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CONTINUATION OF QUASI-PERIODIC SOLUTIONS WITH TWO-FREQUENCY HARMONIC BALANCE METHOD

Louis Guillot[1], **Pierre Vigue**[1], **Christophe Vergez**[1] **and Bruno Cochelin**[1]

[1]Laboratoire de Mecanique et d'Acoustique
CNRS UPR 7051
4 impasse Nikola Tesla, CS 40006, F-13453 Marseille Cedex 13 - France
e-mail : vigue,vergez,cochelin@lma.cnrs-mrs.fr

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Abstract. *Nonlinear systems can have periodic solutions evolving with the parameters of the system. Studying this evolution (numerical continuation of solutions) uncovers sought-after regimes in musical acoustics : many musical instruments rely on auto-oscillation, that is, the excitation of a nonlinear system coupled with a linear resonator, where some parameters may be adjusted by the player. Periodic solutions can be approximated as truncated Fourier series (Harmonic Balance Method) ; the period is one of the unknowns. Several stable or unstable solutions can be found for the same playing parameters thanks to continuation.*
An important challenge is the continuation of quasi-periodic solutions, also called multiphonic sounds by musicians. Depending on the context, these oscillation regimes are considered pleasant (jazz or contemporary music for instance) or unpleasant (classical music). We developed a method based on double Fourier series, coupled with a continuation technique. The two base frequencies are unknowns and incommensurable. The system is reformulated as quadratic in order to allow straight interface with previous work on periodic harmonic balance.
This method is illustrated on simple models relevant to musical acoustics, though the method can be applied to many nonlinear problems, without a priori knowledge of the solutions.

1 INTRODUCTION

Nonlinear systems can have periodic solutions evolving with the parameters of the system. Studying this evolution (numerical continuation of solutions) uncovers sought-after regimes in musical acoustics : many musical instruments rely on auto-oscillation, that is, the excitation of a nonlinear system coupled with a linear resonator, where some parameters may be adjusted by the player. Periodic solutions can be approximated as truncated Fourier series (Harmonic Balance Method) ; the period is one of the unknowns. Then, using a continuation technique, solutions can be continued ; they can be either stable or unstable, and different solutions may occur for the same playing parameters.

However, other solutions can arise, namely quasi-periodic solutions. They are well-known by musicians, who may call them multiphonics. These solutions can be undesirable : for instance, the wolf note on bowed string instruments is a rough, beating sound, and it is an example of a quasi-periodic regime of an autonomous system. These solutions can also be produced voluntarily, using a forced system: singing at a frequency f_1 while playing at a frequency f_2 on a brass instrument can create stunning effects.

Our aim is the continuation of two-frequencies, quasi-periodic solutions. It is important to notice that even the direct computation of quasi-periodic solutions can be difficult. Because of dependence on initial conditions, some solutions may be overlooked when using numerical integration. Moreover, compared to the periodic case, it is not relevant anymore to perform integration on long intervals to get rid of transient solutions : one cannot determine easily if the steady-state solution is reached. These drawbacks have lead to specific algorithms to compute quasi-periodic solutions, first as a response to a quasi-periodic drive [1]. More recently, the computation of quasi-periodic solutions for forced or autonomous systems, based on the Alternating Frequency/Time Domain Method (AFT [2]), was performed with good qualitative agreement on a disc brake model [3]. Peletan *et al.* [4] designed a continuation method, coupling AFT with pseudo-arclength continuation, and applied it to a Jeffcott rotor. For this system, one of the two frequencies is known ; and an harmonic selection procedure improves the efficiency of computations.

For periodic solutions, Cochelin and Vergez [5] showed that given a quadratic reformulation, a coupling of the Harmonic Balance Method and the Asymptotic Numerical Method was straightforward and allowed computations with high number of harmonics. The method developed here is an extension of this idea with double Fourier series. The two base frequencies are unknowns and incommensurable. The system is reformulated as quadratic in order to allow straight interface with previous work on periodic harmonic balance.

This method is illustrated on simple models, with a forced system and an autonomous one. System parameters could be chosen to present results more closely related to musical acoustics. However, simple values underline that the method can be applied to many nonlinear problems, without *a priori* knowledge of the solutions.

2 TWO-FREQUENCY HARMONIC BALANCE METHOD

2.1 Principle : quadratic formulation

Instead of one Fourier series, a variable x is sought after in the form

$$x(t) = \sum_{k_1=-H}^{H} \sum_{k_2=-H}^{H} x_{k_1,k_2} e^{i(k_1\omega_1 + k_2\omega_2)t} \tag{1}$$

where ω_1 and ω_2 are the two unknown pulsations. Adding auxiliary variables, a smooth nonlinear differential system can be transformed into a first-order differential system, with nonlinearities being only products, either of two variables, or a variable and the continuation parameter λ. Let U denote the vector of variables in the time domain, the following system is called quadratic formulation :

$$m(U') = c_0 + \lambda c_1 + l_0(U) + \lambda l_1(U) + q(U, U) \tag{2}$$

where c_0, c_1 are constant vectors, m, l_0 and l_1 are constant linear operators, and q is a constant quadratic operator. Like in the periodic case [5], since eq. (2) is quadratic, and due to the decomposition of variables assumed in eq. (1), substituting this double series in eq. (2) leads to a (larger) quadratical system where the unknowns are Fourier coefficients plus the two pulsations ω_1, ω_2. Note that in the case of a forced system one of these pulsations is the forcing pulsation (see section 2.2). This larger system reads as a quadratical residual function R :

$$R : \mathbf{R}^{N+1} \longrightarrow \mathbf{R}^N, \quad (X, \lambda) \mapsto C_0 + \lambda C_1 + L_0(X) + \lambda L_1(X) + Q(X, X) \tag{3}$$

where X contains Fourier coefficients of U, ω_1 and ω_2. The solution branch $R(X, \lambda) = 0$ can then be followed thanks to the Asymptotic Numerical Method.

2.2 Forced system

An example of a forced system that exhibits a quasi-periodic behaviour is a forced Van der Pol oscillator :

$$x'' - \mu_1 x' + \mu_2 x x' + \mu_3 x^2 x' + a_1 x = \cos(\lambda t) \tag{4}$$

with $\mu_1 = \mu_2 = 0.1$, $\mu_3 = a_1 = 1$. A Neimark-Sacker bifurcation occurs at $\lambda \simeq 1.79$ [6], and the periodic solution at pulsation $\omega = \lambda$ becomes unstable. A quadratic formulation, emphasizing constant, linear and quadratic parts in the right-hand side, is

$$\begin{aligned} x' &= 0 &&+ y &&+ 0 && (5)\\ y' &= \cos(\lambda t) &&+ \mu_1 y - a_1 x &&- \mu_2 xy - \mu_3 yz && (6)\\ \underbrace{0}_{mU'} &= \underbrace{0}_{c_0} &&\underbrace{+ z}_{l_0 U} &&\underbrace{- x^2}_{q(U,U)} && (7) \end{aligned}$$

The forcing term $\cos(\lambda t)$ is placed in the constant operator, similarly to the periodic version of the method (see [5], example 4).

The continuation of the quasi-periodic solution branch can be performed efficiently and precisely : in this example, Fourier series were truncated with $H = 5$. A plot of L^2 norm of x is shown in figure 2.2. Dots indicate the beginning of each continuation step : the ANM provides smooth continuation with an automatic step size determination. The solution obtained through this quasi-periodic Harmonic Balance is qualitatively good with $H = 2$ (figure 2.2, left) : peak-to-peak amplitude and general shape of the curve in the phase space are reached. But areas left blank are not actually correct, while with $H = 5$ (right), its pointwise agreement with a time integration scheme [7] is excellent.

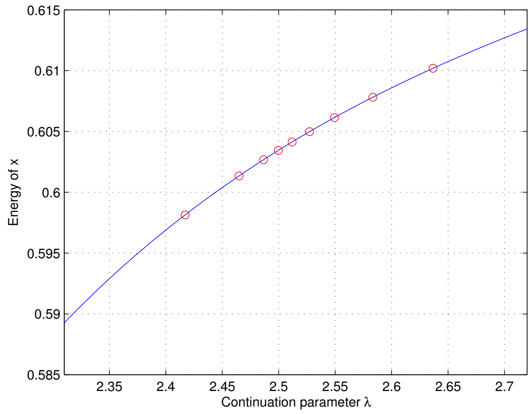

Figure 1: Energy (L^2 norm) of x with respect to the continuation parameter λ. Red dots indicate the beginning of each continuation step.

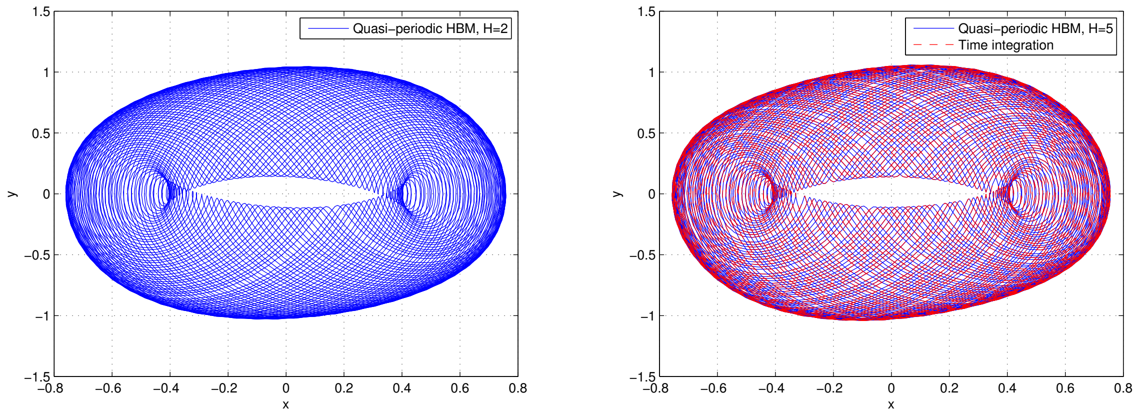

Figure 2: Phase diagrams (x, y), for $\lambda = 2.58$. Left : quasi-periodic Harmonic Balance, $H = 2$. Right : comparison of quasi-periodic Harmonic Balance ($H = 5$, blue solid line) and time integration (red dashed line).

2.3 Autonomous system

The equations used for two coupled Van der Pol oscillators are :

$$x_1'' + a_1 x_1' + \Omega_1^2 x_1 = a_2\lambda(x_1' + x_2') - a_3\lambda(x_1' + x_2')(x_1 + x_2) - a_4\lambda(x_1' + x_2')(x_1 + x_2)^2 \quad (8)$$
$$x_2'' + b_1 x_2' + \Omega_2^2 x_2 = b_2\lambda(x_1' + x_2') - b_3\lambda(x_1' + x_2')(x_1 + x_2) - b_4\lambda(x_1' + x_2')(x_1 + x_2)^2 \quad (9)$$

and the quadratic formulation is :

$$
\begin{aligned}
x_1' &= y_1 + 0 & &+0 \quad (10)\\
y_1' &= 0 \; - a_1 y_1 - \Omega_1^2 x_1 + \lambda\,(a_2(y_1+y_2) - a_3 v - a_4 w) & &+0 \quad (11)\\
x_2' &= y_2 + 0 & &+0 \quad (12)\\
y_2' &= 0 \; - b_1 y_2 - \Omega_2^2 x_2 + \lambda\,(b_2(y_1+y_2) - b_3 v - b_4 w) & &+0 \quad (13)\\
0 &= 0 \; + r & &-(x_1+x_2)^2 \quad (14)\\
0 &= 0 \; + v & &-(x_1+x_2)(y_1+y_2) \quad (15)\\
0 &= 0 \; + w & &-r(y_1+y_2) \quad (16)
\end{aligned}
$$

Parameters values are chosen as : $\Omega_1 = 1$, $a_1 = 0.01$, $a_2 = 0.5$, $a_3 = a_4 = 2$; $\Omega_2 = 2.5$, $b_1 = 0.025$, $b_2 = 1$, $b_3 = b_4 = 4$. For these equations, the quasi-periodic solution branch requires higher orders of truncation of Fourier series H than the forced Van der Pol above. For example, around $\lambda = 0.36$, a good agreement with time integration is achieved with $H = 10$ (figure 2.3), and some differences are noticeable if H is too low (areas left empty are not correct with $H = 4$). For this system the continuation process takes roughly 6 seconds per step for $H = 4$, 43 seconds for $H = 6$, 220 seconds for $H = 8$.

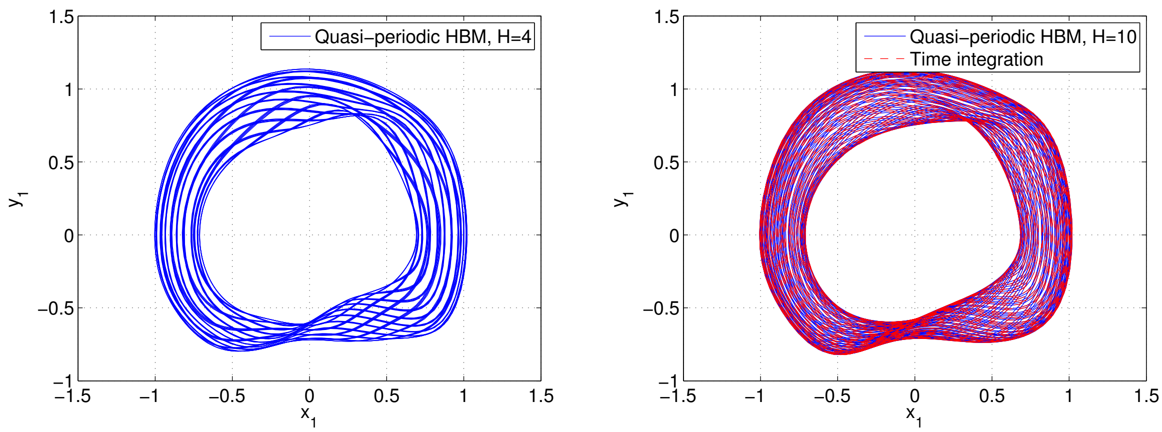

Figure 3: Phase diagrams (x_1, y_1), for $\lambda = 0.36$. Left : quasi-periodic Harmonic Balance, $H = 4$. Right : comparison of quasi-periodic Harmonic Balance ($H = 10$, blue solid line) and time integration (red dashed line).

3 CONCLUSION

The coupling of two-frequencies harmonic balance with the Asymptotic Numerical Method, a robust continuation technique, is performed automatically thanks to the quadratic framework. It proves very efficient to continue quasi-periodic solutions, without any *a priori* knowledge nor optimization. As one could expect, better accuracy is obtained using more Fourier coefficients, and comparison with time integration is successful. Future works will focus on musical examples, though this method is relevant for many nonlinear systems.

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