a)\nabla^2u^\ell(\tau)\|^\ell_{\dot B^{-s_0}_{2,\infty}}\,d\tau\\ &\lesssim \Big(\sup_{\tau\in[0,t]}\langle\tau\rangle^{\frac12}\|a(\tau)\|_{\dot B^{1-\frac dp}_{p,1}}\Big)\Big(\sup_{\tau\in[0,t]}\langle\tau\rangle^{\frac dp}\|\nabla^2u^\ell(\tau)\|_{\dot B^{\frac d2-1}_{2,1}}\Big)\int_0^t \langle t-\tau\rangle^{-\frac{s_0+s}{2}}\langle\tau\rangle^{-(\frac dp+\frac12)}d\tau\\ &\lesssim \langle t\rangle^{-\frac{s_0+s}{2}}\Big(\sup_{\tau\in[0,t]}\langle\tau\rangle^{\frac12}\|a(\tau)\|_{\dot B^{1-\frac dp}_{p,1}}\Big)\Big(\sup_{\tau\in[0,t]}\langle\tau\rangle^{\frac dp}\|u^\ell(\tau)\|_{\dot B^{\frac d2}_{2,1}}\Big) \lesssim \langle t\rangle^{-\frac{s_0+s}{2}}\mathcal{D}_p^2(t).\end{aligned}$$

Bounding $\nabla u^\ell\otimes\nabla F(a)$ requires inequality (3.30) and Proposition A.3: we have

$$\int_0^t\langle t-\tau\rangle^{-\frac{s_0+s}{2}}\|\nabla u^\ell\cdot\nabla F(a)\|^\ell_{\dot B^{-\frac dp}_{2,\infty}}\,d\tau\lesssim\int_0^t\langle t-\tau\rangle^{-\frac{s_0+s}{2}}\|\nabla u^\ell\|_{\dot B^{1-\frac dp}_{2,1}}\|\nabla F(a)\|_{\dot B^{\frac dp-1}_{p,1}}\,d\tau$$
$$\lesssim\int_0^t\langle t-\tau\rangle^{-\frac{s_0+s}{2}}\|u^\ell\|_{\dot B^{2-\frac dp}_{2,1}}\|a\|_{\dot B^{\frac dp}_{p,1}}\,d\tau.$$

Of course, as $s_0\le d/p$, we have

$$\|\nabla u^\ell\cdot\nabla F(a)\|^\ell_{\dot B^{-s_0}_{2,\infty}}\lesssim\|\nabla u^\ell\cdot\nabla F(a)\|^\ell_{\dot B^{-\frac dp}_{2,\infty}}.$$

Now, because the definition of $\mathcal D_p$ ensures that

$$\|u^\ell\|_{\dot B^{2-\frac dp}_{2,1}}\le\langle\tau\rangle^{-(1+\frac{d}{2p}-\frac d4)}\mathcal D_p(\tau),$$

we get, using (3.32) and the fact that $1+\frac{3d}{2p}-\frac d4\ge1+\frac{s_0}{2}$,

$$\int_0^t\langle t-\tau\rangle^{-\frac{s_0+s}{2}}\|\nabla u^\ell\cdot\nabla F(a)\|^\ell_{\dot B^{-s_0}_{2,\infty}}\,d\tau\lesssim\langle t\rangle^{-\frac{s_0+s}{2}}\mathcal D_p^2(t).$$

Bounding the terms corresponding to

$$u\cdot\nabla a^h,\quad a\mathrm{div}\,u^h,\quad u\cdot\nabla u^h,\quad k(a)\nabla a^h\quad\text{and}\quad K(a)\nabla^2u^h$$

requires our using Inequality (A.3) with $\sigma=1-d/p$, namely

$$\|FG^h\|^\ell_{\dot B^{-s_0}_{2,\infty}}\lesssim\bigl(\|F\|_{\dot B^{1-\frac dp}_{p,1}}+\|\dot S_{k_0+N_0}F\|_{L^{p^*}}\bigr)\|G^h\|_{\dot B^{\frac dp-1}_{p,1}}\quad\text{with}\quad\frac1{p^*}\triangleq\frac12-\frac1p,$$

for some universal integer N_0, which implies, owing to the embedding $\dot B^{\frac dp}_{2,1}\hookrightarrow L^{p^*}$ and to Bernstein inequality (note that $p^*\ge p$),

$$\|FG^h\|^\ell_{\dot B^{-s_0}_{2,\infty}}\lesssim\bigl(\|F^\ell\|_{\dot B^{\frac dp}_{2,1}}+\|F\|_{\dot B^{1-\frac dp}_{p,1}}\bigr)\|G^h\|_{\dot B^{\frac dp-1}_{p,1}}.\tag{3.38}$$

As an example, let us show how (3.38) allows to bound the term corresponding to $a\,\mathrm{div}\,u^h$. We start with the inequality

$$\int_0^t\langle t-\tau\rangle^{-\frac{s_0+s}{2}}\|a\,\mathrm{div}\,u^h\|^\ell_{\dot B^{-s_0}_{2,\infty}}\,d\tau\lesssim\int_0^t\langle t-\tau\rangle^{-\frac{s_0+s}{2}}\bigl(\|a^\ell\|_{\dot B^{\frac dp}_{2,1}}+\|a\|_{\dot B^{1-\frac dp}_{p,1}}\bigr)\|\mathrm{div}u^h\|_{\dot B^{\frac dp-1}_{p,1}}d\tau$$
$$=\Bigl(\int_0^1+\int_1^t\Bigr)(\cdots)d\tau\triangleq\widetilde I_1+\widetilde I_2.$$

Because $d/p-1<1-d/p\le d/p$ and $d/p\ge d/2-1$, we have

$$\|a\|_{\dot B^{1-\frac dp}_{p,1}}\lesssim\|a\|_{\dot B^{\frac dp}_{p,1}\cap\dot B^{\frac dp-1}_{p,1}}\lesssim\|a\|^\ell_{\dot B^{\frac d2-1}_{2,1}}+\|a\|^h_{\dot B^{\frac dp}_{p,1}}\quad\text{and}\quad\|a^\ell\|_{\dot B^{\frac dp}_{2,1}}\lesssim\|a\|^\ell_{\dot B^{\frac d2-1}_{2,1}},$$

and thus

$$\widetilde I_1\lesssim\langle t\rangle^{-\frac{s_0+s}{2}}\mathcal D_p(1)\mathcal X_p(1).$$

Furthermore, we notice that

$$\|(a,u)^\ell\|_{\dot B^{\frac dp}_{2,1}}\le\langle\tau\rangle^{-\frac12(s_0+\frac dp)}\mathcal D_p(t).\tag{3.39}$$

Hence, using also (3.31), the fact that
$$\sup_{\tau\in[1,t]}\tau\|\mathrm{div}u^h(\tau)\|_{\dot B^{\frac dp-1}_{p,1}}\lesssim\mathcal{D}_p(t),$$
and that for all $s\leq 2$, we have $\frac{s_0+s}{2}\leq\min(\frac32,\frac12(s_0+\frac dp+2))$, we conclude that
$$\begin{aligned}\widetilde{I}_2&\lesssim\mathcal{D}_p^2(t)\int_0^t\langle t-\tau\rangle^{-\frac{s_0+s}{2}}\big(\langle\tau\rangle^{-\frac12(s_0+\frac dp+2)}+\langle\tau\rangle^{-\frac32}\big)\,d\tau\\&\lesssim\langle t\rangle^{-\frac{s_0+s}{2}}\mathcal{D}_p^2(t),\end{aligned}$$
The term $u\cdot\nabla u^h$ being completely similar (thanks to (3.31) and (3.39)), we get
$$\int_0^t\langle t-\tau\rangle^{-\frac{s_0+s}{2}}\|(a\,\mathrm{div}u^h,u\cdot\nabla u^h)\|^\ell_{\dot B^{-s_0}_{2,\infty}}\,d\tau\lesssim\langle t\rangle^{-\frac{s_0+s}{2}}\Big(\mathcal{D}_p(t)\mathcal{X}_p(t)+\mathcal{D}_p^2(t)\Big)\cdot\tag{3.40}$$
That (3.40) also holds for $t\leq 2$ is just a consequence of the definition of $\mathcal{D}_p$ and $\mathcal{X}_p$.

Bounding the term with $u\cdot\nabla a^h$ works the same since $\alpha\geq1$ and
$$\|a^h(t)\|_{\dot B^{\frac dp}_{p,1}}\lesssim\langle\tau\rangle^{-\alpha}\mathcal{D}_p(t).\tag{3.41}$$

In order to handle the term with $k(a)\nabla a^h$, we use (3.38), Proposition A.3 and proceed as follows:
$$\begin{aligned}\|k(a)\nabla a^h\|^\ell_{\dot B^{-s_0}_{2,\infty}}&\lesssim\big(\|k(a)\|_{\dot B^{1-\frac dp}_{p,1}}+\|\dot S_{k_0+N_0}k(a)\|_{L^{p^*}}\big)\|\nabla a^h\|_{\dot B^{\frac dp-1}_{p,1}}\\&\lesssim\big(\|a\|_{\dot B^{1-\frac dp}_{p,1}}+\|a\|_{L^{p^*}}\big)\|\nabla a^h\|_{\dot B^{\frac dp-1}_{p,1}}.\end{aligned}$$

Using the embeddings $\dot B^{\frac dp}_{2,1}\hookrightarrow L^{p^*}$ and $\dot B^{s_0}_{p,1}\hookrightarrow L^{p^*}$, and decomposing a into low and high frequencies, we discover that
$$\|a\|_{L^{p^*}}\lesssim\|a\|^\ell_{\dot B^{\frac dp}_{2,1}}+\|a\|^h_{\dot B^{\frac dp}_{p,1}}.$$

Hence
$$\|k(a)\nabla a^h\|^\ell_{\dot B^{-s_0}_{2,\infty}}\lesssim\big(\|a\|^\ell_{\dot B^{\frac dp}_{2,1}}+\|a\|^h_{\dot B^{\frac dp}_{p,1}}\big)\|\nabla a^h\|_{\dot B^{\frac dp-1}_{p,1}},$$
and one can thus bound the term corresponding to $k(a)\nabla a^h$ exactly as $u\cdot\nabla a^h$.

Likewise, according to (3.38) and proposition A.3 and arguing as above, we get
$$\|K(a)\nabla^2u^h\|^\ell_{\dot B^{-s_0}_{2,\infty}}\lesssim\big(\|a\|^\ell_{\dot B^{\frac d2-1}_{2,1}}+\|a\|^h_{\dot B^{\frac dp}_{p,1}}\big)\|\nabla^2u^h\|_{\dot B^{\frac dp-1}_{p,1}}.$$

As we have for all $t\geq0$,
$$\int_0^t\|\nabla^2u\|^h_{\dot B^{\frac dp-1}_{p,1}}\,d\tau\lesssim\mathcal{X}_p(t)\quad\text{and}\quad t\|\nabla^2u^h(t)\|_{\dot B^{\frac dp-1}_{p,1}}\lesssim\mathcal{D}_p(t),$$
one can conclude exactly as for the previous term $k(a)\nabla a^h$ that
$$\int_0^t\langle t-\tau\rangle^{-\frac{s_0+s}{2}}\|K(a)\nabla^2u^h\|^\ell_{\dot B^{-s_0}_{2,\infty}}\,d\tau\lesssim\langle t\rangle^{-\frac{s_0+s}{2}}\Big(\mathcal{D}_p(t)\mathcal{X}_p(t)+\mathcal{D}_p^2(t)\Big)\cdot$$

Finally, to bound $\nabla F(a)\otimes\nabla u^h$, we have to resort to (A.4) with $\sigma=1-d/p$, namely

$$\|\nabla F(a)\otimes\nabla u^h\|^{\ell}_{\dot B^{-s_0}_{2,\infty}}\lesssim\Big(\|\nabla F(a)\|_{\dot B^{\frac dp-1}_{p,1}}+\sum_{k=k_0}^{k_0+N_0-1}\|\dot\Delta_k\nabla F(a)\|_{L^{p^*}}\Big)\|\nabla u^h\|_{\dot B^{1-\frac dp}_{p,1}}.$$

As $p^*\geq p$, Bernstein inequality ensures that $\|\dot\Delta_k\nabla F(a)\|_{L^{p^*}}\lesssim\|\dot\Delta_kF(a)\|_{L^p}$ for $k_0\leq k<k_0+N_0$. Hence, thanks to Proposition A.3, we have

$$\|\nabla F(a)\otimes\nabla u^h\|^{\ell}_{\dot B^{-s_0}_{2,\infty}}\lesssim\|a\|_{\dot B^{\frac dp}_{p,1}}\|\nabla u^h\|_{\dot B^{1-\frac dp}_{p,1}}.$$

Therefore, if $t\geq2$ then

$$\begin{aligned}\int_0^t\langle t-\tau\rangle^{-\frac{s_0+s}{2}}\|\nabla F(a)\otimes\nabla u^h\|^{\ell}_{\dot B^{-s_0}_{2,\infty}}d\tau&\lesssim\int_0^t\langle t-\tau\rangle^{-\frac{s_0+s}{2}}\|a\|_{\dot B^{\frac dp}_{p,1}}\|\nabla u^h\|_{\dot B^{1-\frac dp}_{p,1}}d\tau\\&=\Big(\int_0^1+\int_1^t\Big)(\cdots)d\tau\triangleq\widetilde K_1+\widetilde K_2.\end{aligned}$$

As $1-d/p\leq d/p$, it is clear that

$$\widetilde K_1\lesssim\langle t\rangle^{-\frac{s_0+s}{2}}\mathcal D_p(1)\mathcal X_p(1)$$

and that, owing to (3.32),

$$\begin{aligned}\widetilde K_2&\lesssim\Big(\sup_{\tau\in[1,t]}\langle\tau\rangle^{\frac dp}\|a(\tau)\|_{\dot B^{\frac dp}_{p,1}}\Big)\Big(\sup_{\tau\in[1,t]}\tau\|\nabla u^h(\tau)\|_{\dot B^{\frac dp}_{p,1}}\Big)\int_0^t\langle t-\tau\rangle^{-\frac{s_0+s}{2}}\langle\tau\rangle^{-1-\frac dp}d\tau\\&\lesssim\langle t\rangle^{-\frac{s_0+s}{2}}\mathcal D_p^2(t).\end{aligned}$$

Finally, if $t\leq2$ then we have

$$\int_0^t\langle t-\tau\rangle^{-\frac{s_0+s}{2}}\|\nabla F(a)\otimes\nabla u^h(\tau)\|^{\ell}_{\dot B^{-s_0}_{2,\infty}}d\tau\lesssim\|a\|_{L^\infty_t(\dot B^{\frac dp}_{p,1})}\|\nabla u^h\|_{L^1_t(\dot B^{\frac dp}_{p,1})}\lesssim\mathcal D_p(t)\mathcal X_p(t),$$

which completes the proof of (3.12) in the case $p>d$.

Combining with (3.9) for bounding the term of (3.14) pertaining to the data, we conclude that

$$\langle t\rangle^{\frac{s_0+s}{2}}\|(a,u)(t)\|^{\ell}_{\dot B^{s}_{2,1}}\lesssim\mathcal D_{p,0}+\mathcal D_p^2(t)+\mathcal X_p^2(t)\quad\text{for all}\quad t\geq0,\tag{3.42}$$

provided that $-s_0<s\leq2$.

Step 2: Decay estimates for the high frequencies of $(\nabla a,u)$. This step is devoted to bounding the second term of $\mathcal D_p(t)$. In contrast with the first step, here one can provide a common proof for all values of p fulfilling (1.5).

Let $\mathcal P\triangleq\mathrm{Id}+\nabla(-\Delta)^{-1}\mathrm{div}$ be the Leray projector onto divergence-free vector fields. It follows from (2.17) that $\mathcal Pu$ satisfies the following ordinary heat equation:

$$\partial_t\mathcal Pu-\mu_\infty\Delta\mathcal Pu=\mathcal Pg.$$

Applying $\dot\Delta_k$ to the above equation yields for all $k\in\mathbb Z$,

$$\partial_t\mathcal Pu_k-\mu_\infty\Delta\mathcal Pu_k=\mathcal Pg_k\quad\text{with}\quad u_k\triangleq\dot\Delta_ku\quad\text{and}\quad g_k\triangleq\dot\Delta_kg.$$

Then, multiplying each component of the above equation by $|(\mathcal Pu_k)^i|^{p-2}(\mathcal Pu_k)^i$ and integrating over $\mathbb R^d$ gives for $i=1,\cdots,d$,

$$\frac1p\frac{d}{dt}\|\mathcal Pu_k^i\|^p_{L^p}-\mu_\infty\int\Delta(\mathcal Pu_k)^i|(\mathcal Pu_k)^i|^{p-2}(\mathcal Pu_k)^i\,dx=\int|(\mathcal Pu_k)^i|^{p-2}(\mathcal Pu_k)^ig_k^i\,dx.$$

The key observation is that the second term of the l.h.s., although not spectrally localized, may be bounded from below as if it were (see Prop. A.4). After summation on $i=1,\cdots,d$, we end up for some constant c_p with

$$\frac{1}{p}\frac{d}{dt}\|\mathcal{P}u_k\|_{L^p}^p+c_p\mu_\infty 2^{2k}\|\mathcal{P}u_k\|_{L^p}^p\leq\|\mathcal{P}g_k\|_{L^p}\|\mathcal{P}u_k\|_{L^p}^{p-1}. \tag{3.43}$$

At this point, following Haspot's method in [13, 14], we introduce the *effective velocity*

$$w=\nabla(-\Delta)^{-1}(a-\mathrm{div}\,u).$$

It is clear that w fulfills

$$\partial_t w-\Delta w=\nabla(-\Delta)^{-1}(f-\mathrm{div}\,g)+w-(-\Delta)^{-1}\nabla a. \tag{3.44}$$

Hence, arguing exactly as for proving (3.43), we get for $w_k\triangleq\dot\Delta_k w$:

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\|w_k\|_{L^p}^p&+c_p2^{2k}\|w_k\|_{L^p}^p\\&\leq\bigl(\|\nabla(-\Delta)^{-1}(f_k-\mathrm{div}\,g_k)\|_{L^p}+\|w_k-(-\Delta)^{-1}\nabla a_k\|_{L^p}\bigr)\|w_k\|_{L^p}^{p-1}.\end{aligned} \tag{3.45}$$

In terms of w, the function a satisfies the following *damped* transport equation:

$$\partial_t a+\mathrm{div}\,(au)+a=-\mathrm{div}\,w. \tag{3.46}$$

Then, applying the operator $\partial_i\dot\Delta_k$ to (3.46) and denoting $R_k^i\triangleq[u\cdot\nabla,\partial_i\dot\Delta_k]a$ gives

$$\partial_t\partial_i a_k+u\cdot\nabla\partial_i a_k+\partial_i a_k=-\partial_i\dot\Delta_k(a\mathrm{div}\,u)-\partial_i\mathrm{div}\,w_k+R_k^i,\quad i=1,\cdots,d. \tag{3.47}$$

Multiplying by $|\partial_i a_k|^{p-2}\partial_i a_k$, integrating on $\mathbb{R}^d$, and performing an integration by parts in the second term of (3.47), we get

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\|\partial_i a_k\|_{L^p}^p+\|\partial_i a_k\|_{L^p}^p=\frac{1}{p}&\int\mathrm{div}\,u\,|\partial_i a_k|^p\,dx\\&+\int\bigl(R_k^i-\partial_i\dot\Delta_k(a\mathrm{div}\,u)-\partial_i\mathrm{div}\,w_k\bigr)|\partial_i a_k|^{p-2}\partial_i a_k\,dx.\end{aligned}$$

Summing up on $i=1,\cdots,d$, and applying Hölder and Bernstein inequalities leads to

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\|\nabla a_k\|_{L^p}^p+\|\nabla a_k\|_{L^p}^p\leq\Bigl(&\frac{1}{p}\|\mathrm{div}u\|_{L^\infty}\|\nabla a_k\|_{L^p}+\|\nabla\dot\Delta_k(a\mathrm{div}u)\|_{L^p}\\&+C2^{2k}\|w_k\|_{L^p}+\|R_k\|_{L^p}\Bigr)\|\nabla a_k\|_{L^p}^{p-1}.\end{aligned} \tag{3.48}$$

Adding up that inequality (multiplied by εc_p) to (3.43) and (3.45) yields

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\bigl(\|\mathcal{P}u_k\|_{L^p}^p+\|w_k\|_{L^p}^p+\varepsilon c_p\|\nabla a_k\|_{L^p}^p\bigr)+c_p2^{2k}\bigl(\mu_\infty\|\mathcal{P}u_k\|_{L^p}^p+\|w_k\|_{L^p}^p\bigr)+\varepsilon c_p\|\nabla a_k\|_{L^p}^p\\ \leq\bigl(\|\mathcal{P}g_k\|_{L^p}+\|\nabla(-\Delta)^{-1}(f_k-\mathrm{div}\,g_k)\|_{L^p}\bigr)\|(\mathcal{P}u_k,w_k)\|_{L^p}^{p-1}\\ +\varepsilon c_p\Bigl(\frac{1}{p}\|\mathrm{div}u\|_{L^\infty}\|\nabla a_k\|_{L^p}+\|\nabla\dot\Delta_k(a\mathrm{div}\,u)\|_{L^p}+\|R_k\|_{L^p}\Bigr)\|\nabla a_k\|_{L^p}^{p-1}\\ +C\varepsilon c_p2^{2k}\|w_k\|_{L^p}\|\nabla a_k\|_{L^p}^{p-1}+\|w_k\|_{L^p}^p+\|(-\Delta)^{-1}\nabla a_k\|_{L^p}\|w_k\|_{L^p}^{p-1}.\end{aligned}$$

Taking advantage of Young inequality, we see that the last line may be absorbed by the l.h.s. if ε is taken small enough. It is also the case of the last two terms according to (A.6) (as $(-\Delta)^{-1}$ is a homogeneous Fourier multiplier of degree -2), if k is large enough. Therefore, remembering that $f_k=\dot\Delta_k\mathrm{div}\,(au)$ and using that $\nabla(-\Delta)^{-1}\mathrm{div}$ is a homogeneous multiplier of degree 0, we conclude that there exist some $k_0 \in \mathbb{Z}$ and $c_0, \varepsilon > 0$ so that for all $k \geq k_0$, we have

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\big(\|\mathcal{P}u_k\|_{L^p}^p + \|w_k\|_{L^p}^p + \varepsilon c_p\|\nabla a_k\|_{L^p}^p\big) + c_0\big(\|\mathcal{P}u_k\|_{L^p}^p + \|w_k\|_{L^p}^p + \varepsilon c_p\|\nabla a_k\|_{L^p}^p\big)\\ \leq C\big(\|g_k\|_{L^p} + \|\dot{\Delta}_k(au)\|_{L^p}\big)\|(\mathcal{P}u_k, w_k)\|_{L^p}^{p-1}\\ +\varepsilon c_p\Big(\frac{1}{p}\|\mathrm{div}u\|_{L^\infty}\|\nabla a_k\|_{L^p} + \|\nabla\dot{\Delta}_k(a\mathrm{div}\, u)\|_{L^p} + \|R_k\|_{L^p}\Big)\|\nabla a_k\|_{L^p}^{p-1}.\end{aligned}$$

Integrating in time, we arrive (taking smaller c_0 as the case may be) at

$$e^{c_0t}\|(\mathcal{P}u_k, w_k, \nabla a_k)(t)\|_{L^p} \lesssim \|(\mathcal{P}u_k, w_k, \nabla a_k)(0)\|_{L^p} + \int_0^t e^{c_0\tau}S_k(\tau)\,d\tau$$

with $S_k \triangleq S_k^1 + \cdots + S_k^5$ and

$$\begin{gathered}S_k^1 \triangleq \|\dot{\Delta}_k(au)\|_{L^p}, \quad S_k^2 \triangleq \|g_k\|_{L^p},\\ S_k^3 \triangleq \|\nabla\dot{\Delta}_k(a\mathrm{div}\, u)\|_{L^p}, \quad S_k^4 \triangleq \|R_k\|_{L^p}, \quad S_k^5 \triangleq \|\mathrm{div}\, u\|_{L^\infty}\|\nabla a_k\|_{L^p}.\end{gathered}$$

It is clear that $(u_k, \nabla a_k)$ satisfies a similar inequality, for we have

$$u = w - \nabla(-\Delta)^{-1}a + \mathcal{P}u \tag{3.49}$$

which leads for $k \geq k_0$ to

$$\|u_k - (w_k + \mathcal{P}u_k)\|_{L^p} \lesssim 2^{-2k_0}\|\nabla a_k\|_{L^p}. \tag{3.50}$$

Therefore, there exists a constant $c_0 > 0$ such that for all $k \geq k_0$ and $t \geq 0$, we have

$$\|(\nabla a_k, u_k)(t)\|_{L^p} \lesssim e^{-c_0t}\|(\nabla a_k(0), u_k(0))\|_{L^p} + \int_0^t e^{-c_0(t-\tau)}S_k(\tau)\,d\tau. \tag{3.51}$$

Now, multiplying both sides by $\langle t\rangle^\alpha 2^{k(\frac{d}{p}-1)}$, taking the supremum on $[0,T]$, and summing up over $k \geq k_0$ yields

$$\begin{aligned}\|\langle t\rangle^\alpha(\nabla a, u)\|^h_{\widetilde{L}^\infty_T(\dot{B}^{\frac{d}{p}-1}_{p,1})} \lesssim \|(\nabla a_0, u_0)\|^h_{\dot{B}^{\frac{d}{p}-1}_{p,1}}\\ + \sum_{k\geq k_0}\sup_{0\leq t\leq T}\bigg(\langle t\rangle^\alpha\int_0^t e^{c_0(\tau-t)}2^{k(\frac{d}{p}-1)}S_k\,d\tau\bigg)\cdot\end{aligned} \tag{3.52}$$

In order to bound the sum, we first notice that

$$\sum_{k\geq k_0}\sup_{0\leq t\leq 2}\bigg(\langle t\rangle^\alpha\int_0^t e^{c_0(\tau-t)}2^{k(\frac{d}{p}-1)}S_k(\tau)\,d\tau\bigg) \lesssim \int_0^2\sum_{k\geq k_0}2^{k(\frac{d}{p}-1)}S_k(\tau)\,d\tau. \tag{3.53}$$

It follows from Propositions A.1 and A.5 that

$$\int_0^2\sum_{k\geq k_0}2^{k(\frac{d}{p}-1)}S_k(\tau)\,d\tau \lesssim \int_0^2\Big(\|au\|^h_{\dot{B}^{\frac{d}{p}-1}_{p,1}} + \|g\|^h_{\dot{B}^{\frac{d}{p}-1}_{p,1}} + \|\nabla u\|_{\dot{B}^{\frac{d}{p}}_{p,1}}\|a\|_{\dot{B}^{\frac{d}{p}}_{p,1}}\Big)d\tau. \tag{3.54}$$

It is clear that the last term of the r.h.s. may be bounded by $C\mathcal{X}_p^2(2)$ and that, owing to Prop. A.1, we have

$$\|au\|^h_{L^1_t(\dot{B}^{\frac{d}{p}-1}_{p,1})} \lesssim \|au\|_{L^1_t(\dot{B}^{\frac{d}{p}}_{p,1})} \lesssim \|a\|_{L^2_t(\dot{B}^{\frac{d}{p}}_{p,1})}\|u\|_{L^2_t(\dot{B}^{\frac{d}{p}}_{p,1})}. \tag{3.55}$$

Furthermore, combining Propositions A.1 and A.3 yields (remembering that $p < 2d$)

$$\|g\|_{L^1_t(\dot B^{\frac dp-1}_{p,1})} \lesssim \Big(\|u\|_{L^\infty_t(\dot B^{\frac dp-1}_{p,1})}\|\nabla u\|_{L^1_t(\dot B^{\frac dp}_{p,1})} + \|a\|_{L^\infty_t(\dot B^{\frac dp}_{p,1})}\|\nabla u\|_{L^1_t(\dot B^{\frac dp}_{p,1})} + \|a\|_{L^2_t(\dot B^{\frac dp}_{p,1})}\|\nabla a\|_{L^2_t(\dot B^{\frac dp-1}_{p,1})}\Big).$$

Now, we observe that

$$\|a\|_{L^2_t(\dot B^{\frac dp}_{p,1})} \leq \|a\|^\ell_{L^2_t(\dot B^{\frac dp}_{p,1})} + \|a\|^h_{L^2_t(\dot B^{\frac dp}_{p,1})}. \tag{3.56}$$

Combining interpolation, Hölder inequality and embedding (here we use that $p \geq 2$), we may write

$$\begin{aligned}\|a\|^\ell_{L^2_t(\dot B^{\frac dp}_{p,1})} &\lesssim \Big(\|a\|^\ell_{L^1_t(\dot B^{\frac dp+1}_{p,1})}\Big)^{\frac12}\Big(\|a\|^\ell_{L^\infty_t(\dot B^{\frac dp-1}_{p,1})}\Big)^{\frac12}\\ &\lesssim \|a\|^\ell_{L^1_t(\dot B^{\frac d2+1}_{2,1})} + \|a\|^\ell_{L^\infty_t(\dot B^{\frac d2-1}_{2,1})} \lesssim \mathcal{X}_p(t).\end{aligned} \tag{3.57}$$

Likewise, we have

$$\|a\|^h_{L^2_t(\dot B^{\frac dp}_{p,1})} \lesssim \Big(\|a\|^h_{L^1_t(\dot B^{\frac dp}_{p,1})}\Big)^{\frac12}\Big(\|a\|^h_{L^\infty_t(\dot B^{\frac dp}_{p,1})}\Big)^{\frac12} \lesssim \mathcal{X}_p(t).$$

Arguing similarly for bounding u, we get

$$\|a\|_{L^2_t(\dot B^{\frac dp}_{p,1})} + \|u\|_{L^2_t(\dot B^{\frac dp}_{p,1})} \lesssim \mathcal{X}_p(t), \tag{3.58}$$

and one can conclude that the first two terms in the r.h.s. of (3.54) may be bounded by $\mathcal{X}_p^2(2)$. We thus have

$$\sum_{k\geq k_0}\sup_{0\leq t\leq 2}\langle t\rangle^\alpha\int_0^t e^{c_0(\tau-t)}2^{k(\frac dp-1)}S_k(\tau)\,d\tau \lesssim \mathcal{X}_p^2(2). \tag{3.59}$$

Let us now bound the supremum for $2 \leq t \leq T$ in the last term of (3.52), assuming (with no loss of generality) that $T \geq 2$. To this end, it is convenient to split the integral on $[0,t]$ into integrals on $[0,1]$ and $[1,t]$. The integral on $[0,1]$ is easy to handle: because $e^{c_0(\tau-t)} \leq e^{-c_0t/2}$ for $2 \leq t \leq T$ and $0 \leq \tau \leq 1$, one can write that

$$\begin{aligned}\sum_{k\geq k_0}\sup_{2\leq t\leq T}\langle t\rangle^\alpha\int_0^1 e^{c_0(\tau-t)}2^{k(\frac dp-1)}S_k(\tau)\,d\tau &\leq \sum_{k\geq k_0}\sup_{2\leq t\leq T}\langle t\rangle^\alpha e^{-\frac{c_0}{2}t}\int_0^1 2^{k(\frac dp-1)}S_k\,d\tau\\ &\lesssim \int_0^1\sum_{k\geq k_0}2^{k(\frac dp-1)}S_k\,d\tau.\end{aligned}$$

Hence, following the procedure leading to (3.59), we end up with

$$\sum_{k\geq k_0}\sup_{2\leq t\leq T}\bigg(\langle t\rangle^\alpha\int_0^1 e^{c_0(\tau-t)}2^{k(\frac dp-1)}S_k(\tau)\,d\tau\bigg) \lesssim \mathcal{X}_p^2(1). \tag{3.60}$$

In order to bound the $[1,t]$ part of the integral for $2 \leq t \leq T$, we notice that (3.1) guarantees that

$$\sum_{k\geq k_0}\sup_{2\leq t\leq T}\bigg(\langle t\rangle^\alpha\int_1^t e^{c_0(\tau-t)}2^{k(\frac dp-1)}S_k(\tau)\,d\tau\bigg) \lesssim \sum_{k\geq k_0}2^{k(\frac dp-1)}\sup_{1\leq t\leq T}t^\alpha S_k(t). \tag{3.61}$$

In what follows, we shall use repeatedly the following inequality

$$\|\tau\nabla u\|_{\widetilde{L}^\infty_t(\dot B^{\frac dp}_{p,1})}\lesssim \mathcal{D}_p(t), \tag{3.62}$$

which just stems from the definition of $\mathcal{D}_p(t)$, as regards the high-frequencies of u, and from Bernstein inequalities for the low frequencies. Indeed: if $d\geq 3$ then $d/2+1>2$ and one can write that

$$\begin{aligned}\|\tau\nabla u\|^\ell_{\widetilde{L}^\infty_t(\dot B^{\frac dp}_{p,1})}&\lesssim \|\tau u\|^\ell_{\widetilde{L}^\infty_t(\dot B^{\frac d2+1}_{2,1})}\lesssim \|\tau u\|^\ell_{L^\infty_t(\dot B^2_{2,1})}\\ &\lesssim \|\langle\tau\rangle^{\frac{s_0}{2}+1}u\|^\ell_{L^\infty_t(\dot B^2_{2,1})}\leq \mathcal{D}_p(t).\end{aligned}$$

In the 2D-case, observing that $p<4$ implies $s_0>0$, we have for all small enough $\varepsilon>0$:

$$\begin{aligned}\|\tau\nabla u\|^\ell_{\widetilde{L}^\infty_t(\dot B^{\frac 2p}_{p,1})}&\lesssim \|\tau u\|^\ell_{\widetilde{L}^\infty_t(\dot B^2_{2,1})}\lesssim \|\tau u\|^\ell_{L^\infty_t(\dot B^{2-2\varepsilon}_{2,1})}\\ &\lesssim \|\langle\tau\rangle^{\frac{s_0}{2}+1-\varepsilon}u\|^\ell_{L^\infty_t(\dot B^{2-2\varepsilon}_{2,1})}\leq \mathcal{D}_p(t).\end{aligned}$$

To bound the contribution of S^1_k and S^2_k in (3.52), we use the fact that

$$\sum_{k\geq k_0}2^{k(\frac dp-1)}\sup_{1\leq t\leq T}t^\alpha(S^1_k(t)+S^2_k(t))\lesssim \|t^\alpha(au,g)\|^h_{\widetilde{L}^\infty_T(\dot B^{\frac dp-1}_{p,1})}. \tag{3.63}$$

Now, product laws adapted to tilde spaces (see Proposition A.1) ensure that

$$\|t^\alpha au^h\|_{\widetilde{L}^\infty_T(\dot B^{\frac dp-1}_{p,1})}\lesssim \|a\|_{\widetilde{L}^\infty_T(\dot B^{\frac dp}_{p,1})}\|t^\alpha u^h\|_{\widetilde{L}^\infty_T(\dot B^{\frac dp-1}_{p,1})}\lesssim \mathcal{X}_p(T)\mathcal{D}_p(T), \tag{3.64}$$

$$\|t^\alpha a^h u^\ell\|_{\widetilde{L}^\infty_T(\dot B^{\frac dp-1}_{p,1})}\lesssim \|t^\alpha a^h\|_{\widetilde{L}^\infty_T(\dot B^{\frac dp}_{p,1})}\|u^\ell\|_{\widetilde{L}^\infty_T(\dot B^{\frac dp-1}_{p,1})}\lesssim \mathcal{D}_p(T)\mathcal{X}_p(T). \tag{3.65}$$

Note that Bernstein inequality (A.5) and embedding imply that

$$\|t^\alpha a^\ell u^\ell\|^h_{\widetilde{L}^\infty_T(\dot B^{\frac dp-1}_{p,1})}\lesssim \|t^\alpha a^\ell u^\ell\|_{\widetilde{L}^\infty_T(\dot B^{\frac d2}_{2,1})}.$$

Hence, using Proposition A.1, we discover that

$$\|t^\alpha a^\ell u^\ell\|_{\widetilde{L}^\infty_T(\dot B^{\frac dp-1}_{p,1})}\lesssim \|t^{\alpha/2}a^\ell\|_{\widetilde{L}^\infty_T(\dot B^{\frac d2}_{2,1})}\|t^{\alpha/2}u^\ell\|_{\widetilde{L}^\infty_T(\dot B^{\frac d2}_{2,1})}. \tag{3.66}$$

Because $\alpha\leq s_0+\min(2,\frac d2-\varepsilon)$, we deduce that

$$\|t^{\alpha/2}(a^\ell,u^\ell)\|_{\widetilde{L}^\infty_T(\dot B^{\frac d2}_{2,1})}\lesssim \|t^{\alpha/2}(a^\ell,u^\ell)\|_{L^\infty_T(\dot B^{\frac d2-\varepsilon}_{2,1})}\leq \mathcal{D}_p(T)\ \text{ if }\ d\leq 4, \tag{3.67}$$

$$\|t^{\alpha/2}(a^\ell,u^\ell)\|_{\widetilde{L}^\infty_T(\dot B^{\frac d2}_{2,1})}\lesssim \|t^{\alpha/2}(a^\ell,u^\ell)\|_{L^\infty_T(\dot B^2_{2,1})}\leq \mathcal{D}_p(T)\ \text{ if }\ d\geq 5. \tag{3.68}$$

Therefore we conclude that

$$\|t^\alpha(au)\|^h_{\widetilde{L}^\infty_T(\dot B^{\frac dp-1}_{p,1})}\lesssim \mathcal{D}_p(T)(\mathcal{D}_p(T)+\mathcal{X}_p(T)). \tag{3.69}$$

To bound the convection term of g, we just write that

$$\|t^\alpha(u\cdot\nabla u)\|^h_{\widetilde{L}^\infty_T(\dot B^{\frac dp-1}_{p,1})}\lesssim \|t^{\alpha-1}u\|_{\widetilde{L}^\infty_T(\dot B^{\frac dp-1}_{p,1})}\|t\nabla u\|_{\widetilde{L}^\infty_T(\dot B^{\frac dp}_{p,1})}.$$

On one hand, it is obvious that $\|t^{\alpha-1}u\|^h_{\widetilde{L}^\infty_T(\dot{B}^{\frac{d}{p}-1}_{p,1})} \leq \mathcal{D}_p(t)$. On the other hand, we have the following estimates for $z = a, u$ and small enough ε:

$$\|t^{\alpha-1}z\|^\ell_{\widetilde{L}^\infty_T(\dot{B}^{\frac{d}{p}-1}_{p,1})} \lesssim \|t^{\alpha-1}z\|^\ell_{L^\infty_T(\dot{B}^{\frac{d}{2}-1-2\varepsilon}_{2,1})} \leq \mathcal{D}_p(T) \ \text{ if } \ d \leq 6, \tag{3.70}$$

$$\|t^{\alpha-1}z\|^\ell_{\widetilde{L}^\infty_T(\dot{B}^{\frac{d}{p}-1}_{p,1})} \lesssim \|t^{\alpha-1}z\|^\ell_{L^\infty_T(\dot{B}^{2}_{2,1})} \leq \mathcal{D}_p(T) \ \text{ if } \ d \geq 7, \tag{3.71}$$

provided $\alpha - 1 \leq \frac{s_0}{2} + \frac{d}{4} - \frac{1}{2} - \varepsilon$ if $d \leq 6$ and $\alpha - 1 \leq \frac{s_0}{2} + 1$ if $d \geq 7$. Hence

$$\|t^\alpha(u\cdot\nabla u)\|^h_{\widetilde{L}^\infty_T(\dot{B}^{\frac{d}{p}-1}_{p,1})} \lesssim \mathcal{D}^2_p(T). \tag{3.72}$$

To bound the term with $k(a)\nabla a$, we use that according to Propositions A.1 and A.3, and to (3.67), (3.68), we have

$$\|t^\alpha(k(a)\nabla a^h)\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{d}{p}-1}_{p,1})} \lesssim \|a\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{d}{p}}_{p,1})} \|t^\alpha a\|^h_{\widetilde{L}^\infty_T(\dot{B}^{\frac{d}{p}}_{p,1})} \leq \mathcal{X}_p(T)\mathcal{D}_p(T), \tag{3.73}$$

$$\|t^\alpha(k(a)\nabla a^\ell)\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{d}{p}-1}_{p,1})} \lesssim \|t^{\alpha/2}a\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{d}{p}}_{p,1})} \|t^{\alpha/2}a\|^\ell_{\widetilde{L}^\infty_T(\dot{B}^{\frac{d}{2}}_{2,1})} \lesssim \mathcal{D}^2_p(T). \tag{3.74}$$

To bound the term containing $I(a)\mathcal{A}u$, we write that

$$\|t^\alpha I(a)\mathcal{A}u\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{d}{p}-1}_{p,1})} \lesssim \|t\nabla^2 u\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{d}{p}-1}_{p,1})} \big(\|t^{\alpha-1}a\|^\ell_{\widetilde{L}^\infty_T(\dot{B}^{\frac{d}{2}}_{2,1})} + \|t^{\alpha-1}a\|^h_{\widetilde{L}^\infty_T(\dot{B}^{\frac{d}{p}}_{p,1})}\big). \tag{3.75}$$

The first term on the right-side may be bounded by virtue of (3.62), and it is clear that the last term is bounded by $\mathcal{D}_p(T)$. As for the second one, we use (3.70) and (3.71).

The last term of g is of the type $\nabla F(a)\otimes\nabla u$ with $F(0) = 0$, and we have

$$\|t^\alpha\nabla F(a)\otimes\nabla u\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{d}{p}-1}_{p,1})} \lesssim \|t^{\alpha-1}a\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{d}{p}}_{p,1})} \|t\nabla u\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{d}{p}}_{p,1})}.$$

So using (3.62), the definition of $\mathcal{D}_p(T)$ and (3.70), (3.71), we see that

$$\|t^\alpha\nabla F(a)\otimes\nabla u\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{d}{p}-1}_{p,1})} \lesssim \mathcal{D}^2_p(T).$$

Reverting to (3.63), we end up with

$$\sum_{k\geq k_0} \sup_{1\leq t\leq T} t^\alpha 2^{k(\frac{d}{p}-1)}(S^1_k + S^2_k)(t) \lesssim \mathcal{D}_p(T)\mathcal{X}_p(T) + \mathcal{D}^2_p(T). \tag{3.76}$$

The term S^3_k is similar to the last two terms of g. As for bounding S^4_k, we notice that a small modification of Proposition A.5 (just include t^α in the definition of the commutator, follow the proof treating the time variable as a parameter, and take the supremum on $[0, T]$ at the end) yields:

$$\sum_{k\in\mathbb{Z}} 2^{k(\frac{d}{p}-1)} \sup_{0\leq t\leq T} t^\alpha\|R_k(t)\|_{L^p} \lesssim \|t\nabla u\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{d}{p}}_{p,1})} \|t^{\alpha-1}\nabla a\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{d}{p}-1}_{p,1})}. \tag{3.77}$$

Hence using (3.62), (3.70) and (3.71) gives

$$\sum_{k\in\mathbb{Z}} 2^{k(\frac{d}{p}-1)} \sup_{0\leq t\leq T} t^\alpha\|R_k(t)\|_{L^p} \lesssim \mathcal{D}^2_p(T).$$

The term with S_k^5 is clearly bounded by the r.h.s. of (3.77). Putting all the above inequalities together, we conclude that

$$\sum_{k\geq k_0} 2^{k(\frac{d}{p}-1)} \sup_{1\leq t\leq T} t^\alpha S_k(t) \lesssim \mathcal{D}_p(T)\mathcal{X}_p(T)+\mathcal{D}_p^2(T). \tag{3.78}$$

Plugging (3.78) in (3.61), and remembering (3.52), (3.59) and (3.60), we end up with

$$\|\langle t\rangle^\alpha(\nabla a,u)\|^h_{\widetilde L^\infty_T(\dot B^{\frac{d}{p}-1}_{p,1})} \lesssim \|(\nabla a_0,u_0)\|^h_{\dot B^{\frac{d}{p}-1}_{p,1}} + \mathcal{X}_p^2(T)+\mathcal{D}_p^2(T). \tag{3.79}$$

Step 3: Decay estimates with gain of regularity for the high frequencies of u. In order to bound the last term in $\mathcal{D}_p(t)$, it suffices to notice that the velocity u satisfies

$$\partial_t u-\mathcal{A}u = F \triangleq -(1+k(a))\nabla a - u\cdot\nabla u - I(a)\mathcal{A}u + \frac{1}{1+a}\mathrm{div}\left(2\widetilde\mu(a)D(u)+\widetilde\lambda(a)\mathrm{div}\,u\,\mathrm{Id}\right).$$

Hence

$$\partial_t(t\mathcal{A}u)-\mathcal{A}(t\mathcal{A}u) = \mathcal{A}u + t\mathcal{A}F. \tag{3.80}$$

We thus deduce from Proposition A.6 and the remark that follows, that

$$\|\tau\nabla^2 u\|^h_{\widetilde L^\infty_t(\dot B^{\frac{d}{p}-1}_{p,1})} \lesssim \|\mathcal{A}u\|^h_{L^1_t(\dot B^{\frac{d}{p}-1}_{p,1})} + \|\tau\mathcal{A}F\|^h_{\widetilde L^\infty_t(\dot B^{\frac{d}{p}-3}_{p,1})}, \tag{3.81}$$

whence, using the bounds given by Theorem 1.1,

$$\|\tau\nabla u\|^h_{\widetilde L^\infty_t(\dot B^{\frac{d}{p}}_{p,1})} \lesssim \|u\|^h_{L^1_t(\dot B^{\frac{d}{p}+1}_{p,1})} + \|\tau F\|^h_{\widetilde L^\infty_t(\dot B^{\frac{d}{p}-1}_{p,1})} \lesssim \mathcal{X}_p(0) + \|\tau F\|^h_{\widetilde L^\infty_t(\dot B^{\frac{d}{p}-1}_{p,1})}. \tag{3.82}$$

In order to bound the first term of F, we notice that, because $\alpha\geq 1$, we have

$$\|\tau\nabla a\|^h_{\widetilde L^\infty_t(\dot B^{\frac{d}{p}-1}_{p,1})} \lesssim \|\langle\tau\rangle^\alpha a\|^h_{\widetilde L^\infty_t(\dot B^{\frac{d}{p}}_{p,1})}. \tag{3.83}$$

Next, product and composition estimates (see Propositions A.1 and A.3) adapted to tilde spaces give

$$\|\tau\, k(a)\nabla a\|^h_{\widetilde L^\infty_t(\dot B^{\frac{d}{p}-1}_{p,1})} \lesssim \|\tau^{\frac12} a\|^2_{\widetilde L^\infty_t(\dot B^{\frac{d}{p}}_{p,1})} \lesssim \mathcal{D}_p^2(t), \tag{3.84}$$

as well as

$$\|\tau\, u\cdot\nabla u\|^h_{\widetilde L^\infty_t(\dot B^{\frac{d}{p}-1}_{p,1})} \lesssim \|u\|_{\widetilde L^\infty_t(\dot B^{\frac{d}{p}-1}_{p,1})} \|\tau\nabla u\|_{\widetilde L^\infty_t(\dot B^{\frac{d}{p}}_{p,1})} \lesssim \mathcal{X}_p(t)\mathcal{D}_p(t) \tag{3.85}$$

and

$$\|\tau I(a)\mathcal{A}u\|^h_{\widetilde L^\infty_t(\dot B^{\frac{d}{p}-1}_{p,1})} \lesssim \|a\|_{\widetilde L^\infty_t(\dot B^{\frac{d}{p}}_{p,1})} \|\tau\nabla^2 u\|_{\widetilde L^\infty_t(\dot B^{\frac{d}{p}-1}_{p,1})} \lesssim \mathcal{X}_p(t)\mathcal{D}_p(t). \tag{3.86}$$

Obviously, the terms $\frac{\widetilde\mu(a)}{1+a}\mathrm{div}\,D(u)$ and $\frac{\widetilde\lambda(a)}{1+a}\nabla\mathrm{div}\,u$ also satisfy (3.86). Finally, we notice that for any smooth function K, we have

$$\|\tau K(a)\nabla a\otimes\nabla u\|^h_{\widetilde L^\infty_t(\dot B^{\frac{d}{p}-1}_{p,1})} \lesssim (1+\|a\|_{\widetilde L^\infty_T(\dot B^{\frac{d}{p}}_{p,1})})\|\nabla a\|_{\widetilde L^\infty_T(\dot B^{\frac{d}{p}-1}_{p,1})} \|\tau\nabla u\|_{\widetilde L^\infty_T(\dot B^{\frac{d}{p}}_{p,1})}.$$

Hence, reverting to (3.82) and remembering (3.62), we get

$$\|\tau\nabla u\|^h_{\widetilde L^\infty_t(\dot B^{\frac{d}{p}}_{p,1})} \lesssim \mathcal{X}_{p,0} + \mathcal{D}_p(t)\mathcal{X}_p(t) + \mathcal{D}_p^2(t) + \|\langle\tau\rangle^\alpha a\|^h_{\widetilde L^\infty_t(\dot B^{\frac{d}{p}}_{p,1})}. \tag{3.87}$$

Finally, bounding the last term on the right-side of (3.87) according to (3.79), and adding up the obtained inequality to (3.42) and (3.79) yields for all $T\geq 0$,

$$\mathcal{D}_p(T)\lesssim \mathcal{D}_{p,0}+\|(a_0,u_0)\|^{\ell}_{\dot B^{\frac d2-1}_{2,1}}+\|(\nabla a_0,u_0)\|^{h}_{\dot B^{\frac dp-1}_{p,1}}+\mathcal{X}_p^2(T)+\mathcal{D}_p^2(T). \tag{3.88}$$

As Theorem 1.1 ensures that $\mathcal{X}_p\lesssim\mathcal{X}_{p,0}\ll 1$ and as $\|(a_0,u_0)\|^{\ell}_{\dot B^{\frac d2-1}_{2,1}}\lesssim\|(a_0,u_0)\|^{\ell}_{\dot B^{-s_0}_{2,\infty}}$, one can conclude that (2.15) is fulfilled for all time if $\mathcal{D}_{p,0}$ and $\|(\nabla a_0,u_0)\|^{h}_{\dot B^{\frac dp-1}_{p,1}}$ are small enough. This completes the proof of Theorem 2.1.

4. More decay estimates

This short section is devoted to pointing out some corollaries of Theorem 2.1.

To start with, let us extend its statement to general values of ϱ_∞, c_∞ and ν_∞. It is based on the change of unknowns (2.13) and on the scaling invariance (A.1) of Besov norms. For example, introducing the *Mach number* $Ma\triangleq 1/c_\infty$ and the *Reynolds number* $Re\triangleq\varrho_\infty/c_\infty$, and denoting

$$\|z\|^{\ell,\zeta}_{\dot B^s_{2,1}}:=\sum_{2^k\leq\zeta 2^{k_0}}2^{ks}\|\dot\Delta_k z\|_{L^2}\quad\text{for}\quad\zeta>0,$$

we easily find that

$$\big\|\langle\widetilde\tau\rangle^{\frac{s_0+s}{2}}(\widetilde a,\widetilde u)\big\|^{\ell,1}_{L^\infty_{\widetilde t}(\dot B^s_{2,1})}=\Big\|\Big\langle\frac{Re}{Ma^2}\,\tau\Big\rangle^{\frac{s_0+s}{2}}\Big(\frac{\varrho-\varrho_\infty}{\varrho_\infty},Ma\,u\Big)\Big\|^{\ell,\frac{Re}{Ma}}_{L^\infty_{\frac{Re}{Ma}\widetilde t}(\dot B^s_{2,1})},$$

and similar relations for the other terms of $\mathcal{D}_p(t)$.

This leads to the following statement:

Theorem 4.1. *Let d, p, α and s_0 be as in Theorem 2.1. There exists a constant c depending only on p and d such that if*

$$\Big\|\frac{\varrho_0-\varrho_\infty}{\varrho_\infty}\Big\|^{\ell,\frac{Re}{Ma}}_{\dot B^{-s_0}_{2,\infty}}+Ma\|u_0\|^{\ell,\frac{Re}{Ma}}_{\dot B^{-s_0}_{2,\infty}}\leq c\Big(\frac{Ma}{Re}\Big)^{\frac{2d}{p}}\quad\textit{and}\quad\frac{1}{\varrho_\infty}\|\nabla\varrho_0\|^{h,\frac{Re}{Ma}}_{\dot B^{\frac dp-1}_{p,1}}+Re\,\|u_0\|^{h,\frac{Re}{Ma}}_{\dot B^{\frac dp-1}_{p,1}}\leq c_0,$$

then System (1.1) has a unique solution (ϱ,u) satisfying the regularity properties of Theorem 1.1. Furthermore, we have for all $t\geq 0$,

$$\sup_{s\in(-s_0,2]}\Big\|\Big\langle\frac{Re}{Ma^2}\,\tau\Big\rangle^{\frac{s_0+s}{2}}\Big(\frac{\varrho-\varrho_\infty}{\varrho_\infty},Ma\,u\Big)\Big\|^{\ell,\frac{Re}{Ma}}_{L^\infty_t(\dot B^s_{2,1})}+\Big\|\Big\langle\frac{Re}{Ma^2}\tau\Big\rangle^{\alpha}\Big(\frac{\nabla\varrho}{\varrho_\infty},Re\,u\Big)\Big\|^{h,\frac{Re}{Ma}}_{\widetilde L^\infty_t(\dot B^{\frac dp-1}_{p,1})}$$
$$+\|\tau\nabla u\|^{h,\frac{Re}{Ma}}_{\widetilde L^\infty_t(\dot B^{\frac dp}_{p,1})}\lesssim\Big(\frac{Re}{Ma}\Big)^{\frac{2d}{p}}\Big\|\Big(\frac{\varrho_0-\varrho_\infty}{\varrho_\infty},Ma\,u_0\Big)\Big\|^{\ell,\frac{Re}{Ma}}_{\dot B^{-s_0}_{2,\infty}}+\Big\|\Big(\frac{\nabla\varrho_0}{\varrho_\infty},Re\,u_0\Big)\Big\|^{h,\frac{Re}{Ma}}_{\dot B^{\frac dp-1}_{p,1}}.$$

Just to compare our results with those of the prior literature on decay estimates, let us now state the L^q-L^r type decay rates that we can get from our main theorem. For notational simplicity, we assume that $\varrho_\infty=1$ and that $Re=Ma=1$.

Corollary 4.1. *The solution (ϱ,u) constructed in Theorem 2.1 satisfies*

$$\|\Lambda^s(\varrho-1)\|_{L^p}\leq C\big(\mathcal{D}_{p,0}+\|(\nabla a_0,u_0)\|^h_{\dot B^{\frac dp-1}_{p,1}}\big)\langle t\rangle^{-\frac{s_0+s}{2}}\quad \textit{if}\quad -s_0<s\leq\min\Big(2,\frac dp\Big),$$

$$\|\Lambda^s u\|_{L^p}\leq C\big(\mathcal{D}_{p,0}+\|(\nabla a_0,u_0)\|^h_{\dot B^{\frac dp-1}_{p,1}}\big)\langle t\rangle^{-\frac{s_0+s}{2}}\quad \textit{if}\quad -s_0<s\leq\min\Big(2,\frac dp-1\Big),$$

where the fractional derivative operator Λ^ℓ is defined by $\Lambda^\ell f\triangleq\mathcal{F}^{-1}(|\cdot|^\ell\mathcal{F}f)$.

Proof. Recall that for functions with compactly supported Fourier transform, one has the embedding $\dot B^s_{2,1}\hookrightarrow\dot B^{s-d(1/2-1/p)}_{p,1}\hookrightarrow\dot B^s_{p,1}$ for $p\geq2$. Hence, we may write

$$\sup_{t\in[0,T]}\langle t\rangle^{\frac{s_0+s}{2}}\|\Lambda^s a\|_{\dot B^0_{p,1}}\lesssim\|\langle t\rangle^{\frac{s_0+s}{2}}a\|^\ell_{L^\infty_T(\dot B^s_{2,1})}+\|\langle t\rangle^{\frac{s_0+s}{2}}a\|^h_{L^\infty_T(\dot B^s_{p,1})}.$$

If follows from Inequality (2.15) and the definition of $\mathcal{D}_p$ and α that

$$\|\langle t\rangle^{\frac{s_0+s}{2}}a\|^\ell_{L^\infty_T(\dot B^s_{2,1})}\lesssim\mathcal{D}_{p,0}+\|(\nabla a_0,u_0)\|^h_{\dot B^{\frac dp-1}_{p,1}}\quad\text{if}\quad -s_0<s\leq2$$

and that, because we have $\alpha\geq\frac{s_0+s}{2}$ for all $s\leq\min(2,d/p)$,

$$\|\langle t\rangle^{\frac{s_0+s}{2}}a\|^h_{L^\infty_T(\dot B^s_{p,1})}\lesssim\mathcal{D}_{p,0}+\|(\nabla a_0,u_0)\|^h_{\dot B^{\frac dp-1}_{p,1}}\quad\text{if}\quad s\leq d/p.$$

This yields the desired result for a. Bounding the velocity u works almost the same, except that we need the stronger condition $s\leq d/p-1$ for the high frequencies. This completes the proof of Corollary 4.1. $\square$

Remark 4.1. *Taking $p=2$ (hence $s_0=d/2$) and $s=0$ in Corollary 4.1 leads back to the standard optimal L^1-L^2 decay rate of (a,u). Note however that our estimates also hold in the general L^p critical framework. Additionally, the regularity index s can take both negative and nonnegative values, rather than only nonnegative integers, which improves the classical decay results in high Sobolev regularity, such as* [24] *or* [27].

One can get more L^q-L^r decay estimates, as a consequence of the following Gagliardo-Nirenberg type inequalities which parallel the work of Sohinger and Strain [28] (see also [1], Chap. 2, and [29]):

Proposition 4.1. *The following interpolation inequality holds true:*

$$\|\Lambda^\ell f\|_{L^r}\lesssim\|\Lambda^m f\|^{1-\theta}_{L^q}\|\Lambda^k f\|^\theta_{L^q},$$

whenever $0\leq\theta\leq1$, $1\leq q\leq r\leq\infty$ and

$$\ell+d\Big(\frac1q-\frac1r\Big)=m(1-\theta)+k\theta.$$

Corollary 4.2. *Let the assumptions of Theorem 2.1 be fulfilled with $p=2$. Then the corresponding solution (ϱ,u) satisfies*

$$\|\Lambda^\ell(\varrho-1,u)\|_{L^r}\leq C\big(\mathcal{D}_{2,0}+\|(\nabla a_0,u_0)\|^h_{\dot B^{\frac d2-1}_{2,1}}\big)\langle t\rangle^{-\frac d2(1-\frac1r)-\frac\ell2},\tag{4.89}$$

for all $2\leq r\leq\infty$ and $\ell\in\mathbb{R}$ satisfying $-\frac d2<\ell+d\big(\frac12-\frac1r\big)<\min\big(2,\frac d2-1\big)\cdot$

Proof. It follows from Corollary 4.1 with $p=2$, and Proposition 4.1 with $q=2$, $m=\min(2,\frac{d}{2}-1)$ and $k=-\frac{d}{2}+\varepsilon$ with ε small enough. Indeed, if we define θ by the relation

$$k\theta+m(1-\theta)=\ell+d\Big(\frac{1}{2}-\frac{1}{r}\Big),$$

then one can take ε so small as θ to be in $(0,1)$. Therefore we have

$$\begin{aligned}\|\Lambda^\ell(a,u)\|_{L^r} &\lesssim \|\Lambda^m(a,u)\|_{L^2}^{1-\theta}\|\Lambda^k(a,u)\|_{L^2}^{\theta}\\ &\lesssim \big(\mathcal{D}_{2,0}+\|(\nabla a_0,u_0)\|^h_{\dot B^{\frac{d}{2}-1}_{2,1}}\big)\Big\{\langle t\rangle^{-\frac{d}{4}-\frac{m}{2}}\Big\}^{1-\theta}\Big\{\langle t\rangle^{-\frac{d}{4}-\frac{k}{2}}\Big\}^{\theta}\\ &= \big(\mathcal{D}_{2,0}+\|(\nabla a_0,u_0)\|^h_{\dot B^{\frac{d}{2}-1}_{2,1}}\big)\langle t\rangle^{-\frac{d}{4}-\frac{m}{2}(1-\theta)-\frac{k}{2}\theta},\end{aligned}\tag{4.90}$$

which completes the proof of the corollary. □

Appendix A. Littlewood-Paley decomposition and Besov spaces

We here recall basic properties of Besov spaces and paradifferential calculus that have been used repeatedly in the paper (more details may be found in e.g. Chap. 2 and 3 of [1]). We also prove some slightly less classical product laws, and the commutator estimate (3.77).

As mentioned in the introduction, homogeneous Besov spaces possess scaling invariance properties. In the $\mathbb{R}^d$ case, they read for any $\sigma\in\mathbb{R}$ and $(p,r)\in[1,+\infty]^2$:

$$C^{-1}\lambda^{\sigma-\frac{d}{p}}\|f\|_{\dot B^\sigma_{p,r}}\leq\|f(\lambda\cdot)\|_{\dot B^\sigma_{p,r}}\leq C\lambda^{\sigma-\frac{d}{p}}\|f\|_{\dot B^\sigma_{p,r}},\qquad\lambda>0,\tag{A.1}$$

where the constant C depends only on σ, p and on the dimension d.

The following embedding properties have been used several times:

- For any $p\in[1,\infty]$ we have the continuous embedding
$$\dot B^0_{p,1}\hookrightarrow L^p\hookrightarrow\dot B^0_{p,\infty}.$$
- If $\sigma\in\mathbb{R}$, $1\leq p_1\leq p_2\leq\infty$ and $1\leq r_1\leq r_2\leq\infty$, then $\dot B^\sigma_{p_1,r_1}\hookrightarrow\dot B^{\sigma-d(\frac{1}{p_1}-\frac{1}{p_2})}_{p_2,r_2}$.
- The space $\dot B^{\frac{d}{p}}_{p,1}$ is continuously embedded in the set of bounded continuous functions (going to 0 at infinity if $p<\infty$).

Let us also mention the following interpolation inequality that is satisfied whenever $1\leq p,r_1,r_2,r\leq\infty$, $\sigma_1\neq\sigma_2$ and $\theta\in(0,1)$:

$$\|f\|_{\dot B^{\theta\sigma_2+(1-\theta)\sigma_1}_{p,r}}\lesssim\|f\|^{1-\theta}_{\dot B^{\sigma_1}_{p,r_1}}\|f\|^{\theta}_{\dot B^{\sigma_2}_{p,r_2}}.$$

The following product estimates in Besov spaces play a fundamental role in our analysis of the bilinear terms of (2.17).

Proposition A.1. *Let $\sigma>0$ and $1\leq p,r\leq\infty$. Then $\dot B^\sigma_{p,r}\cap L^\infty$ is an algebra and*

$$\|fg\|_{\dot B^\sigma_{p,r}}\lesssim\|f\|_{L^\infty}\|g\|_{\dot B^\sigma_{p,r}}+\|g\|_{L^\infty}\|f\|_{\dot B^\sigma_{p,r}}.$$

Let the real numbers σ_1, σ_2, p_1 and p_2 be such that

$$\sigma_1+\sigma_2>0,\quad\sigma_1\leq\frac{d}{p_1},\quad\sigma_2\leq\frac{d}{p_2},\quad\sigma_1\geq\sigma_2,\quad\frac{1}{p_1}+\frac{1}{p_2}\leq1.$$

Then we have

$$\|fg\|_{\dot B^{\sigma_2}_{q,1}} \lesssim \|f\|_{\dot B^{\sigma_1}_{p_1,1}}\|g\|_{\dot B^{\sigma_2}_{p_2,1}} \quad \textit{with} \quad \frac{1}{q}=\frac{1}{p_1}+\frac{1}{p_2}-\frac{\sigma_1}{d}\cdot$$

Finally, for exponents $\sigma>0$ and $1\le p_1,p_2,q\le\infty$ satisfying

$$\frac{d}{p_1}+\frac{d}{p_2}-d\le\sigma\le\min\left(\frac{d}{p_1},\frac{d}{p_2}\right) \quad \textit{and} \quad \frac{1}{q}=\frac{1}{p_1}+\frac{1}{p_2}-\frac{\sigma}{d},$$

we have

$$\|fg\|_{\dot B^{-\sigma}_{q,\infty}} \lesssim \|f\|_{\dot B^{\sigma}_{p_1,1}}\|g\|_{\dot B^{-\sigma}_{p_2,\infty}}.$$

Proof. The first inequality is classical (see e.g. [1], Chap. 2). For proving the second item, we need the following so-called Bony decomposition for the product of two tempered distributions f and g:

$$fg=T_fg+R(f,g)+T_gf, \tag{A.2}$$

where the *paraproduct* between f and g is defined by

$$T_fg:=\sum_j \dot S_{j-1}f\dot\Delta_jg \quad \text{with} \quad \dot S_{j-1}\triangleq\chi(2^{-(j-1)}D),$$

and the *remainder* $R(f,g)$ is given by the series:

$$R(f,g):=\sum_j \dot\Delta_jf\left(\dot\Delta_{j-1}g+\dot\Delta_jg+\dot\Delta_{j+1}g\right).$$

In the case $\sigma_2\ge0$ then we use the embeddings $\dot B^{\sigma_1}_{p_1,1}\hookrightarrow L^{q_1}$ with $\frac{1}{q_1}=\frac{1}{p_1}-\frac{\sigma_1}{d}$ and $\dot B^{\sigma_2}_{p_2,1}\hookrightarrow L^{q_2}$ with $\frac{1}{q_2}=\frac{1}{p_2}-\frac{\sigma_2}{d}$ and the fact that:

- T maps $L^{q_1}\times\dot B^{\sigma_2}_{p_2,1}$ to $\dot B^{\sigma_2}_{q,1}$ with $\frac1q=\frac{1}{q_1}+\frac{1}{p_2}=\frac{1}{p_1}+\frac{1}{p_2}-\frac{\sigma_1}{d}$;
- T maps $L^{q_2}\times\dot B^{\sigma_1}_{p_1,1}$ to $\dot B^{\sigma_1}_{q_3,1}$ with $\frac{1}{q_3}=\frac{1}{q_2}+\frac{1}{p_1}=\frac{1}{p_1}+\frac{1}{p_2}-\frac{\sigma_2}{d}\cdot$

As $\sigma_2\le\sigma_1$, we have $q_3\le q$, and thus $\dot B^{\sigma_1}_{q_3,1}\hookrightarrow\dot B^{\sigma_2}_{q,1}$. Therefore the two paraproduct terms of (A.2) fulfill the desired inequality.

To bound the remainder term $R(f,g)$, we use the continuity result $\dot B^{\sigma_1}_{p_1,1}\times\dot B^{\sigma_2}_{p_2,1}\to\dot B^{\sigma_1+\sigma_2}_{p,1}$ with $\frac1p=\frac{1}{p_1}+\frac{1}{p_2}$, and the embedding $\dot B^{\sigma_1+\sigma_2}_{p,1}\hookrightarrow\dot B^{\sigma_2}_{q,1}$.

In the case $\sigma_2<0$ we use the fact that T maps $\dot B^{\sigma_2}_{p_2,1}\times\dot B^{\sigma_1}_{p_1,1}$ to $\dot B^{\sigma_1+\sigma_2}_{p,1}$ and, again, the embedding $\dot B^{\sigma_1+\sigma_2}_{p,1}\hookrightarrow\dot B^{\sigma_2}_{q,1}$.

Let us finally prove the last item. To this end, we use the fact that both R and T map $\dot B^{-\sigma}_{p_2,\infty}\times\dot B^{\sigma}_{p_1,1}$ to $\dot B^{0}_{p,\infty}$, with $\frac1p=\frac{1}{p_1}+\frac{1}{p_2}\cdot$ As $\dot B^0_{p,\infty}$ continuously embeds in $\dot B^{-\sigma}_{q,\infty}$, the last two terms of (A.2) satisfy the desired inequality. Regarding the first term in (A.2), it suffices to notice that, as $0<\sigma\le d/p_1$, we have $\dot B^{\sigma}_{p_1,1}\hookrightarrow L^{q_1}$ with $\frac{d}{q_1}=\frac{d}{p_1}-\sigma$, and that T maps $L^{q_1}\times\dot B^{-\sigma}_{p_2,\infty}$ to $\dot B^{-\sigma}_{q,\infty}$. □

To handle the case $p>d$ in the proof of Theorem 2.1, just resorting to the above proposition does not allow to get suitable bounds for the low frequency part of some nonlinear terms. We had to take advantage of the following result.

Proposition A.2. *Let $k_0\in\mathbb{Z}$, and denote $z^\ell\triangleq\dot S_{k_0}z$, $z^h\triangleq z-z^\ell$ and, for any $s\in\mathbb{R}$,*

$$\|z\|^{\ell}_{\dot B^s_{2,\infty}}\triangleq\sup_{k\le k_0}2^{ks}\|\dot\Delta_kz\|_{L^2}.$$

There exists a universal integer N_0 such that for any $2\leq p\leq 4$ and $\sigma>0$, we have

$$\|fg^h\|^{\ell}_{\dot B^{-s_0}_{2,\infty}}\leq C\big(\|f\|_{\dot B^{\sigma}_{p,1}}+\|\dot S_{k_0+N_0}f\|_{L^{p^*}}\big)\|g^h\|_{\dot B^{-\sigma}_{p,\infty}}\tag{A.3}$$

$$\|f^hg\|^{\ell}_{\dot B^{-s_0}_{2,\infty}}\leq C\big(\|f^h\|_{\dot B^{\sigma}_{p,1}}+\|(\dot S_{k_0+N_0}-\dot S_{k_0})f\|_{L^{p^*}}\big)\|g\|_{\dot B^{-\sigma}_{p,\infty}}\tag{A.4}$$

with $s_0\triangleq\frac{2d}{p}-\frac d2$ and $\frac{1}{p^*}\triangleq\frac12-\frac1p$, and C depending only on k_0, d and σ.

Proof. To prove the first inequality, we start with Bony's decomposition:

$$fg^h=T_{g^h}f+R(g^h,f)+T_fg^h.$$

As $-\sigma<0$, the first two terms are in $\dot B^0_{p/2,\infty}$ and thus in $\dot B^{-s_0}_{2,\infty}$ by embedding. Moreover,

$$\|T_{g^h}f+R(g^h,f)\|_{\dot B^{-s_0}_{2,\infty}}\lesssim\|f\|_{\dot B^{\sigma}_{p,1}}\|g^h\|_{\dot B^{-\sigma}_{p,\infty}}.$$

To handle the last term, we notice that the definition of the spectral truncation operators $\dot\Delta_k$ and $\dot S_k$ implies that $\dot\Delta_kg^h\equiv0$ if $k<k_0-1$. As in addition $\dot\Delta_{k'}(\dot S_{k-1}f\,\dot\Delta_kg^h)\equiv0$ if $|k-k'|>4$, we deduce that

$$\begin{aligned}\|T_fg^h\|^{\ell}_{\dot B^{-s_0}_{2,\infty}}&=\sup_{k\leq k_0}2^{-ks_0}\bigg\|\dot\Delta_k\bigg(\sum_{k'\geq k_0-1}\dot S_{k'-1}f\,\dot\Delta_{k'}g^h\bigg)\bigg\|_{L^2}\\&\leq C2^{-k_0s_0}\sup_{k_0-1\leq k'\leq k_0+4}\|\dot S_{k'-1}f\|_{L^{p^*}}\|\dot\Delta_{k'}g^h\|_{L^p}.\end{aligned}$$

This gives (A.3).

Proving the second inequality is similar. On one hand, as above, we have

$$\|T_gf^h+R(g,f^h)\|_{\dot B^{-s_0}_{2,\infty}}\lesssim\|f^h\|_{\dot B^{\sigma}_{p,1}}\|g\|_{\dot B^{-\sigma}_{p,\infty}}.$$

On the other hand, owing to the definition of f^h and of $\dot S_{k'-1}$,

$$T_{f^h}g=\sum_{k'\geq k_0+1}\dot S_{k'-1}f^h\,\dot\Delta_{k'}g$$

and thus $\dot\Delta_kT_{f^h}g\equiv0$ for $k<k_0-4$.

Therefore we have

$$\|T_{f^h}g\|^{\ell}_{\dot B^{-s_0}_{2,\infty}}\lesssim\|(\dot S_{k_0+4}-\dot S_{k_0})f\|_{L^{p^*}}\sum_{|k-k_0|\leq4}\|\dot\Delta_kg\|_{L^p},$$

whence (A.4). □

System (2.17) also involves compositions of functions (through $I(a)$, $\widetilde\lambda(a)$, $\widetilde\mu(a)$ and $G'(a)$) that are bounded thanks to the following classical result:

Proposition A.3. *Let $F:\mathbb R\to\mathbb R$ be smooth with $F(0)=0$. For all $1\leq p,r\leq\infty$ and $\sigma>0$ we have $F(f)\in\dot B^{\sigma}_{p,r}\cap L^\infty$ for $f\in\dot B^{\sigma}_{p,r}\cap L^\infty$, and*

$$\|F(f)\|_{\dot B^{\sigma}_{p,r}}\leq C\|f\|_{\dot B^{\sigma}_{p,r}}$$

with C depending only on $\|f\|_{L^\infty}$, F' (and higher derivatives), σ, p and d.

Let us now recall the following classical *Bernstein inequality*:

$$\|D^k f\|_{L^b} \leq C^{1+k}\lambda^{k+d(\frac{1}{a}-\frac{1}{b})}\|f\|_{L^a} \tag{A.5}$$

that holds for all function f such that $\operatorname{Supp}\mathcal{F}f \subset \{\xi \in \mathbb{R}^d : |\xi| \leq R\lambda\}$ for some $R > 0$ and $\lambda > 0$, if $k \in \mathbb{N}$ and $1 \leq a \leq b \leq \infty$.

More generally, if assume f to satisfy $\operatorname{Supp}\mathcal{F}f \subset \{\xi \in \mathbb{R}^d : R_1\lambda \leq |\xi| \leq R_2\lambda\}$ for some $0 < R_1 < R_2$ and $\lambda > 0$, then for any smooth homogeneous of degree m function A on $\mathbb{R}^d \setminus \{0\}$ and $1 \leq a \leq \infty$, we have (see e.g. Lemma 2.2 in [1]):

$$\|A(D)f\|_{L^a} \lesssim \lambda^m\|f\|_{L^a}. \tag{A.6}$$

An obvious consequence of (A.5) and (A.6) is that $\|D^k f\|_{\dot{B}^s_{p,r}} \approx \|f\|_{\dot{B}^{s+k}_{p,r}}$ for all $k \in \mathbb{N}$.

We also need the following nonlinear generalization of (A.6) (see Lemma 8 in [8]):

Proposition A.4. *If* $\operatorname{Supp}\mathcal{F}f \subset \{\xi \in \mathbb{R}^d : R_1\lambda \leq |\xi| \leq R_2\lambda\}$ *then there exists* c *depending only on* d, R_1 *and* R_2 *so that for all* $1 < p < \infty$,

$$c\lambda^2\left(\frac{p-1}{p}\right)\int_{\mathbb{R}^d}|f|^p\,dx \leq (p-1)\int_{\mathbb{R}^d}|\nabla f|^2|f|^{p-2}\,dx = -\int_{\mathbb{R}^d}\Delta f\,|f|^{p-2}f\,dx.$$

A time-dependent version of the following commutator estimate has been used in the second step of the proof of Theorem 2.1.

Proposition A.5. *Let* $1 \leq p, p_1 \leq \infty$ *and*

$$-\min\left(\frac{d}{p_1}, \frac{d}{p'}\right) < \sigma \leq 1 + \min\left(\frac{d}{p}, \frac{d}{p_1}\right)\cdot \tag{A.7}$$

There exists a constant $C > 0$ *depending only on* σ *such that for all* $j \in \mathbb{Z}$ *and* $\ell \in \{1, \cdots, d\}$, *we have*

$$\|[v\cdot\nabla, \partial_\ell\dot{\Delta}_j]a\|_{L^p} \leq Cc_j 2^{-j(\sigma-1)}\|\nabla v\|_{\dot{B}^{\frac{d}{p_1}}_{p_1,1}}\|\nabla a\|_{\dot{B}^{\sigma-1}_{p,1}}, \tag{A.8}$$

where the commutator $[\cdot,\cdot]$ *is defined by* $[f,g] = fg - gf$ *and* $(c_j)_{j\in\mathbb{Z}}$ *denotes a sequence such that* $\|(c_j)\|_{\ell^1} \leq 1$.

Proof. We just sketch the proof as it is very similar to that of Lemma 2.100 in [1]. Decomposing the two terms of $[v\cdot\nabla, \partial_\ell\dot{\Delta}_j]a$ according to (A.2), we see that, with the summation convention over repeated indices:

$$\begin{aligned}[v\cdot\nabla, \partial_\ell\dot{\Delta}_j]a = [T_{v^k}, \partial_\ell\dot{\Delta}_j]\partial_k a + T_{\partial_k\partial_\ell\dot{\Delta}_j a}v^k \\ - \partial_\ell\dot{\Delta}_j T_{\partial_k a}v^k + R(v^k, \partial_k\partial_\ell\dot{\Delta}_j a) - \partial_\ell\dot{\Delta}_j R(v^k, \partial_k a).\end{aligned} \tag{A.9}$$

To bound the first term, we notice that owing to the properties of spectral localization of the Littlewood-Paley decomposition, we have

$$[T_{v^k}, \partial_\ell\dot{\Delta}_j]\partial_k a = 2^j\sum_{|j-j'|\leq 4}[\dot{S}_{j'-1}v^k, \widetilde{\Delta}^\ell_j]\partial_k\dot{\Delta}_{j'}a \quad \text{with} \quad \widetilde{\Delta}^\ell_j \triangleq i(\xi^\ell\varphi)(2^{-j}D).$$

Now, setting $h^\ell := \mathcal{F}^{-1}(i\xi^\ell\varphi)$, we get

$$[\dot{S}_{j'-1}v^k, \widetilde{\Delta}^\ell_j]\partial_k\dot{\Delta}_{j'}a(x) = 2^{jd}\int_{\mathbb{R}^d}h^\ell(2^j(x-y))\big(\dot{S}_{j'-1}v^k(x) - \dot{S}_{j'-1}v^k(y)\big)\partial_k\dot{\Delta}_{j'}a(y)\,dy.$$

Hence using the mean value formula and Bernstein inequalities yields

$$\|[T_{v^k},\partial_\ell\dot\Delta_j]\partial_k a\|_{L^p}\lesssim\|\nabla v\|_{L^\infty}\sum_{|j'-j|\leq4}\|\dot\Delta_{j'}\nabla a\|_{L^p}.$$

As $\dot B^{\frac{d}{p_1}}_{p_1,1}\hookrightarrow L^\infty$, we get (A.8) for that term.

Bounding the third and last term in (A.9) follows from standard results of continuity for the remainder and paraproduct operators. Here we need Condition (A.7). To estimate the second term of (A.9), let us write that

$$T_{\partial_k\partial_\ell\dot\Delta_j a}v^k=\sum_{j'\geq j-3}\dot S_{j'-1}\partial_k\partial_\ell\dot\Delta_j a\,\dot\Delta_{j'}v^k.$$

Using Bernstein inequality, this yields

$$\|T_{\partial_k\partial_\ell\dot\Delta_j a}v^k\|_{L^p}\lesssim\sum_{j'\geq j-3}2^{j-j'}\|\nabla\dot\Delta_j a\|_{L^p}\,\|\nabla\dot\Delta_{j'}v\|_{L^\infty},$$

and convolution inequality for series and embedding thus ensures (A.8).

Finally, we have (because $\dot\Delta_{j'}\dot\Delta_j=0$ if $|j'-j|>1$),

$$R(v^k,\partial_k\partial_\ell\dot\Delta_j a)=\sum_{|j'-j|\leq1}(\dot\Delta_{j'-1}+\dot\Delta_{j'}+\dot\Delta_{j'+1})v^k\,\dot\Delta_{j'}\dot\Delta_j\partial_\ell\partial_k a.$$

Hence, by virtue of Bernstein inequality,

$$\|R(v^k,\partial_k\partial_\ell\dot\Delta_j a)\|_{L^p}\lesssim\sum_{|j'-j|\leq1}\|\dot\Delta_j\partial_k a\|_{L^p}\|\nabla v^k\|_{L^\infty},$$

which completes the proof of (A.8). □

When localizing PDE's by means of Littlewood-Paley decomposition, one naturally ends up with bounds for each dyadic block in spaces of type $L^\rho_T(L^p)\triangleq L^\rho(0,T;L^p(\mathbb{R}^d))$. To get an information on Besov norms, we then have to perform a summation on $\ell^r(\mathbb{Z})$. However, this does not quite yield a bound for the norm in $L^\rho_T(\dot B^\sigma_{p,r})$, as the time integration has been performed *before* the ℓ^r summation. This leads to the definition of the following norms first introduced by J.-Y. Chemin in [4] (see also [5] for the particular case of Sobolev spaces) for $0\leq T\leq+\infty$, $\sigma\in\mathbb{R}$ and $1\leq r,p,\rho\leq\infty$:

$$\|f\|_{\widetilde L^\rho_T(\dot B^\sigma_{p,r})}\triangleq\Big\|\big(2^{j\sigma}\|\dot\Delta_j f\|_{L^\rho_T(L^p)}\big)\Big\|_{\ell^r(\mathbb{Z})}.$$

For notational simplicity, index T is omitted if $T=+\infty$.

We also used the following functional space:

$$\widetilde{\mathcal C}_b(\mathbb{R}_+;\dot B^\sigma_{p,r})\triangleq\big\{f\in\mathcal C(\mathbb{R}_+;\dot B^\sigma_{p,r})\ s.t.\ \|f\|_{\widetilde L^\infty(\dot B^\sigma_{p,r})}<\infty\big\}.\tag{A.10}$$

The above norms may be compared with those of the more standard Lebesgue-Besov spaces $L^\rho_T(\dot B^\sigma_{p,r})$ via Minkowski's inequality:

$$\|f\|_{\widetilde L^\rho_T(\dot B^\sigma_{p,r})}\leq\|f\|_{L^\rho_T(\dot B^\sigma_{p,r})}\ \text{ if }\ r\geq\rho,\qquad\|f\|_{\widetilde L^\rho_T(\dot B^\sigma_{p,r})}\geq\|f\|_{L^\rho_T(\dot B^\sigma_{p,r})}\ \text{ if }\ r\leq\rho.\tag{A.11}$$

The general principle is that all the properties of continuity for the product, commutators and composition which are true for Besov norms extend to the above norms: the time exponent ρ just behaves according to Hölder inequality.

Using the norms defined in (A.10) leads to optimal regularity estimates for the heat equation, as is recalled in the proposition below.

Proposition A.6. *Let $\sigma \in \mathbb{R}$, $(p,r) \in [1,\infty]^2$ and $1 \le \rho_2 \le \rho_1 \le \infty$. Let u satisfy*

$$\begin{cases} \partial_t u - \mu \Delta u = f, \\ u_{|t=0} = u_0. \end{cases}$$

Then for all $T > 0$ the following a priori estimate is fulfilled:

$$\mu^{\frac{1}{\rho_1}} \|u\|_{\widetilde{L}^{\rho_1}_T(\dot{B}^{\sigma+\frac{2}{\rho_1}}_{p,r})} \lesssim \|u_0\|_{\dot{B}^{\sigma}_{p,r}} + \mu^{\frac{1}{\rho_2}-1} \|f\|_{\widetilde{L}^{\rho_2}_T(\dot{B}^{\sigma-2+\frac{2}{\rho_2}}_{p,r})}. \tag{A.12}$$

Remark A.1. *The solutions to the following* Lamé system

$$\begin{cases} \partial_t u - \mu \Delta u - (\lambda+\mu)\nabla \mathrm{div}\, u = f, \\ u_{|t=0} = u_0, \end{cases} \tag{A.13}$$

where λ and μ are constant coefficients such that $\mu > 0$ and $\lambda + 2\mu > 0$, also fulfill (A.12) *(up to the dependence w.r.t. the viscosity). Indeed: both $\mathcal{P}u$ and $\mathcal{Q}u$ satisfy a heat equation, as may be easily seen by applying $\mathcal{P}$ and $\mathcal{Q}$ to* (A.13)*.*

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